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$\mathfrak{c}$-Continuity, $\mathfrak{c}$-Compact and $\mathfrak{c}$-Separation Axioms via Soft Sets

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Abstract. This paper focuses on the concept of $S\mathfrak{c}$-open sets as a generalization of classical $\mathfrak{c}$-open sets in topology. The reason behind introducing $S\mathfrak{c}$-open sets is to overcome the limitations of traditional open sets in handling uncertainty and vagueness prevalent in decision-making processes. Moreover, the paper makes significant contributions to the discussion of the concepts of soft topological spaces (STS) by utilizing $S\mathfrak{c}$-open sets that investigate the theoretical foundations and mathematical properties of $S\mathfrak{c}$-open sets, exploring their relationships with other soft open sets and soft topological concepts. Overall, the paper aims to provide a comprehensive understanding of STS and their properties and theorems utilizing the concept of $S\mathfrak{c}$-open set and explores the theoretical foundations, mathematical properties, and relationships of these sets while extending their application to domains such as $S\mathfrak{c}$-continuity, $S\mathfrak{c}$-separation axiom and $S\mathfrak{c}$ compactness.

Keywords: $S\mathfrak{c}$-open set; $S\mathfrak{c}$-regular space; $S\mathfrak{c}$-normal space; $S\mathfrak{c}$-compact space; $S\mathfrak{c}^*$-compact space; $S\mathfrak{c}$-continuous function; $S\mathfrak{c}^*$-continuous function.

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1. **Introduction**

In 1999, Molodtsov introduced soft sets as a mathematical tool for handling uncertainty. Since then, soft set theory has found applications in diverse fields such as smoothness of functions, game theory, operations research, integration theory, probability, and measurement theory [1–14]. The properties and applications of soft set theory have been extensively studied by researchers [4], [10], [12], [14], [18], [20], [23], [24], [25]. In recent years, the combination of soft set theory with fuzzy set theory has led to interesting developments and applications [3], [5], [12], [13], [14], [15] , [17], [21]. This integration has involved redefining operations on soft sets and constructing decision-making methods using these new concepts [6]. The study of soft topological spaces was initiated by Shabir and Naz in 2011 [16]. They introduced soft

topology $\Im$ defined on the collection of soft sets over a set Λ. They established fundamental concepts such as soft open sets, soft closed sets, soft subspaces, soft closures, soft neighborhoods of a point, soft separation axioms, soft regular spaces, and soft normal spaces. They also investigated the properties of these concepts. Hussain and Ahmad further expanded the study of soft topological spaces in 2011 [7]. They focused on properties related to open soft sets, closed soft sets, soft neighborhoods, and soft closures. Additionally, they introduced and discussed the properties of soft interior, soft exterior, and soft boundary within the context of soft topological spaces. The introduction of soft set theory and the subsequent development of soft topological spaces have provided valuable tools for handling uncertainty and vagueness in various mathematical and applied domains. These theories have been successfully applied in numerous fields, and researchers continue to explore their properties and applications.
In 2023, [22] Alqahtani and Saleh introduced a novel type of open sets known as $\mathfrak{c}$-open sets. Their work focused on exploring the relationship between these sets and other types of open sets existing in classical topology. Specifically, a set $\aleph$ in a classical topological space is defined to be $\mathfrak{c}$-open if and only if $cl(\aleph) \setminus \aleph$ is countable set. This concept generalizes the traditional notion of open sets and offers a fresh perspective on openness in classical topology. This research aims to expand the investigation of the concept introduced by Alqahtani and Saleh by incorporating soft sets and $\mathfrak{c}$-open. Fundamental properties and theories are discussed in soft topological spaces via the $\mathfrak{c}$-open sets. This article is organized as follows: it begins by introducing a category of $S\mathfrak{c}$-open sets and outlining their fundamental properties. It then proceeds to explore various concepts such as $S\mathfrak{c}$-regular, $S\mathfrak{c}$-normal, and $S\mathfrak{c}$-T_i-spaces for $i \in 0, 1, 2, 3, 4$, as well as $S\mathfrak{c}$-compact and $S\mathfrak{c}^*$-compact spaces, utilizing $S\mathfrak{c}$-open sets. The article then delves into the examination of concepts such as $S\mathfrak{c}$-continuous, $S\mathfrak{c}^*$-continuous, $S\mathfrak{c}$-homeomorphism, and $S\mathfrak{c}^*$-homeomorphism functions through the perspective of $S\mathfrak{c}$-open sets. Additionally, it reviews essential properties related to $S\mathfrak{c}$-compact and $S\mathfrak{c}^*$-compact spaces, supported by numerous illustrative examples. Finally, the article concludes by providing insights and suggesting potential future research directions.

2. **Preliminaries**

Throughout this paper, we revisit some concepts in soft sets and soft topological spaces and soft topological spaces. Let Λ be an universe set and Π be a set of parameters. Let $\Gamma \subseteq \Pi$ and $P(\Lambda)$ denotes the collection of all subsets of Λ. The family of all soft sets over Λ with set of parameters $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ will be denoted by $SS(\Lambda)_\Gamma$. A pair $(\hat{\aleph}, \Gamma)$ is called a soft set (SS) on Λ, where $\hat{\aleph}$ is a mapping since Γ into P(Λ) is defined by $(\hat{\aleph}, \Gamma) = \{(\gamma, \hat{\aleph}(\gamma)) : \gamma \in \Gamma, \hat{\aleph}(\gamma) \subseteq \Lambda\}$. A relative null soft set (Φ, Γ) is a soft set $(\hat{\aleph}, \Gamma)$ if $\hat{\aleph}(\gamma) = \emptyset$ for all $\gamma \in \Gamma$. Similarly, a relative absolute soft set $(\hat{\Lambda}, \Gamma)$ is a soft set $(\hat{\aleph}, \Gamma)$ if $\hat{\aleph}(\gamma) = \Lambda$ for all $\gamma \in \Gamma$.

We include herein some definitions utilized in this study: [1, 2, 4]

Let $(\hat{\aleph}_1, \Gamma_1)$ and $(\hat{\aleph}_2, \Gamma_2)$ be *SSs* over Λ, we define

(1) $(\hat{\aleph}_1, \Gamma_1)$ is a soft subset of $(\hat{\aleph}_2, \Gamma_2)$ if: $\Gamma_1 \subseteq \Gamma_2$ and, $\hat{\aleph}_1(\gamma) \subseteq \hat{\aleph}_2(\gamma)$ for all $\gamma \in \Gamma$. This relationship is denoted by $(\hat{\aleph}_1, \Gamma_1) \hat{\sqsubseteq} (\hat{\aleph}_2, \Gamma_2)$. So, $(\hat{\aleph}_1, \Gamma_1)$ is a soft superset of $(\hat{\aleph}_2, \Gamma_2)$, denoted by $(\hat{\aleph}_1, \Gamma_1) \hat{\sqsupseteq} (\hat{\aleph}_2, \Gamma_2)$, if $(\hat{\aleph}_2, \Gamma_2)$ is a soft subset of $(\hat{\aleph}_1, \Gamma_1)$. Also, $(\hat{\aleph}_1, \Gamma_1)$ is a soft equal of $(\hat{\aleph}_2, \Gamma_2)$ if $(\hat{\aleph}_1, \Gamma_1)$ it is a soft subset and a soft superset of $(\hat{\aleph}_2, \Gamma_2)$.

(2) The soft union of two soft sets $(\hat{\aleph}_1, \Gamma_1)$ and $(\hat{\aleph}_2, \Gamma_2)$ over the common universe Λ is denoted by $(\hat{\aleph}_1, \Gamma_1) \hat{\sqcup} (\hat{\aleph}_2, \Gamma_2)$ and is the soft set $(\hat{\aleph}, \Gamma)$, where $\Gamma = \Gamma_1 \cup \Gamma_2$ for all $\gamma \in \Gamma$ defined as: $\hat{\aleph}(\gamma) = \hat{\aleph}_1(\gamma)$, if $\gamma \in \Gamma_1 \setminus \Gamma_2$, $\hat{\aleph}_2(\gamma)$, if $\gamma \in \Gamma_2 \setminus \Gamma_1$, and $\hat{\aleph}_1(\gamma) \cup \hat{\aleph}_2(\gamma)$ if $\gamma \in \Gamma_1 \cap \Gamma_2$. The soft intersection of two soft sets $(\hat{\aleph}_1, \Gamma_1)$ and $(\hat{\aleph}_2, \Gamma_2)$ over the common universe Λ is denoted by $(\hat{\aleph}_1, \Gamma_1) \hat{\sqcap} (\hat{\aleph}_2, \Gamma_2)$ and is the soft set $(\hat{\aleph}, \Gamma)$, where $\Gamma = \Gamma_1 \cap \Gamma_2 \neq \phi$ for all $\gamma \in \Gamma$, $\hat{\aleph}(\gamma) = \hat{\aleph}_1(\gamma) \cap \hat{\aleph}_2(\gamma)$.

(3) The complement of a soft set $(\hat{\aleph}, \Gamma)$ is denoted by $(\hat{\aleph}, \Gamma)^c$ and defined by $(\hat{\aleph}, \Gamma)^c = (\hat{\aleph}^c, \Gamma)$, where $\hat{\aleph}^c$ is a mapping given by $\hat{\aleph}^c(\gamma) = \Lambda \setminus \hat{\aleph}(\gamma)$ for all $\gamma \in \Gamma$. The soft difference between the two soft sets $(\hat{\aleph}_1, \Gamma_1)$ and $(\hat{\aleph}_2, \Gamma_2)$ is the soft set $(\hat{\aleph}, \Gamma)$ where $\Gamma = \Gamma_1 \cup \Gamma_2$ is defined as: $(\hat{\aleph}_1, \Gamma_1) \hat{\setminus} (\hat{\aleph}_2, \Gamma_2) = (\hat{\aleph}_1, \Gamma_1) \hat{\sqcap} (\hat{\aleph}_2, \Gamma_2)^c$.

(4) A soft set $(\hat{\aleph}, \Gamma)$ is said to be a soft countable set, denoted by S-countable if $\hat{\aleph}(\gamma)$ is countable.

