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A Study on $(\lambda - \mu)$ Zweier Sequences and Their Behaviour in Neutrosophic Normed Spaces

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Abstract. Ideal convergence of sequences in neutrosophic normed spaces is defined by Ömer Kişi [12]. This paper defines new sequence spaces using the Zweier matrix and neutrosophic norm. We explore (λ, μ)-Zweier convergence, (λ, μ) ideal convergence of double sequences in neutrosophic norm. We show that a double sequence that is (λ, μ)-Zweier convergent or ideal convergent has a unique limit with respect to neutrosophic norm. Additionally, we prove that (λ, μ)-Zweier ideal convergence is equivalent to (λ, μ)-Zweier ideal Cauchy for a double sequence in neutrosophic normed space.

Keywords: Neutrosophic normed space, Statistical convergence, $(\lambda - \mu)$ Zweier convergence, I-convergence.

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced by Lotfi A. Zadeh [24] in 1965 as a mathematical framework to deal with uncertainty and vagueness in data. In a fuzzy set, each element is assigned a membership value ranging from 0 to 1, indicating the degree of membership of that element in the set. Kaleva, O. and Seikkala, S. (1984) [7] defined fuzzy metric space and Felbin, C. (1992) [4], Bag, T. and Samanta, S. K. (2003) [2] studied fuzzy normed linear space. Krassimir Atanassov [1] generalized the notion of fuzzy set to intuitionistic fuzzy set. Park [19] defined intuitionistic fuzzy metric space In 1995, Florentin Smarandache [23], [22] introduced the concept of neutrosophic sets as an extension of intuitionistic fuzzy sets. Neutrosophic sets aim to handle three types of indeterminacy: membership, non-membership and indeterminacy, represented by the values of truth-membership, falsehood-membership, and indeterminacy-membership respectively. Kirişci, M. and Şimşek, N. [10], [11] defined metric on neutrosophic set and also studied statistical convergence.

Concept of Statistical convergence was given H. Fast [5]. λ statistical convergence was introduced by Mursaleen [16]. λ statistical convergence is the extension of $[V, \lambda]$ summability [14]. Mursaleen et al. [17] defined the concept of (λ, μ) convergence for double sequence. The I-convergence [13] gives a unifying look on several types of convergence related to the statistical convergence. Mursaleen and Mohiuddine [18] defined I-convergence for double sequences in intuitionistic fuzzy normed spaces. In neutrosophic normed spaces, statistically convergent and statistically Cauchy double sequences are defined and studied by Granados C and Dhital [6]. Khan V. A. and Faisal M. [8], Khan V. A. and Ahmad M. [9] defined (λ, μ) Zweier ideal convergence on neutrosophic normed space and studied Cesaro summability in neutrosophic normed spaces respectively.. Zweier-Verfahren is a German word, in which Zwei means two and Verfahren means method. Zweier matrix or operator is denoted as $\mathfrak{Z}^{\varrho}$, where $\varrho \neq 1$. Let $x = (\xi_n) \in \omega$ where $\omega = \{x = (\xi_n) : \xi_n \in \mathbb{R}/\mathbb{C}\}$, then $\mathfrak{Z}^{\varrho}$ is defined as

$$\mathfrak{Z}^{\varrho} = \mathfrak{Z}_{ik} = \begin{cases} \varrho, & i = k \\ 1 - \varrho, & i - 1 = k \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Şengönül [21], define new sequence $y = (y_n)$ as $\mathfrak{Z}^{\varrho}$ transform of the sequence $x = (\xi_n)$,

$$y_n = \mathfrak{Z}^{\varrho}(\xi_n) = \varrho\xi_n + (1 - \varrho)\xi_{n-1} \tag{2}$$

where $\xi_{-1} = 0$ and $\varrho \neq 1$. Matrix

$$\mathfrak{Z}^{\varrho} = \begin{bmatrix} \varrho & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots \\ 1-\varrho & \varrho & 0 & 0 & 0 & 0 & 0 & \ldots\ldots \\ 0 & 1-\varrho & \varrho & 0 & 0 & 0 & 0 & \ldots\ldots \\ 0 & 0 & 1-\varrho & \varrho & 0 & 0 & 0 & \ldots\ldots \\ . & . & 0 & 1-\varrho & \varrho & 0 & 0 & \ldots\ldots \\ . & . & . & . & . & . & . & \ldots\ldots \\ . & . & . & . & . & . & . & \ldots\ldots \\ . & . & . & . & . & . & . & \ldots\ldots \end{bmatrix}.$$

