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The Theoretical Framework on Approximation of Neutrosophic Numbers and Their Application

Augus Kurian[1], Sumathi I R[2,*] and Omaima Al-Shanqiti[3]
[1,2,*]Department of Mathematics, Amrita School of Physical Sciences, Coimbatore,
Amrita Vishwa Vidyapeetham, India [1]auguskurian@gmail.com
[3] Umm Al-Qura University, Department of Mathematics, Mekkah, Saudia Arabia; omshanqiti@uqu.edu.sa
*Correspondence: ir_sumathi@cb.amrita.edu

Abstract. Neutrosophic sets are effectual logic represented to understand ambiguous and inconsistent information. They are frequently used to explain many types of partial or incomplete information. Researchers have given much attention to the decision-making theory and its associated methodologies based on uncertain linguistic factors. This article emphasizes the novel neutrosophic number approximations to handle linguistic variables and their application in multiple-attribute decision-making. Different approximation techniques are introduced in neutrosophic sets, but substantial data loss may occur. Hence, a hexagonal neutrosophic number was proposed to deal with information loss during approximation. Also, the comparison study with existing techniques is explored to show the effectiveness of the proposed approximation. The expected interval criterion was retained although an approximation was made to give more desirable features. An MCDM (Multi-Criteria Decision Making) problem is presented to demonstrate efficiency and simplicity with uncertain parameters.

Keywords: Neutrosophic Number; Generalized Nonlinear Hexagonal Neutrosophic Number with asymmetry; Approximation; Expected Interval; Values; MCDM.

1. Introduction

Fuzzy logic enables greater flexibility when dealing with imprecise or uncertain data. Zadeh [1] initiated the fuzzy logic to deal with ambiguity and uncertainty in a flexible and enhanced way of reasoning, which allows the truth value to range between 0 and 1, that helps to model imprecise data in various fields like artificial intelligence, control systems, and decision-making. Fuzzy numbers have been introduced to deal with imprecise numerical quantities in decision analysis, risk assessment, finance, etc., In [2–4] various fuzzy numbers, such as triangular, trapezoidal, pentagonal, and hexagonal, were discussed along with their arithmetic operators.

A new concept of generalized 'n' gonal linear fuzzy numbers, which encompasses triangular, trapezoidal, hexagonal, octagonal, and decagonal fuzzy numbers was introduced in [5], and a novel total ordering method was presented. Initially, interval approximations [6] were provided. Later in [7], an interval approximation operator that preserves this nonspecificity measure is provided, along with an uncertainty measure known as entropy-like nonspecificity. Various forms of approximation of fuzzy numbers were extensively discussed in [8–19]. Later, using the Karush Kuhn Tucker theorem [20–22], LR fuzzy representations and the L fuzzy rough set were more thoroughly analyzed, and their approximations were described.

Compared to traditional fuzzy sets, Intuitionistic fuzzy sets (IFS) [23] provide a more comprehensive representation of uncertainty, which can be helpful in decision-making where a lack of information or conflicting evidence is present. IFS captures not only an element's degree of membership but also its degree of non-membership. Similarly, various Intuitionistic Fuzzy Numbers (IFN) were developed for MCDM, and their approximation operators [24, 25] were determined for multiple polygonal IFNs.

Neutrosophic sets [26, 27] are extensions of fuzzy sets and IFS to address even more complex aspects of uncertainty. The importance of neutrosophic sets lies in their ability to handle both the degrees of truth, falsehood, and indeterminacy. The added dimension allows for better representation in areas where the boundary between membership and non-membership is poorly defined. Similarly, different neutrosophic numbers [28–40] were developed to approach MCDM problems systematically. To systematically deal with decision-making problems, various arithmetic operations are required. The trapezoidal approximation [41] of neutrosophic numbers is defined to deal with transportation problems. Since neutrosophic is an effective tool to represent indeterminacy, and inspired by the numerous aspects of neutrosophic set, in this paper, we describe the approximations of neutrosophic numbers using GNHNNA. In many problem-solving scenarios, particularly those involving uncertainty and imprecision, it can be challenging to find accurate arithmetic operations. Traditional approximation methods frequently give rise to data loss and sometimes insufficient generalization of the (α, β, γ)-cuts. Due to these limitations, we investigated new approximation methods, and the hexagonal neutrosophic approximation method can be a more suitable alternative. An example is given to show the comparison with other existing approximations. We provided various theorems pertaining to the approximation points and their values. The Karush Kuhn Tucker (KKT) theorem and the expected interval criterion are applied in approximation. The result obtained approximates efficiently any linear and nonlinear neutrosophic numbers to generalized hexagonal neutrosophic numbers.

Contributions

- Using Lagrange's method, a novel approach for the approximation of neutrosophic numbers using hexagonal neutrosophic numbers is introduced.
- The paper significantly contributes by employing advanced distance measures and expected interval criteria in the approximation process. This enhances the accuracy and reliability of the neutrosophic number approximation, addressing practical concerns in real-world applications.
- Providing a solid theoretical foundation, the paper establishes and discusses theorems related to the values, ambiguity, and approximation of neutrosophic numbers. This contributes to the understanding of the mathematical principles underlying the proposed approximation method.
- A numerical example is given to show the effectiveness of the proposed method in solving real-world problems. This application demonstrates the applicability and efficiency of the hexagonal neutrosophic number approximation in addressing complex scenarios.
- The paper makes a noteworthy contribution by conducting a comparative analysis, demonstrating that Hexagonal neutrosophic numbers yield the least error compared to other neutrosophic numbers.

The systematic framework of this article is as follows: The preliminaries provide the basic definitions relevant to this paper. In the next section, the distance measure and expected interval are discussed in the case of neutrosophic sets. Then, by satisfying the Karush Kuhn Tucker theorem and the expected interval criterion, the distance measure is minimized. In section 3, the approximation of any neutrosophic number to GNHNNA is explored along with graphical interpretation. As one of the particular cases of the previous section, the approximation of linear hexagonal neutrosophic numbers with symmetry is explored. The comparison with an approximation of various neutrosophic numbers was given to show that the proposed approximation has minimal data loss. In Quantitative Analysis of Decision-Making Research, a theoretical framework was given to deal with MCDM problems effectively. Finally, a decision-making problem was given in the stock market with vague criteria, and the future scope is given in the conclusion.

2. Preliminaries

Definition 2.1. [36] Generalized Nonlinear Hexagonal Neutrosophic Numbers with Asymmetry (GNHNNA) is defined as, $A_{GNHNNA} = \Big\{ T\left(a_1, a_2, a_3, a_4, a_5, a_6; r, s; \omega\right)_{(n_1,n_2,n_3,n_4)},$
$I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1, s_1; \rho\right)_{(m_1,m_2,m_3,m_4)}, F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2, s_2; \delta\right)_{(p_1,p_2,p_3,p_4)} \Big\}.$
where the membership function is defined as,

$$T_{A_{GNHNNA}} = \begin{cases} r\left(\frac{x-a_1}{a_2-a_1}\right)^{n_1} & , \text{ if } a_1 \leq x \leq a_2 \\ r+(\omega-r)\left(\frac{x-a_2}{a_3-a_2}\right)^{n_2} & , \text{ if } a_2 \leq x \leq a_3 \\ \omega & , \text{ if } a_3 \leq x \leq a_4 \\ s+(\omega-s)\left(\frac{x-a_5}{a_4-a_5}\right)^{n_3} & , \text{ if } a_4 \leq x \leq a_5 \\ s\left(\frac{x-a_6}{a_5-a_6}\right)^{n_4} & , \text{ if } a_5 \leq x \leq a_6 \\ 0 & , \text{ otherwise.} \end{cases}$$

The indeterminacy function can be described as,

$$I_{A_{GNHNNA}} = \begin{cases} 1-r_1\left(\frac{x-b_1}{b_2-b_1}\right)^{m_1} & , \text{ if } b_1 \leq x \leq b_2 \\ 1-r_1+(r_1-\rho)\left(\frac{x-b_2}{b_3-b_2}\right)^{m_2} & , \text{ if } b_2 \leq x \leq b_3 \\ 1-\rho & , \text{ if } b_3 \leq x \leq b_4 \\ 1-s_1+(s_1-\rho)\left(\frac{x-b_5}{b_4-b_5}\right)^{m_3} & , \text{ if } b_4 \leq x \leq b_5 \\ 1-s_1\left(\frac{x-b_6}{b_5-b_6}\right)^{m_4} & , \text{ if } b_5 \leq x \leq b_6 \\ 1 & , \text{ otherwise.} \end{cases}$$

Non-membership function can be described as,

$$F_{A_{GNHNNA}} = \begin{cases} 1-r_2\left(\frac{x-c_1}{c_2-c_1}\right)^{p_1} & , \text{ if } c_1 \leq x \leq c_2 \\ 1-r_2+(r_2-\delta)\left(\frac{x-c_2}{c_3-c_2}\right)^{p_2} & , \text{ if } c_2 \leq x \leq c_3 \\ 1-\delta & , \text{ if } c_3 \leq x \leq c_4 \\ 1-s_2+(s_2-\delta)\left(\frac{x-c_5}{c_4-c_5}\right)^{p_3} & , \text{ if } c_4 \leq x \leq c_5 \\ 1-s_2\left(\frac{x-c_6}{c_5-c_6}\right)^{p_4} & , \text{ if } c_5 \leq x \leq c_6 \\ 1 & , \text{ otherwise.} \end{cases}$$

where $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$, $b_1 < b_2 < b_3 < b_4 < b_5 < b_6$ and $c_1 < c_2 < c_3 < c_4 < c_5 < c_6$ $\forall$ a_i, b_i and $c_i\,(i = 1, ..., 6)$ are real constants and $0 < r, s < \omega$, $1-\rho < r_1, s_1 < 1$ and $1-\delta < r_2, s_2 < 1$, ω, ρ, $\delta \in [0, 1]$.

Definition 2.2. [36] The (α, β, γ)-cut form of GNHNNA is as follows, $A_{(\alpha,\beta,\gamma)} = \{x \in X/\ T_{A_{GNHNNA}} \geq \alpha,\ I_{A_{GNHNNA}} \leq \beta,\ F_{A_{GNHNNA}} \leq \gamma\}$. Let $T_\alpha = \{x \in X/\ T_{A_{GNHNNA}} \geq \alpha\}$ where $\alpha \in (0, \omega]$.

If $r \leq s$ then,
$$T_\alpha = \begin{cases} \left[a_1 + \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}}(a_2-a_1), a_6 + \left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}}(a_5-a_6)\right] & , \text{ if } 0 < \alpha \leq r \\ \left[a_2 + \left(\frac{\alpha-r}{\omega-r}\right)^{\frac{1}{n_2}}(a_3-a_2), a_6 + \left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}}(a_5-a_6)\right] & , \text{ if } r \leq \alpha \leq s \\ \left[a_2 + \left(\frac{\alpha-r}{\omega-r}\right)^{\frac{1}{n_2}}(a_3-a_2), a_5 + \left(\frac{\alpha-s}{\omega-s}\right)^{\frac{1}{n_3}}(a_4-a_5)\right] & , \text{ if } s \leq \alpha \leq \omega \\ [a_3, a_4] & , \text{ if } \alpha = \omega. \end{cases}$$

If $s \le r$, then, $T_\alpha = \begin{cases} \left[a_1 + \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}}(a_2 - a_1), a_6 + \left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}}(a_5 - a_6)\right] & \text{, if } 0 < \alpha \le s \\ \left[a_1 + \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}}(a_2 - a_1), a_5 + \left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}}(a_4 - a_5)\right] & \text{, if } s \le \alpha \le r \\ \left[a_2 + \left(\frac{\alpha - r}{\omega - r}\right)^{\frac{1}{n_2}}(a_3 - a_2), a_5 + \left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}}(a_4 - a_5)\right] & \text{, if } r \le \alpha \le \omega \\ [a_3, a_4] & \text{, if } \alpha = \omega. \end{cases}$

Let $I_\beta = \{x \in X / \ I_{A_{GNHNNA}} \le \beta\}$, where $\beta \in [1-\rho, 1)$. If $r_1 \le s_1$,

$$I_\beta = \begin{cases} [b_3, b_4] & \text{, if } \beta = 1 - \rho \\ \left[b_2 + \left(\frac{1-\beta-r_1}{\rho - r_1}\right)^{\frac{1}{m_2}}(b_3 - b_2), b_5 + \left(\frac{1-\beta-s_1}{\rho - s_1}\right)^{\frac{1}{m_3}}(b_4 - b_5)\right] & \text{, if } 1 - \rho \le \beta \le 1 - s_1 \\ \left[b_2 + \left(\frac{1-\beta-r_1}{\rho - r_1}\right)^{\frac{1}{m_2}}(b_3 - b_2), b_6 + \left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(b_5 - b_6)\right] & \text{, if } 1 - s_1 \le \beta \le 1 - r_1 \\ \left[b_1 + \left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(b_2 - b_1), b_6 + \left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(b_5 - b_6)\right] & \text{, if } 1 - r_1 \le \beta < 1. \end{cases}$$

If $s_1 \le r_1$, then,

$$I_\beta = \begin{cases} [b_3, b_4] & \text{, if } \beta = 1 - \rho \\ \left[b_2 + \left(\frac{1-\beta-r_1}{\rho - r_1}\right)^{\frac{1}{m_2}}(b_3 - b_2), b_5 + \left(\frac{1-\beta-s_1}{\rho - s_1}\right)^{\frac{1}{m_3}}(b_4 - b_5)\right] & \text{, if } 1 - \rho \le \beta \le 1 - r_1 \\ \left[b_1 + \left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(b_2 - b_1), b_5 + \left(\frac{1-\beta-s_1}{\rho - s_1}\right)^{\frac{1}{m_3}}(b_4 - b_5)\right] & \text{, if } 1 - r_1 \le \beta \le 1 - s_1 \\ \left[b_1 + \left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}(b_2 - b_1), b_6 + \left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}(b_5 - b_6)\right] & \text{, if } 1 - s_1 \le \beta < 1. \end{cases}$$