Definition 2.1. [22] Let $(\Lambda, \Im)$ be a classical topological space. Below are some important definitions used in the study:

(1) An open subset $\aleph$ of a topological space $(\Lambda, \Im)$ is called $\mathfrak{c}$-*open* set if $cl(\aleph) \setminus \aleph$ is a countable set. That is, $\aleph$ is an open set. So, a closed subset $\aleph$ of a topological space $(\Lambda, \Im)$ is called $\mathfrak{c}$-closed set if $\aleph \setminus int(\aleph)$ is a countable set. That is, $\aleph$ is a closed set.

(2) $(\Lambda, \Im)$ is called a $\mathfrak{c}$-*regular* space if for each closed subset $\Omega \subseteq \Lambda$ and each point $\mathfrak{x} \notin \Omega$, there exist disjoint $\mathfrak{c}$-open sets $\aleph_1$ and $\aleph_2$ such that $\mathfrak{x} \in \aleph_1$ and $\Omega \subseteq \aleph_2$. So, $(\Lambda, \Im)$ is called a $\mathfrak{c}$-normal space if for each pair of closed disjoint subsets Ω_1 and Ω_2 of Λ, there exist disjoint $\mathfrak{c}-$open sets $\aleph_1$ and $\aleph_2$ such that $\Omega_1 \subseteq \aleph_1$ and $\Omega_2 \subseteq \aleph_2$.

(3) Let $(\Lambda, \Im)$ be a topology space, we say that Λ is $\mathfrak{c}-T_0$-space, if given $\mathfrak{x}_1, \mathfrak{x}_2 \in \Lambda$, $\mathfrak{x}_1 \neq \mathfrak{x}_2$, then there is either a $\mathfrak{c}$-open set containing $\mathfrak{x}_1$ but not $\mathfrak{x}_2$ or a $\mathfrak{c}$-open set containing $\mathfrak{x}_2$ but not $\mathfrak{x}_1$. So, Λ is $\mathfrak{c}-T_1$-space, if given $\mathfrak{x}_1, \mathfrak{x}_2 \in \Lambda$, $\mathfrak{x}_1 \neq \mathfrak{x}_2$, then there are two $\mathfrak{c}$-open subsets $\aleph_1$ and $\aleph_2$ of Λ, such that $\mathfrak{x}_1 \in \aleph_1$, $\mathfrak{x}_2 \notin \aleph_1$, and $\mathfrak{x}_1 \notin \aleph_2$, $\mathfrak{x}_2 \in \aleph_2$. Moreover, Λ is $\mathfrak{c}-T_2$-space, if given $\mathfrak{x}_1, \mathfrak{x}_2 \in \Lambda$, $\mathfrak{x}_1 \neq \mathfrak{x}_2$, then there are two disjoint $\mathfrak{c}$-open subsets $\aleph_1$ and $\aleph_2$ of Λ, such that $\mathfrak{x}_1 \in \aleph_1$ and $\mathfrak{x}_2 \in \aleph_2$.

(4) Let $(\Lambda, \Im)$ be a topological space, then Λ is $\mathfrak{c}$-compact (resp., $\mathfrak{c}^*$-compact) if for each open (resp., $\mathfrak{c}$-open) cover of Λ has a finite subcover of $\mathfrak{c}$-open (resp., open) sets.

(5) A function $\varphi : (\Lambda, \Im) \rightarrow (\zeta, \Im')$ is said to be $\mathfrak{c}$-continuous (resp., $\mathfrak{c}^*$-continuous) if $\varphi^{-1}(\aleph)$ is $\mathfrak{c}$-open (resp., open) in Λ for each open (resp., $\mathfrak{c}$-open) subset $\aleph$ in ζ. A function φ is said to be $\mathfrak{c}$-open function if for each $\mathfrak{c}$-open subset $\aleph \subseteq \Lambda$, we have $\varphi(\aleph)$ is an open subset in ζ. So, φ is said to be $\mathfrak{c}$-closed function if for each $\mathfrak{c}$-closed subset $\hat{\Theta} \subseteq \Lambda$, we have $\varphi(\hat{\Theta})$ is closed subset in ζ. Moreover A bijection function $\varphi : (\Lambda, \Im) \rightarrow (\zeta, \Im')$ is said to be $\mathfrak{c}$-homeomorphism (resp., $\mathfrak{c}^*$-homeomorphism) if φ and φ^{-1} are $\mathfrak{c}$-continuous (resp., $\mathfrak{c}^*$-continuous).

Definition 2.2. [2] Let $\hat{\Im}$ be a family of soft sets on Λ, then $\hat{\Im}$ is said to be soft topology (ST) on Λ if null soft set and absolute soft set in $\hat{\Im}$, the union of any members of soft sets in $\hat{\Im}$ belongs to $\hat{\Im}$ and intersection of any two soft sets in $\hat{\Im}$ belongs to $\hat{\Im}$. The triple $(\Lambda, \hat{\Im}, \Gamma)$ is called a soft topological space (STS) on Λ. Every member of $\hat{\Im}$ is called a soft open set, denoted by, S-open. The complement of an S-open set is a soft closed set, denoted by, S-closed.

Definition 2.3. [26], [9] Let $(\hat{\Lambda}, \Gamma)$ and $(\hat{\Lambda}', \Gamma')$ be two SSs, $\sigma : \Lambda \longrightarrow \Lambda'$ and $\alpha : \Gamma \longrightarrow \Gamma'$ be two mappings. Then $\hat{\varphi}_{\sigma\alpha} : (\hat{\Lambda}, \Gamma) \longrightarrow (\hat{\Lambda}', \Gamma')$ is called a soft mapping, S mapping. The image of $(\hat{\aleph}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$ under $\hat{\varphi}_{\sigma\alpha}$, $\hat{\varphi}_{\sigma\alpha}\big((\hat{\aleph}, \Gamma)\big) = \big(\hat{\varphi}_{\alpha}(\aleph), \Gamma'\big)$ is a SS in $(\hat{\Lambda}', \Gamma')$ given as, for all $\gamma' \in \Gamma'$, $\hat{\varphi}_{\sigma\alpha}(\aleph)(\gamma') = \hat{\sigma}\big(\bigcup_{\gamma \in \alpha^{-1}(\gamma') \cap \Gamma} \aleph(\gamma)\big)$, if $\alpha^{-1}(\gamma') \cap \Gamma \neq \phi$ and ϕ for otherwise. For $\gamma' \in \Gamma'$, $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma) = (\sigma(\aleph), \alpha(\Gamma))$ is called a soft image of $(\hat{\aleph}, \Gamma)$. The inverse image of $(\hat{\Theta}, \Gamma')$ $\hat{\sqsubseteq}$ $(\hat{\Lambda}', \Gamma)$ under $\hat{\varphi}_{\sigma\alpha}$, $\hat{\varphi}^{-1}_{\hat{\varphi}\alpha}\big((\hat{\Theta}, \Gamma')\big) = \big(\hat{\varphi}^{-1}_{\hat{\varphi}\alpha}(\hat{\Theta}), \Gamma\big)$ is a SS in $(\hat{\Lambda}, \Gamma)$ given as, for all $\gamma \in \Gamma$, $\hat{\varphi}^{-1}_{\hat{\varphi}\alpha}(\hat{\Theta})(\gamma) = \hat{\varphi}^{-1}_{\sigma\alpha}\big(\hat{\Theta}(\alpha(\gamma))\big)$, if $\alpha(\gamma) \in \Gamma'$, ϕ, if $\alpha(\gamma) \notin \Gamma'$. The S mapping, $\hat{\varphi}_{\sigma\alpha}$ is called a

(1) S surjective mapping if σ and α are surjective mappings.
(2) S injective mapping if σ and α are injective mappings.
(3) S bijective mapping if σ and α are bijective mappings.

3. $S\mathfrak{c}$-Open and $S\mathfrak{c}$-Closed Sets in $STSs$

In this section, we introduce the definitions of $S\mathfrak{c}$-open and $S\mathfrak{c}$-closed sets, and discuss the theorems and properties derived from them, supported by relevant counterexamples.

Definition 3.1. A S-open subset $(\hat{\aleph}, \Gamma)$ of a STS $(\Lambda, \hat{\Im}, \Gamma)$ is called soft $\mathfrak{c}$-open set, $S\mathfrak{c}$-open set if $cl((\hat{\aleph}, \Gamma)) \hat{\setminus} (\hat{\aleph}, \Gamma)$ is S-countable set. That is, $(\hat{\aleph}, \Gamma)$ is a S-open set.

Definition 3.2. A S-closed subset $(\hat{\aleph}, \Gamma)$ of a STS $(\Lambda, \hat{\Im}, \Gamma)$ is called soft $\mathfrak{c}$-closed set, $S\mathfrak{c}$-closed set if $(\hat{\aleph}, \Gamma) \hat{\setminus} int(\hat{\aleph}, \Gamma)$ is S-countable set. That is, $(\hat{\aleph}, \Gamma)$ is a S-closed set.

Theorem 3.3. *The complement of any $S\mathfrak{c}$-open (resp., $S\mathfrak{c}$-closed) subset of a STS $(\Lambda, \hat{\Im}, \Gamma)$ is a $S\mathfrak{c}$-closed (resp., $S\mathfrak{c}$-open) set.*

Proof. Let $(\hat{\aleph}, \Gamma)$ be any $S\mathfrak{c}$-open subset of a STS $(\Lambda, \hat{\Im}, \Gamma)$. Then its complement, $(\hat{\aleph}, \Gamma)^c = (\hat{\aleph}^c, \Gamma)$ is S-closed, and satisfy the following:

$$\begin{aligned}(\hat{\aleph}, \Gamma)^c \hat{\setminus} int(\hat{\aleph}, \Gamma)^c =& (\hat{\aleph}, \Gamma)^c \hat{\setminus} (cl(\hat{\aleph}, \Gamma))^c \\ &= (\hat{\aleph}, \Gamma)^c \hat{\sqcup} cl(\hat{\aleph}, \Gamma) \\ &= cl(\hat{\aleph}, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma),\end{aligned}$$

which is S-countable by Definition 3.1. Consequently, $(\hat{\aleph}, \Gamma)^c \hat{\setminus} int(\hat{\aleph}, \Gamma)^c$ is also S-countable. Therefore, $(\hat{\aleph}, \Gamma)^c$ is a $S\mathfrak{c}$-closed set. On the other hand, if $(\hat{\aleph}, \Gamma)$ is any $S\mathfrak{c}$-closed subset of a STS $(\Lambda, \hat{\Im}, \Gamma)$, following the same method, $(\hat{\aleph}, \Gamma)^c$ is a $S\mathfrak{c}$-open set.