Definition 1.1. [20] A double sequence $x = (\xi_{gh})$ is said to be convergent(Pringsheim's Sense) to $\mathfrak{p} > 0$, if for every $\kappa > 0$, there exist $N \in \mathbb{N}$ such that

$$|\ \xi_{gh} - \mathfrak{p}\ | < k, \text{ whenever } g, h \geq N.$$

Definition 1.2. [20] A double sequence $x = (\xi_{gh})$ is said to be Cauchy (Pringsheim's Sense) , if for every $\kappa > 0$ there exist $\mathfrak{N} \in \mathbb{N}$ such that

$$|\ \xi_{gh} - \xi_{pq}\ | < \kappa, \text{ whenever } g \geq p \geq \mathfrak{N} \text{ and } h \geq q \geq \mathfrak{N}.$$

Definition 1.3. Let $\mathcal{K} \subseteq \mathbb{N}\times\mathbb{N}$, if the sequence $\dfrac{|\ \mathcal{K}(g,h)\ |}{nm}$ is convergent (Pringsheim's Sense), then we say that $\mathcal{K}$ has double natural density and defined as

$$\delta_2(\mathcal{K}) = (P)\lim_{g,h} \frac{|\ \mathcal{K}(g,h)\ |}{gh}$$

where $\mathcal{K}(g,h) = \{(p,q) \in \mathbb{N}\times\mathbb{N} : p \leq g \text{ and } q \leq h\}$. If, $g = h$ Christopher's [3] two-dimensional natural density is obtained.

Definition 1.4. [20] A double sequence $x = (\xi_{gh})$ is said to be statistically convergent to $\mathfrak{p} > 0$, if for every $\kappa > 0$,

$$\mathcal{A} = \{(g,h) : g \leq n, h \leq m \text{ such that } \ |\ \xi_{gh} - \mathfrak{p}\ |\geq \kappa\}$$

has $\delta_2(\mathcal{A}) = 0$.

Definition 1.5. [20] A double sequence $x = (\xi_{gh})$ is said to be statistically Cauchy, if for every $\kappa > 0$, there exist $\mathfrak{N} = \mathfrak{N}(\kappa)$ and $\mathfrak{M} = \mathfrak{M}(\kappa)$ such that for all $g, p \geq \mathfrak{N}$ and $h, q \geq \mathfrak{M}$ the set

$$\mathcal{A} = \{(g,h) : g \leq \mathfrak{N}, h \leq \mathfrak{M} \text{ such that } \ |\ \xi_{gh} - \xi_{pq}\ |\geq \kappa\}$$

has $\delta_2(\mathcal{A}) = 0$.

Definition 1.6. [15] If continuous mapping $\star : [0,1] \times [0,1] \longrightarrow [0,1]$ meets the following requirements, it is called a continuous $t - norm$:

(a) $\star(\mathfrak{y},\mathfrak{z}) = \star(\mathfrak{z},\mathfrak{y})$ and $\star\big(\mathfrak{y},\star(\mathfrak{z},\mathfrak{c})\big) = \star\big(\star(\mathfrak{y},\mathfrak{z}),\mathfrak{c}\big)$, for all $\mathfrak{y},\mathfrak{z},\mathfrak{c} \in [0,1]$,

(b) $\star(\mathfrak{y},1) = \mathfrak{y}$, $\forall\ \mathfrak{y} \in [0,1]$,

(c) $\mathfrak{y} \leq \mathfrak{c}$ and $\mathfrak{z} \leq \mathfrak{d} \Longrightarrow \star(\mathfrak{y},\mathfrak{z}) \leq \star(\mathfrak{c},\mathfrak{d})$, for each $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{c}$, $\mathfrak{d} \in [0,1]$.

Definition 1.7. [15] If continuous mapping $\bullet : [0,1] \times [0,1] \longrightarrow [0,1]$ meets the following requirements, it is called continuous $t - conorm$ if:

(a)$\bullet(\mathfrak{y},\mathfrak{z}) = \bullet(\mathfrak{z},\mathfrak{y})$ and $\bullet\big(\mathfrak{y},\bullet(\mathfrak{z},\mathfrak{c})\big) = \bullet\big(\bullet(\mathfrak{y},\mathfrak{z}),\mathfrak{c}\big)$, for all $\mathfrak{y},\mathfrak{z},\mathfrak{c} \in [0,1]$

(b) $\bullet(\mathfrak{y},0) = \mathfrak{y}$, $\forall\ \mathfrak{y} \in [0,1]$,

(c) $\mathfrak{y} \leq \mathfrak{c}$ and $\mathfrak{y} \leq \mathfrak{d} \Longrightarrow \bullet(\mathfrak{y},\mathfrak{y}) \leq \bullet(\mathfrak{c},\mathfrak{d})$ for each $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{c}$, $\mathfrak{d} \in [0,1]$.

