NSS

Neutrosophic Sets and Systems, Vol. 73, 2024

University of New Mexico

More on neutrosophic topology

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Abstract. The aim of this paper is to introduce the notion of neutrosophic singletons and the induced neutrosophic topology. We also study some of its basic properties.

Keywords: Neutrosophic topological space, neutrosophic singleton, induced neutrosophic topology.

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1. Notations and Terminology

In 1965, L. A. Zadeh [16] introduced the concept of *fuzzy sets*, which revolutionized our understanding of set theory. Since then, fuzzy set theory has influenced almost every branch of pure and applied mathematics, as well as fields such as physics, engineering, information theory, and control theory. Three years later, Chang [6] introduced the notion of *fuzzy topological spaces*, sparking numerous research projects that extended classical topological concepts into the fuzzy context.

Nineteen years after the introduction of fuzzy sets, Atanassov [1] proposed the idea of *intuitionistic fuzzy sets*. Subsequently, he and his colleagues [2–5] advanced this concept, yielding several interesting and important results. Eleven years later, Coker [7] introduced the notion of *intuitionistic fuzzy topological spaces*.

Smarandache [14, 15] later introduced the concepts of *neutrosophy* and *neutrosophic sets*. In 2002, Smarandache [14] expanded this framework by proposing the notion of *neutrosophic topology* on the non-standard interval. Notably, Lupiáñez [8–10] established several properties of neutrosophic topological spaces, showing, for instance, that an intuitionistic fuzzy topology is generally not a neutrosophic topology [8]. In 2012, Salama and Alblowi [12] introduced the concepts of *neutrosophic crisp sets* and *neutrosophic topological spaces*. V. L. Nayagam and G. Sivaraman [11] later explored properties of induced topology on intuitionistic fuzzy singletons.

In this paper, we introduce and investigate the concepts of *neutrosophic singletons* and the corresponding *induced neutrosophic topology* along similar lines. We also study some of their fundamental properties. Below, we summarize some well-known notions that will be used in subsequent sections.

Definition 1.1. Let $\mathcal{X}$ be a nonempty fixed set. A neutrosophic set (briefly NS) A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in \mathcal{X}\}$ where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) respectively of each element $x \in \mathcal{X}$ to the set A.

Remark 1.2.
(1) A neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in \mathcal{X}\}$ can be identified to an ordered triple $\langle \mu_A, \sigma_A, \gamma_A\rangle$ in $]0^-, 1^+[$ on $\mathcal{X}$.
(2) For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \sigma_A, \gamma_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in \mathcal{X}\}$.

Definition 1.3. Let $\mathcal{X}$ be a nonempty set and the neutrosophic sets A and B in the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in \mathcal{X}\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \gamma_B(x)\rangle : x \in \mathcal{X}\}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in \mathcal{X}$;
(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(c) $\bar{A} = \{\langle x, \gamma_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in \mathcal{X}\}$; [Complement of A]
(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in \mathcal{X}\}$;
(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in \mathcal{X}\}$;

Definition 1.4. Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in $\mathcal{X}$. Then

(a) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in \mathcal{X}\}$;
(b) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in \mathcal{X}\}$.

Since our main purpose is to construct the tools for developing neutrosophic topological spaces, we must introduce the neutrosophic sets 0_N and 1_N in $\mathcal{X}$ as follows:

Definition 1.5. $0_N = \{\langle x, 0, 0, 1\rangle : x \in \mathcal{X}\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in \mathcal{X}\}$.

Definition 1.6. [13] A neutrosophic topology (briefly NT) on a nonempty set $\mathcal{X}$ is a family $\mathcal{T}$ of neutrosophic sets in X satisfying the following axioms:

(i) $0_N, 1_N \in \mathcal{T}$,

(ii) $G_1 \cap G_2 \in \mathcal{T}$ for any $G_1, G_2 \in \mathcal{T}$,

(iii) $\cup G_i \in \mathcal{T}$ for arbitrary family $\{G_i \mid i \in \Lambda\} \subseteq \mathcal{T}$.