Corollary 3.4. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a STS and $(\hat{\aleph}, \Gamma) \hat{\in} SS(\Lambda)_\Gamma$, then*

(1) *Every S-clopen (S-closed and S-open) subset of a STS is $S\mathfrak{c}$-clopen ($S\mathfrak{c}$-open and $S\mathfrak{c}$-closed) set.*
(2) *Every S-countable S-closed set is $S\mathfrak{c}$-closed.*

Proof.

(1) Utilizing Definitions 3.1 and 3.2, alongside the properties of S-open and S-closed sets, incorporating their S-interior and S-closure, we can demonstrate it directly.
(2) Suppose $(\hat{\aleph}, \Gamma)$ is S-countable and S-closed. Then, we have $cl(\hat{\aleph}, \Gamma) = (\hat{\aleph}, \Gamma) = bd(\hat{\aleph}, \Gamma)$. Consequently,

$$\begin{aligned}(\hat{\aleph}, \Gamma) \setminus int(\hat{\aleph}, \Gamma) &= cl(\hat{\aleph}, \Gamma) \hat{\sqcap} (int(\hat{\aleph}, \Gamma))^c \\ &= cl(\hat{\aleph}, \Gamma) \hat{\sqcap} cl(\hat{\aleph}, \Gamma)^c \\ &= bd(\hat{\aleph}, \Gamma) \\ &= (\hat{\aleph}, \Gamma).\end{aligned}$$

Since $(\hat{\aleph}, \Gamma)$ is S-countable, it follows that $(\hat{\aleph}, \Gamma) \setminus int(\hat{\aleph}, \Gamma)$ is also S-countable. Therefore, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-closed.

Remark 3.5. Indeed, every S-countable S-open set may not necessarily be $S\mathfrak{c}$-open.

Example 3.6. Let $\Re$ be a set of real number and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$ be soft point (SP). The collection

$$\hat{\Im} = \{(\hat{\aleph}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma) : \hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \in (\hat{\aleph}, \Gamma)\} \hat{\sqcup} \{(\Phi, \Gamma)\},$$

is the particular soft point topology on Λ and $(\hat{\mathcal{N}}, \Gamma)$ be a S-countable S-open set but $cl(\hat{\mathcal{N}}, \Gamma) \hat{\setminus} (\hat{\mathcal{N}}, \Gamma) = (\hat{\Re}, \Gamma) \setminus (\hat{\mathcal{N}}, \Gamma)$ is S-uncountable set.

Proposition 3.7. *In STS, every $S\mathfrak{c}$-open (resp. $S\mathfrak{c}$-closed) set is a S-open (resp. S-closed) set.*

Proof. Obvious, by the Definitions 3.1 and 3.2.

Remark 3.8. In general, every S-open (resp. S-closed) set is not necessarily a $S\mathfrak{c}$-open (resp. $S\mathfrak{c}$-closed) set.

Example 3.9. Let $(\hat{\Lambda}, \Gamma)$ be a non-empty SS, and let (I, Γ) be a soft subset such that (I, Γ) $\hat{\sqsubseteq}$ $(\hat{\Lambda}, \Gamma)$. We define the soft subset $\hat{\Im}_{(I,\Gamma)}$ of SSs as follows:

$$\hat{\Im}_{(I,\Gamma)} = \{(\mu, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma) : (\mu, \Gamma) \hat{\sqcap} (I, \Gamma) = (\Phi, \Gamma)\} \hat{\sqcup} (\hat{\Lambda}, \Gamma).$$

All soft subsets of $(\hat{\Lambda}, \Gamma)$ that are soft disjoint from (I, Γ), along with $(\hat{\Lambda}, \Gamma)$ itself, constitute $\hat{\Im}_{(I,\Gamma)}$. This SS forms a STS called the excluded set ST on $(\hat{\Lambda}, \Gamma)$ with respect to (I, Γ). Consequently, $(\hat{Q}, \Gamma)$ is S-open in $(\Re, \hat{\Im}_I, \Gamma)$. However, $cl(\hat{Q}, \Gamma) \hat{\setminus} (\hat{Q}, \Gamma) = (\hat{\Re}, \Gamma) \hat{\setminus} (\hat{Q}, \Gamma) = (\hat{IQ}, \Gamma)$ is a S-uncountable set. Thus, $(\hat{Q}, \Gamma)$ is not $S\mathfrak{c}$-open. Furthermore, $(\hat{IQ}, \Gamma)$ exemplifies a S-closed set that is not $S\mathfrak{c}$-closed.

Remark 3.10. An example exists where a $S\mathfrak{c}$-open (resp., $S\mathfrak{c}$-closed) set is not a S-open (resp., S-closed) domain set.

Example 3.11. Consider $(\Re, \hat{\Im}_{\mathcal{U}}, \Gamma)$ as a STS on $\Re$ with $\Gamma = \{\gamma_1, \gamma_2\}$. Let $(\hat{\aleph}, \Gamma) = \{(\gamma_1, (2, 5)), (\gamma_2, (2, 5))\} \hat{\sqcup} \{(\gamma_1, (5, 9)), (\gamma_2, (5, 9))\}$, which is a $S\mathfrak{c}$-open subset in $(\Re, \hat{\Im}_{\mathcal{U}}, \Gamma)$. However, $(\hat{\aleph}, \Gamma)$ is not S-open domain. Furthermore, $(\hat{\aleph}, \Gamma)^c$ is a $S\mathfrak{c}$-closed set, but $(\hat{\aleph}, \Gamma)^c$ is not S-closed domain.

Definition 3.12. [27] In a STS, a SS $(\hat{\aleph}, \Gamma)$ is considered soft regular open ($S\mathfrak{r}$-open) if it $int(cl(\hat{\aleph}, \Gamma)) = (\hat{\aleph}, \Gamma)$. Similarly, a SS $(\hat{\aleph}, \Gamma)$ is considered soft regular closed ($S\mathfrak{r}$-closed) if $cl(int(\hat{\aleph}, \Gamma)) = (\hat{\aleph}, \Gamma)$, or if its soft complement is an $S\mathfrak{r}$-open.

Remark 3.13. In general, $S\mathfrak{c}$-open (resp., $S\mathfrak{c}$-closed) sets and $S\mathfrak{r}$-open (resp., $S\mathfrak{r}$-closed) sets are not comparable, as illustrated by the following example.

Example 3.14. By Example 3.9, let $(\hat{\aleph}, \Gamma) = \{(\gamma_1, \{1, 2, 3\}), (\gamma_2, \{2, 3\})\}$. Then $int(cl(\hat{\aleph}, \Gamma))$ $= int(\hat{\aleph}, \Gamma) \hat{\sqcup} (\hat{I}, \Gamma) = (\hat{\aleph}, \Gamma)$ is $S\mathfrak{r}$-open. However, $cl((\hat{\aleph}, \Gamma)) \hat{\setminus} (\hat{\aleph}, \Gamma) = (\hat{\aleph}, \Gamma) \sqcup (\hat{I}, \Lambda) = (\hat{I}, \Gamma)$ is S-uncountable set, hence $(\hat{\aleph}, \Gamma)$ is not $S\mathfrak{c}$-open.

Moreover, let $(\hat{\aleph}_1, \Gamma) = (\hat{\hat{\Re}}, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma)$. Then

$$\begin{aligned} cl(int((\hat{\aleph}_1, \Gamma))) &= cl(int((\hat{\hat{\Re}}, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma))) \\ &= cl((Q, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma)) \\ &= (\hat{I}, \Gamma) \hat{\sqcup} ((Q, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma)) \\ &= (\hat{\hat{\Re}}, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma) \\ &= (\hat{\aleph}_1, \Gamma), \end{aligned}$$

s $S\mathfrak{r}$-closed. Also,

$$\begin{aligned}(\hat{\aleph}_1,\Gamma)\,\hat{\setminus}\,int((\hat{\aleph}_1,\Gamma)) &= ((\hat{\hat{\Re}},\Gamma)\,\hat{\setminus}\,(\hat{\aleph},\Gamma))\hat{\setminus}int((\hat{\hat{\Re}},\Gamma)\,\hat{\setminus}\,(\hat{\aleph},\Gamma))\\ &= ((\hat{\hat{\Re}},\Gamma)\,\hat{\setminus}\,(\hat{\aleph},\Gamma))\hat{\setminus}((\hat{Q},\Gamma)\,\hat{\setminus}\,(\hat{\aleph},\Gamma))\\ &= (\hat{I},\Gamma),\end{aligned}$$

is S-uncountable set, then $(\hat{\aleph}_1,\Gamma)$ is not $S\mathfrak{c}$-closed set.

Remark 3.15. The definitions of S-open, S-closed, $S\mathfrak{c}$-open, and $S\mathfrak{c}$-closed sets lead to the following diagram:

$$S\mathfrak{c}\text{-open set} \longrightarrow S\text{-open set}$$
$$S\mathfrak{c}\text{-closed set} \longrightarrow S\text{-closed set}$$

Diagram (i)

None of the above implications are reversible.