Definition 1.8. [11] Let V, $\star$, and $\bullet$ be linear space, continuous $t - norm$ and continuous t-conorm respectively. A four tuple of the form $\{V, \phi(z,.), \psi(z,.), \gamma(z,.) : z \in V\}$, is called neutrosophic normed space where ϕ, ψ, and γ are fuzzy sets on $V \times \mathbb{R}^+$ which satisfy the following conditions:

(i) $0 \leq \phi(\nu,p), \psi(\nu,p), \gamma(\nu,p) \leq 1$ for all $p \in \mathbb{R}^+$,

(ii) $0 \leq \phi(\nu,p) + \psi(\nu,p) + \gamma(\nu,p) \leq 3$ for all $p \in \mathbb{R}^+$,

(iii) $\phi(\nu,p) = 1$ (for $p > 0$) if and only if $\nu = 0$,

(iv) $\phi(\alpha\nu, p) = \phi(\nu, \frac{p}{|\alpha|})$ for $\alpha \neq 0$

(v) $\star\big(\phi(\nu, p), \phi(y, q)\big) \leq \phi(\nu + y, p + q)$

(vi) $\phi(\nu, .)$ is continuous and non-decreasing

(vii) $\lim_{p\to\infty} \phi(\nu, p) = 1,$

$(viii)$ $\psi(\nu, p) = 0$ for $(p > 0)$ if and only if $\nu = 0,$

(ix) $\psi(\alpha\nu, p) = \psi(\nu, \frac{p}{|\alpha|})$ for $\alpha \neq 0,$

(x) $\bullet\big(\psi(\nu, p), \psi(y, q)\big) \geq \psi(\nu + y, p + q)$

(xi) $\psi(\nu, .)$ is continuous and non-increasing,

(xii) $\lim_{p\to\infty} \psi(\nu, p) = 0,$

$(xiii)$ $\gamma(\nu, p) = 0$ (for $p > 0$) if and only if $\nu = 0,$

(xiv) $\gamma(\alpha\nu, p) = \gamma(\nu, \frac{p}{|\alpha|})$ if $\alpha \neq 0,$

(xv) $\bullet\big(\gamma(\nu, p), \gamma(y, q)\big) \geq \gamma(\nu + y, p + q),$

(xvi) $\gamma(\nu, .)$ is continuous and non-increasing,

$(xvii)$ $\lim_{p\to\infty} \gamma(\nu, p) = 0,$

$(xviii)$ If $p \leq 0$ then $\phi(\nu, p) = 0, \psi(\nu, p) = 1,$ and $\gamma(\nu, \mathbf{p}) = 1$

Then $\mathcal{N} = \big(\phi, \psi, \gamma\big)$ is called neutrosophic norm. Throughout the paper we will use usual t-norm and usual t-conorm i.e; $\star(\tau, \varrho) = min\{\tau, \varrho\}$ and $\bullet(\tau, \varrho) = max\{\tau, \varrho\}$.

Example 1.9. 11 Let $\big(V, ||.||\big)$ be a normed space. Let ϕ, ψ, γ be Fuzzy sets on $V \times \mathbb{R}^+$ such that, for $\mathfrak{t} \geq ||\mathfrak{V}||$

$$\phi(\mathfrak{V}, \mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} \leq 0 \\ \frac{\mathfrak{t}}{\mathfrak{t}+||\mathfrak{V}||}, & \mathfrak{t} > 0, \end{cases} \quad \psi(\mathfrak{V}, \mathfrak{t}) = \begin{cases} 0, & \mathfrak{t} \leq 0 \\ \frac{||\mathfrak{V}||}{\mathfrak{t}+||\mathfrak{V}||}, & \mathfrak{t} > 0, \end{cases} \text{ and } \gamma(\mathfrak{V}, \mathfrak{t}) = \frac{||\mathfrak{V}||}{\mathfrak{t}}$$

for all $\mathfrak{V} \in V$ and $\mathfrak{t} \geq 0$. If $\mathfrak{t} \leq ||\mathfrak{V}||$ then $\phi(\mathfrak{V}, \mathfrak{t}) = 0$, $\psi(\mathfrak{V}, \mathfrak{t}) = 1$ and $\gamma(\mathfrak{V}, \mathfrak{t}) = 1$. Then $\big(V, \mathcal{N}, \star, \bullet\big)$ is neutrosophic normed space.