Let $F_\gamma = \{x \in X / F_{A_{GNHNNA}} \le \gamma\}$, where $\gamma \in [1-\delta, 1)$. If $r_2 \le s_2$, then,

$$F_\gamma = \begin{cases} [c_3, c_4] & \text{, if } \gamma = 1 - \delta \\ \left[c_2 + \left(\frac{1-\gamma-r_2}{\delta - r_2}\right)^{\frac{1}{p_2}}(c_3 - c_2), c_5 + \left(\frac{1-\gamma-s_2}{\delta - s_2}\right)^{\frac{1}{p_3}}(c_4 - c_5)\right] & \text{, if } 1 - \delta \le \gamma \le 1 - s_2 \\ \left[c_2 + \left(\frac{1-\gamma-r_2}{\delta - r_2}\right)^{\frac{1}{p_2}}(c_3 - c_2), c_6 + \left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(c_5 - c_6)\right] & \text{, if } 1 - s_2 \le \gamma \le 1 - r_2 \\ \left[c_1 + \left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(c_2 - c_1), c_6 + \left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(c_5 - c_6)\right] & \text{, if } 1 - r_2 \le \gamma < 1. \end{cases}$$

If $s_2 \le r_2$, then

$$F_\gamma = \begin{cases} [c_3, c_4] & \text{, if } \gamma = 1 - \delta \\ \left[c_2 + \left(\frac{1-\gamma-r_2}{\delta - r_2}\right)^{\frac{1}{p_2}}(c_3 - c_2), c_5 + \left(\frac{1-\gamma-s_2}{\delta - s_2}\right)^{\frac{1}{p_3}}(c_4 - c_5)\right] & \text{, if } 1 - \delta \le \gamma \le 1 - r_2 \\ \left[c_1 + \left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(c_2 - c_1), c_5 + \left(\frac{1-\gamma-s_2}{\delta - s_2}\right)^{\frac{1}{p_3}}(c_4 - c_5)\right] & \text{, if } 1 - r_2 \le \gamma \le 1 - s_2 \\ \left[c_1 + \left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}(c_2 - c_1), c_6 + \left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}(c_5 - c_6)\right] & \text{, if } 1 - s_2 \le \gamma < 1. \end{cases}$$

Definition 2.3. [36] Let $A_{GNHNNA} = \Big\{T(a_1, a_2, a_3, a_4, a_5, a_6; r, s; \omega)_{(n_1, n_2, n_3, n_4)}$, $I(b_1, b_2, b_3, b_4, b_5, b_6; r_1, s_1; \rho)_{(m_1, m_2, m_3, m_4)}, F(c_1, c_2, c_3, c_4, c_5, c_6; r_2, s_2; \delta)_{(p_1, p_2, p_3, p_4)}\Big\}$ be the GNHNNA and $T_\alpha = [L(\alpha), R(\alpha)]$, $I_\beta = [L_1(\beta), R_1(\beta)]$ and $F_\gamma = [L_2(\gamma), R_2(\gamma)]$ be the α, β and γ - cut respectively.

(1) The values of the A_{GNHNNA} corresponding to α - cut set, denoted by $V_T(A_{GNHNNA})$, is defined as, $V_T(A_{GNHNNA}) = \int_0^{\omega}[L(\alpha)+R(\alpha)]f_1(\alpha)\,d\alpha$. where $f_1(\alpha) \in [0,1]\,(\alpha\in[0,\omega])\,,f_1(0)=0$ and $f_1(\alpha)$ is increasing and monotonic in $\alpha\in[0,\omega]$.

(2) The values of the A_{GNHNNA} corresponding to β - cut set, denoted by $V_I(A_{GNHNNA})$, is defined as, $V_I(A_{GNHNNA}) = \int_{1-\rho}^{1}[L_1(\beta)+R_1(\beta)]f_2(\beta)\,d\beta$. where $f_2(\beta) \in [0,1]\,(\beta\in[\rho,1])\,,f_2(1)=0$ and $f_2(\beta)$ is decreasing and monotonic in $\beta\in[\rho,1]$.

(3) The values of the A_{GNHNNA} corresponding to γ - cut set, denoted by $V_F(A_{GNHNNA})$, is defined as, $V_F(A_{GNHNNA}) = \int_{1-\delta}^{1}[L_2(\gamma)+R_2(\gamma)]f_3(\gamma)\,d\gamma$. where $f_3(\gamma) \in [0,1]\,(\gamma\in[\delta,1])\,,f_3(1)=0$ and $f_3(\gamma)$ is decreasing and monotonic in $\gamma\in[\delta,1]$.

Without loss of generality, we choose $f_1(\alpha)=\alpha\ (\alpha\in[0,\omega])$, $f_2(\beta)=1-\beta\ (\beta\in[1-\rho,1])$ and $f_3(\gamma)=1-\gamma\ (\gamma\in[1-\delta,1])$.

Theorem 2.4. *[36] Let* $A_{GNHNNA}=\Big\{T(a_1,a_2,a_3,a_4,a_5,a_6;r,s;\omega)_{(n_1,n_2,n_3,n_4)}\,,$ $I(b_1,b_2,b_3,b_4,b_5,b_6;r_1,s_1;\rho)_{(m_1,m_2,m_3,m_4)}\,,F(c_1,c_2,c_3,c_4,c_5,c_6;r_2,s_2;\delta)_{(p_1,p_2,p_3,p_4)}\Big\}$. *be the GNHNNA. Then,*

(1) *The α-cut set of the A_{GNHNNA} for the truth-membership is computed as $T_\alpha = [L(\alpha),R(\alpha)]$ where $\alpha\in[0,\omega]$. If $f_1(\alpha)=\alpha$, we obtain the value and ambiguity of the GNHNNA number A_{GNHNNA} as,*

$$V_T(A_{GNHNNA}) = \left[\frac{1}{2}-\frac{n_1}{1+2n_1}\right]r^2a_1+\left[\frac{\omega^2-r^2}{2}-n_2\frac{(\omega-r)^2}{1+2n_2}-rn_2\frac{\omega-r}{1+n_2}\right]a_2$$
$$+\left[\frac{n_2(\omega-r)^2}{1+2n_2}+rn_2\frac{\omega-r}{1+n_2}\right]a_3+\left[\frac{n_3(\omega-s)^2}{1+2n_3}+sn_3\frac{\omega-s}{1+n_3}\right]a_4$$
$$+\left[\frac{\omega^2-s^2}{2}-n_3\frac{(\omega-s)^2}{1+2n_3}-sn_3\frac{\omega-s}{1+n_3}\right]a_5+\left[\frac{1}{2}-\frac{n_4}{1+2n_4}\right]s^2a_6$$

(2) *The β-cut set of A_{GNHNNA} for indeterminacy -membership is calculated as $I_\beta = [L_1(\beta),R_1(\beta)]$ where $\beta\in[1-\rho,1]$. When $f_2(\beta)=1-\beta$, we obtain the value and ambiguity of A_{GNHNNA}, respectively as,*

$$V_I(A_{GNHNNA}) = \left[\frac{1}{2}-\frac{m_1}{1+2m_1}\right]r_1^2b_1+\left[\frac{\rho^2-r_1^2}{2}-m_2\frac{(\rho-r_1)^2}{1+2m_2}-r_1m_2\frac{\rho-r_1}{1+m_2}\right]b_2$$
$$+\left[\frac{m_2(\rho-r_1)^2}{1+2m_2}+r_1m_2\frac{\rho-r_1}{1+m_2}\right]b_3+\left[\frac{m_3(\rho-s_1)^2}{1+2m_3}+s_1m_3\frac{\rho-s_1}{1+m_3}\right]b_4$$
$$+\left[\frac{\rho^2-s_1^2}{2}-m_3\frac{(\rho-s_1)^2}{1+2m_3}-s_1m_3\frac{\rho-s_1}{1+m_3}\right]b_5+\left[\frac{1}{2}-\frac{m_4}{1+2m_4}\right]s_1^2b_6$$

(3) *The γ-cut set of A_{GNHNNA} for falsity-membership is computed as $F_\gamma=[L_2(\gamma),R_2(\gamma)]$ where $\gamma\in[1-\delta,1]$. If $f_3(\gamma)=1-\beta$, we obtain the value and ambiguity of A_{GNHNNA},*

by the following steps,

$$V_F\left(A_{GNHNNA}\right)=\left[\frac{1}{2}-\frac{p_1}{1+2p_1}\right]r_2^2c_1+\left[\frac{\delta^2-r_2^2}{2}-p_2\frac{\left(\delta-r_2\right)^2}{1+2p_2}-r_2p_2\frac{\delta-r_2}{1+p_2}\right]c_2$$
$$+\left[\frac{p_2\left(\delta-r_2\right)^2}{1+2p_2}+r_2p_2\frac{\delta-r_2}{1+p_2}\right]c_3+\left[\frac{p_3\left(\delta-s_2\right)^2}{1+2p_3}+s_2p_3\frac{\delta-s_2}{1+p_3}\right]c_4$$
$$+\left[\frac{\delta^2-s_2^2}{2}-p_3\frac{\left(\delta-s_2\right)^2}{1+2p_3}-s_2p_3\frac{\delta-s_2}{1+p_3}\right]c_5+\left[\frac{1}{2}-\frac{p_4}{1+2p_4}\right]s_2^2c_6$$

Definition 2.5 ([38])**.** The score function of a SVNS($\mathscr{N}$) is defined as
$\tilde{S}(S)=\theta(x)-\iota(x)-\phi(x)$.

3. Hexagonal Approximation of a Neutrosophic Number

The first two definitions will provide the structure to approximate any curve to a GNHNNA.

Definition 3.1. Let A and B be two neutrosophic numbers with (α, β, γ)-cuts ($[A_{T-},\ A_{T+}]$, $[A_{I-},\ A_{I+}]$, $[A_{F-},\ A_{F+}]$) and ($[B_{T-},\ B_{T+}]$, $[B_{I-},\ B_{I+}]$, $[B_{F-},\ B_{F+}]$), then the distance measure between A and B is defined as,

$$d(A,B)=\sqrt{\frac{1}{3}(M+N+O)}\quad \text{where,} \tag{1}$$

$M=\int_0^{\omega}(A_{T+}-B_{T+})^2d\alpha+\int^{\omega}(A_{T-}-B_{T-})^2d\alpha$, $N=\int_{1-\rho}^{1}(A_{I+}-B_{I+})^2d\beta+\int_{1-\rho}^{1}(A_{I-}-B_{I-})^2d\beta$ and $O=\int_{1-\delta}^{1}(A_{F+}-B_{F+})^2d\gamma+\int_{1-\delta}^{1}(A_{F-}-B_{F-})^2d\gamma$

Definition 3.2. Let A be a neutrosophic number with (α, β, γ)-cut ($[A_{T-},\ A_{T+}]$, $[A_{I-},\ A_{I+}]$, $[A_{F-},\ A_{F+}]$), then the expected interval of A is defined as,

$$EI(A)=[X,Y,Z]\quad \text{where,} \tag{2}$$

$X=[\int_0^{\omega}A_{T-}d\alpha,\int_0^{\omega}A_{T+}\mathrm{d}\alpha]$, $Y=[\int_0^{\omega}A_{I-}d\beta,\int_0^{\omega}A_{I+}d\beta]$ and $Z=[\int_0^{\omega}A_{F-}d\gamma,\int_0^{\omega}A_{F+}d\gamma]$.

3.1. *Approximation of GNHNNA*

In this subsection, we approximate any neutrosophic number by preserving expected intervals, and the distance measure defined in definition 3.1 to a GNHNNA by using an approximation operator H: $\mathscr{N}(\mathscr{R})\to\mathscr{N}^H(\mathscr{R})$ where $\mathscr{N}(\mathscr{R})$ represent the set of all neutrosophic numbers defined on the real domain and $\mathscr{N}^H(\mathscr{R})$ is the set of all GNHNNA.