Proposition 3.16. *A finite soft union of $S\mathfrak{c}$-open sets remains $S\mathfrak{c}$-open.*

Proof. Let $\{(\hat{\aleph}_{\mathsf{j}},\Gamma)\}_{\mathsf{j}\in\mathfrak{J}}$ be a finite collection of $S\mathfrak{c}$-open sets. Then $(\hat{\aleph}_{\mathsf{j}},\Gamma)$ is an S-open set and $cl((\hat{\aleph}_{\mathsf{j}},\Gamma))\ \hat{\setminus}(\hat{\aleph}_{\mathsf{j}},\Gamma)$ is S-countable for all $\mathfrak{j}$. Now, consider $\hat{\bigsqcup}_{\mathsf{j}=1}^{n}(\hat{\aleph}_{\mathsf{j}},\Gamma)$ is S-open, then to show the other condition of $S\mathfrak{c}$-open set. Now, we have, $cl(\hat{\bigsqcup}_{\mathsf{j}=1}^{n}\ (\hat{\aleph}_{\mathsf{j}},\Gamma))\ \hat{\setminus}\ \hat{\bigsqcup}_{\mathsf{j}=1}^{n}\ (\hat{\aleph}_{\mathsf{j}},\Gamma)$ $=\hat{\bigsqcup}_{\mathsf{j}=1}^{n}(cl((\hat{\aleph}_{\mathsf{j}},\Gamma))\ \hat{\setminus}\ (\hat{\aleph}_{\mathsf{j}},\Gamma))$, since the finite soft union of S-countable sets is S-countable. Then, $cl(\hat{\bigsqcup}_{\mathsf{j}=1}^{n}(\hat{\aleph}_{\mathsf{j}},\Gamma))\ \hat{\setminus}\ \hat{\bigsqcup}_{\mathsf{j}=1}^{n}(\hat{\aleph}_{\mathsf{j}},\Gamma)$ is S-countable. Therefore, $\hat{\bigsqcup}_{\mathsf{j}=1}^{n}(\hat{\aleph}_{\mathsf{j}},\Gamma)$ is $S\mathfrak{c}$-open.

Example 3.17. Let $(\hat{\Lambda}_i,\Gamma)=\{\{\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i},\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}_i}\}:\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i}\neq\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}_i},i\in I\}$ be a family of pairwise disjoint soft spaces, where I is an uncountable index set. Let $\hat{\Im}_i$ be a particular soft point topology on $(\hat{\Lambda}_i,\Gamma)$ at $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i}$. Let the $\hat{\Im}=\{(\hat{\aleph},\Gamma)\,\hat{\sqsubseteq}\,\hat{\sqcup}_{i\in I}(\hat{\Lambda}_i,\Gamma):(\hat{\aleph},\Gamma)\,\hat{\sqcap}\,(\hat{\Lambda}_i,\Gamma)\,\hat{\sqsubseteq}\,(\hat{\Lambda}_i,\Gamma)$ S-open for all $i\}$ be a ST on the disjoint soft union $(\hat{\Lambda},\Gamma)=\hat{\sqcup}_{i\in I}\ (\hat{\Lambda}_i,\Gamma)$. We call $(\Lambda,\hat{\Im},\Gamma)$ soft topological sum of the $(\hat{\Lambda}_i,\Gamma)$. for all $i\in I$, pick $(\hat{\mu}_i,\Gamma)=\{\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i}\}\,\hat{\sqsubseteq}\,(\hat{\Lambda},\Gamma)$, where $(\hat{\mu}_i,\Gamma)=\{\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}_i}\}$ for all i and $(\hat{\mu}_i,\Gamma)=(\Phi,\Gamma)$ for all $i\neq j$ in I. Then, $(\hat{\mu}_i,\Gamma)$ is S-open and $cl((\hat{\mu}_i,\Gamma))\ \hat{\setminus}\ (\hat{\mu}_i,\Gamma)$ is finite. Therefore, $(\hat{\mu}_i,\Gamma)$ is $S\mathfrak{c}$-open in $(\hat{\Lambda},\Gamma)$ for all $i\in I$. However, $cl(\hat{\sqcup}_{i\in I}\,(\hat{\mu}_i,\Gamma))\ \hat{\setminus}\ \hat{\sqcup}_{i\in I}\,(\hat{\mu}_i,\Gamma)$ is S-uncountable. Hence, $\hat{\sqcup}_{i\in I}\,(\hat{\mu}_i,\Gamma)$ is not $S\mathfrak{c}$-open in $(\hat{\Lambda},\Gamma)$.

Corollary 3.18. *A finite soft intersection of $S\mathfrak{c}$-closed sets remains $S\mathfrak{c}$-closed.*

Proof. This is evident from the Theorems 3.16, Theorem 3.3 and by Morgan's Laws.

Theorem 3.19. *A finite soft union of $S\mathfrak{c}$-closed sets remains $S\mathfrak{c}$-closed.*

Proof.

Let $(\hat{\aleph}_{\mathsf{j}},\Gamma)$ be a collection of $S\mathfrak{c}$-closed sets for all $\{j=1,2,...,n\}$, it follows that $(\hat{\aleph}\mathsf{j},\Gamma)$ is S-closed, and the soft difference $(\hat{\aleph}_{\mathsf{j}},\Gamma)\ \hat{\setminus}\ \mathrm{int}((\hat{\aleph}_{\mathsf{j}},\Gamma))$ is S-countable. Considering the union

$\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)$, which is also S-closed, demonstrating the other condition suffices to prove the $S\mathfrak{c}$-closed property.

Claim

$$\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)\,\hat{\setminus}\, int\big(\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)\big)\,\hat{\sqsubseteq}\,\hat{\bigsqcup}_{j=1}^{n}\big((\hat{\aleph}_j,\Gamma)\,\hat{\setminus}\, int((\hat{\aleph}_j,\Gamma))\big).$$

Let $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ \hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)\ \hat{\setminus}\ int(\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma))$ be arbitrary. Since $\hat{\bigsqcup}_{j=1}^{n}int((\hat{\aleph}_j,\Gamma))\ \hat{\sqsubseteq}\ int(\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma))$, then there exists $j'\ \hat{\in}\ \{1,2,3,\ldots,n\}$ such that $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ (\hat{\aleph}_1,\Gamma)_{j'}$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\notin}\ int((\hat{\aleph}_j,\Gamma))$ for all $j\in\{1,2,3,\ldots,n\}$. Then $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ ((\hat{\aleph}_1,\Gamma)_{j'})\ \hat{\setminus}\ int\big((\hat{\aleph}_1,\Gamma)_{j'}\big)$, then $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ \bigsqcup_{j=1}^{n}\big((\hat{\aleph}_j,\Gamma)\ \hat{\setminus}\ int\big((\hat{\aleph}_j,\Gamma)\big)\big)$. Claim is proved.

Since the finite soft union of S-countable sets is S-countable. Then, $\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)\ \hat{\setminus}\ int(\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma))$ is S-countable set. Therefore, $\hat{\bigsqcup}_{j=1}^{n}(\hat{\aleph}_j,\Gamma)$ is $S\mathfrak{c}$-closed.

Corollary 3.20. *a finite soft intersections of $S\mathfrak{c}$-open sets remains $S\mathfrak{c}$-open.*

Proof. This is evident from the Theorems 3.3, Theorem 3.19 and by Morgan's Laws.

We suggest referring to soft infra-topological spaces using the term "$S\mathfrak{c}$-open sets," emphasizing their notable characteristic. Furthermore, we can define soft generalized infra-topological spaces as those that may not necessarily encompass the entire absolute soft set $(\hat{\Lambda},\Gamma)$.

Definition 3.21. Let $\hat{\mathfrak{I}}_{S\mathfrak{c}}$ be a collection of $S\mathfrak{c}$-open sets over Λ under a fixed set of parameters Γ, then $\hat{\mathfrak{I}}_{S\mathfrak{c}}$ is said to be infra soft c-topological space ($ISTS_{\mathfrak{c}}$) on Λ if it satisfies the following axioms:

(1) (Φ,Γ), $(\hat{\Lambda},\Gamma)\ \hat{\in}\ \hat{\mathfrak{I}}_{S\mathfrak{c}}$,

(2) Closed under finite soft intersection.

Then, the triple $(\Lambda,\hat{\mathfrak{I}}_{S\mathfrak{c}},\Gamma)$ is called an $ISTS_{\mathfrak{c}}$. Every member of $\hat{\mathfrak{I}}_{S\mathfrak{c}}$ is called an $IS\mathfrak{c}$-open set, and its soft complement is called an $IS\mathfrak{c}$-closed set.

4. Separation Axioms and Compactness are Explored Through a Novel Category of Soft Open Sets.

This section presents the definitions of regular, normal and T_i-spaces for $i = 1,2,3,4$ utilizing $S\mathfrak{c}$-open sets. Furthermore, we define compactness within this category of soft open sets, namely $\mathfrak{c}$-compact space, and $\mathfrak{c}^*$-compact space, and explores their principal properties.

Definition 4.1. A STS $(\Lambda,\hat{\Im},\Gamma)$ on Λ is said to be

(1) $S\mathfrak{c}$-Regular if for each S-closed $(\hat{\mu},\Gamma)$; $(\hat{\mu},\Gamma)\ \hat{\sqsubseteq}\ (\hat{\Lambda},\Gamma)$ and each SP $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\notin}\ (\hat{\mu},\Gamma)$, there exist soft disjoint $S\mathfrak{c}$-open sets $(\hat{\aleph}_1,\Gamma)$ and $(\hat{\aleph}_2,\Gamma)$ such that $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\ \hat{\in}\ (\hat{\aleph}_1,\Gamma)$ and $(\hat{\mu},\Gamma)\ \hat{\sqsubseteq}\ (\hat{\aleph}_2,\Gamma)$.

(2) $S\mathfrak{c}$-Normal if for each pair of S-closed disjoint soft subsets $(\hat{\mu}_1, \Gamma)$ and $(\hat{\mu}_2, \Gamma)$ of $(\hat{\Lambda}, \Gamma)$, there exist soft disjoint $S\mathfrak{c}$-open sets $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ such that $(\hat{\mu}_1, \Gamma) \hat{\sqsubseteq} (\hat{\aleph}_1, \Gamma)$ and $(\hat{\mu}_2, \Gamma) \hat{\sqsubseteq} (\hat{\aleph}_2, \Gamma)$.

(3) $S\mathfrak{c}$-$\hat{T}_0$, if given $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}, \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\Lambda}, \Gamma)$, $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \neq \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$, then there is either a $S\mathfrak{c}$-open set containing $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}$ but not $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$ or a $S\mathfrak{c}$-open set containing $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$ but not $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}$.