Now we will discuss about the convergence of the sequence (ς_k) in neutrosophic normed space $\big(V, \mathcal{N}, \star, \bullet\big)$.

Definition 1.10. [11] Let (ξ_g) be sequence in NNS $\big(V, \mathcal{N}, \star, \bullet\big)$ Then (ξ_g) is said to be convergent to $\mathfrak{p} \in V$ if for each $t > 0$ and $\kappa \in$ (0,1) there exists $g_0 \in \mathbb{N}$ such that

$$\phi(\xi_g - \mathfrak{p}, t) > 1 - \kappa,\ \psi(\xi_g - \mathfrak{p}, t) < \kappa \text{ and } \gamma(\xi_n - \mathfrak{p}, t) < c \tag{3}$$

for all $g \geq g_0$.

Definition 1.11. [11] Let (ξ_g) be sequence in NNS $\big(V, \mathcal{N}, \star, \bullet\big)$ Then (ξ_g) is said to be Cauchy if, for each $t > 0$ and $\kappa \in$ (0,1) there exists $g_0 \in \mathbb{N}$ such that

$$\phi(\xi_k - \xi_h, t) > 1 - \kappa,\ \psi(\xi_k - \xi_h, t) < \kappa \text{ and } \gamma(\xi_k - \xi_h, t) < \kappa \tag{4}$$

for all $h, k \geq g_0$.

Definition 1.12. [6] Let $(V, \phi, \psi, \gamma, \star, \bullet)$ be a NNS. A double sequence $x = (\xi_{gh})$ in V is said to be convergent statistically to $\mathfrak{p}$ if, for each $\kappa > 0$ and $t > 0$

$$\delta_2\{(g,h) \in \mathbb{N} \times \mathbb{N} : \phi(\xi_{gh} - \mathfrak{p}, t) \leq 1 - \kappa,\ \psi(\xi_{gh} - \mathfrak{p}, t) \geq \kappa \text{ and } \gamma(\xi_{gh} - \mathfrak{p}, t) \geq \kappa\} = 0.$$

Definition 1.13. Let $\Upsilon = (\Upsilon_g)$ and $\Pi = (\Pi_h)$ be two positive non-decreasing sequences and $\lim_{g\to\infty} \Upsilon_g = \infty$, $\lim_{h\to\infty} \Pi_h = \infty$, defined as

$$\Upsilon_{g+1} \leq \Upsilon_g + 1, \qquad \text{where } \Upsilon_1 = 0. \tag{5}$$

$$\Pi_{h+1} \leq \Pi_h + 1, \qquad \text{where } \mu_1 = 0. \tag{6}$$

Let $I_g = [g - \Upsilon_g + 1, g]$ and $I_h = [h - \Pi_h + 1, h]$. Then

$$\delta_{\Upsilon,\Pi}(\mathcal{K}) = \lim_{g,h\to\infty} \frac{1}{\Upsilon_g \mu_h} \Big| \{(i,j) \in \mathbb{N} \times \mathbb{N} : (i,j) \in \mathcal{K}\} \Big|$$

is the (Υ, Π) density of the set $\mathcal{K} \subseteq \mathbb{N} \times \mathbb{N}$. If $\Upsilon_g = g$ and $\Pi_h = h$, the (Υ, Π) density reduces to natural double density of $\mathcal{K}$. The generalized double Valée Poussin mean is

$$t_{g,h} = \frac{1}{\Upsilon_g \Pi_h} \sum_{i\in I_g} \sum_{j \in I_h} \xi_{ij}.$$

Throughout the paper we will denote the sequences 5,6 by $\lambda = (\lambda_i)$ and $\mu = (\mu_i)$ respectively.

2. Main Results

Definition 2.1. Let $(V, \phi, \psi, \gamma, \star, \bullet)$ be a NNS and I^2 be an admissible ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (\xi_{mn})$ in V is said to be I^2- convergent to ξ if, for each $\epsilon > 0$ and $t > 0$ the set

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \phi(\xi_{mn} - \xi, t) \leq 1 - \epsilon,\ \psi(\xi_{mn} - \xi, t) \geq \epsilon \text{ and } \gamma(\xi_{mn} - \xi, t) \geq \epsilon\} \in I^2.$$

Definition 2.2. Let $(V, \phi, \psi, \gamma, \star, \bullet)$ be a NNS. A double sequence $x = (\xi_{mn})$ in V is said to be (λ, μ)- Zweier convergent to ξ with respect to neutrosophic norm (ϕ, ψ, γ) if, for every $\epsilon > 0$ and $t > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \leq 1 - \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon$$

for all $m, n \geq n_0$ and write it as $(\lambda, \mu) - \lim_{m,n\to\infty} \mathcal{Z}^q \xi_{mn} = \xi$.