In this case the ordered pair $(\mathcal{X}, \mathcal{T})$ or simply $\mathcal{X}$ is called a neutrosophic topological space (briefly NTS($\mathcal{X}$)) and each neutrosophic set in $\mathcal{T}$ is called a neutrosophic open set (briefly NOS). The complement $\overline{A}$ of a NOS A in $\mathcal{X}$ is called a neutrosophic closed set (briefly NCS) in $\mathcal{X}$.

Definition 1.7. Let $\mathcal{X}$ be a nonempty set. If r, t, s are real standard or non standard subsets of $]0^-, 1^+[$ then the neutrosophic set $x_{r,t,s}$ is called a neutrosophic point or singleton in $\mathcal{X}$ given by

$$x_{r,t,s}(x_p) = \begin{cases} (r,t,s), & \text{if } x = x_p \\ (0,0,1), & \text{if } x \neq x_p \end{cases}$$

for $x_p \in \mathcal{X}$ is called the support of $x_{r,t,s}$, where r denotes the degree of membership value, t denotes the degree of indeterminacy and s is the degree of non-membership value of $x_{r,t,s}$.

From now on, we denote a neutrosophic singleton by $p = (\mu_p, \sigma_p, \gamma_p)$.

Definition 1.8. Let $\mathcal{X}$ be any non-empty set. A neutrosophic singleton p defined on x is said to belong to a neutrosophic set $A = (\mu_A, \sigma_A, \gamma_A)$ $(p \in A)$ if $\mu_p \leq \mu_A$, $\sigma_p \leq \sigma_A$ and $\gamma_A \leq \gamma_p$.

Definition 1.9. A neutrosophic topological space $(\mathcal{X}, \mathcal{T})$ is said to be neutrosophic Hausdorff if for every distinct points $x, y \in \mathcal{X}$, there exist neutrosophic open sets $A = (\mu_A, \sigma_A, \gamma_A)$, $B = (\mu_B, \sigma_B, \gamma_B) \in \tau$ such that $\mu_A(x) = 1$, $\mu_B(y) = 1$ and $A \cap B = 0_{_N}$.

Definition 1.10. A neutrosophic topological space $(\mathcal{X}, \mathcal{T})$ is said to be neutrosophic compact if for every cover ϱ by neutrosophic open sets $1_{_N} = \cup_{A \in \varrho}$, there exists a finite subcover A_1, A_2, A_3, ...,A_n of ϱ such that $1_{_N} = \cup_{i=1}^{n} A_i$.

Definition 1.11. A neutrosophic topological space $(\mathcal{X}, \mathcal{T})$ is said to be neutrosophic connected if $1_{_N}$ can not be written as the union of two neutrosophic open sets $A = (\mu_A, \sigma_A, \gamma_A)$, $B = (\mu_B, \sigma_B, \gamma_B) \in \mathcal{T}$ such that $A \cap B = 0_{_N}$.

2. Induced neutrosophic topology and some properties

Here we introduce the induced neutrosophic topology on neutrosophic singletons.

Definition 2.1. Let $(\mathcal{X}, \mathcal{T})$ be a neutrosophic topological space and $\mathcal{P}(\mathcal{X})$ the collection of all neutrosophic singletons of $\mathcal{X}$. The induced neutrosophic topology $\sigma_{\mathcal{T}}$ on $\mathcal{P}(\mathcal{X})$ is defined as the topology generated by $\mathcal{B}$= $\{\mathcal{V}_A \mid A \in \mathcal{T}\}$, where $\mathcal{V}_A = \{p \in \mathcal{P}(\mathcal{X}) \mid p \in A\}$. We call $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ the induced neutrosophic topological space.