(4) $S\mathfrak{c}$-$\hat{T}_1$, if given $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}, \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\Lambda}, \Gamma)$, $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \neq \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$, then there are two $S\mathfrak{c}$-open soft subsets $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ of $(\hat{\Lambda}, \Gamma)$, such that $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \hat{\in} (\hat{\aleph}_1, \Gamma)$, $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\notin} (\hat{\aleph}_1, \Gamma)$, and $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \hat{\notin} (\hat{\aleph}_2, \Gamma)$, $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\aleph}_2, \Gamma)$.

(5) $S\mathfrak{c}$-$\hat{T}_2$, if given $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}}, \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\Lambda}, \Gamma)$, $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \neq \hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}}$, then there are two soft disjoint $S\mathfrak{c}$-open subsets $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ of $(\hat{\Lambda}, \Gamma)$, such that $\hat{\mathcal{P}}^{\gamma}_{\mathfrak{a}} \hat{\in} (\hat{\aleph}_1, \Gamma)$ and $\hat{\mathcal{P}}^{\gamma'}_{\mathfrak{b}} \hat{\in} (\hat{\aleph}_2, \Gamma)$.

(6) $S\mathfrak{c}$-$\hat{T}_3$ if it is a $S\mathfrak{c}$-$\hat{T}_1$ and $S\mathfrak{c}$-regular space.

(7) $S\mathfrak{c}$-$\hat{T}_4$ if it is a $S\mathfrak{c}$-$\hat{T}_1$ and $S\mathfrak{c}$-normal space.

Proposition 4.2. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a STS on Λ, then any $S\mathfrak{c}$-regular (resp. $S\mathfrak{c}$-normal) space is a S-regular (resp. S-normal) space.*

Proof. Obvious from the definitions 4.1 and 3.2.

Remark 4.3. In general, the converse of Proposition 4.2 does not hold as illustrated by the following examples.

Example 4.4. Consider $(\hat{\Re} \times \hat{\Re}, \hat{\Im}_{\mathcal{U} \times \mathcal{U}}, \Gamma)$ represents the usual soft topology on $\hat{\Re} \times \hat{\Re}$, and $\Gamma = \{\gamma_1, \gamma_2\}$ is a set of parameters. Then, $(\hat{\Re} \times \hat{\Re}, \hat{\Im}_{\mathcal{U} \times \mathcal{U}}, \Gamma)$ is a S-normal. However, if we select $(\hat{\mu_1}, \Gamma)$ and $(\hat{\mu_2}, \Gamma)$ as two disjoint S-closed sets defined by:

$$(\hat{\mu_1}, \Gamma) = \{(\gamma_1, [2,3] \times [2,3]), (\gamma_2, [-2,-3] \times [2,3])\},$$

$$(\hat{\mu_2}, \Gamma) = \{(\gamma_1, [-2,-3] \times [2,3]), (\gamma_2, [2,3] \times [2,3])\}.$$

Can not be separated by two disjoint $S\mathfrak{c}$-open sets. Consequently, $(\hat{\Re} \times \hat{\Re}, \hat{\Im}_{\mathcal{U} \times \mathcal{U}}, \Gamma)$ is S-normal space, which is not $S\mathfrak{c}$-normal space.

Example 4.5. The Niemytzki Plane. Let $L = \{(a, b) \in \Re : b \geq 0\}$, be the upper half-plane with the X-axis. Define a soft set $(\hat{\aleph}, \Gamma)$ over L, where Γ is a set of parameters and $\hat{\aleph}$ is a mapping from Γ into the set of subsets of L. Let $L_1 = \{(a, 0) : a \in \Re\}$, i.e., the X-axis, and $\mathsf{L}_2 = L \setminus L_1$. Define the *SS* $(\hat{\aleph}, \Gamma)$ such that for every $(a, 0) \in L_1$ and $r \in \Re, r > 0$, the *SS* $(\hat{\aleph}, \Gamma)\big((a, 0), r\big)$ is the set of all *SP* of L inside the circle of radius r tangent to L_1 at $(a, 0)$. Furthermore, let $(\hat{\aleph}, \Gamma)_i\big((a, 0)\big) = (\hat{\aleph}, \Gamma)\big((a, 0), \frac{1}{i}\big) \hat{\sqcup} \{(a, 0)\}$ for $i \in \mathcal{N}$. For every $(a, b) \in L_2$ and $r > 0$, let $(\hat{\aleph}, \Gamma)\big((a, b), r\big)$ be the set of all *SPs* of L inside the circle of radius r centered at (a, b), and define $(\hat{\aleph}, \Gamma)_i\big((a, b)\big) = (\hat{\aleph}, \Gamma)\big((a, b), \frac{1}{i}\big)$ for $i \in \mathcal{N}$. The Niemytzki Plane over *SS* is S-regular. Let $(\hat{\mu}, \Gamma) = cl\Big((\hat{\aleph}, \Gamma)\big((2, 2), \frac{1}{2}\big)\Big)$ be a S-closed set. Since $(-5, 2) \hat{\notin} (\hat{\mu}, \Gamma)$, it cannot be separated by two disjoint $S\mathfrak{c}$-open sets because the smallest $S\mathfrak{c}$-open set containing $(\hat{\mu}, \Gamma)$ is L_2. Hence, the Niemytzki Plane is not a $S\mathfrak{c}$-regular space.

Remark 4.6. It is evident from the definitions that every $S\mathfrak{c}$-$\hat{T}_i$-space is an S-$\hat{\mathtt{T}}_{\mathtt{i}}$-space for $i \in 0, 1, 2$. However, the converse may not hold, as the following examples demonstrate.

Example 4.7. From Example 4.5, The Niemytzki Plane in soft set theory is S-$\hat{T}_3$-space, but is not $S\mathfrak{c}$-$\hat{T}_0$-space.

Proposition 4.8. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a STS on Λ, then any $S\mathfrak{c}$-$\hat{T}_4$-space is a $S\mathfrak{c}$-$\hat{T}_3$-space.*

Proof. Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-$\hat{T}_4$-space. Then, it is $S\mathfrak{c}$-normal and $S\mathfrak{c}$-$\hat{T}_1$-space. Let $(\hat{\mu}, \Gamma)$ be any S-closed subset of $(\hat{\Lambda}, \Gamma)$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$ be a SP in $(\hat{\Lambda}, \Gamma)$ with $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \hat{\notin} (\hat{\mu}, \Gamma)$. Since $S\mathfrak{c}$-$\hat{T}_1$-space is S-$\hat{T}_1$-space. Hence, $\{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\}$ is S-closed. Thus, $(\hat{\mu}, \Gamma)$ and $\{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\}$ are disjoint S-closed, by $S\mathfrak{c}$-normality of $(\Lambda, \hat{\Im}, \Gamma)$, there exists two disjoint $S\mathfrak{c}$-open sets $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ containing $(\hat{\mu}, \Gamma)$ and $\{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}\}$, respectively. Therefore, $(\Lambda, \hat{\Im}, \Gamma)$ is a $S\mathfrak{c}$-$\hat{T}_3$-space

Remark 4.9. In general, the converse of Proposition 4.8 does not hold.

Example 4.10. Let $(\hat{\Re}^2, \hat{\Im}, \Gamma)$ be a STS over $\Re^2$ and $(\hat{\Theta}, \Gamma)$ be a SS, where $(\hat{\Theta}, \Gamma) = \{(\gamma, \hat{\Theta}(\gamma))$: $\hat{\Theta}(\gamma) = (x, y), y \geq 0, \gamma \in \Gamma\}$, the upper half soft plane with the $\hat{X}$-axis. Let $(\hat{\Theta}_1, \Gamma) = \{(\gamma, \hat{\Theta}_1(\gamma))$: $\hat{\Theta}_1(\gamma) = (x, 0), \gamma \in \Gamma\}$, i.e., the $\hat{X}$-axis. Let $(\hat{P}', \Gamma) = (\hat{\Theta}, \Gamma) \hat{\setminus} (\hat{\Theta}_1, \Gamma)$ is S-$\hat{T}_2$-space, but is not $S\mathfrak{c}$-$\hat{T}_0$-space. There exist two SPs $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} = \hat{\mathcal{P}}_{(-1,2)}^{\gamma} \neq \hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} = \hat{\mathcal{P}}_{(1,2)}^{\gamma'}$ such that there is not $S\mathfrak{c}$-open set containing either $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$ but not $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ or containing $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ but not $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, because the smallest $S\mathfrak{c}$-open set containing either $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$ or $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ is $(\hat{P}', \Gamma)$.

Based on the previous theorems and examples, we can derive the following diagram:

$$\begin{array}{ccccccccc} S\mathfrak{c}\text{–}\hat{T}_4 & \to & S\mathfrak{c}\text{–}\hat{T}_3 & \to & S\mathfrak{c}\text{–}\hat{T}_2 & \to & S\mathfrak{c}\text{–}\hat{T}_1 & \to & S\mathfrak{c}\text{–}\hat{T}_0. \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S\text{-}\hat{T}_4 & \longrightarrow & S\text{-}\hat{T}_3 & \longrightarrow & S\text{-}\hat{T}_2 & \longrightarrow & S\text{-}\hat{T}_1 & \longrightarrow & S\text{-}\hat{T}_0. \end{array}$$

Diagram (ii)

None of the above implications is reversible.

Definition 4.11. Let $(\Lambda, \hat{\Im}, \Gamma)$ be a STS, then a SS $(\hat{\aleph}, \Gamma)$ is a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) set if for any S-open (resp., $S\mathfrak{c}$-open) soft cover of $(\hat{\aleph}, \Gamma)$ has a finite soft subcover of $S\mathfrak{c}$-open (resp., S-open) sets. In particular, a STS is said to be a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) space if $(\hat{\Lambda}, \Gamma)$ is a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) set.

Theorem 4.12. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a STS, every $S\mathfrak{c}$-compact set (space) is S-compact.*

Proof. Let $(\hat{\aleph}, \Gamma)$ be a $S\mathfrak{c}$-compact set in STS $(\Lambda, \hat{\Im}, \Gamma)$, then $(\hat{\aleph}, \Gamma)$ is S-open, by definition 4.11, $(\hat{\aleph}, \Gamma)$ is S-compact.

Remark 4.13. The converse does not always hold. In general, every S-compact space which is not $S\mathfrak{c}$-compact.