Definition 2.3. Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS and I^2 is an admissible ideal. A double sequence $x = (\xi_{mn})$ in V is said to be (λ, μ)- Zweier ideal convergent to ξ with respect to neutrosophic norm (ϕ, ψ, γ), if for every $\epsilon > 0$ and for all $t > 0$, the set

$$\Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \leq 1 - \epsilon, \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon,$$
$$\text{and } \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon, \Big\} \in I^2.$$

In short we write $(\phi, \psi, \gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n \to \infty} \mathcal{Z}^q \xi_{mn} = \xi$

Definition 2.4. Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS. A double sequence $x = (\xi_{mn})$ in V is said to be (λ, μ)- Zweier Cauchy sequence with respect to neutrosophic norm (ϕ, ψ, γ), if for every $\epsilon > 0$ and for all $t > 0$, there exists N such that

$$\frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \leq 1 - \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon.$$

for all $m, n, k, p \geq N$.

Definition 2.5. Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS and I^2 is an admissible ideal. A double sequence $\xi = (\xi_{mn})$ in V is said to be (λ, μ)-Zweier ideal Cauchy sequence if for each $\epsilon > 0$ and $t > 0$ there exists a positive integer N such that

$$\Big\{(i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \leq 1 - \epsilon,$$
$$\frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon,$$
$$\text{and } \frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon \Big\} \in I^2.$$

Lemma 2.6. *Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS and $I^2 \subseteq \mathbb{N} \times \mathbb{N}$ be an admissible ideal. Let $x = (\xi_{mn})$ be a double sequence in V, then the following are equivalent*

(i) $(\phi, \psi, \gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n \to \infty} \mathcal{Z}^q(\xi_{mn}) = \mathcal{L}$.

(ii) $\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big) \leq 1-\kappa\right\} \in I^2$,
$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big) \geq \kappa\right\} \in I^2$ *and*
$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big) \geq \kappa\right\} \in I^2$ *for all* $\kappa > 0$ *and* $t > 0$.

(iii) $\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big) > 1-\kappa\right\} \in F(I^2)$,
$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big) < \kappa\right\} \in F(I^2)$ *and*
$\left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t\big) < \kappa\right\} \in F(I^2)$ *for all* $\kappa > 0$ *and* $t > 0$.

(iv) $(\phi,\psi,\gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n\to\infty}\phi(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t) = 1$,
$(\phi,\psi,\gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n\to\infty}\phi(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t) = 0$ *and*
$(\phi,\psi,\gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n\to\infty}\phi(\mathcal{Z}^q\xi_{mn}-\mathcal{L},t) = 0$ *for all* $\kappa > 0$ *and* $t > 0$.

Theorem 2.7. *Let* $(V,\mathcal{N},\star,\bullet)$ *be a NNS and* I^2 *is an admissible ideal. If a double sequence* $x = (\xi_{mn})$ *in* V *is* (λ,μ)*- Zweier ideal convergent with respect to neutrosophic norm* (ϕ,ψ,γ)*, then its limit is unique.*

Proof. We will prove it by contradiction i.e. Let on contrary that the sequence $x = (\xi_{mn})$ converges to two limits (say) ℓ_1 and ℓ_2. By definition 2.3, for a given $\epsilon > 0$, choose $\kappa > 0$ such that $(1-\kappa)\star(1-\kappa) > 1-\epsilon$ and $\kappa\star\kappa < \epsilon$. Then for $t > 0$

$$A_{\phi,1}(\kappa,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right) \leq 1-\kappa\right\} \in I^2.$$

$$A_{\phi,2}(\kappa,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right) \leq 1-\kappa\right\} \in I^2.$$

$$A_{\psi,1}(\kappa,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right) \geq \kappa\right\} \in I^2.$$

$$A_{\psi,2}(\kappa,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right) \geq \kappa\right\} \in I^2.$$

$$A_{\gamma,1}(\kappa,t) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right) \geq \kappa\right\} \in I^2.$$

$$A_{\gamma,2}(\kappa,t)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right)\geq\kappa\right\}\in I^2.$$

Now the following set

$$A_{\phi,\psi,\gamma}(\kappa,t)=\Big[A_{\phi,1}(\kappa,t)\cup A_{\phi,2}(\kappa,t)\Big]\cap\Big[A_{\psi,1}(\kappa,t)\cup A_{\psi,2}(\kappa,t)\Big]\cap\Big[A_{\gamma,1}(\kappa,t)\cup A_{\gamma,2}(\kappa,t)\Big]\in I^2$$

$$A_{\phi,\psi,\gamma}(\kappa,t)\in I^2\implies A^{\complement}_{\phi,\psi,\gamma}(\kappa,t)\in F(I^2).$$