Remark 2.2. It should be mentioned that obviously $\mathcal{B}$ is a basis for the topology $\mathcal{P}(\mathcal{X})$. We know that $1_N \in \tau$ and $\mathcal{V}_{1_N} = \mathcal{P}(\mathcal{X}) \in \mathcal{B}$. Thus for every neutrosophic singleton p, $p \in \mathcal{P}(\mathcal{X}) \in \mathcal{B}$. Moreover, if $\mathcal{V}_A$, $\mathcal{V}_B \in \mathcal{B}$, then it is clear that $\mathcal{V}_A \cap \mathcal{V}_B = \mathcal{V}_{A\cap B}$ and $\mathcal{V}_A \cap \mathcal{V}_B \in \mathcal{B}$.

Example 2.3. Let $\mathcal{X} = \{a, b, c\}$ and

$$\mathcal{T} = \big\{\{(x, y, 0), (r, s, 0), (n, m, 1)\}, \{(0, y, z), (0, r, s), (1, m, d)\}, \{(0, y, 0), (0, s, 0), (1, m, 1)\}\big\},$$

where $z < x < y$. Here $\{(x, y, z), (r, s, t), (n, m, d)\} \in \mathcal{I}^{\mathcal{X}} \times \mathcal{I}^{\mathcal{X}} \times \mathcal{I}^{\mathcal{X}}$. Observe that $(\mathcal{X}, \mathcal{T})$ is a neutrosophic topological space. Take $\sigma_{\mathcal{T}}= (\{\mathcal{V}_{0_N}, \mathcal{V}_{1_N},\ \mathcal{V}_{\{(x,y,0),(r,s,0),(n,m,1)\}}$, $\mathcal{V}_{\{(0,y,z),(0,r,s),(1,m,d)\}}, \mathcal{V}_{\{(0,y,0),(0,s,0),(1,m,1)\}})$. As it can be seen the induced neutrosophic topology $\sigma_{\mathcal{T}}$ on $\mathcal{P}(\mathcal{X})$ is not necessarily discrete.

Now we introduce the notion of the induced neutrosophic function.

Definition 2.4. Let $f : \mathcal{X} \to \mathcal{Y}$ be any function, where $\mathcal{X} \neq \emptyset$ and $\mathcal{Y} \neq \emptyset$. If the function $i_f : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{Y})$ is defined for any neutrosophic singleton $p = (\mu_p, \sigma_p, \gamma_p)$ defined on $x \in \mathcal{X}$ by $i_f(p) = q$, where $q = (\mu_q, \sigma_q, \gamma_q)$ is a neutrosophic singleton defined on $f(x) \in \mathcal{Y}$ with $\mu_q(f(x)) = \mu_p(x)$, $\sigma_q(f(x)) = \sigma_p(x)$ and $\gamma_q(f(x)) = \gamma_p(x)$, then i_f is called the induced neutrosophic function of f.

Lemma 2.5. *If $f : \mathcal{X} \to \mathcal{Y}$ is any function and $i_f : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{Y})$ the induced neutrosophic function of f, then the following statements hold:*
a) For any neutrosophic set $A = (\mu_A, \sigma_A, \gamma_A)$ of $\mathcal{Y}$, $\mathcal{V}_{f^{-1}(A)} = i_{f^{-1}}(\mathcal{V}_A)$, where $f^{-1}(A) = (\mu_{f^{-1}(A)}, \sigma_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$, where $\mu_{f^{-1}(A)}(x) = \mu_A(f(x))$, $\sigma_{f^{-1}(A)}(x) = \sigma_A(f(x))$ and $\gamma_{f^{-1}(A)}(x) = \gamma_A(f(x))$.
b) For any neutrosophic set $A = (\mu_A, \sigma_A, \gamma_A)$ of X, if $\mathcal{V}_A = \cup_{A_\gamma} \mathcal{V}_{A_\gamma}$, where A_γ belongs to the neutrosophic sets of $\mathcal{X}$, then $A = \cup A_\gamma$.