Example 4.14. Let $(\hat{\aleph}, \Gamma) = \{(\gamma, [-1, 1])\}$ be a SS, and let $(\hat{\aleph}, \hat{\Im}_{\aleph}, \Gamma)$ be a STS over $\aleph$ and parameterize $\Gamma = \{\gamma\}$ generated by $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$, where

$$(\hat{\aleph}_1, \Gamma) = \{(\gamma, (a, 1]) : -1 < a < 0\},$$
$$(\hat{\aleph}_2, \Gamma) = \{(\gamma, [-1, b)]) : 0 < b < 1\}.$$

Then all SSs of the form $\{(\gamma, (a, b)) : a < 0 < b\})$ are S-open sets in $(\hat{\aleph}, \hat{\Im}_{\aleph}, \Gamma)$. Clearly, $(\hat{\aleph}, \Gamma)$ is a S-compact space because any S-open covering of the two SSs $(\hat{\aleph}_1, \Gamma)$ and $(\hat{\aleph}_2, \Gamma)$ which include $\hat{\mathcal{P}}^{\gamma}_{\{1\}}$ and $\hat{\mathcal{P}}^{\gamma}_{\{-1\}}$ will finite S-open cover $(\hat{\aleph}, \Gamma)$, but it is not $S\mathfrak{c}$-compact because if we take

$$(\hat{\aleph}_3, \Gamma) = \{(\gamma, [-1, 0.5))\},$$
$$(\hat{\aleph}_4, \Gamma) = \{(\gamma, (-0.5, 1])\},$$

is an S-open cover for $(\hat{\aleph}, \Gamma)$ which no finite S-subcover of $S\mathfrak{c}$-open due to $(\hat{\aleph}_3, \Gamma)$ and $(\hat{\aleph}_4, \Gamma)$ are not $S\mathfrak{c}$-open sets such that

$$\begin{aligned} cl(\hat{\aleph}_3, \Gamma)\hat{\setminus}(\hat{\aleph}_3, \Gamma) &= cl(\gamma, [-1, 0.5))\hat{\setminus}(\gamma, [-1, 0.5)) \\ &= (\gamma, [-1, 1])\hat{\setminus}(\gamma, [-1, 0.5)) \\ &= (\gamma, [0.5, 1]), \\ cl(\hat{\aleph}_4, \Gamma)\hat{\setminus}(\hat{\aleph}_4, \Gamma) &= cl(\gamma, (-0.5, 1])\hat{\setminus}(\gamma, (-0.5, 1]) \\ &= (\gamma, [-1, 1])\hat{\setminus}(\gamma, (-0.5, 1]) \\ &= (\gamma, [-1, -0.5]). \end{aligned}$$

are not S-countable sets.

Theorem 4.15. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a STS, every S-compact set (space) is $S\mathfrak{c}^*$-compact.*

Proof. By Definition 4.11 any S-compact set(space) is $S\mathfrak{c}^*$-compact.

Remark 4.16. In general, every $S\mathfrak{c}^*$-compact which is not S-compact.

Example 4.17. Consider $(\Re, \hat{\Im}_{\hat{r}}, \Gamma)$, where $\hat{\Im}_{\hat{r}}$ is the right order ST on $\Re$. Then, $(\hat{\Re}, \hat{\Im}_{\hat{r}}, \Gamma)$ is not S-compact space. Alternatively, the collection of $S\mathfrak{c}$-open set in $(\Re, \hat{\Im}_{\hat{r}}, \Gamma)$ is $\{(\Phi, \Gamma), (\hat{\Re}, \Gamma)\}$. Hence, $(\hat{\Re}, \hat{\Im}_{\hat{r}}, \Gamma)$ is $S\mathfrak{c}^*$-compact.

Remark 4.18. From Theorems 4.12, 4.15 and Examples 4.14, 4.17 the following diagram is acquired:

$$S\mathfrak{c}\text{-compact} \longrightarrow S\text{-compact} \longrightarrow S\mathfrak{c}^*\text{-compact}.$$

Diagram (iii)

Remark 4.19. None of these relationships can be reversed.

Remark 4.20. In a *STS* $(\Lambda, \hat{\Im}, \Gamma)$, the $S\mathfrak{c}^*$-compactness is not hereditary.

Example 4.21. Let $(\Re, \hat{\Im}_{(\gamma,\{2\})}, \Gamma)$ be the included soft point topological space on $\Re$ by $(\gamma, \{2\})$. Then $(\Re, \hat{\Im}_{(\gamma,\{2\})}, \Gamma)$ is $S\mathfrak{c}^*$-compact space, because $(\hat{\Re}, \Gamma)$ is $S\mathfrak{c}$-open finite subcover for any S-open cover of $(\hat{\Re}, \Gamma)$. However, $(\hat{\Re}, \Gamma)\hat{\setminus}(\gamma, \{2\})$ is a soft infinite discrete subspace. Hence, $(\hat{\Re}, \Gamma)\hat{\setminus}(\gamma, \{2\})$ with soft discrete topology is not $S\mathfrak{c}^*$-compact.

Remark 4.22. The $S\mathfrak{c}^*$-compactness is hereditary with respect to $S\mathfrak{c}$-closed subspace.

Theorem 4.23. *Every $S\mathfrak{c}$-closed subset of $S\mathfrak{c}$-compact (resp. $S\mathfrak{c}^*$-compact) space is $S\mathfrak{c}$-compact (resp. $S\mathfrak{c}^*$-compact).*

Proof. Let $(\hat{\Theta}, \Gamma)$ be a S-closed subset in a *STS* $(\Lambda, \hat{\Im}, \Gamma)$ and $\hat{\Psi} = \{(\hat{\aleph}, \Gamma)_\zeta : \zeta \in \Delta\}$ be a S-open (resp., $S\mathfrak{c}$-open) cover of $(\hat{\Theta}, \Gamma)$, where $(\hat{\aleph}, \Gamma)_\zeta \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$ is S-open (resp., $S\mathfrak{c}$-open) in $(\hat{\Lambda}, \Gamma)$, for each $\zeta \in \Psi$. Since $(\hat{\Theta}, \Gamma)$ is $S\mathfrak{c}$-closed in $(\hat{\Lambda}, \Gamma)$, then $(\hat{\Lambda}, \Gamma)\hat{\setminus}(\hat{\Theta}, \Gamma)$ is $S\mathfrak{c}$-open in $(\hat{\Lambda}, \Gamma)$. However, any $S\mathfrak{c}$-open set is S-open, then $(\hat{\Theta}, \Gamma)^c = (\hat{\Lambda}, \Gamma)\hat{\setminus}(\hat{\Theta}, \Gamma)$ is S-open in $(\hat{\Lambda}, \Gamma)$. Thus $\{(\hat{\aleph}, \Gamma)_\zeta : \zeta \in \Omega\} \hat{\sqcup} (\hat{\Theta}, \Gamma)^c$ is a S-open (resp., $S\mathfrak{c}$-open) cover for $(\hat{\Lambda}, \Gamma)$. Since $(\hat{\Lambda}, \Gamma)$ is $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact), then there exist $\zeta_1, \zeta_2, ..., \zeta_n$ such that $(\hat{\Lambda}, \Gamma) \hat{\sqsubseteq} \left(\hat{\sqcup}_{i=1}^n (\hat{\aleph}, \Gamma)_i\right) \hat{\sqcup} (\hat{\Theta}, \Gamma)^c$. Thus $(\hat{\Theta}, \Gamma) \hat{\sqsubseteq} \hat{\sqcup}_{i=1}^n (\hat{\aleph}, \Gamma)_i$. Therefore, $(\hat{\Theta}, \Gamma)$ is $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact).

Corollary 4.24. *If $(\hat{\Lambda}, \Gamma)$ is $S\mathfrak{c}$-compact and $(\hat{\Theta}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$ is S-closed, then $(\hat{\Theta}, \Gamma)$ is S-compact.*

Proof. Obviously by 4.12, since any $S\mathfrak{c}$-compact space is S-compact

Theorem 4.25. *A $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) subset of $S\mathfrak{c}$-$\hat{T}_2$-space is $S\mathfrak{c}$-closed.*

Proof.

Let $(\hat{\aleph}, \Gamma)$ be a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) subset of $S\mathfrak{c}$-$\hat{T}_2$-space over Λ. If $(\hat{\aleph}, \Gamma) = (\hat{\Lambda}, \Gamma)$, then $(\hat{\Lambda}, \Gamma)$ is S-closed and $(\hat{\Lambda}, \Gamma)\hat{\setminus} int(\hat{\Lambda}, \Gamma) = (\Phi, \Gamma)$ is S-countable set. Hence $(\hat{\Lambda}, \Gamma)$ is $S\mathfrak{c}$-closed. If $(\hat{\aleph}, \Gamma) \neq (\hat{\Lambda}, \Gamma)$. Suppose $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} SP(\Lambda)_\Gamma$, $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \neq \hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ such that $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \hat{\in} (\hat{\aleph}, \Gamma)^c$ and $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} (\hat{\aleph}, \Gamma)$. Then there are two disjoint $S\mathfrak{c}$-open sets $(\hat{\aleph}_1, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}}$ and $(\hat{\aleph}_2, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}}$ such that $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} (\hat{\aleph}_2, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}}$, $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma'} \hat{\in} (\hat{\aleph}_1, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}}$ with $(\hat{\aleph}_2, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}} \hat{\sqcap} (\hat{\aleph}_1, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}} = (\Phi, \Gamma)$. Let $\hat{\Psi} = \{(\hat{\mu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}_i}^{\gamma'}} : i \in I\}$, then $\hat{\Psi}$ is S-open (resp., $S\mathfrak{c}$-open) cover of $(\hat{\aleph}, \Gamma)$. Since $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-compact (resp.,

$S\mathfrak{c}^*$-compact), then there are $\hat{\mathcal{P}}_{\mathfrak{b}_1}^{\gamma'}, \hat{\mathcal{P}}_{\mathfrak{b}_2}^{\gamma'}, ..., \hat{\mathcal{P}}_{\mathfrak{b}_n}^{\gamma'}$, such that $(\hat{\aleph}, \Gamma) \hat{\sqsubseteq} \hat{\sqcup}_{i=1}^{n} (\hat{\mu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}_i}^{\gamma'}} = (\hat{\mu}, \Gamma)$. Let $(\hat{\nu}, \Gamma) = \hat{\sqcap}_{i=1}^{n} (\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_i}^{\gamma}}$, by Corollary 3.20, $(\hat{\nu}, \Gamma)$ is a $S\mathfrak{c}$-open set containing $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, clearly $(\hat{\nu}, \Gamma)$ $\hat{\sqcap}$ $(\hat{\mu}, \Gamma) = (\Phi, \Gamma)$, so that $(\hat{\nu}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma)$, hence $\hat{\mathcal{P}}_{\mathfrak{a}_i}^{\gamma}$ is a $S\mathfrak{c}$-interior point of $(\hat{\Lambda}, \Gamma) \hat{\setminus}$ $(\hat{\aleph}, \Gamma)$, Therefore $(\hat{\Lambda}, \Gamma) \hat{\setminus} (\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-open. Hence, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-closed.