If $(i,j)\in A^{\complement}_{\phi,\psi,\gamma}(\kappa,t)$, then there are three cases. Either $(i,j)\in A^{\complement}_{\phi,1}(\kappa,t)\cap A^{\complement}_{\phi,2}(\kappa,t)$ or $(i,j)\in A^{\complement}_{\psi,1}(\kappa,t)\cap A^{\complement}_{\psi,2}(\kappa,t)$ or $(i,j)\in A^{\complement}_{\gamma,1}(\kappa,t)\cap A^{\complement}_{\gamma,2}(\kappa,t)$. Firstly, if $(i,j)\in A^{\complement}_{\phi,1}(\kappa,t)\cap A^{\complement}_{\phi,2}(\kappa,t)$, we have

$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right)>1-\kappa$$

$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right)>1-\kappa$$

Now take $(a,b)\in\mathbb{N}\times\mathbb{N}$, such that

$$\phi\left(\mathcal{Z}^q\xi_{ab}-\ell_1,\frac{t}{2}\right)>\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\right)>1-\kappa$$

and

$$\phi\left(\mathcal{Z}^q\xi_{ab}-\ell_2,\frac{t}{2}\right)>\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\left(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\right)>1-\kappa.$$

Select (a,b) for (m,n) such that the maximum hold i.e. max $\Big\{\phi\Big(\mathcal{Z}^q\xi_{mn}-\ell_1,\frac{t}{2}\Big),\psi\Big(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\Big),\gamma\Big(\mathcal{Z}^q\xi_{mn}-\ell_2,\frac{t}{2}\Big):m\in\Lambda_i,n\in\Lambda_j\Big\}$. Then, we have

$$\phi(\ell_1-\ell_2,t)\geq\phi\Big(\mathcal{Z}^q\xi_{ab}-\ell_1,\frac{t}{2}\Big)\star\phi\Big(\mathcal{Z}^q\xi_{ab}-\ell_2,\frac{t}{2}\Big)>(1-\kappa)\star(1-\kappa)>1-\epsilon$$

Since $\epsilon>0$ was arbitrary, for every $t>0$, we get $\phi\Big(\ell_1-\ell_2,t\Big)=1$, that means $\ell_1=\ell_2$. For other cases if, $(i,j)\in A^{\complement}_{\psi,1}(\kappa,t)\cap A^{\complement}_{\psi,2}(\kappa,t)$ or $(i,j)\in A^{\complement}_{\gamma,1}(\kappa,t)\cap A^{\complement}_{\gamma,2}(\kappa,t)$, in similar fashion we get $\psi(\ell_1-\ell_2,t)>\epsilon$ and $\gamma(\ell_1-\ell_2,t)>\epsilon$ for $t>0$. Hence for each case, we get $\ell_1=\ell_2$.Hence limit is unique. □

Theorem 2.8. *Let $(V,\mathcal{N},\star,\bullet)$ be a NNS. If a double sequence $x=(\xi_{mn})$ in V is (λ,μ)- Zweier convergent with respect to neutrosophic norm (ϕ,ψ,γ), then its limit is unique.*

Proof. Let $(\lambda, \mu) - \lim_{m,n\to\infty} \mathcal{Z}^q \xi_{mn} = \xi_1$ and $(\lambda, \mu) - \lim_{m,n\to\infty} \mathcal{Z}^q \xi_{mn} = \xi_2$. Given $\epsilon > 0$, select $\kappa > 0$ such that $(1-\kappa) \star (1-\kappa) > 1-\epsilon$ and $\kappa \bullet \kappa < \epsilon$. Then for any $t > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n\in\Lambda_i} \sum_{m\in\Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi_1, t\big) > 1-\epsilon$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n\in\Lambda_i} \sum_{m\in\Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \xi_1, t\big) < \epsilon$$

and

$$\frac{1}{\lambda_i \mu_j} \sum_{n\in\Lambda_i} \sum_{m\in\Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \xi_1, t\big) < \epsilon$$

for all $i, j \geq n_1$. There also exists n_2 such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n\in\Lambda_i} \sum_{m\in\Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi_2, t\big) > 1-\epsilon$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n\in\Lambda_i} \sum_{m\in\Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \xi_2, t\big) < 1-\epsilon$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n\in\Lambda_i} \sum_{m\in\Lambda_j} \lambda\big(\mathcal{Z}^q \xi_{mn} - \xi_2, t\big) < \epsilon$$

for all $i, j \geq n_2$. Now chose $n_0 = max\{n_1, n_2\}$. Then for all $n \geq n_0$, we have$(a, b) \in \mathbb{N} \times \mathbb{N}$ such that

$$\phi\big(\mathcal{Z}^q \xi_{ab} - \xi_1, t\big) > \frac{1}{\lambda_i \mu_j} \sum_{n\in\Lambda_i} \sum_{m\in\Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi_1, t\big) > 1-\kappa$$