Proof. a) $p \in \mathcal{V}_{f^{-1}(A)}$ iff $\mu_p(x) \leq \mu_{f^{-1}(A)}(x)$, $\sigma_p(x) \leq \sigma_{f^{-1}(A)}(x)$ and $\gamma_p(x) \geq \gamma_{f^{-1}(A)}(x)$. This means that $\mu_p(x) \leq \mu_A(f(x))$, $\sigma_p(x) \leq \sigma_A(f(x))$ and $\gamma_p(x) \geq \gamma_A(f(x))$ iff $\mu_{i_{f(p)}}(f(x)) \leq \mu_A(f(x))$, $\sigma_{i_{f(p)}}(f(x)) \leq \sigma_A(f(x))$ and $\gamma_{i_{f(p)}}(f(x)) \geq \gamma_A(f(x))$ iff $i_f(p) \in \mathcal{V}_A$ iff $p \in i_{f^{-1}}(\mathcal{V}_A)$.

b) Let $\mathcal{V}_A = \cup_{A_\gamma} \mathcal{V}_{A_\gamma}$, where $A = (\mu_A, \sigma_A, \gamma_A)$. If we want to show $A = \cup A_\gamma$, it is the same to prove $A(x) = \cup_{A_\gamma}(x)$ for any $x \in \mathcal{X}$. Let us define a neutrosophic singleton p on x such that $\mu_p(x) = \mu_A(x)$, $\sigma_p(x) = \sigma_A(x)$ and $\gamma_p(x) = \gamma_A(x)$. It is obvious that $p \in \mathcal{V}_A$. Thus if $p \in \cup A_\gamma$ implies that there exists a γ such that $p \in \mathcal{V}_{A_\gamma}$. Therefore $\mu_p(x) \leq \mu_A(x)$, $\sigma_p(x) \leq \sigma_A(x)$ and $\gamma_p(x) \geq \gamma_A(x)$. Hence $\mu_A(x) \leq \mu_{A_\gamma}(x)$, $\sigma_A(x) \leq \sigma_{A_\gamma}(x)$ and $\gamma_A(x) \geq \gamma_{A_\gamma}(x)$. Observe that $\mu_A(x) \leq sup_{A_\gamma} \mu_{A_\gamma}(x)$, $\sigma_A(x) \leq sup_{A_\gamma} I_{A_\gamma}(x)$ and $\gamma_A(x) \geq inf_{A_\gamma} \gamma_{A_\gamma}(x)$. If $\mu_A(x) \leq sup_{A_\gamma} \mu_{A_\gamma}(x)$, then there exists some A_α in the set of neutrosophic sets of $\mathcal{X}$ such that $\mu_A(x) < \mu_{A_\alpha}(x) \leq sup_{A_\gamma}\mu_{A_\gamma}(x)$. Moreover, define a neutrosophic singleton q on x such that $\mu_q(x) = \mu_{A_\alpha}(x)$, $\sigma_q(x) = \sigma_{A_\alpha}(x)$ and $\gamma_q(x) = \gamma_{A_\alpha}(x)$. Thus $q \in \mathcal{V}_{A_\alpha}$ and obviously $q \in \cup_{A_\gamma}\mathcal{V}_{A_\gamma}$. Here we come to a contradiction since $q \notin \mathcal{V}_A$. Therefore $\mu_A(x) = sup_{A_\gamma}\mu_{A_\gamma}(x)$. By the same token, we obtain $\sigma_A(x) = sup_{A_\gamma}\sigma_{A_\gamma}(x)$ and $\gamma_A(x) = inf_{A_\gamma}\gamma_{A_\gamma}(x)$. This shows that $A = \cup A_\gamma$. □

Theorem 2.6. *A function $f : (\mathcal{P}(\mathcal{X}), \mathcal{T}) \to (\mathcal{Y}, \varrho)$ is a neutrosophic continuous function iff the induced neutrosophic function $i_f : (P(\mathcal{X}), \sigma_\mathcal{T}) \to (P(\mathcal{X}), \sigma_\varrho)$ is continuous.*