Suppose A be any neutrosophic number with (α, β, γ)-cut ($[A_{T-},\ A_{T+}]$, $[A_{I-},\ A_{I+}]$, $[A_{F-},\ A_{F+}]$). Now we will try to find the nearest GNHNNA H(A), the closest to A that satisfies the distance measure defined in definition 3.1. Let (α, β, γ)-cut of H(A) be ($[H_{T-},\ H_{T+}]$, $[H_{I-},\ H_{I+}]$, $[H_{F-},\ H_{F+}]$), then the distance measure between A and $B=H(A)$ is defined as,

$$d(A,B)=\sqrt{\frac{1}{3}(M+N+O)}\quad \text{where,} \tag{3}$$

$M = \int_0^\omega (A_{T+} - H_{T+})^2 d\alpha + \int_0^\omega (A_{T-} - H_{T-})^2 d\alpha$, $N = \int_{1-\rho}^1 (A_{I+} - H_{I+})^2 d\beta + \int_{1-\rho}^1 (A_{I-} - H_{I-})^2 d\beta$ and $O = \int_{1-\delta}^1 (A_{F+} - H_{F+})^2 d\gamma + \int_{1-\delta}^1 (A_{F-} - H_{F-})^2 d\gamma$

By using definition 2.2, equation 3 reduces to,

$$d(A,B) = \sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,} \tag{4}$$

$$M = \int_0^s (A_{T+} - (a_6 + \left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}} (a_5 - a_6))^2 d\alpha + \int_s^\omega (A_{T+} - (a_5 + \left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}} (a_4 - a_5)))^2 d\alpha$$
$$+ \int_0^r (A_{T-} - (a_1 + \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}} (a_2 - a_1)))^2 d\alpha + \int_r^\omega (A_{T-} - (a_2 + \left(\frac{\alpha - r}{\omega - r}\right)^{\frac{1}{n_2}} (a_3 - a_2)))^2 d\alpha$$

$N = \int_{1-\rho}^{1-s_1} (A_{I+} - (b_5 + \left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}} (b_4 - b_5)))^2 d\beta + \int_{1-s_1}^1 (A_{I+} - (b_6 + \left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}} (b_5 - b_6)))^2 d\beta +$
$\int_{1-\rho}^{1-r_1} (A_{I-} - (b_2 + \left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}} (b_3 - b_2)))^2 d\beta + \int_{1-r_1}^1 (A_{I-} - (b_1 + \left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}} (b_2 - b_1)))^2 d\beta$ and
$O = \int_{1-\delta}^{1-s_2} (A_{F+} - (c_5 + \left(\frac{1-\gamma-s_2}{\delta-s_2}\right)^{\frac{1}{p_3}} (c_4 - c_5)))^2 d\gamma + \int_{1-s_2}^1 (A_{F+} - (c_6 + \left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}} (c_5 - c_6)))^2 d\gamma +$
$\int_{1-\delta}^{1-r_2} (A_{F-} - (c_2 + \left(\frac{1-\gamma-r_2}{\delta-r_2}\right)^{\frac{1}{p_2}} (c_3 - c_2)))^2 d\gamma + \int_{1-r_2}^1 (A_{F-} - (c_1 + \left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}} (c_2 - c_1)))^2 d\gamma$

For GNHNNA, $\int_0^r (A_{T-}) =$

$$\begin{cases} r\frac{a_1 + a_2 n_1}{n_1 + 1} & , \text{ if } 0 < n_1 \\ a_1 r - \lim_{\alpha \to 0} a_1 \alpha - \dfrac{\alpha n_1 (a_1 - a_2)(\frac{\alpha}{r})^{\frac{1}{n_1}}}{n_1 + 1} & , \text{ if } n_1 \leq 0 \end{cases} \tag{5}$$

In a similar way we can compute the other integrals, $\int_0^\omega A_{T+} \mathrm{d}\alpha$, $\int_0^\omega A_{I-} d\beta$, $\int_0^\omega A_{I+} d\beta$, $\int_0^\omega A_{F-} d\gamma$, $\int_0^\omega A_{F+} d\gamma$ given in Definition 3.2.

Minimizing D($a_1, a_2, a_3, a_4, a_5, a_6$,$b_1, b_2, b_3, b_4, b_5, b_6$,$c_1, c_2, c_3, c_4, c_5, c_6$)= $d^2(A, H(A))$ in considering the following constraints is sufficient for finding the approximation.

$$\int_0^\omega H_{T-}\, d\alpha - \int_0^\omega A_{T-}\, d\alpha = 0 \qquad \int_0^\omega H_{T+}\, d\alpha - \int_0^\omega A_{T+}\, d\alpha = 0 \tag{6}$$

$$\int_0^\omega H_{I-}\, d\beta - \int_0^\omega A_{I-}\, d\beta = 0 \qquad \int_0^\omega H_{I+}\, d\beta - \int_0^\omega A_{I+}\, d\beta = 0 \tag{7}$$

$$\int_0^\omega H_{F-}\, d\gamma - \int_0^\omega A_{F-}\, d\gamma = 0 \qquad \int_0^\omega H_{F+}\, d\gamma - \int_0^\omega A_{F+}\, d\gamma = 0 \tag{8}$$

Using the Lagrangian multiplier method, we can find the value of real numbers $a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6$ such that $a_1 \leq a_2 \leq ... \leq a_6, b_1 \leq b_2 \leq ... \leq b_6$ and

$c_1 \leq c_2 \leq ... \leq c_6$ that minimize the function,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + \lambda_1(\int_0^\omega H_{T-} d\alpha - \int_0^\omega A_{T-} d\alpha)$$
$$+ \lambda_2(\int_0^\omega H_{T+} d\alpha - \int_0^\omega A_{T+} d\alpha) + \lambda_3(\int_0^\omega H_{I-} d\beta - \int_0^\omega A_{I-} d\beta) + \lambda_4(\int_0^\omega HI{+}d\beta - \int_0^\omega A_{I+} d\beta)$$
$$+ \lambda_5(\int_0^\omega H_{F-} d\gamma - \int_0^\omega A_{F-} d\gamma) + \lambda_6(\int_0^\omega H_{F+} d\gamma - \int_0^\omega A_{F+} d\gamma) \quad (9)$$

where λ_1, λ_2,..., λ_6 are Lagrangian multipliers.

Now we have to find the partial derivatives. The minimization problem is rewritten as follows using the KKT theorem.

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_1} = 0 \quad (10)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_2} = 0 \quad (11)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_3} = 0 \quad (12)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_4} = 0 \quad (13)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_5} = 0 \quad (14)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial a_6} = 0 \quad (15)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_1} = 0 \quad (16)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_2} = 0 \quad (17)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_3} = 0 \quad (18)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_4} = 0 \quad (19)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_5} = 0 \quad (20)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial b_6} = 0 \quad (21)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_1} = 0 \quad (22)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_2} = 0 \quad (23)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_3} = 0 \quad (24)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_4} = 0 \quad (25)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_5} = 0 \quad (26)$$

$$\frac{\partial L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6)}{\partial c_6} = 0 \tag{27}$$

By solving 10, 11 and 12 we get

$$a_1 = \frac{\begin{array}{l} -(2X^-\omega - 2X^-r + 5X^-n_1\omega - 4X^-n_1r - Y^-n_1r + Z^-n_1r + 4X^-n_1^2\omega + X^-n_1^3\omega - 2X^-n_1^2r - Y^-n_1^2r + \\ Z^-n_1^2r + 2X^-n_1n_2r - 2Y^-n_1n_2r + Z^-n_1n_2r + X^-n_1n_2^2r + 4X^-n_1^2n_2r + 2X^-n_1^3n_2r - Y^-n_1n_2^2r - \\ 2Y^-n_1^2n_2r + Z^-n_1^2n_2r + 2X^-n_1^2n_2^2r + X^-n_1^3n_2^2r - Y^-n_1^2n_2^2r) \end{array}}{2r(n_1rn_2^2 + 2n_1rn_2 + 2\omega - 2r + 2n_1\omega - n_1r)}$$

$$a_2 = \frac{\begin{array}{l} 2Z^- - 2Y^- + X^-n_1 - 2Y^-n_1 - 4Y^-n_2 + 2Z^-n_1 + 2Z^-n_2 + X^-n_1^2 - 2Y^-n_2^2 + X^-n_1^2n_2^2 + \\ 2X^-n_1n_2 - 4Y^-n_1n_2 + 2Z^-n_1n_2 + X^-n_1n_2^2 + 2X^-n_1^2n_2 - 2Y^-n_1n_2^2 \end{array}}{2n_1rn_2^2 + 4n_1rn_2 + 4\omega - 4r + 4n_1\omega - 2n_1r}$$

$$a_3 = \frac{\begin{array}{l} -(4Z^-\omega - 4Z^-r - 2Y^-n_2\omega + 4Z^-n_1\omega + 4Z^-n_2\omega + 2Y^-n_2r - 2Z^-n_1r - 4Z^-n_2r - 2Y^-n_2^2\omega + 2Y^-n_2^2r + \\ X^-n_1n_2\omega - 2Y^-n_1n_2\omega + 4Z^-n_1n_2\omega - X^-n_1n_2r + 2Y^-n_1n_2r + Z^-n_1n_2r + X^-n_1n_2^2\omega + X^-n_1^2n_2\omega - \\ 2Y^-n_1n_2^2\omega - X^-n_1n_2^2r - X^-n_1^2n_2r + 2Y^-n_1n_2^2r + 4Z^-n_1n_2^2r + Z^-n_1n_2^3r + X^-n_1^2n_2^2\omega - X^-n_1^2n_2^2r) \end{array}}{2n_2(\omega - r)(n_1rn_2^2 + 2n_1rn_2 + 2\omega - 2r + 2n_1\omega - n_1r)}$$

By solving 13, 14 and 15 we get

$$a_4 = \frac{\begin{array}{l} -(4Z^+\omega - 4Z^+s - 2Y^+n_3\omega + 4Z^+n_4\omega + 4Z^+n_3\omega + 2Y^+n_3s - 2Z^+n_4s - 4Z^+n_3s - 2Y^+n_3^2\omega + 2Y^+n_3^2s + \\ X^+n_4n_3\omega - 2Y^+n_4n_3\omega + 4Z^+n_4n_3\omega - X^+n_4n_3s + 2Y^+n_4n_3s + Z^+n_4n_3s + X^+n_4n_3^2\omega + X^+n_4^2n_3\omega - \\ 2Y^+n_4n_3^2\omega - X^+n_4n_3^2s - X^+n_4^2n_3s + 2Y^+n_4n_3^2s + 4Z^+n_4n_3^2s + Z^+n_4n_3^3s + X^+n_4^2n_3^2\omega - X^+n_4^2n_3^2s) \end{array}}{2n_3(\omega - s)(n_4sn_3^2 + 2n_4sn_3 + 2\omega - 2s + 2n_4\omega - n_4s)}$$

$$a_5 = \frac{\begin{array}{l} 2Z^+ - 2Y^+ + X^+n_4 - 2Y^+n_4 - 4Y^+n_3 + 2Z^+n_4 + 2Z^+n_3 + X^+n_4^2 - 2Y^+n_3^2 + X^+n_4^2n_3^2 + \\ 2X^+n_4n_3 - 4Y^+n_4n_3 + 2Z^+n_4n_3 + X^+n_4n_3^2 + 2X^+n_4^2n_3 - 2Y^+n_4n_3^2 \end{array}}{2n_4sn_3^2 + 4n_4sn_3 + 4\omega - 4s + 4n_4\omega - 2n_4s}$$

$$a_6 = \frac{\begin{array}{l} -(2X^+\omega - 2X^+s + 5X^+n_4\omega - 4X^+n_4s - Y^+n_4s + Z^+n_4s + 4X^+n_4^2\omega + X^+n_4^3\omega - 2X^+n_4^2s - \\ Y^+n_4^2s + Z^+n_4^2s + 2X^+n_4n_3s - 2Y^+n_4n_3s + Z^+n_4n_3s + X^+n_4n_3^2s + 4X^+n_4^2n_3s + 2X^+n_4^3n_3s - \\ Y^+n_4n_3^2s - 2Y^+n_4^2n_3s + Z^+n_4^2n_3s + 2X^+n_4^2n_3^2s + X^+n_4^3n_3^2s - Y^+n_4^2n_3^2s) \end{array}}{2s(n_4sn_3^2 + 2n_4sn_3 + 2\omega - 2s + 2n_4\omega - n_4s)},$$

where

$$X^{-} = 2\int_0^r \left[\left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}} - 1\right] A_{T^-}\,d\alpha,$$

$$Y^{-} = 2\int_r^\omega \left[\left(\frac{\alpha - r}{\omega - r}\right)^{\frac{1}{n_2}} - 1\right] A_{T^-}\,d\alpha - 2\int_0^r \left(\frac{\alpha}{r}\right)^{\frac{1}{n_1}} A_{T^-}\,d\alpha,$$

$$Z^{-} = -2\int_r^\omega \left(\frac{\alpha - r}{\omega - r}\right)^{\frac{1}{n_2}} A_{T^-}\,d\alpha, X^{+} = 2\int_0^s \left[\left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}} - 1\right] A_{T^+}\,d\alpha,$$

$$Y^{+} = 2\int_s^\omega \left[\left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}} - 1\right] A_{T^+}\,d\alpha - 2\int_0^s \left(\frac{\alpha}{s}\right)^{\frac{1}{n_4}} A_{T^+}\,d\alpha,$$

$$Z^{+} = -2\int_s^\omega \left(\frac{\alpha - s}{\omega - s}\right)^{\frac{1}{n_3}} A_{T^+}\,d\alpha.$$