. and

$$\phi\big(\mathcal{Z}^q \xi_{ab} - \xi_2, t\big) > \frac{1}{\lambda_i \mu_j} \sum_{n\in\Lambda_i} \sum_{m\in\Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi_2, t\big) > 1-\kappa.$$

Hence, we get

$$\phi(\xi_1 - \xi_2, t) \geq \phi\big(\mathcal{Z}^q \xi_{mn} - \xi_1, t\big) \star \phi\big(\mathcal{Z}^q \xi_{mn} - \xi_2, t\big) > (1-\kappa) \star (1-\kappa) > 1-\epsilon.$$

Since ϵ was arbitrary. Hence we get $\phi(\xi_1 - \xi_2, t) = 1$ for all $t > 0$. Hence $\xi_1 = \xi_2$. Similarly, we will prove for ψ and γ that $\psi(\xi_1 - \xi_2, t) < \epsilon$ and $\gamma(\xi_1 - \xi_2, t) < \epsilon$ for any $\epsilon > 0$. Hence we get $\psi(\xi_1 - \xi_2, t) = 0$ and $\gamma(\xi_1 - \xi_2, t) = 0$ for $t > 0$, this implies that $\xi_1 = \xi_2$. □

Theorem 2.9. *Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS. (λ, μ)-Zweier convergence of a double sequence $x = (\xi_{mn})$ in V implies (λ, μ)-Zweier ideal convergence of $x = (\xi_{mn})$ with respect to neutrosophic norm and $(\lambda, \mu) - \lim_{m,n\to\infty} \mathcal{Z}^q \xi_{mn} = (\phi, \psi, \gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n\to\infty} \mathcal{Z}^q \xi_{mn} = \xi$.*

Proof. Let sequence $x = (\xi_{mn})$ is (λ, μ)- Zweier convergent to ξ. Then for $\epsilon > 0$ and $t > 0$, there exists positive integer N such that

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) > 1 - \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) < \epsilon,$$

$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) < \epsilon$$

for all $i, j \geq N$. Then the set

$$R(\epsilon, t) = \Bigg\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m \in \Lambda_i} \sum_{n \in \Lambda_j} \phi\Bigg(\mathcal{Z}^q \xi_{mn} - \xi, t \Bigg) \leq 1 - \epsilon,$$
$$\frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon, \text{and } \frac{1}{\lambda_i \mu_j} \sum_{n \in \Lambda_i} \sum_{m \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \xi, t\big) \geq \epsilon, \Bigg\}.$$

is contained in the set $Q = \{(1,1), (1,2), (2,1), (2,2), ..., (N-1, N-1)\}$. Since I^2 is an admissible ideal. We get, $R(\epsilon, t) \subseteq Q \in I^2$. Hence (λ, μ)-Zweier convergence of a double sequence implies (λ, μ)-Zweier ideal convergence and $(\phi, \psi, \gamma) - I^2_{(\lambda,\mu)} - \lim_{m,n \to \infty} \mathcal{Z}^q \xi_{mn} = \xi$. □

Theorem 2.10. *Let $(V, \mathcal{N}, \star, \bullet)$ be a NNS. A double sequence $x = (\xi_{mn})$ in V is (λ, μ)-Zweier ideal convergent if and only if it is (λ, μ)-Zweier ideal Cauchy.*

Proof. Let $x = (\xi_{mn})$ in V is (λ, μ)-Zweier ideal Cauchy sequence but not (λ, μ)-Zweier ideal convergent in V with respect to neutrosophic norm (ϕ, ψ, γ). Now, by the definition of (λ, μ)-Zweier ideal Cauchy for each $\epsilon > 0$ and $t > 0$, there exists positive integer N such that

$$G(\epsilon, t) = \Bigg\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \phi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \leq 1 - \epsilon,$$
$$\frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \psi\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon,$$
$$\text{and } \frac{1}{\lambda_i \mu_j} \sum_{m,k \in \Lambda_i} \sum_{n,p \in \Lambda_j} \gamma\big(\mathcal{Z}^q \xi_{mn} - \mathcal{Z}^q \xi_{kp}, t\big) \geq \epsilon \Bigg\} \in I^2.$$

and

$$H(\epsilon,t)=\Bigg\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\leq 1-\epsilon,$$
$$\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\geq\epsilon,\ \text{and}$$
$$\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\geq\epsilon\Bigg\}\in F(I^2)$$