Proof. Let f be a neutrosophic continuous function. Now we will prove that i_f is continuous. Suppose $\mathcal{V}_A$ is an open set in σ_ϱ and thus $A \in \varrho$, By statement (a) in Lemma 2.5, $i_{f^{-1}}(\mathcal{V}_A) = \mathcal{V}_{f^{-1}(A)}$. By the fact that f is neutrosophic continuous, then $f^{-1}(A)$ is a neutrosophic open set in $\mathcal{T}$. This means that $i_{f^{-1}}(\mathcal{V}_A)$ is an open set in $\sigma_\mathcal{T}$.
Conversely, let i_f be continuous and $A \in \varrho$. We know that A is neutrosophic open, thus $\mathcal{V}_A$ is an open set in σ_ϱ. Since i_f is continuous , then $i_{f^{-1}}(\mathcal{V}_A)$ is open in $\sigma_\mathcal{T}$. This implies that $i_{f^{-1}}(\mathcal{V}_A) = \cup_{A_\gamma}\mathcal{V}_{A_\gamma}$, where $A_\gamma \in \sigma_\mathcal{T}$. By (a) in Lemma 2.5, this means that $\mathcal{V}_{f^{-1}(A)} = \cup_{A_\gamma}\mathcal{V}_{A_\gamma}$. By (b) in Lemma 2.5, $f^{-1}(A) = \cup A_\gamma$. This shows that $f^{-1}(A) \in \mathcal{T}$. □

Theorem 2.7. *Let $(\mathcal{X}, \mathcal{T})$ be a neutrosophic compact space. The induced neutrosophic topological space $(\mathcal{P}(\mathcal{X}), \sigma_\mathcal{T})$ is compact but the converse is not true.*

Proof. Suppose that $(\mathcal{X}, \mathcal{T})$ is a neutrosophic compact space. For some $B_0 \subseteq \mathcal{B}$ and $\mathcal{B}$ is a base for $\sigma_\mathcal{T}$, let $\mathcal{P}(\mathcal{X}) = \cup_{\mathcal{V}_{A\in B_0}}\mathcal{V}_A$. Take any $x \in \mathcal{X}$, $(1_x, 0_x) \in \mathcal{P}(\mathcal{X})$, where 1_x and 0_x are the characteristic functions on $\{x\}$ and on $A \setminus \{x\}$, respectively. Observe that $(1_x, 0_x) \in \mathcal{V}_{A_x}$ for some $\mathcal{V}_{A_x} \in B_0$. Then, we have $1_x \leq T_{A_x}$, $1_x \leq I_{A_x}$ and $0_x \geq F_{A_x}$. This means that $T_{A_x}(x) = 1$, $I_{A_x}(x) = 1$ and $F_{A_x}(x) = 0$. Hence $1_N = \vee_{x\in X} A_x$. Since $(\mathcal{X}, \mathcal{T})$ is a neutrosophic compact space, then $1_N = \vee_i^n A_{x_i}$ for $i = 1, 2, ..., n$. Thus $\mathcal{P}(X) = \vee_i^n A_{x_i}$. This means that $(\mathcal{P}(\mathcal{X}), \sigma_\mathcal{T})$ is compact.
The converse is not true. Take $(\mathcal{X}, \mathcal{T})$ such that $\mathcal{T} = \{0_N, 1_N, (1 - \frac{1}{n}, 1 - \frac{1}{n}, \frac{1}{n})\}$. This space is a neutrosophic topological space. Let $A_n = (1 - \frac{1}{n}, 1 - \frac{1}{n}, \frac{1}{n})$. An open covering of $\mathcal{P}(\mathcal{X})$ which contains $\mathcal{V}_{\cup A_n}$ and hence $(\mathcal{P}(\mathcal{X}), \sigma_\mathcal{T})$ is compact but $(\mathcal{X}, \mathcal{T})$ is not compact. □

Remark 2.8. It is worth-noticing that if we take two neutrosophic singletons which are defined on the same point with different values are distinct in $\mathcal{P}(\mathcal{X})$, then it is impossible to talk about Hausdorffness since we can not separate then by two open sets. Mind that any two neutrosophic singletons defined on distinct points can be separated by open sets. Thus this suggests to introduce the notion of neutrosophic pseudo Hausdorff space on $(\mathcal{P}(\mathcal{X}), \sigma_\mathcal{T})$.