By solving 16, 17 and 18 we get

$$b_1 = \frac{\begin{array}{l} -(2X^-\rho - 2X^-r_1 - 4X^-m_1r_1 + 5X^-m_1\rho - Y^-m_1r_1 + Z^-m_1r_1 - 2X^-m_1^2r_1 + 4X^-m_1^2\rho + X^-m_1^3\rho) \\ -(Y^-m_1^2r_1 - Z^-m_1^2r_1 + 2X^-m_1m_2r_1 - 2Y^-m_1m_2r_1 + Z^-m_1m_2r_1 + X^-m_1m_2^2r_1) - (4X^-m_1^2m_2r_1 + \\ 2X^-m_1^3m_2r_1 - Y^-m_1m_2^2r_1 - 2Y^-m_1^2m_2r_1 + Z^-m_1^2m_2r_1 + 2X^-m_1^2m_2^2r_1 + X^-m_1^3m_2^2r_1 - Y^-m_1^2m_2^2r_1) \end{array}}{2r_1(m_1r_1m_2^2 + 2m_1r_1m_2 - 2r_1 + 2\rho - m_1r_1 + 2m_1\rho)}$$

$$b_2 = \frac{\begin{array}{l} 2Z^- - 2Y^- + X^-m_1 - 2Y^-m_1 - 4Y^-m_2 + 2Z^-m_1 + 2Z^-m_2 + X^-m_1^2 - 2Y^-m_2^2 \\ + X^-m_1^2m_2^2 + 2X^-m_1m_2 - 4Y^-m_1m_2 + 2Z^-m_1m_2 + X^-m_1m_2^2 + 2X^-m_1^2m_2 - 2Y^-m_1m_2^2 \end{array}}{2m_1r_1m_2^2 + 4m_1r_1m_2 - 4r_1 + 4\rho - 2m_1r_1 + 4m_1\rho}$$

$$b_3 = \frac{\begin{array}{l} 4Z^-\rho - 4Z^-r_1 + 2Y^-m_2r_1 - 2Y^-m_2\rho - 2Z^-m_1r_1 - 4Z^-m_2r_1 + 4Z^-m_1\rho + 4Z^-m_2\rho \\ + 2Y^-m_2^2r_1 - 2Y^-m_2^2\rho - X^-m_1m_2r_1 + X^-m_1m_2\rho + 2Y^-m_1m_2r_1 - 2Y^-m_1m_2\rho \\ + Z^-m_1m_2r_1 + 4Z^-m_1m_2\rho - X^-m_1m_2^2r_1 - X^-m_1^2m_2r_1 + X^-m_1m_2^2\rho + X^-m_1^2m_2\rho \\ + 2Y^-m_1m_2^2r_1 - 2Y^-m_1m_2^2\rho + 4Z^-m_1m_2^2r_1 + Z^-m_1m_2^3r_1 - X^-m_1^2m_2^2r_1 + X^-m_1^2m_2^2\rho \end{array}}{2m_2(r_1 - \rho)(m_1r_1m_2^2 + 2m_1r_1m_2 - 2r_1 + 2\rho - m_1r_1 + 2m_1\rho)}$$

By solving 19, 20 and 21 we get

$$b_4 = \frac{\begin{array}{l} 4Z^+\rho - 4Z^+s_1 + 2Y^+m_3s_1 - 2Y^+m_3\rho - 2Z^+m_4s_1 - 4Z^+m_3s_1 + 4Z^+m_4\rho + 4Z^+m_3\rho \\ + 2Y^+m_3^2s_1 - 2Y^+m_3^2\rho - X^+m_4m_3s_1 + X^+m_4m_3\rho + 2Y^+m_4m_3s_1 - 2Y^+m_4m_3\rho \\ + Z^+m_4m_3s_1 + 4Z^+m_4m_3\rho - X^+m_4m_3^2s_1 - X^+m_4^2m_3s_1 + X^+m_4m_3^2\rho + X^+m_4^2m_3\rho \\ + 2Y^+m_4m_3^2s_1 - 2Y^+m_4m_3^2\rho + 4Z^+m_4m_3^2s_1 + Z^+m_4m_3^3s_1 - X^+m_4^2m_3^2s_1 + X^+m_4^2m_3^2\rho \end{array}}{2m_3(s_1 - \rho)(m_4s_1m_3^2 + 2m_4s_1m_3 - 2s_1 + 2\rho - m_4s_1 + 2m_4\rho)}$$

$$b_5 = \frac{\begin{array}{l} 2Z^+ - 2Y^+ + X^+m_4 - 2Y^+m_4 - 4Y^+m_3 + 2Z^+m_4 + 2Z^+m_3 + X^+m_4^2 - 2Y^+m_3^2 \\ + X^+m_4^2m_3^2 + 2X^+m_4m_3 - 4Y^+m_4m_3 + 2Z^+m_4m_3 + X^+m_4m_3^2 + 2X^+m_4^2m_3 - 2Y^+m_4m_3^2 \end{array}}{2m_4s_1m_3^2 + 4m_4s_1m_3 - 4s_1 + 4\rho - 2m_4s_1 + 4m_4\rho}$$

$$b_6 = \frac{\begin{gathered}-(2X^+\rho - 2X^+s_1 - 4X^+m_4s_1 + 5X^+m_4\rho - Y^+m_4s_1 + Z^+m_4s_1 - 2X^+m_4^2s_1 + 4X^+m_4^2\rho + X^+m_4^3\rho)\\ -(Y^+m_4^2s_1 - Z^+m_4^2s_1 + 2X^+m_4m_3s_1 - 2Y^+m_4m_3s_1 + Z^+m_4m_3s_1 + X^+m_4m_3^2s_1) - (4X^+m_4^2m_3s_1+\\ 2X^+m_4^3m_3s_1 - Y^+m_4m_3^2s_1 - 2Y^+m_4^2m_3s_1 + Z^+m_4^2m_3s_1 + 2X^+m_4^2m_3^2s_1 + X^+m_4^3m_3^2s_1 - Y^+m_4^2m_3^2s_1)\end{gathered}}{2s_1(m_4s_1m_3^2 + 2m_4s_1m_3 - 2s_1 + 2\rho - m_4s_1 + 2m_4\rho)}$$

where

$$X^- = 2\int_{1-r_1}^{1}\left[\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}} - 1\right]A_{I-}\,d\beta,$$

$$Y^- = -2\int_{1-r_1}^{1}\left[\left(\frac{1-\beta}{r_1}\right)^{\frac{1}{m_1}}\right]A_{I-}\,d\beta - 2\int_{1-\rho}^{1-r_1}\left[\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}} - 1\right]A_{I-}\,d\beta,$$

$$Z^- = -2\int_{1-\rho}^{1-r_1}\left(\frac{1-\beta-r_1}{\rho-r_1}\right)^{\frac{1}{m_2}}A_{I-}\,d\beta, X^+ = 2\int_{1-s_1}^{1}\left[\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}} - 1\right]A_{I+}\,d\beta,$$

$$Y^+ = -2\int_{1-s_1}^{1}\left[\left(\frac{1-\beta}{s_1}\right)^{\frac{1}{m_4}}\right]A_{I+}\,d\beta - 2\int_{1-\rho}^{1-s_1}\left[\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}} - 1\right]A_{I+}\,d\beta,$$

$$Z^+ = -2\int_{1-\rho}^{1-s_1}\left(\frac{1-\beta-s_1}{\rho-s_1}\right)^{\frac{1}{m_3}}A_{I+}\,d\beta.$$

By solving 20, 21 and 22 we get

$$c_1 = \frac{\begin{gathered}-(2X^-\delta - 2X^-r_2 - 4X^-p_4r_2 + 5X^-p_4\delta - Y^-p_4r_2 + Z^-p_4r_2 - 2X^-p_4^2r_2 + 4X^-p_4^2\delta + X^-p_4^3\delta)\\ -(Y^-p_4^2r_2 - Z^-p_4^2r_2 + 2X^-p_4p_2r_2 - 2Y^-p_4p_2r_2 + Z^-p_4p_2r_2 + X^-p_4p_2^2r_2) - (4X^-p_4^2p_2r_2+\\ 2X^-p_4^3p_2r_2 - Y^-p_4p_2^2r_2 - 2Y^-p_4^2p_2r_2 + Z^-p_4^2p_2r_2 + 2X^-p_4^2p_2^2r_2 + X^-p_4^3p_2^2r_2 - Y^-p_4^2p_2^2r_2)\end{gathered}}{2r_2(p_4r_2p_2^2 + 2p_4r_2p_2 - 2r_2 + 2\delta - p_4r_2 + 2p_4\delta)}$$

$$c_2 = \frac{\begin{gathered}2Z^- - 2Y^- + X^-p_4 - 2Y^-p_4 - 4Y^-p_2 + 2Z^-p_4 + 2Z^-p_2 + X^-p_4^2 - 2Y^-p_2^2\\ + X^-p_4^2p_2^2 + 2X^-p_4p_2 - 4Y^-p_4p_2 + 2Z^-p_4p_2 + X^-p_4p_2^2 + 2X^-p_4^2p_2 - 2Y^-p_4p_2^2\end{gathered}}{2p_4r_2p_2^2 + 4p_4r_2p_2 - 4r_2 + 4\delta - 2p_4r_2 + 4p_4\delta}$$

$$c_3 = \frac{\begin{gathered}4Z^-\delta - 4Z^-r_2 + 2Y^-p_2r_2 - 2Y^-p_2\delta - 2Z^-p_4r_2 - 4Z^-p_2r_2 + 4Z^-p_4\delta + 4Z^-p_2\delta\\ + 2Y^-p_2^2r_2 - 2Y^-p_2^2\delta - X^-p_4p_2r_2 + X^-p_4p_2\delta + 2Y^-p_4p_2r_2 - 2Y^-p_4p_2\delta\\ + Z^-p_4p_2r_2 + 4Z^-p_4p_2\delta - X^-p_4p_2^2r_2 - X^-p_4^2p_2r_2 + X^-p_4p_2^2\delta + X^-p_4^2p_2\delta\\ + 2Y^-p_4p_2^2r_2 - 2Y^-p_4p_2^2\delta + 4Z^-p_4p_2^2r_2 + Z^-p_4p_2^3r_2 - X^-p_4^2p_2^2r_2 + X^-p_4^2p_2^2\delta\end{gathered}}{2p_2(r_2-\delta)(p_4r_2p_2^2 + 2p_4r_2p_2 - 2r_2 + 2\delta - p_4r_2 + 2p_4\delta)}$$

By solving 23, 24 and 25 we get

$$c_4 = \frac{\begin{array}{l} 4Z^{+}\delta - 4Z^{+}s_2 + 2Y^{+}p_3s_2 - 2Y^{+}p_3\delta - 2Z^{+}p_4s_2 - 4Z^{+}p_3s_2 + 4Z^{+}p_4\delta + 4Z^{+}p_3\delta \\ + 2Y^{+}p_3^2s_2 - 2Y^{+}p_3^2\delta - X^{+}p_4p_3s_2 + X^{+}p_4p_3\delta + 2Y^{+}p_4p_3s_2 - 2Y^{+}p_4p_3\delta \\ + Z^{+}p_4p_3s_2 + 4Z^{+}p_4p_3\delta - X^{+}p_4p_3^2s_2 - X^{+}p_4^2p_3s_2 + X^{+}p_4p_3^2\delta + X^{+}p_4^2p_3\delta \\ + 2Y^{+}p_4p_3^2s_2 - 2Y^{+}p_4p_3^2\delta + 4Z^{+}p_4p_3^2s_2 + Z^{+}p_4p_3^3s_2 - X^{+}p_4^2p_3^2s_2 + X^{+}p_4^2p_3^2\delta \end{array}}{2p_3(s_2-\delta)(p_4s_2p_3^2 + 2p_4s_2p_3 - 2s_2 + 2\delta - p_4s_2 + 2p_4\delta)}$$

$$c_5 = \frac{\begin{array}{l} 2Z^{+} - 2Y^{+} + X^{+}p_4 - 2Y^{+}p_4 - 4Y^{+}p_3 + 2Z^{+}p_4 + 2Z^{+}p_3 + X^{+}p_4^2 - 2Y^{+}p_3^2 \\ + X^{+}p_4^2p_3^2 + 2X^{+}p_4p_3 - 4Y^{+}p_4p_3 + 2Z^{+}p_4p_3 + X^{+}p_4p_3^2 + 2X^{+}p_4^2p_3 - 2Y^{+}p_4p_3^2 \end{array}}{2p_4s_2p_3^2 + 4p_4s_2p_3 - 4s_2 + 4\delta - 2p_4s_2 + 4p_4\delta}$$

$$c_6 = \frac{\begin{array}{l} -(2X^{+}\delta - 2X^{+}s_2 - 4X^{+}p_4s_2 + 5X^{+}p_4\delta - Y^{+}p_4s_2 + Z^{+}p_4s_2 - 2X^{+}p_4^2s_2 + 4X^{+}p_4^2\delta + X^{+}p_4^3\delta) \\ -(Y^{+}p_4^2s_2 - Z^{+}p_4^2s_2 + 2X^{+}p_4p_3s_2 - 2Y^{+}p_4p_3s_2 + Z^{+}p_4p_3s_2 + X^{+}p_4p_3^2s_2) - (4X^{+}p_4^2p_3s_2 + \\ 2X^{+}p_4^3p_3s_2 - Y^{+}p_4p_3^2s_2 - 2Y^{+}p_4^2p_3s_2 + Z^{+}p_4^2p_3s_2 + 2X^{+}p_4^2p_3^2s_2 + X^{+}p_4^3p_3^2s_2 - Y^{+}p_4^2p_3^2s_2) \end{array}}{2s_2(p_4s_2p_3^2 + 2p_4s_2p_3 - 2s_2 + 2\delta - p_4s_2 + 2p_4\delta)}$$

$$X^{-} = 2\int_{1-r_2}^{1}\left[\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}} - 1\right]A_{F-}\,d\gamma,$$

$$Y^{-} = -2\int_{1-r_2}^{1}\left[\left(\frac{1-\gamma}{r_2}\right)^{\frac{1}{p_1}}\right]A_{F-}\,d\gamma - 2\int_{1-\rho}^{1-r_2}\left[\left(\frac{1-\gamma-r_2}{\rho-r_2}\right)^{\frac{1}{p_2}} - 1\right]A_{F-}\,d\gamma,$$

$$Z^{-} = -2\int_{1-\rho}^{1-r_2}\left(\frac{1-\gamma-r_2}{\rho-r_2}\right)^{\frac{1}{p_2}}A_{F-}\,d\gamma, X^{+} = 2\int_{1-s_2}^{1}\left[\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}} - 1\right]A_{F+}\,d\gamma,$$

$$Y^{+} = -2\int_{1-s_2}^{1}\left[\left(\frac{1-\gamma}{s_2}\right)^{\frac{1}{p_4}}\right]A_{F+}\,d\gamma - 2\int_{1-\rho}^{1-s_2}\left[\left(\frac{1-\gamma-s_2}{\rho-s_2}\right)^{\frac{1}{p_3}} - 1\right]A_{F+}\,d\gamma,$$

$$Z^{+} = -2\int_{1-\rho}^{1-s_2}\left(\frac{1-\gamma-s_2}{\rho-s_2}\right)^{\frac{1}{p_3}}A_{F+}\,d\gamma.$$

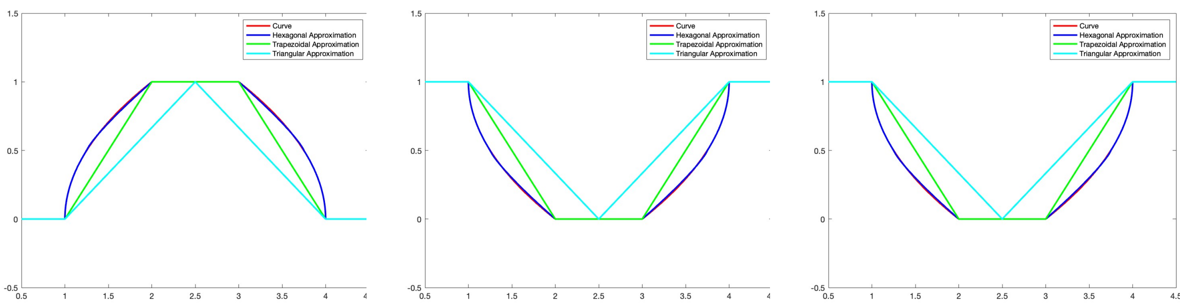
Suppose A be a neutrosophic number defined as given below.