Since

$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\geq\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)>1-\epsilon,$$

$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\leq\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<\epsilon,$$

and

$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\leq\frac{2}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<\epsilon.$$

If

$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)>\frac{1-\epsilon}{2},\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<\frac{\epsilon}{2},$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<\frac{\epsilon}{2}.$$

Then, we have

$$\delta_(\lambda,\mu)\Big(\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\leq 1-\epsilon,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon,$$
$$\text{and}\ \frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon\Big\}\Big)=0$$

That is $G(\epsilon,t)\in F(I^2)$, which is a contradiction. Hence, sequence $\xi=(\xi_{mn})$ is (λ,μ)-Zweier ideal convergent with respect to neutrosophic norm.

Conversely, suppose that $(\phi,\psi,\gamma)-I^2_{(\lambda,\mu)}-\lim_{m,n\to\infty}\mathcal{Z}^q\xi_{mn}=\xi$. Choose $r>0$, such that $(1-r)\star(1-r)>1-\epsilon$ and $r\bullet r<\epsilon$.For $t>0$ define

$$W(r,t)=\Bigg\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\leq 1-r,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\geq r,\text{ and}$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m\in\Lambda_i}\sum_{n\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)\geq r\Bigg\}\in I^2$$

or $W(r,t)^{\complement}\in F(I^2)$. Assume $(k,p)\in W(r,t)^{\complement}$. Then, we get

$$\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)>1-r$$
$$\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<r$$
$$\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\gamma\Big(\mathcal{Z}^q\xi_{mn}-\xi,\frac{t}{2}\Big)<r$$

For every $\epsilon>0$, we take

$$U(\epsilon,t)=\Bigg\{(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\leq 1-\epsilon,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon,$$
$$\text{and }\frac{1}{\lambda_i\mu_j}\sum_{m,k\in\Lambda_i}\sum_{n,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{mn}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon\Bigg\}\in I^2.$$

Now we will show that $U(\epsilon,t)\subset W(r,t)$. Let $(u,v)\in U(\epsilon,t)$

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)\leq 1-\epsilon,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon,$$
$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon.$$

Here we are going to divide it into three cases as follows

Case1 :

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)\leq 1-\epsilon.$$

Then

$$\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)\leq 1-r \implies (u,v)\in W(r,t).$$

Otherwise, if $\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)>1-r$. Then, we have

$$1-\epsilon\geq\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)$$

$$\geq\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)\star\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{kp}-\xi,\frac{t}{2}\Big)$$

$$>(1-r)\star(1-r)$$

$$>1-\epsilon.$$

We reach at a contradiction. Hence $U(\epsilon,t)\subset W(r,t)$.

Case2 : Consider

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\psi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon.$$

We get

$$\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)\geq r \implies (u,v)\in W(r,t).$$

Otherwise, if

$$\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\psi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)<r.$$

Then we obtain

$$1-\epsilon\leq\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\phi\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)$$

$$\leq\frac{1}{\lambda_i\mu_j}\sum_{u\in\Lambda_i}\sum_{v\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{uv}-\xi,\frac{t}{2}\Big)\bullet\frac{1}{\lambda_i\mu_j}\sum_{k\in\Lambda_i}\sum_{p\in\Lambda_j}\phi\Big(\mathcal{Z}^q\xi_{kp}-\xi,\frac{t}{2}\Big)$$

$$<(1-r)\bullet(1-r)$$

$$<1-\epsilon.$$

Which is a contradiction. Hence, $U(\epsilon,t)\subset W(r,t)$.

Case 3 is similar to the case 2 , in this case we will consider that

$$\frac{1}{\lambda_i\mu_j}\sum_{u,k\in\Lambda_i}\sum_{v,p\in\Lambda_j}\gamma\big(\mathcal{Z}^q\xi_{uv}-\mathcal{Z}^q\xi_{kp},t\big)\geq\epsilon.$$

And we obtain that, $U(\epsilon,t)\subset W(r,t)$.

In each case we obtain that $U(\epsilon,t)\subset W(r,t)$. This implies that $U(\epsilon,t)\in I^2$. Hence double sequence is a Zweier ideal Cauchy sequence with respect to th neutrosophic norm (ϕ,ψ,γ). □

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