Definition 2.9. The induced neutrosophic topological space $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is called neutrosophic pseudo Hausdorff if for any two neutrosophic singletons $p = (\mu_p, \sigma_p, \gamma_p)$, $q = (\mu_q, \sigma_q, \gamma_q)$ defined on distinct points, there exists two disjoint open sets $\mathcal{V}_A$ and $\mathcal{V}_B$ such that $p \in \mathcal{V}_A$ and $q \in \mathcal{V}_B$.

Theorem 2.10. *A neutrosophic topological space $(\mathcal{X}, \mathcal{T})$ is neutrosophic Hausdorff iff $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is neutrosophic pseudo Hausdorff.*

Proof. Suppose that $(\mathcal{X}, \mathcal{T})$ is neutrosophic Hausdorff. Proving $(\Rightarrow)$, let us take two neutrosophic singletons $p = (\mu_p, \sigma_p, \gamma_p)$, $q = (\mu_q, \sigma_q, \gamma_q)$ with distinct supports, i.e., $\mu_p = \{x\} \neq \mu_q = \{y\}$. Since $(\mathcal{X}, \mathcal{T})$ is neutrosophic Hausdorff, there exist $A, B \in \mathcal{T}$ such that $\mu_A(x) = 1$, $\mu_B(y) = 1$ such that $A \cap B = 0_N$. By the fact that A and B are neutrosophic subsets, $\sigma_A(x) = 1$, $\sigma_B(x) = 1$ and $\gamma_A(x) = 0$, $\gamma_B(x) = 0$. Therefore, $p \in \mathcal{V}_A$, $q \in \mathcal{V}_B$. By the fact that $A \cap B = 0_N$, then $\mathcal{V}_{A\cap B} = \emptyset$. Thus $\mathcal{V}_A \cap \mathcal{V}_B = \mathcal{V}_{A\cap B} = \emptyset$. So there exist disjoint open sets $\mathcal{V}_A$ and $\mathcal{V}_B$ belonging to $\sigma_{\mathcal{T}}$ such that $p \in \mathcal{V}_A$ and $q \in \mathcal{V}_B$. For proving $(\Leftarrow)$, Let $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ be neutrosophic pseudo Hausdorff. Let x, y be two distinct points of $\mathcal{X}$. Let us define neutrosophic singletons p and q on x and y, respectively with $\mu_p(x) = \mu_q(y) = 1$. Since $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is neutrosophic pseudo Hausdorff, there exist $\mathcal{V}_A$ and $\mathcal{V}_B$ belonging to $\sigma_{\mathcal{T}}$ such that for $A, B \in \mathcal{T}$, $p \in \mathcal{V}_A$ and $q \in \mathcal{V}_B$ and $\mathcal{V}_A \cap \mathcal{V}_B = \emptyset$. It is obvious that by definition of $\mu_p(x) = 1 \leq \mu_A(x)$ and we have $\mu_A(x) = 1$. By the same token, $\mu_B(y) = 1$. So $(V)_{A\cap B} = \mathcal{V}_A \cap \mathcal{V}_B = \emptyset$. This means that $A \cap B = 0_N$. This shows that $(\mathcal{X}, \mathcal{T})$ is neutrosophic Hausdorff. □

Theorem 2.11. *The space $(\mathcal{X}, \mathcal{T})$ is neutrosophic connected iff $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is connected.*