Where the membership function is defined as, $T_A = \begin{cases} (x-1)^{\frac{1}{2}} & , \text{ if } 1 \le x \le 2 \\ 1 & , \text{ if } 2 \le x \le 3 \\ (4-x)^{\frac{1}{2}} & , 3 \le x \le 4 \\ 0 & , \text{ otherwise.} \end{cases}$

Indeterminacy function is described as, $I_A = \begin{cases} 1-(x-1)^{\frac{1}{2}} & , \text{ if } 1 \le x \le 2 \\ 0 & , \text{ if } 2 \le x \le 3 \\ 1-(4-x)^{\frac{1}{2}} & , 3 \le x \le 4 \\ 1 & , \text{ otherwise.} \end{cases}$

The non-membership function is described as, $F_A = \begin{cases} 1-(x-1)^{\frac{1}{2}} & \text{, if } 1 \leq x \leq 2 \\ 0 & \text{, if } 2 \leq x \leq 3 \\ 1-(4-x)^{\frac{1}{2}} & ,3 \leq x \leq 4 \\ 1 & \text{, otherwise.} \end{cases}$

The comparison of the Membership,Indeterminacy and falsity function with GNHNNA, Trapezoidal and triangular neutrosophic numbers is given in FIGURE 1a,1b and 1c.



(a) Membership function (b) Indeterminacy function (c) Non-membership function

Figure 1. Approximation of Neutrosophic Numbers

4. **Approximation of Linear Hexagonal Neutrosophic Numbers with Symmetry**

In this section, the approximation of Linear Hexagonal Neutrosophic Numbers with Symmetry (LHNNS) is given by putting $r = s, n_1 = n_2 = n_3 = n_4 = 1, r_1 = s_1, m_1 = m_2 = m_3 = m_4 = 1, r_2 = s_2, p_1 = p_2 = p_3 = p_4 = 1$. The LHNNS is defined as, $A_{LHNNS} = \left\{T\left(a_1, a_2, a_3, a_4, a_5, a_6; r, r; 1\right)_{(1,1,1,1)}, I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1, r_1; 1\right)_{(1,1,1,1)}, F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2, r_2; 1\right)_{(1,1,1,1)}\right\} = \{T\left(a_1, a_2, a_3, a_4, a_5, a_6; r\right), I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1\right), F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2\right)\}$

Lemma 4.1. *Let* $\{T\left(a_1, a_2, a_3, a_4, a_5, a_6; r\right), I\left(b_1, b_2, b_3, b_4, b_5, b_6; r_1\right), F\left(c_1, c_2, c_3, c_4, c_5, c_6; r_2\right)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry then approximation of the truth function of any curve A which preserves the expected interval criterion is given by,*

$$a_1 = -\frac{6A - 6B + 6Ar - 2Br - 6Cr + 4Dr}{2r} \tag{28}$$

$$a_2 = 6A - 2B - 6C + 4D \quad \text{and} \tag{29}$$

$$a_3 = \frac{6A - 2B - 12C + 4D - 6Ar + 2Br + 6Cr - 4Dr}{2(r-1)}, \tag{30}$$

$$a_4 = \frac{6A_1 - 2B_1 - 12C_1 + 4D_1 - 6A_1r + 2B_1r + 6C_1r - 4D_1r}{2(r-1)},$$

$$a_5 = 6A_1 - 2B_1 - 6C_1 + 4D_1 \quad \text{and } a_6 = -\frac{6A_1 - 6B_1 + 6Ar - 2B_1 r - 6C_1 r + 4D_1 r}{2r}$$

where

$$A = \int_0^r [(\frac{\alpha}{r})A_{T-}]d\alpha, B = \int_0^r [A_{T-}]d\alpha, C = \int_r^1 [(\frac{\alpha - r}{1-r})A_{T-}]d\alpha, \text{ and } D = \int_r^1 [A_{T-}]d\alpha$$

$$A_1 = \int_0^r [(\frac{\alpha}{r})A_{T+}]d\alpha, B_1 = \int_0^r [A_{T+}]d\alpha, C_1 = \int_r^1 [(\frac{\alpha - r}{1-r})A_{T+}]d\alpha, \text{ and } D_1 = \int_r^1 [A_{T+}]d\alpha$$

Proof. By using equation 3.2 the lower limit of expected interval for truth value is given by

$$\int_0^\omega H_{T-}d\alpha = \frac{r(a_1 + a_2) - (a_2 + a_3)(r-1)}{2} \tag{31}$$

By equation 3 we get,

$$d(A, H(A)) = \sqrt{\frac{1}{3}(M + N + O)}$$

where

$$M = \int_0^r (A_{T+} - (a_6 + \left(\frac{\alpha}{r}\right)(a_5 - a_6))^2 d\alpha + \int_r^1 (A_{T+} - (a_5 + \left(\frac{\alpha - r}{1-r}\right)(a_4 - a_5)))^2 d\alpha$$
$$+ \int_0^r (A_{T-} - (a_1 + \left(\frac{\alpha}{r}\right)(a_2 - a_1)))^2 d\alpha + \int_r^1 (A_{T-} - (a_2 + \left(\frac{\alpha - r}{1-r}\right)(a_3 - a_2)))^2 d\alpha,$$

$N = N(b_1, b_2, b_3, b_4, b_5, b_6)$ and $O = O(c_1, c_2, c_3, c_4, c_5, c_6)$

By Equation 9 we get,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + P_1(a_4, a_5, a_6) +$$
$$\lambda_1(\frac{r(a_1 + a_2) - (a_2 + a_3)(r-1)}{2} - \int_0^1 A_{T-}d\alpha) + K(b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6)$$

Using Equations 10, 11, 12 and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2A - 2B + \frac{\lambda_1 r}{2} + \frac{r(2a_1 + a_2)}{3} = 0 \tag{32}$$

$$2C - 2A - 2E + \frac{\lambda_1}{2} + \frac{r(a_1 + 2a_2)}{3} - \frac{(2a_2 + a_3)(r-1)}{3} = 0 \tag{33}$$

$$-2C - \frac{(a_2 + 2a_3)(r-1)}{3} - \lambda_1 \frac{r-1}{2} = 0 \tag{34}$$

By solving equations 31, 32 and 34 we will get

$$a_1 = -\frac{6A - 6B + 6Ar - 2Br - 6Cr + 4Dr}{2r}, a_2 = 6A - 2B - 6C + 4D \quad \text{and}$$

$$a_3 = \frac{6A - 2B - 12C + 4D - 6Ar + 2Br + 6Cr - 4Dr}{2(r-1)}$$

By using equation 3.2 the lower limit of expected interval for truth value is given by

$$\int_0^\omega H_{T+}d\alpha = \frac{r(a_6 + a_5) - (a_5 + a_4)(r-1)}{2} \tag{35}$$

By equation 3 we get,

$$d(A, H(A)) = \sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,}$$

$$M = \int_0^r (A_{T+} - (a_6 + \left(\frac{\alpha}{r}\right)(a_5 - a_6))^2 d\alpha + \int_r^1 (A_{T+} - (a_5 + \left(\frac{\alpha - r}{1-r}\right)(a_4 - a_5)))^2 d\alpha$$
$$+ \int_0^r (A_{T-} - (a_1 + \left(\frac{\alpha}{r}\right)(a_2 - a_1)))^2 d\alpha + \int_r^1 (A_{T-} - (a_2 + \left(\frac{\alpha - r}{1-r}\right)(a_3 - a_2)))^2 d\alpha,$$

$N = N(b_1, b_2, b_3, b_4, b_5, b_6)$ and $O = O(c_1, c_2, c_3, c_4, c_5, c_6)$. By Equation 9 we get,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + P_2(a_1, a_2, a_3)+$$
$$\lambda_2(\frac{r(a_6 + a_5) - (a_5 + a_4)(r-1)}{2} - \int_0^1 A_{T+} d\alpha) + K(b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5, c_6)$$

Using Equations 13, 14, 15 and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2A_1 - 2B_1 + \frac{\lambda_2 r}{2} + \frac{r(2a_6 + a_5)}{3} = 0 \tag{36}$$

$$2C_1 - 2A_1 - 2E_1 + \frac{\lambda_2}{2} + \frac{r(a_6 + 2a_5)}{3} - \frac{(2a_5 + a_4)(r-1)}{3} = 0 \tag{37}$$

$$-2C_1 - \frac{(a_5 + 2a_4)(r-1)}{3} - \lambda_1 \frac{r-1}{2} = 0 \tag{38}$$

By solving equations 35, 36 and 38 we will get

$$a_4 = \frac{6A_1 - 2B_1 - 12C_1 + 4D_1 - 6A_1 r + 2B_1 r + 6C_1 r - 4D_1 r}{2(r-1)},$$

$$a_5 = 6A_1 - 2B_1 - 6C_1 + 4D_1 \quad \text{and } a_6 = -\frac{6A_1 - 6B_1 + 6Ar - 2B_1 r - 6C_1 r + 4D_1 r)}{2r}$$

□

Lemma 4.2. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r), I(b_1, b_2, b_3, b_4, b_5, b_6; r_1),$ $F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry then approximation of the indeterminacy function of any curve A which preserves the expected interval criterion is given by,*

$$b_1 = -\frac{6CI - 6DI + 4AIr_1 - 6BIr_1 + 6CIr_1 - 2DIr_1}{2r_1}, b_2 = 4AI - 6BI + 6CI - 2DI \tag{39}$$

$$b_3 = \frac{4AI - 12BI + 6CI - 2DI - 4AIr_1 + 6BIr_1 - 6CIr_1 + 2DIr_1}{2(r_1 - 1)} \tag{40}$$

$$b_4 = \frac{4AI_1 - 12BI_1 + 6CI_1 - 2DI_1 - 4AI_1 r_1 + 6BI_1 r_1 - 6CI_1 r_1 + 2DI_1 r_1}{2(r_1 - 1)}, \tag{41}$$

$$b_5 = 4AI_1 - 6BI_1 + 6CI_1 - 2DI_1 \quad \text{and } b_6 = -\frac{6CI_1 - 6DI_1 + 4AI_1 r_1 - 6BI_1 r_1 + 6CI_1 r_1 - 2DI_1 r_1}{2r_1} \tag{42}$$

where $AI= \int_0^{1-r_1}[A_{I-}]d\beta$, $BI= \int_0^{1-r_1}[(\frac{1-\beta-r_1}{1-r_1})A_{I-}]d\beta$, $CI= \int_{1-r_1}^{1}[(\frac{1-\beta}{r_1})A_{I-}]d\beta$ *and* $DI= \int_{1-r_1}^{1}[A_{I-}]d\beta$. $AI_1= \int_0^{1-r_1}[A_{I+}]d\beta$, $BI_1= \int_0^{1-r_1}[(\frac{1-\beta-r_1}{1-r_1})A_{I+}]d\beta$, $CI_1= \int_{1-r_1}^{1}[(\frac{1-\beta}{r_1})A_{I+}]d\beta$ *and* $DI_1= \int_{1-r_1}^{1}[A_{I+}]d\beta$

Proof. By using equation 3.2 the upper limit of the expected interval for indeterminacy value is given by

$$\int_0^1 H_{I-}d\beta = \frac{r_1(b_1+b_2)-(b_2+b_3)(r_1-1)}{2} \tag{43}$$

By equation 3 we get,

$$d(A,H(A)) = \sqrt{\frac{1}{3}(M+N+O)}$$

where $M = M(a_1,a_2,a_3,a_4,a_5,a_6)$,

$$N = \int_0^{1-r_1}(A_{I+}-(b_5+\left(\frac{1-\beta-r_1}{1-s_1}\right)(b_4-b_5)))^2 d\beta + \int_{1-r_1}^{1}(A_{I+}-(b_6+\left(\frac{1-\beta}{r_1}\right)(b_5-b_6)))^2 d\beta$$
$$+\int_0^{1-r_1}(A_{I-}-(b_2+\left(\frac{1-\beta-r_1}{1-r_1}\right)(b_3-b_2)))^2 d\beta + \int_{1-r_1}^{1}(A_{I-}-(b_1+\left(\frac{1-\beta}{r_1}\right)(b_2-b_1)))^2 d\beta$$

and $O = O(c_1,c_2,c_3,c_4,c_5,c_6)$. By Equation 9 we get,

$$L(a_1,a_2,...,a_6,b_1,b_2,...,b_6,c_1,c_2,...,c_6) = d^2(A,H(A)) + P_3(b_4,b_5,b_6)+$$
$$\lambda_3(\frac{r_1(b_1+b_2)-(b_2+b_3)(r_1-1)}{2} - \int_0^1 A_{I-}d\alpha) + K(a_1,a_2,a_3,a_4,a_5,a_6,c_1,c_2,c_3,c_4,c_5,c_6)$$