Theorem 4.26. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-compact $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-regular.*

Proof. Let $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \hat{\in} SP(\Lambda)_{\Gamma}$ and $(\hat{\aleph}, \Gamma)$ be a $S\mathfrak{c}$-closed set not containing $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$. Since $S\mathfrak{c}$-$\hat{T}_2$, for each $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} (\hat{\aleph}, \Gamma)$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma} \hat{\in} (\hat{\aleph}, \Gamma)^c$ there are two disjoint $S\mathfrak{c}$-open sets $(\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}}$ and $(\hat{\mu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}}$ containing $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, respectively. Suppose $\hat{\Psi} = \{(\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}} \colon \hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'} \hat{\in} (\hat{\aleph}, \Gamma)\}$ is a $S\mathfrak{c}$-open cover of $(\hat{\aleph}, \Gamma)$. By Theorem 4.25, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-compact, and therefore there is a $S\mathfrak{c}$-open finite subcover $\{(\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_1}^{\gamma}}, (\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_2}^{\gamma}}, ..., (\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_n}^{\gamma}}\} \hat{\sqsubseteq} \hat{\Psi}$. Thus, $(\hat{\nu}, \Gamma) = \hat{\sqcup}_{i}^{n} (\hat{\nu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{b}_i}^{\gamma'}}$ and $(\hat{\mu}, \Gamma)$ $= \hat{\sqcap}_{i}^{n} (\hat{\mu}, \Gamma)_{\hat{\mathcal{P}}_{\mathfrak{a}_i}^{\gamma}}$ are disjoint $S\mathfrak{c}$-open sets containing $\hat{\mathcal{P}}_{\mathfrak{b}}^{\gamma'}$ and $\hat{\mathcal{P}}_{\mathfrak{a}}^{\gamma}$, hence $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-regular.

Corollary 4.27. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-compact $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-$\mathtt{T}_3$-space.*

Proof. Obviously.

Theorem 4.28. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}^*$-compact $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is S-regular.*

Proof. The same of Theorem 4.26.

Corollary 4.29. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}^*$-compact $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is S-$\mathtt{T}_3$-space.*

Theorem 4.30. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-compact (resp., $S\mathfrak{c}^*$-compact) $S\mathfrak{c}$-$\hat{T}_2$-space. Then $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-normal (resp., S-normal).*

Proof. The same of Theorem 4.26.

5. $S\mathfrak{c}$-Continuous and $S\mathfrak{c}^*$-Continuous Functions

This section introduces two definitions of continuity utilizing $S\mathfrak{c}$-open set namely, $S\mathfrak{c}$-continuous, $S\mathfrak{c}^*$-continuous. Moreover, the concept of $S\mathfrak{c}$-homeomorphism and $S\mathfrak{c}^*$-homeomorphism via the concept of $S\mathfrak{c}$-open sets are studied. In addition, some of their properties with $S\mathfrak{c}$-compact and $S\mathfrak{c}^*$-compact spaces are discussed.

Definition 5.1. A S-function $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is said to be $S\mathfrak{c}$-*continuous* (resp., $S\mathfrak{c}^*$-continuous) if $\hat{\varphi}_{\sigma\alpha}^{-1}(\hat{\Theta}, \Gamma)$ is $S\mathfrak{c}$-open (resp., S-open) in $(\hat{\Lambda}, \Gamma)$ for any S-open (resp., $S\mathfrak{c}$-open) subset $(\hat{\Theta}, \Gamma)$ in $(\hat{\Delta}, \Gamma')$.

Theorem 5.2. *Every $S\mathfrak{c}$-continuous function is S-continuous, and every S-continuous function is $S\mathfrak{c}^*$-continuous.*

Proof. Obviously, from the definition 5.1

Remark 5.3. The converse may not be true as shown by the following two examples.

Example 5.4. Let $(\Re, \hat{\Im}_I, \Gamma)$ be the excluded soft set topological space on $\Re$ by I. Then the identity S-function $\hat{\mathcal{I}}_{\sigma\alpha} : (\Re, \hat{\Im}_I, \Gamma) \to (\Re, \hat{\Im}_I, \Gamma)$ is S-continuous function, which is not $S\mathfrak{c}$-continuous, because $(\hat{Q}, \Gamma) \hat{\in} \hat{\Im}_I$ is S-open and $\hat{\mathcal{I}}^{-1}_{\sigma\alpha}(\hat{Q}, \Gamma) = (\hat{Q}, \Gamma)$ is not $S\mathfrak{c}$-open, because $cl(\hat{Q}, \Gamma) \hat{\setminus} (\hat{Q}, \Gamma) = (\hat{\Re}, \Gamma) \hat{\setminus} (\hat{Q}, \Gamma) = (\hat{I}, \Gamma)$ is S-uncountable set.

Example 5.5. In Example 4.17, $(\Re, \hat{\Im}_r, \Gamma')$ is the right order soft topology on the set of all soft real numbers $(\hat{\Re}, \Gamma')$. The family of all $S\mathfrak{c}$-open set in $(\Re, \hat{\Im}_r, \Gamma')$ is $\{(\Phi, \Gamma'), (\hat{\Re}, \Gamma')\}$ only. Consider $(\hat{\Re}, \hat{\Im}_{cof}, \Gamma)$, where $\hat{\Im}_{cof}$ is the finite complement soft topology on the soft set of all soft real numbers $(\hat{\Re}, \Gamma')$. Then the identity S-function $\hat{\mathcal{I}}_{\sigma\alpha} : (\hat{\Re}, \hat{\Im}_{cof}, \Gamma) \to (\hat{\Re}, \hat{\Im}_r, \Gamma')$ is $S\mathfrak{c}^*$-continuous function, which is not S-continuous,

Remark 5.6. It is clear, from the definitions of S-continuous, , and Examples 5.4, 5.5, the following diagram is obtained:

$S\mathfrak{c}$-continuity $\longrightarrow$ S-continuity $\longrightarrow$ $S\mathfrak{c}'$-continuity. **Diagram (iv)**

None of the above implications is reversible.

Definition 5.7. Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ be a S-continuous function. Then, $\hat{\varphi}_{\sigma\alpha}$ is a $S\mathfrak{c}$-open function if for any $S\mathfrak{c}$-open subset $(\hat{\Theta}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$, we have $\hat{\varphi}_{\sigma\alpha}(\hat{\Theta}, \Gamma)$ is a S-open subset in $(\hat{\Delta}, \Gamma')$. Moreover, $\hat{\varphi}_{\sigma\alpha}$ is said to be $S\mathfrak{c}$-closed function if any $S\mathfrak{c}$-closed subset $(\hat{\Sigma}, \Gamma)$ $\hat{\sqsubseteq}$ $(\hat{\Lambda}, \Gamma)$, we have $\hat{\varphi}_{\sigma\alpha}$ $(\hat{\Sigma}, \Gamma)$ is S-closed subset in $(\hat{\Delta}, \Gamma')$.

Theorem 5.8. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ be $S\mathfrak{c}$-continuous (resp., $S\mathfrak{c}^*$-continuous), onto S-function and $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}^*$-compact, then $(\hat{\Delta}, \Gamma')$ is $S\mathfrak{c}^*$-compact.*