Proof. Suppose that $(\mathcal{X}, \mathcal{T})$ is neutrosophic connected. Assume that $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ is not connected. Then $\mathcal{P}(\mathcal{X}) = \mathcal{V}_A \cup \mathcal{V}_B$, for some $A, B \in \mathcal{T}$ for which $\mathcal{V}_A \cap \mathcal{V}_B = \emptyset$. It is obvious that $A \cap B = 0_N$. For every $x \in \mathcal{X}$, we have $(1_x, 0_x) \in \mathcal{P}(\mathcal{X})$. This implies that $(1_x, 0_x) \in \mathcal{V}_A$ or $(1_x, 0_x) \in \mathcal{V}_B$. Thus $\mu_A(x) = 1$, $\sigma_A(x) = 1$ and $\gamma_A(x) = 0$ or $\mu_B(x) = 1$, $\sigma_B(x) = 1$ and $\gamma_B(x) = 0$. Therefore $\mu_A \vee \mu_B = 1$, $\sigma_A \vee \sigma_B = 1$ and $\gamma_A \vee \gamma_B = 0$. So we have $A \cup B = 1_N$ which is a contradiction. To prove the converse implication, let $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ be connected and that $(\mathcal{X}, \mathcal{T})$ is not neutrosophic connected.. Thus, for some $A, B \in \mathcal{T}$, $1_N = A \cup B$ and $0_N = A \cap B$. Therefore $1 = \mu_A \vee \mu_B$, $1 = \sigma_A \vee \sigma_B$ and $0 = \gamma_A \wedge \gamma_B$ and $0 = \mu_A \wedge \mu_B$, $0 = \sigma_A \wedge \sigma_B$ and $1 = \gamma_A \vee \gamma_B$. Therefore for any $x \in \mathcal{X}$, $\mu_A(x) = 1$, $\mu_B(x) = 0$, $\sigma_A(x) = 1$, $\sigma_B(x) = 0$, $\gamma_A(x) = 0$, $\gamma_B(x) = 1$ or $\mu_A(x) = 0$, $\mu_B(x) = 1$, $\sigma_A(x) = 0$, $\sigma_B(x) = 1$, $\gamma_A(x) = 1$, $\gamma_B(x) = 0$. Thus, for any neutrosophic singleton $p = (\mu_A, \sigma_A, \gamma_A)$, we have $\mu_p(x) \leq \mu_A(x)$, $\sigma_p(x) \leq \sigma_A(x)$ and $\gamma_p(x) \geq \gamma_A(x)$ or $\mu_p(x) \leq \mu_B(x)$, $\sigma_p(x) \leq \sigma_B(x)$ and $\gamma_p(x) \geq \gamma_B(x)$. Therefore, $p \in \mathcal{V}_A$ or $\mathcal{V}_B$. Observe that $\mathcal{V}_A \cap \mathcal{V}_B = \mathcal{V}_{A\cap B} = \emptyset$ which is contradiction to our hypothesis. □

Theorem 2.12. *Let $(\mathcal{X}, \mathcal{T})$ be a neutrosophic topological space and $E \subseteq \mathcal{X}$. The subspace on $(\mathcal{P}(E))$ inherited from $(\mathcal{P}(\mathcal{X}), \sigma_{\mathcal{T}})$ equals to the topological space on $\mathcal{P}(E)$ induced by neutrosophic subspace on E inherited from $(\mathcal{X}, \sigma_{\mathcal{T}})$.*