Using Equations 16, 17, 18, and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2CI - 2EI + \frac{\lambda_3 r_1}{2} + \frac{r_1(2b_1+b_2)}{3} = 0 \tag{44}$$

$$2BI - 2AI - 2CI + \frac{\lambda_3}{2} + \frac{r_1(b_1+2b_2)}{3} - \frac{(2b_2+b_3)(r_1-1)}{3} = 0 \tag{45}$$

$$-2BI - \frac{(b2+2b3)(r1-1)}{3} - \lambda_3(\frac{r}{2}-\frac{1}{2}) = 0 \tag{46}$$

By solving equations 43, 44, 45 and 46 we will get

$$b_1 = -\frac{6CI - 6DI + 4AIr_1 - 6BIr_1 + 6CIr_1 - 2DIr_1}{2r_1},$$
$$b_2 = 4AI - 6BI + 6CI - 2DI \quad \text{and}$$
$$b_3 = (4AI - 12BI + 6CI - 2DI - 4AIr_1 + 6BIr_1 - 6CIr_1 + 2DIr_1)/(2(r_1-1)).$$

By using equation 3.2 the upper limit of the expected interval for indeterminacy value is given by

$$\int_0^1 H_{I+}d\beta = \frac{r_1(b_6+b_5)-(b_5+b_4)(r_1-1)}{2} \tag{47}$$

By equation 3 we get,

$$d(A,H(A)) = \sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,}$$

$M = M(a_1, a_2, a_3, a_4, a_5, a_6),$

$$N = \int_0^{1-r_1} (A_{I+} - (b_5 + \left(\frac{1-\beta-r_1}{1-s_1}\right)(b_4 - b_5)))^2 d\beta + \int_{1-r_1}^1 (A_{I+} - (b_6 + \left(\frac{1-\beta}{r_1}\right)(b_5 - b_6)))^2 d\beta$$
$$+ \int_0^{1-r_1} (A_{I-} - (b_2 + \left(\frac{1-\beta-r_1}{1-r_1}\right)(b_3 - b_2)))^2 d\beta + \int_{1-r_1}^1 (A_{I-} - (b_1 + \left(\frac{1-\beta}{r_1}\right)(b_2 - b_1)))^2 d\beta$$

By Equation 9 we get,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + P(b_1, b_2, b_3) +$$
$$\lambda_4(\frac{r_1(b_6 + b_5) - (b_5 + b_4)(r_1 - 1)}{2} - \int_0^1 A_{I+} d\beta) + K(a_1, a_2, a_3, a_4, a_5, a_6, c_1, c_2, c_3, c_4, c_5, c_6)$$

Using Equations 19, 20, 21, and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2CI_1 - 2EI_1 + \frac{\lambda_4 r_1}{2} + \frac{r_1(2b_6 + b_5)}{3} = 0 \tag{48}$$

$$2BI_1 - 2AI_1 - 2CI_1 + \frac{\lambda_4}{2} + \frac{r_1(b_6 + 2b_5)}{3} - \frac{(2b_5 + b_4)(r_1 - 1)}{3} = 0 \tag{49}$$

$$-2BI_1 - \frac{(b_5 + 2b_4)(r1 - 1)}{3} - \lambda_4(\frac{r}{2} - \frac{1}{2}) = 0 \tag{50}$$

By solving equations 47, 48, 49 and 50 we will get

$$b_4 = \frac{4AI_1 - 12BI_1 + 6CI_1 - 2DI_1 - 4AI_1r_1 + 6BI_1r_1 - 6CI_1r_1 + 2DI_1r_1}{2(r_1 - 1)},$$

$$b_5 = 4AI_1 - 6BI_1 + 6CI_1 - 2DI_1 \quad \text{and}\; b_6 = -\frac{6CI_1 - 6DI_1 + 4AI_1r_1 - 6BI_1r_1 + 6CI_1r_1 - 2DI_1r_1}{2r_1}$$

□

Lemma 4.3. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r), I(b_1, b_2, b_3, b_4, b_5, b_6; r_1),$ $F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry then approximation of the falsity function of any curve A which preserves the expected interval criterion is given by,*

$$c_1 = -\frac{6CF - 6DF + 4AFr_2 - 6BFr_2 + 6CFr_1 - 2DFr_2}{2r_2}, c_2 = 4AF - 6BF + 6CF - 2DF \tag{51}$$

$$c_3 = \frac{4AF - 12BF + 6CF - 2DF - 4AFr_2 + 6BFr_2 - 6CFr_2 + 2DFr_2}{2(r_2 - 1)} \tag{52}$$

$$c_4 = \frac{4AF_1 - 12BF_1 + 6CF_1 - 2DF_1 - 4AF_1r_2 + 6BF_1r_2 - 6CF_1r_2 + 2DF_1r_2}{2(r_2 - 1)}, \tag{53}$$

$$c_5 = 4AF_1 - 6BF_1 + 6CF_1 - 2DF_1 \quad \text{and} \tag{54}$$

$$c_6 = -\frac{6CF_1 - 6DF_1 + 4AF_1r_2 - 6BF_1r_2 + 6CF_1r_1 - 2DF_1r_2}{2r_2} \tag{55}$$

where $AF= \int_0^{1-r_2}[A_{F-}]d\gamma$, $BF= \int_0^{1-r_2}[(\frac{1-\gamma-r_2}{1-r_2})A_{F-}]d\gamma$,$CF= \int_{1-r_2}^{1}[(\frac{1-\gamma}{r_2})A_{F-}]d\gamma$ *and* $DF= \int_{1-r_2}^{1}[A_{F-}]d\gamma$. $AF_1= \int_0^{1-r_2}[A_{F+}]d\gamma$, $BF_1= \int_0^{1-r_2}[(\frac{1-\gamma-r_2}{1-r_2})A_{F+}]d\gamma$, $CF_1= \int_{1-r_2}^{1}[(\frac{1-\gamma}{r_2})A_{F+}]d\gamma$ *and* $DF_1= \int_{1-r_2}^{1} d\gamma$.

Proof. By using equation 3.2 the upper limit of expected interval for falsity value is given by

$$\int_0^1 H_{F-}d\gamma = \frac{r_2(c_1+c_2)-(c_2+c_3)(r_2-1)}{2} \tag{56}$$

By equation 3 we get,

$$d(A,H(A)) = \sqrt{\frac{1}{3}(M+N+O)} \text{ where,}$$

$M = M(a_1,a_2,a_3,a_4,a_5,a_6)$, $N = N(b_1,b_2,b_3,b_4,b_5,b_6)$ and

$$O = \int_0^{1-r_2}(A_{F+}-(c_5+\left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_4-c_5)))^2d\gamma + \int_{1-r_2}^{1}(A_{F+}-(c_6+\left(\frac{1-\gamma}{r_2}\right)(c_5-c_6)))^2d\gamma$$
$$+\int_0^{1-r_2}(A_{F-}-(c_2+\left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_3-c_2)))^2d\gamma + \int_{1-r_2}^{1}(A_{F-}-(c_1+\left(\frac{1-\gamma}{r_2}\right)(c_2-c_1)))^2d\gamma$$

By Equation 9 we get,

$$L(a_1,a_2,...,a_6,b_1,b_2,...,b_6,c_1,c_2,...,c_6) = d^2(A,H(A)) + P(c_4,c_5,c_6)+$$
$$\lambda_5(\frac{r_2(c_1+c_2)-(c_2+c_3)(r_2-1)}{2} - \int_0^1 A_{F-}d\gamma) + K(a_1,a_2,a_3,a_4,a_5,a_6,b_1,b_2,b_3,b_4,b_5,b_6)$$

Using Equations 22, 23, 24, and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2CF - 2EI + \frac{\lambda_4 r_2}{2} + \frac{r_2(2c_1+c_2)}{3} = 0 \tag{57}$$

$$2BF - 2AF - 2CF + \frac{\lambda_4}{2} + \frac{r_2(c_1+2c_2)}{3} - \frac{(2c_2+c_3)(r_2-1)}{3} = 0 \tag{58}$$

$$-2BI - \frac{(c_2+2c_3)(r1-1)}{3} - \lambda_4(\frac{r}{2}-\frac{1}{2}) = 0 \tag{59}$$

By solving equations 56, 57, 58 and 59 we will get

$$c_1 = -\frac{6CF-6DF+4AFr_2-6BFr_2+6CFr_1-2DFr_2}{2r_2}, c_2 = 4AF-6BF+6CF-2DF$$

$$c_3 = \frac{4AF-12BF+6CF-2DF-4AFr_2+6BFr_2-6CFr_2+2DFr_2}{2(r_2-1)}$$

. By using equation 3.2 the upper limit of expected interval for falsity value is given by

$$\int_0^1 H_{F+}d\alpha = \frac{r_2(c_6+c_5)-(c_5+c_4)(r_2-1)}{2} \tag{60}$$

By equation 3 we get,

$$d(A,H(A)) = \sqrt{\frac{1}{3}(M+N+O)} \quad \text{where,}$$

$M = M(a_1, a_2, a_3, a_4, a_5, a_6)$, $N = N(b_1, b_2, b_3, b_4, b_5, b_6)$ and

$$O = \int_0^{1-r_2} (A_{F+} - (c_5 + \left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_4 - c_5)))^2 d\gamma + \int_{1-r_2}^{1} (A_{F+} - (c_6 + \left(\frac{1-\gamma}{r_2}\right)(c_5 - c_6)))^2 d\gamma$$
$$+ \int_0^{1-r_2} (A_{F-} - (c_2 + \left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_3 - c_2)))^2 d\gamma + \int_{1-r_2}^{1} (A_{F-} - (c_1 + \left(\frac{1-\gamma}{r_2}\right)(c_2 - c_1)))^2 d\gamma$$

By Equation 9 we get,

$$L(a_1, a_2, ..., a_6, b_1, b_2, ..., b_6, c_1, c_2, ..., c_6) = d^2(A, H(A)) + P(c_1, c_2, c_3)+$$
$$\lambda_6(\frac{r_2(c_6 + c_5) - (c_5 + c_4)(r_2 - 1)}{2} - \int_0^1 A_{F+} d\gamma) + K(a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6)$$

Using Equations 25, 26, 27, and KKT theorem we form the partial derivatives corresponding to the Lagrangian multipliers

$$2CF_1 - 2DF_1 + \frac{\lambda_6 r_2}{2} + \frac{r_2(2c_6 + c_5)}{3} = 0 \tag{61}$$

$$2BF_1 - 2AF_1 - 2CF_1 + \frac{\lambda_6}{2} + \frac{r_2(c_6 + 2c_5)}{3} - \frac{(2c_5 + c_4)(r_2 - 1)}{3} = 0 \tag{62}$$

$$-2BF_1 - \frac{(c_5 + 2c_4)(r1 - 1)}{3} - \lambda_6(\frac{r}{2} - \frac{1}{2}) = 0 \tag{63}$$

By solving equations 47, 61, 62 and 63 we will get

$$c_4 = \frac{4AF_1 - 12BF_1 + 6CF_1 - 2DF_1 - 4AF_1 r_2 + 6BF_1 r_2 - 6CF_1 r_2 + 2DF_1 r_2}{2(r_2 - 1)}, \tag{64}$$

$$c_5 = 4AF_1 - 6BF_1 + 6CF_1 - 2DF_1 \quad \text{and} \tag{65}$$

$$c_6 = -\frac{6CF_1 - 6DF_1 + 4AF_1 r_2 - 6BF_1 r_2 + 6CF_1 r_1 - 2DF_1 r_2}{2r_2} \tag{66}$$

□

Note 4.4. *The solution of the equations in Lemma 4.1 to Lemma 4.3 are solved using Matlab.*

Example 4.5. Suppose A be a neutrosophic number defined as given below.