Proof. Let $\{(\hat{\Theta}, \Gamma')_\alpha : \alpha \in \Omega\}$ be a $S\mathfrak{c}$-open cover of $(\hat{\Delta}, \Gamma')$. Since $\hat{\varphi}_{\sigma\alpha}$ is $S\mathfrak{c}$-continuous (resp., $S\mathfrak{c}^*$-continuous) and any $S\mathfrak{c}$-open set is S-open, then $\hat{\varphi}^{-1}_{\alpha}((\hat{\Theta}, \Gamma')_\alpha)$ is $S\mathfrak{c}$-open (resp., S-open) in $(\hat{\Lambda}, \Gamma)$ for each $\alpha \in \Omega$. Since $(\hat{\Delta}, \Gamma') \hat{\sqsubseteq} \hat{\bigsqcup}_{\alpha\in\Omega}(\hat{\Theta}, \Gamma')_\alpha$, then $(\hat{\Lambda}, \Gamma)$ $= \hat{\varphi}^{-1}_{\sigma\alpha}(\hat{\Delta}, \Gamma') \hat{\sqsubseteq} \hat{\varphi}^{-1}_{\sigma\alpha}(\bigcup_{\alpha\in\Omega}(\hat{\Theta}, \Gamma')_\alpha) = \hat{\bigsqcup}_{\alpha\in\Omega}\hat{\varphi}^{-1}_{\alpha}((\hat{\Theta}, \Gamma')_\alpha)$, that is means $\{\hat{\varphi}^{-1}_{\alpha}((\hat{\Theta}, \Gamma')_\alpha) : \alpha \in \Omega\}$ is $S\mathfrak{c}$-open cover of $(\hat{\Lambda}, \Gamma)$. Then, by the $S\mathfrak{c}^*$-compactness of $(\hat{\Lambda}, \Gamma)$, there exist $\alpha_1, \alpha_2, \cdots \alpha_n \in \Omega$ such that $\hat{\varphi}^{-1}_{\sigma\alpha}((\hat{\Theta}, \Gamma')_{\alpha_1}) \hat{\sqcup} \hat{\varphi}^{-1}_{\sigma\alpha}((\hat{\Theta}, \Gamma')_{\alpha_2}) \hat{\sqcup} \ldots \hat{\sqcup} \hat{\varphi}^{-1}_{\sigma\alpha}((\hat{\Theta}, \Gamma')_{\alpha_n}) = (\hat{\Lambda}, \Gamma)$, then $\hat{\varphi}_{\sigma\alpha}[\hat{\varphi}^{-1}_{\sigma\alpha}((\hat{\Theta}, \Gamma')_{\alpha_1}) \hat{\sqcup} \hat{\varphi}^{-1}_{\sigma\alpha}((\hat{\Theta}, \Gamma')_{\alpha_2}) \hat{\sqcup} \ldots \hat{\sqcup} \hat{\varphi}^{-1}_{\sigma\alpha}((\hat{\Theta}, \Gamma')_{\alpha_n})] = \hat{\varphi}_{\sigma\alpha}(\hat{\Lambda}, \Gamma)$, thus $\hat{\varphi}_{\sigma\alpha}$ $\big(\hat{\varphi}^{-1}_{\alpha}((\hat{\Theta}, \Gamma')_{\alpha_1})\big) \hat{\sqcup} \hat{\varphi}_{\sigma\alpha}\big(\hat{\varphi}^{-1}_{\alpha}((\hat{\Theta}, \Gamma')_{\alpha_2})\big) \hat{\sqcup} \ldots \hat{\sqcup} \hat{\varphi}_{\sigma\alpha}\big(\hat{\varphi}^{-1}_{\sigma\alpha}((\hat{\Theta}, \Gamma')_{\alpha_n})\big) = (\hat{\Delta}, \Gamma')$, then $(\hat{\Theta}, \Gamma')_{\alpha_1}$ $\hat{\sqcup}$ $(\hat{\Theta}, \Gamma')_{\alpha_2} \hat{\sqcup} \ldots \hat{\sqcup} (\hat{\Theta}, \Gamma')_{\alpha_n} = (\hat{\Delta}, \Gamma')$. Hence, $\{(\hat{\Theta}, \Gamma')_{\alpha_1} \hat{\sqcup} (\hat{\Theta}, \Gamma')_{\alpha_2} \hat{\sqcup} \ldots \hat{\sqcup} (\hat{\Theta}, \Gamma')_{\alpha_n}\}$ is a finite S-subcover of S-open sets for $(\hat{\Delta}, \Gamma')$. Therefore, $(\Delta, \hat{\Im}', \Gamma')$ is a $S\mathfrak{c}^*$-compact space.

Corollary 5.9. *$S\mathfrak{c}^*$-compactness is a topological property.*

From the previous theorem and the Diagram (iii), we have the following corollary:

Corollary 5.10.

i) *$S\mathfrak{c}$-continuous image of S-compact (resp.,$S\mathfrak{c}$-compact, $S\mathfrak{c}^*$-compact) is S-compact (resp.,$S\mathfrak{c}$-compact, $S\mathfrak{c}^*$-compact);*

ii) *$S\mathfrak{c}^*$-continuous image of S-compact (resp.,$S\mathfrak{c}$-compact, $S\mathfrak{c}^*$-compact) is S-compact (resp.,$S\mathfrak{c}$-compact, $S\mathfrak{c}^*$-compact);*

iii) *S-continuous image of $S\mathfrak{c}$-compact is $S\mathfrak{c}$-compact.*

Theorem 5.11. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is onto $S\mathfrak{c}$-continuous function and $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}^*$-compact, then $(\hat{\Delta}, \Gamma')$ is S-compact.*

Proof. Same the proof of Theorem 5.8.

Theorem 5.12. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ be $S\mathfrak{c}$-continuous, $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}^*$-compact, and $(\Delta, \hat{\Im}', \Gamma')$ is $S\mathfrak{c}$-$\hat{T}_2$-space, then $\hat{\varphi}_{\sigma\alpha}$ is $S\mathfrak{c}$-closed function.*

Proof. Let $(\hat{\aleph}, \Gamma)$ be a $S\mathfrak{c}$-closed subset in $(\hat{\Lambda}, \Gamma)$. Since $(\hat{\Lambda}, \Gamma)$ is $S\mathfrak{c}^*$-compact, then from Theorem 5.9, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}^*$-compact. Since the image of a $S\mathfrak{c}^*$-compact space is $S\mathfrak{c}^*$-compact under a $S\mathfrak{c}$-continuous function (see Corollary 5.10). Hence, $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}^*$-compact. Since every $S\mathfrak{c}^*$-compact subspace of a $S\mathfrak{c}$-$\hat{T}_2$-space is $S\mathfrak{c}$-closed (see Theorem 4.25) this implies that $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-closed. But any $S\mathfrak{c}$-closed set is S-closed. Therefore, $\hat{\varphi}_{\sigma\alpha}$ is a $S\mathfrak{c}$-closed function.

From Theorem 5.12, and Diagram (iv), we have the following corollaries:

Corollary 5.13. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is $S\mathfrak{c}$-continuous, $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-compact (resp., S-compact), and $(\Delta, \hat{\Im}', \Gamma')$ is $S\mathfrak{c}$-$\hat{T}_2$-space, then $\hat{\varphi}_{\sigma\alpha}$ is $S\mathfrak{c}$-closed function.*

Corollary 5.14. *Let $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is S-continuous, $(\Lambda, \hat{\Im}, \Gamma)$ is $S\mathfrak{c}$-compact, and $(\Delta, \hat{\Im}', \Gamma')$ is $S\mathfrak{c}$-$\hat{T}_2$-space, then $\hat{\varphi}_{\sigma\alpha}$ is $S\mathfrak{c}$-closed function.*

Definition 5.15. A bijection S-function $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is said to be $S\mathfrak{c}$-*homeomorphism* (resp., $S\mathfrak{c}^*$-homeomorphism) if and $\hat{\varphi}_{\sigma\alpha}$ and $\hat{\varphi}_{\alpha}^{-1}$ are $S\mathfrak{c}$-continuous (resp., $S\mathfrak{c}^*$-continuous).

$$S\mathfrak{c}\text{-homeomorphism} \longrightarrow S\text{-homeomorphism} \longrightarrow S\mathfrak{c}^*\text{-homeomorphism}.$$

Diagram (v)

Proposition 5.16. *Every $S\mathfrak{c}$-homeomorphism function is S-homeomorphism, and every S-homeomorphism function is $S\mathfrak{c}^*$-homeomorphism.*

Proof. It is clear, from the definition 5.15.

Remark 5.17. The converse of Proposition 5.16 may not be hold in general as shown by the following two examples.

Example 5.18. See Example 5.4.

Example 5.19. Consider $(\hat{\Re}, \hat{\Im}_r, \Gamma')$ is the right order soft topology on the soft set of all soft real numbers $(\hat{\Re}, \Gamma')$, and $(\Re, \hat{\Im}_{ID}, \Gamma)$, where $\hat{\Im}_{ID}$ is the indiscrete soft topology on the soft set of all soft real numbers $(\hat{\Re}, \Gamma)$. Then the identity S-function $\hat{\mathcal{I}}_{\sigma\alpha} : (\Re, \hat{\Im}_{ID}, \Gamma) \to (\hat{\Re}, \hat{\Im}_r, \Gamma')$ is $S\mathfrak{c}^*$-homeomorphism function, which is not S-continuous because The family of all $S\mathfrak{c}$-open sets in $(\hat{\Re}, \hat{\Im}_r, \Gamma')$ and $(\Re, \hat{\Im}_{ID}, \Gamma)$ are $\{(\Phi, \Gamma), (\hat{\Re}, \Gamma)\}$ only.

Theorem 5.20. *Let $(\Lambda, \hat{\Im}, \Gamma)$ be a $S\mathfrak{c}$-compact soft topological space and let $(\Delta, \hat{\Im}', \Gamma')$ be a $S\mathfrak{c}$-$\hat{T}_2$-space. Then any $S\mathfrak{c}^*$-continuous S-bijection $\hat{\varphi}_{\sigma\alpha} : (\Lambda, \hat{\Im}, \Gamma) \to (\Delta, \hat{\Im}', \Gamma')$ is a $S\mathfrak{c}^*$-homeomorphism.*

Proof. Let $(\hat{\aleph}, \Gamma) \hat{\sqsubseteq} (\hat{\Lambda}, \Gamma)$ be a $S\mathfrak{c}$-closed set. By Theorem 5.9, $(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-compact, and therefore, $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-compact by Corollary 5.10. By Theorem 4.25, we have that $\hat{\varphi}_{\sigma\alpha}(\hat{\aleph}, \Gamma)$ is $S\mathfrak{c}$-closed, as required.

6. Conclusions

This article contributes to expanding the literature on new soft topological properties with a new class of soft open sets. The results show that some soft topological properties, such as compactness, continuity and others, can be generalized, leading to new examples and properties that enhance the understanding of soft topological spaces. On the other hand, the paper introduces the definitions of novel types of soft sets namely, $S\mathfrak{c}$-open and $S\mathfrak{c}$-closed sets and discusses their fundamental properties. So, it presents a study about soft separation axioms defined by these new soft sets. Additionally, the paper defines $S\mathfrak{c}$-compact and $S\mathfrak{c}^*$-compact in STS accompanied by properties and counterexamples related to these definitions. Furthermore, it introduces the concept of $S\mathfrak{c}$-continuous, $S\mathfrak{c}$-homeomorphism and $S\mathfrak{c}^*$-homeomorphism. Moreover, it investigate new kinds of regularity and normality in$STSs$ using $S\mathfrak{c}$-open(closed) sets. According to the obtained results, many properties of these concepts in TSs are still valid for soft topological structures. The aim is to further develop the soft topological concepts utilizing $S\mathfrak{c}$-open and $S\mathfrak{c}$-closed sets, including $S\mathfrak{c}$-paracompact spaces, $S\mathfrak{c}$-connected spaces, as well as hyper $S\mathfrak{c}$-open and hyper $S\mathfrak{c}$-closed sets.

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