Proof. For any neutrosophic open set $A \in \mathcal{T}$, we prove $\mathcal{V}_A \cap \mathcal{P}(E) = \mathcal{V}_{A|E}$. Take $p = (\mu_p, \sigma_p, \gamma_p) \in \mathcal{V}_A \cap \mathcal{P}(E)$. By the fact that $p \in \mathcal{V}_A$, we have $\mu_p(x) \leq \mu_A(x)$, $\sigma_p(x) \leq \sigma_A(x)$ and $\gamma_p(x) \geq \gamma_A(x)$. Also for $p \in \mathcal{P}(E)$, supp $\mu_p = x$ which is in E. Thus $\mu_p(x) \leq (\mu_A \mid E)(x)$, $\sigma_p(x) \leq (\sigma_A \mid E)(x)$ and $\gamma_p(x) \geq (\gamma_A \mid E)(x)$. Therefore $p \in \mathcal{V}_{A|E}$. It follows that $\mathcal{V}_A \cap \mathcal{P}(E) \subseteq \mathcal{V}_{A|E}$. Suppose $p \in \mathcal{V}_{A|E}$. Thus $\mu_p(x) \leq (\mu_A \mid E)(x)$, $\sigma_p(x) \leq (\sigma_A \mid E)(x)$ and $\gamma_p(x) \geq (\gamma_A \mid E)(x)$. If supp $\mu_p = x$ and supp $\sigma_p = x$ does not belong to E, then $(\mu_A \mid E)(x) = 0$ and $\sigma_A \mid E)(x) = 0$ which means that $\mu_A(x) = 0$ and $\sigma_A(x) = 0$. But this is a contradiction to the fact that $p \in \mathcal{V}_{A|E}$. Hence supp $\mu_p = x$ and supp $\sigma_p = x$. Thus $\mu_p(x) \leq \mu_A(x)$, $\sigma_p(x) \leq \sigma_A(x)$ and $\gamma_p(x) \geq \gamma_A(x)$. This shows that $p \in \mathcal{V}_A \cap \mathcal{P}(E)$. □

Remark 2.13. Suppose that $W \cap \mathcal{P}(E)$ is an arbitrary open set in $(\mathcal{P}(E), \sigma_\tau \cap \mathcal{P}(E))$ where W is an open set in the induced neutrosophic topology σ_T.Since $W = \cup_{A \in B} \mathcal{V}_A$ for some $B \subseteq T$, then $W \cap \mathcal{P}(E) = (\cup_{A \in B} \mathcal{V}_A) \cap \mathcal{P}(E) = \cup_{A \in B}(\mathcal{V}_A) \cap \mathcal{P}(E))$. Hence by the above theorem, $W \cap \mathcal{P}(E) = \cup_{A \in B}(\mathcal{V}_{A|E}) \in (\mathcal{P}(E), \sigma_{T|E})$. By the same token, one can show that any basic open set $\mathcal{V}_{A|E}$ in $(\mathcal{P}(E), \sigma_{T|E})$ is also a basic open set in $(\mathcal{P}(E), \sigma_T \cap \mathcal{P}(E))$.

Theorem 2.14. *Suppose that $(\mathcal{Y}, \delta)$ and $(\mathcal{Z}, \rho)$ are neutrosophic topological spaces. The induced neutrosophic topological space on $\mathcal{P}(\mathcal{Y} \times \mathcal{Z})$ defined by product neutrosophic topological space $(Y \times \mathcal{Z}, \delta \times \rho)$ is embedded in $(\mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Z}), \sigma_\delta \times \sigma_\rho)$*

Proof. The proof is done by proving that there exists a function $f : (\mathcal{P}(\mathcal{Y} \times \mathcal{Z}), \sigma_{\delta \times \rho}) \rightarrow \mathcal{P}(\mathcal{Y}) \times \mathcal{P}(\mathcal{Z}), \sigma_\delta \times \sigma_\rho)$ which is bijective and f and its inverse are continuous. □

Conclusions

In this paper, we introduce and investigate the concept of induced neutrosophic topology through neutrosophic singletons. Additionally, we present several important properties of the induced neutrosophic function. These notions provide fertile ground for further research on neutrosophic separation axioms, neutrosophic weak separation axioms, and neutrosophic generalized open and closed sets.

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Received: July 11, 2024. Accepted: September 05, 2024