Where the membership function is defined as, $T_A = \begin{cases} (x-1)^{\frac{1}{2}} & \text{, if } 1 \leq x \leq 2 \\ 1 & \text{, if } 2 \leq x \leq 3 \\ (4-x)^{\frac{1}{2}} & \text{,} 3 \leq x \leq 4 \\ 0 & \text{, otherwise.} \end{cases}$

Indeterminacy function is described as, $I_A = \begin{cases} 1-(x-1)^{\frac{1}{2}} & \text{, if } 1 \leq x \leq 2 \\ 0 & \text{, if } 2 \leq x \leq 3 \\ 1-(4-x)^{\frac{1}{2}} & \text{,} 3 \leq x \leq 4 \\ 1 & \text{, otherwise.} \end{cases}$

The non-membership function is described as, $F_A = \begin{cases} 1-(x-1)^{\frac{1}{2}} & \text{, if } 1 \leq x \leq 2 \\ 0 & \text{, if } 2 \leq x \leq 3 \\ 1-(4-x)^{\frac{1}{2}} & ,3 \leq x \leq 4 \\ 1 & \text{, otherwise.} \end{cases}$

The Hexagonal approximation of this curve H(A) is given by, (here $r = r_1 = r_2 = .5$)

By using Lemma 4.1 - Lemma 4.3 we get $a_1 = 1$, $a_2 = \frac{5}{4}$, $a_3 = 2$, $a_4 = 3$, $a_5 = \frac{15}{4}$, $a_6 = 4$, $b_1 = 1$, $b_2 = \frac{5}{4}$, $b_3 = 2$, $b_4 = 3$, $b_5 = \frac{15}{4}$, $b_6 = 4$, $c_1 = 1$, $c_2 = \frac{5}{4}$, $c_3 = 2$, $c_4 = 3$, $c_5 = \frac{15}{4}$ $c_6 = 4$, where the membership function is defined as,

$$T_{H(A)} = \begin{cases} .5\left(\frac{x-1}{\frac{5}{4}-1}\right) & \text{, if } 1 \leq x \leq \frac{5}{4} \\ .5+.5\left(\frac{x-\frac{5}{4}}{2-\frac{5}{4}}\right) & \text{, if } \frac{5}{4} \leq x \leq 2 \\ 1 & \text{, if } 2 \leq x \leq 3 \\ .5+.5\left(\frac{x-\frac{15}{4}}{3-\frac{15}{4}}\right) & \text{, if } 3 \leq x \leq \frac{15}{4} \\ .5\left(\frac{x-4}{\frac{15}{4}-4}\right) & \text{, if } \frac{15}{4} \leq x \leq 4 \\ 0 & \text{, otherwise.} \end{cases}$$

The comparison of membership function with LHNNS, Trapezoidal (TI) and triangular (Tr) neutrosophic numbers is given in FIGURE 2a. Indeterminacy function can be described as,

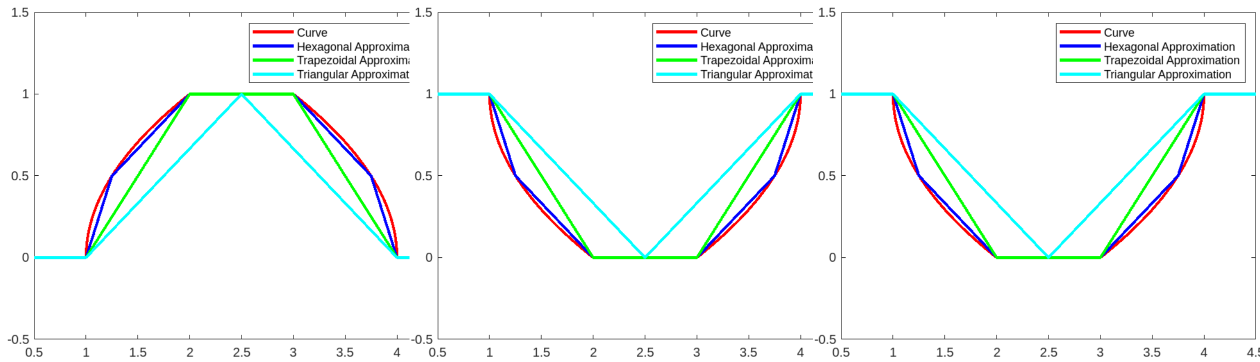
$$I_{H(A)} = \begin{cases} 1-.5\left(\frac{x-1}{\frac{5}{4}-1}\right) & \text{, if } 1 \leq x \leq \frac{5}{4} \\ 1-(.5+.5\left(\frac{x-\frac{5}{4}}{2-\frac{5}{4}}\right)) & \text{, if } \frac{5}{4} \leq x \leq 2 \\ 0 & \text{, if } 2 \leq x \leq 3 \\ 1-(.5+.5\left(\frac{x-\frac{15}{4}}{3-\frac{15}{4}}\right)) & \text{, if } 3 \leq x \leq \frac{15}{4} \\ 1-(.5\left(\frac{x-4}{\frac{15}{4}-4}\right)) & \text{, if } \frac{15}{4} \leq x \leq 4 \\ 1 & \text{, otherwise.} \end{cases}$$

The comparison of Indeterminacy function with LHNNS, Trapezoidal and triangular neutrosophic numbers is given in FIGURE 2b. Non-membership function can be described as,

$$F_{H(A)} = \begin{cases} 1-.5\left(\frac{x-1}{\frac{5}{4}-1}\right) & \text{, if } 1 \leq x \leq \frac{5}{4} \\ 1-(.5+.5\left(\frac{x-\frac{5}{4}}{2-\frac{5}{4}}\right)) & \text{, if } \frac{5}{4} \leq x \leq 2 \\ 0 & \text{, if } 2 \leq x \leq 3 \\ 1-(.5+.5\left(\frac{x-\frac{15}{4}}{3-\frac{15}{4}}\right)) & \text{, if } 3 \leq x \leq \frac{15}{4} \\ 1-(.5\left(\frac{x-4}{\frac{15}{4}-4}\right)) & \text{, if } \frac{15}{4} \leq x \leq 4 \\ 1 & \text{, otherwise.} \end{cases}$$

The comparison of non-membership function with LHNNS, Trapezoidal $(a_2 = a_3, a_4 = a_5, r = s = 1, b_2 = b_3, b_4 = b_5, r_1 = s_1 = 1, c_2 = c_3, c_4 = c_5, r_2 = s_2 = 1)$ and triangular $(a_2 = a_3 = a_4 = a_5, r = s = 1, b_2 = b_3 = b_4 = b_5, r_1 = s_1 = 1, c_2 = c_3 = c_4 = c_5, r_2 = s_2 = 1)$

neutrosophic numbers is given in the FIGURE 2c.



(a) Membership function (b) Indeterminacy function (c) Non-membership function

Figure 2. Approximation of Neutrosophic Numbers

$$d(A, H(A)) = \sqrt{\frac{1}{3}(M + N + O)} \text{ where,}$$

$M = \int_0^r (A_{T+} - (a_6 + \left(\frac{\alpha}{r}\right)(a_5 - a_6))^2 d\alpha + \int_r^1 (A_{T+} - (a_5 + \left(\frac{\alpha - r}{1-r}\right)(a_4 - a_5)))^2 d\alpha + \int_0^r (A_{T-} - (a_1 + \left(\frac{\alpha}{r}\right)(a_2 - a_1)))^2 d\alpha + \int_r^1 (A_{T-} - (a_2 + \left(\frac{\alpha - r}{1-r}\right)(a_3 - a_2)))^2 d\alpha,$

$N = \int_0^{1-r_1} (A_{I+} - (b_5 + \left(\frac{1-\beta-r_1}{1-s_1}\right)(b_4 - b_5)))^2 d\beta + \int_{1-r_1}^1 (A_{I+} - (b_6 + \left(\frac{1-\beta}{r_1}\right)(b_5 - b_6)))^2 d\beta + \int_0^{1-r_1} (A_{I-} - (b_2 + \left(\frac{1-\beta-r_1}{1-r_1}\right)(b_3 - b_2)))^2 d\beta + \int_{1-r_1}^1 (A_{I-} - (b_1 + \left(\frac{1-\beta}{r_1}\right)(b_2 - b_1)))^2 d\beta$ and

$O = \int_0^{1-r_2} (A_{F+} - (c_5 + \left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_4 - c_5)))^2 d\gamma + \int_{1-r_2}^1 (A_{F+} - (c_6 + \left(\frac{1-\gamma}{r_2}\right)(c_5 - c_6)))^2 d\gamma + \int_0^{1-r_2} (A_{F-} - (c_2 + \left(\frac{1-\gamma-r_2}{1-r_2}\right)(c_3 - c_2)))^2 d\gamma + \int_{1-r_2}^1 (A_{F-} - (c_1 + \left(\frac{1-\gamma}{r_2}\right)(c_2 - c_1)))^2 d\gamma$

d(A, H(A))= 0.0645; d(A, Tl(A))= 0.3651; d(A, Tr(A))= 0.6831.

Example 4.6. Suppose A be a neutrosophic number defined as given below.

Where the membership function is defined as, $T_A = \begin{cases} (x-1)^2 & , \text{ if } 1 \leq x \leq 2 \\ (4-x)^2 & , 2 \leq x \leq 3 \\ 0 & , \text{ otherwise.} \end{cases}$

The indeterminacy function is be described as, $I_A = \begin{cases} 1-(x-1)^2 & , \text{ if } 1 \leq x \leq 2 \\ 1-(4-x)^2 & , 2 \leq x \leq 3 \\ 1 & , \text{ otherwise.} \end{cases}$

The non-membership function is described as, $F_A = \begin{cases} 1-(x-1)^2 & , \text{ if } 1 \leq x \leq 2 \\ 1-(4-x)^2 & , 2 \leq x \leq 3 \\ 1 & , \text{ otherwise.} \end{cases}$

The Hexagonal approximation of this curve H(A) is given by, (here $r = r_1 = r_2 = .5$)

By using Lemma 4.1 - Lemma 4.3 we get

$a_1 = 1,\ a_2 = 1 + \frac{1}{\sqrt{2}},\ a_3 = 2,\ a_4 = 2,\ a_5 = 3 - \frac{1}{\sqrt{2}},\ a_6 = 3, b_1 = 1,\ b_2 = 1 + \frac{1}{\sqrt{2}},\ b_3 = 2,\ b_4 = 2,$

$b_5 = 3 - \frac{1}{\sqrt{2}}$, $b_6 = 3$, $c_1 = 1$, $c_2 = 1 + \frac{1}{\sqrt{2}}$, $c_3 = 2$, $c_4 = 2$, $c_5 = 3 - \frac{1}{\sqrt{2}}$, $c_6 = 3$, where the membership function is defined as,

$$T_{H(A)} = \begin{cases} \frac{1}{\sqrt{2}}(x-1) & , \text{ if } 1 \leq x \leq 1 + \frac{1}{\sqrt{2}} \\ 1 + \dfrac{x-2}{2-\sqrt{2}} & , \text{ if } 1 + \frac{1}{\sqrt{2}} \leq x \leq 2 \\ 1 + \dfrac{2-x}{2-\sqrt{2}} & , \text{ if } 2 \leq x \leq 3 - \frac{1}{\sqrt{2}} \\ \dfrac{3-x}{\sqrt{2}} & , \text{ if } 3 - \frac{1}{\sqrt{2}} \leq x \leq 3 \\ 0 & , \text{ otherwise.} \end{cases}$$

The comparison of membership function with LHNNS, Trapezoidal and triangular neutrosophic numbers is given in the FIGURE 3a. The indeterminacy function can be described as,

$$I_{H(A)} = \begin{cases} 1 - \frac{1}{\sqrt{2}}(x-1) & , \text{ if } 1 \leq x \leq 1 + \frac{1}{\sqrt{2}} \\ \dfrac{2-x}{2-\sqrt{2}} & , \text{ if } 1 + \frac{1}{\sqrt{2}} \leq x \leq 2 \\ \dfrac{x-2}{2-\sqrt{2}} & , \text{ if } 2 \leq x \leq 3 - \frac{1}{\sqrt{2}} \\ 1 - \dfrac{3-x}{\sqrt{2}} & , \text{ if } 3 - \frac{1}{\sqrt{2}} \leq x \leq 3 \\ 1 & , \text{ otherwise.} \end{cases}$$

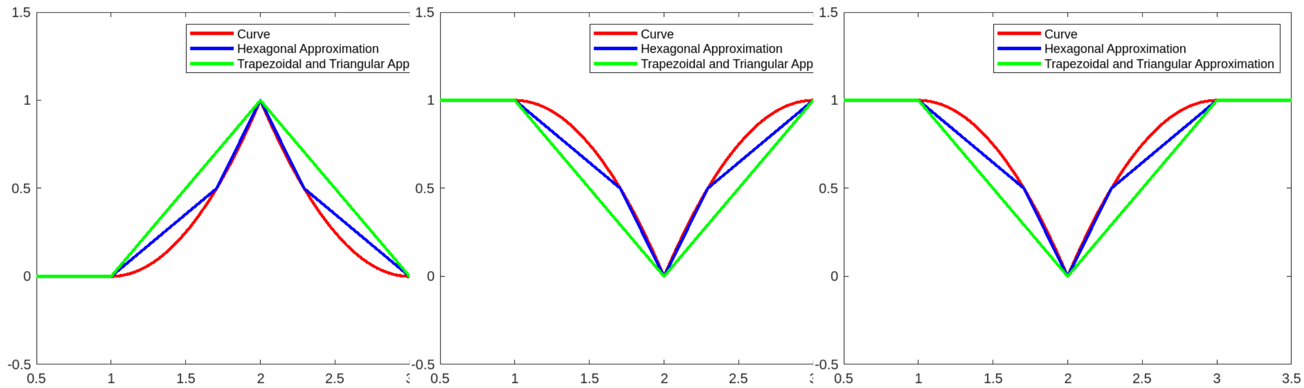
The comparison of the Indeterminacy function with LHNNS, Trapezoidal and triangular neutrosophic numbers is given in FIGURE 3b. Non-membership function can be described as,

$$F_{H(A)} = \begin{cases} 1 - \frac{1}{\sqrt{2}}(x-1) & , \text{ if } 1 \leq x \leq 1 + \frac{1}{\sqrt{2}} \\ \dfrac{2-x}{2-\sqrt{2}} & , \text{ if } 1 + \frac{1}{\sqrt{2}} \leq x \leq 2 \\ \dfrac{x-2}{2-\sqrt{2}} & , \text{ if } 2 \leq x \leq 3 - \frac{1}{\sqrt{2}} \\ 1 - \dfrac{3-x}{\sqrt{2}} & , \text{ if } 3 - \frac{1}{\sqrt{2}} \leq x \leq 3 \\ 1 & , \text{ otherwise.} \end{cases}$$

The comparison of non-membership function with LHNNS, Trapezoidal and triangular neutrosophic numbers is given in FIGURE 3c. d(A, H(A))= 0.1294; d(A, Tl(A))= 0.2582; d(A, Tr(A))= 0.2582.

From the above two examples, we can conclude that the distance in the hexagonal case is minimal.

Theorem 4.7. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r), I(b_1, b_2, b_3, b_4, b_5, b_6; r_1), F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry and T(A) denotes the approximation with LHNNS then,* $V_T(T(A)) = ((0.3333(r-1)^2 - 0.5000r(r-1))(6A_1 - 2B_1 - 12C_1 + 4D_1 - 6A_1r + 2B_1r + 6C_1r - 4E_1r))/(2*r-2) - 0.0833r(6A - 6B + 6Ar - 2Br - 6Cr + 4rD) - 0.0833r(6A_1 - 6B_1 + 6A_1r - 2B_1r - 6C_1r + 4E_1r) - (6A - 2B - 6C + 4E)(0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000) - (0.3333(r-1)^2 - 0.5000r(r-$



(a) Membership function (b) Indeterminacy function (c) Non-membership function

Figure 3. Functions for Neutrosophic Numbers

$1) + 0.5000r^2 - 0.5000)(6A_1 - 2B_1 - 6C_1 + 4E_1) + ((0.3333(r-1)^2 - 0.5000r(r-1))(6A - 2B - 12C + 4E - 6Ar + 2Br + 6Cr - 4rE))/(2r-2)$

Proof. We know,

$$a_1 = -\frac{6A - 6B + 6Ar - 2Br - 6Cr + 4Dr}{2r}, a_2 = 6A - 2B - 6C + 4D,$$

$$a_3 = \frac{6A - 2B - 12C + 4D - 6Ar + 2Br + Cr - 4Dr}{2(r-1)},$$

$$a_4 = \frac{6A_1 - 2B_1 - 12C_1 + 4D_1 - 6A_1r + 2B_1r + 6C_1r - 4D_1r}{2(r-1)},$$

$$a_5 = 6A_1 - 2B_1 - 6C_1 + 4D_1 \text{ and } a_6 = -\frac{6A_1 - 6B_1 + 6Ar - 2B_1r - 6C_1r + 4D_1r}{2r},$$

from definition 2.3 we get

$V_T(T(A)) = \left[\frac{1}{2} - \frac{1}{3}\right] r^2 a_1 + \left[\frac{1-r^2}{2} - \frac{(1-r)^2}{3} - r\frac{1-r}{2}\right] a_2 + \left[\frac{(1-r)^2}{3} + r\frac{1-r}{2}\right] a_3 + \left[\frac{(1-r)^2}{3} + r\frac{1-r}{2}\right] a_4 + \left[\frac{1-r^2}{2} - \frac{(1-r)^2}{3} - r\frac{1-r}{2}\right] a_5 + \left[\frac{1}{2} - \frac{1}{3}\right] r^2 a_6$

Substituting the values of $a_1 - a_6$ and simplifying we get, $V_T(T(A)) = ((0.3333(r-1)^2 - 0.5000r(r-1))(6A_1 - 2B_1 - 12C_1 + 4D_1 - 6A_1r + 2B_1r + 6C_1r - 4E_1r))/(2*r-2) - 0.0833r(6A - 6B + 6Ar - 2Br - 6Cr + 4rD) - 0.0833r(6A_1 - 6B_1 + 6A_1r - 2B_1r - 6C_1r + 4E_1r) - (6A - 2B - 6C + 4E)(0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000) - (0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000)(6A_1 - 2B_1 - 6C_1 + 4E_1) + ((0.3333(r-1)^2 - 0.5000r(r-1))(6A - 2B - 12C + 4E - 6Ar + 2Br + 6Cr - 4rE))/(2r-2)$ □

Theorem 4.8. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r), I(b_1, b_2, b_3, b_4, b_5, b_6; r_1), F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry and T(A) denotes the approximation with LHNNS then,*

$V_I(T(A)) = ((0.3333(r-1)^2 - 0.5000r(r-1))(2AI - 6BI + 3CI - DI - 2AIr_1 + 3BIr_1 - 3CIr_1 + DIr_1))/(r_1 - 1) - (0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000)(4AI_1 - 6BI_1 +$

$6CI_1 - 2DI_1) - (0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000)(4AI - 6BI + 6CI - 2DI) + ((0.3333(r-1)^2 - 0.5000r(r-1))(2AI_1 - 6BI_1 + 3CI_1 - DI_1 - 2AI_1r_1 + 3BI_1r_1 - 3CI_1r_1 + DI_1r_1))/(r_1 - 1) - (0.1667r^2(3CI - 3DI + 2AIr_1 - 3BIr_1 + 3CIr_1 - DIr_1))/r_1 - (0.1667r^2(3CI_1 - 3DI_1 + AI_1r_1 - 3BI_1r_1 + 3CI_1r_1 - DI_1r_1))/r_1$

Proof. The proof is similar to theorem 4.7 □

Theorem 4.9. *Let* $\{T(a_1, a_2, a_3, a_4, a_5, a_6; r), I(b_1, b_2, b_3, b_4, b_5, b_6; r_1), F(c_1, c_2, c_3, c_4, c_5, c_6; r_2)\}$ *be a Linear Hexagonal Neutrosophic Number with Symmetry and T(A) denotes the approximation with LHNNS then,*
$V_F(T(A)) = ((0.3333(r-1)^2 - 0.5000r(r-1))(2AF - 6BF + 3CF - DF - 2AFr_1 + 3BFr_1 - 3CFr_1 + DFr_1))/(r_1 - 1) - (0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000)(4AF_1 - 6BF_1 + 6CF_1 - 2DF_1) - (0.3333(r-1)^2 - 0.5000r(r-1) + 0.5000r^2 - 0.5000)(4AF - 6BF + 6CF - 2DF) + ((0.3333(r-1)^2 - 0.5000r(r-1))(2AF_1 - 6BF_1 + 3CF_1 - DF_1 - 2AF_1r_1 + 3BF_1r_1 - 3CF_1r_1 + DF_1r_1))/(r_1 - 1) - (0.1667r^2(3CF - 3DF + 2AFr_1 - 3BFr_1 + 3CFr_1 - DFr_1))/r_1 - (0.1667r^2(3CF_1 - 3DF_1 + AF_1r_1 - 3BF_1r_1 + 3CF_1r_1 - DF_1r_1))/r_1$

Proof. The proof is similar to theorem 4.7 □

5. Quantitative Analysis of Decision-Making Research

This section discusses a decision-making problem to show the effectiveness of approximation in LHNNS. Here decision-makers can use any linear or non-linear neutrosophic number as parameters. The Neutrosophic numbers, which are employed to express linguistic variables, are approximated to LHNNS. Then we will find an LHNNS for each characteristic using weighted arithmetic addition and use the values of LHNNS to accomplish single-valued neutrosophic numbers. By using the score function, deneutrosophication is done. Finally, we will compare the score corresponding to each available alternative and choose the best one. This approach can compare any decision-making issue using any neutrosophic number. This hexagonal approximation will get a better outcome if two separate data sets need to be analyzed. The following is an algorithm for determining the ideal result.

(1) Approximate each linguistic variable to LHNNS.
(2) Calculate the LHNNS corresponding to each attribute using scalar multiplication and addition operations.
(3) Find the values corresponding to each attribute which converts them into a single-valued neutrosophic number.
(4) Find the score function.
(5) Ranking is given according to score, and the best outcome is selected.

5.1. *Numerical Example*

Consider a problem for investing in a stock market that will give excellent yield. The stock markets were shortlisted into five $x_i (1 \leq i \leq 5)$ categories by considering the feasibility. Now, the investor has to choose the best stock market by evaluating the criteria of c_1 - Earnings Growth, c_2 - Revenue Growth, c_3 - Profit Margins, c_4 - Dividend Yield, and c_5 - Valuation Metrics.

Step:1 Let table 1 denote the decision-making matrix with linguistic variables given below. SU-Subdued; RE-Reasonable; EX-Exceptional; SUB-Substantial; SC-Scant.

Step:2 Corresponding to each criterion, the linguistic variables are given in terms of non-linear trapezoidal numbers in table 2.

Step:3 The weights $w_1 = 0.4$, $w_2 = 0.2$, $w_3 = 0.15$, $w_4 = 0.17$ and $w_5 = 0.08$ are assigned for each criteria c_1 to c_5 respectively.

Step:4 The approximation is applied to each linguistic variable and is given in Table 3.

Step:5 Table 4 gives hexagonal values corresponding to each x_i using addition and scalar multiplication.

Step:6 Table 5 shows the single-valued neutrosophic number obtained using Values (Definition 2.3).

Step:7 The score function is evaluated and shown in Table 6. Also, we get $x_2 > x_3 > x_4 > x_5 > x_1$. Therefore x_2 is the best possible outcome.

Table 1. Decision making Matrix

	c_1	c_2	c_3	c_4	c_5
x_1	SU	RE	SU	EX	RE
x_2	EX	EX	EX	RE	RE
x_3	SUB	SU	RE	EX	SC
x_4	SUB	SC	EX	SU	RE
x_5	SU	EX	SUB	RE	SC

Table 2. Linguistic Variable (LV) and the corresponding Nonlinear Trapezoidal Number (NLTN)

LV	NLTN
Scant	[T(0,0,0,0;0.5,0.5),I(3,5,6,8;0.5,0.5), F(3,5,6,8;0.5,0.5)]
Subdued	[T(0,2,3,5;0.5,0.5), I(2,4,5,7;0.5,0.5),F(2,4,5,7;0.5,0.5)]
Reasonable	[T(1,3,4,6;0.5,0.5), I(1,3,4,6;0.5,0.5), F(1,3,4,6;0.5,0.5)]
Substantial	[T(2,4,5,7;0.5,0.5), I(0,2,3,5;0.5,0.5), F(0,2,3,5;0.5,0.5)]
Exceptional	[T(3,5,6,8;0.5,0.5), I(0,0,0,0;0.5,0.5), F(0,0,0,0;0.5,0.5)]

Table 3. Approximation of NLTN to LHNNS

LV	LHNNS
Scant	[T(0,0,0,0,0,0;0.5,0.5), I(3,3.5,5,6,7.5,8;0.5,0.5),F(3,3.5,5,6,7.5,8;0.5,0.5)]
Subdued	[T(0,0.5,2,3,4.5,5;0.5,0.5),I(2,2.5,4,5,6.5,7;0.5,0.5), F(2,2.5,4,5,6.5,7;0.5,0.5)]
Reasonable	[T(1,1.5,3,4,5.5,6;0.5,0.5), I(1,1.5,3,4,5.5,6;0.5,0.5), F(1,1.5,3,4,5.5,6;0.5,0.5)]
Substantial	[T(2,2.5,4,5,6.5,7;0.5,0.5),I(0,0.5,2,3,4.5,5;0.5,0.5), F(0,0.5,2,3,4.5,5;0.5,0.5)]
Exceptional	[T(3,3.5,5,6,7.5,8;0.5,0.5), I(0,0,0,0,0,0;0.5,0.5), F(0,0,0,0,0,0;0.5,0.5)]

Table 4. Hexagonal Neutrosophic number corresponding to each x_i

Attribute	LHNNS
x_1	[T(0.79,1.29,2.79,3.79,5.29,5.79), I(1.38,1.79,3.04,3.87,5.11,5.53), F(1.38,1.79,3.04,3.87,5.11,5.53)]
x_2	[T(2.50,3.00,4.50,5.50,7.00,7.50), I(0.25,0.37,0.75,1.00,1.37,1.50), F(0.25,0.37,0.75,1.00,1.37,1.50)]
x_3	[T(1.46,1.92,3.30,4.22,5.60,6.06), I(0.79,1.20,2.45,3.28,4.52,4.94), F(0.79,1.20,2.45,3.28,4.52,4.94)]
x_4	[T(1.33,1.73,2.93,3.73,4.93,5.33), I(1.02,1.44,2.72,3.57,4.84,5.27), F(1.02,1.44,2.72,3.57,4.84,5.27)]
x_5	[T(1.07,1.53,2.91,3.83,5.21,5.67), I(1.21,1.61,2.81,3.61,4.81,5.21), F(1.21,1.61,2.81,3.61,4.81,5.21)]

Table 5. Value of LHNNS

Attribute	Neutrosophic Number
x_1	(2.7417,0.8797,0.8797)
x_2	(4.167, 0.7292, 0.7292)
x_3	(3.1333, 2.3875, 2.3875)
x_4	(2.775, 2.6208, 2.6208)
x_5	(2.8083,2.675,2.675)

Table 6. Score of table 5

Attribute	Score
x_1	-3.0167
x_2	2.7083
x_3	-1.6417
x_4	-2.4667
x_5	-2.542

6. **Conclusion**

In this study, we explored the approximation of generalized neutrosophic numbers with asymmetry and provided a comparison with other well-known approaches for approximating neutrosophic numbers. The results emphasize the significance of taking uneven membership degrees into account within the neutrosophic context since it more precisely captures the complexities of actual uncertainty. The complexity and subjectivity associated with depicting uncertainty using neutrosophic numbers have come to light due to research of numerous theories relating to value assignment and approximation points.

We have illustrated the benefits and limitations of several approximation strategies using numerical examples. When nonlinearity in membership degrees is typical, the suggested method for approximating generalized neutrosophic numbers with asymmetry shows accuracy and broad application.

The neutrosophic theory offers a versatile framework for modelling uncertainty in various domains. Future research may focus on hybrid strategies that combine neutrosophic numbers with alternative uncertainty representations and broaden the application of these models to more challenging decision-making scenarios.

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