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Combining Two Auxiliary Variables for Elevated Estimation of Finite Population Mean under Neutrosophic Framework

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Abstract: Typically, most researchers rely on precise data for estimating population parameters using classical statistical methods. However, there are scenarios where dealing with uncertain and imprecise data in the form of intervals becomes necessary. To tackle this challenge, various adaptations of classical estimators, such as the neutrosophic ratio estimator and their improved ones, have emerged. This article introduces a novel estimator known as the neutrosophic ratio-cum-product exponential estimator combining two auxiliary variables, specifically designed for the elevated estimation of population mean in such situations. Performance evaluation is conducted using metrics like Mean Square Error (MSE) and Percentage Relative Efficiency (PRE). The effectiveness of the proposed estimator is demonstrated through both empirical and simulation studies. Additionally, its practical applicability is showcased using agricultural data. The results illustrate that the proposed estimator surpasses all other estimators discussed in this paper. To justify the usefulness of neutrosophic estimators over their classical ones, a simulation study is also made. Simulation results obtained for classical estimators have also been compared with their neutrosophic adaptations.

Keywords: Neutrosophic Ratio-cum-Product exponential type estimator; Bias; Mean square error; Neutrosophic Simulation; Percentage relative efficiency.

1. Introduction

In daily life, we often encounter many situations where data is imprecise, vague, and uncertain. To address the vagueness in logic L.A. Zadeh made his very first attempt and proposed the concept of fuzzy logic [1]. Subsequently, many extensions of fuzzy logic have been proposed to address the vagueness in the data. To address the problem of indeterminacy in the data Florentin Smarandache proposed the concept of neutrosophy which is the generalization of fuzzy logic [2].

When a data or part of it has some indeterminacy, it leads to neutrosophic statistics, whereas when data are completely deterministic or precise, it leads us to classical statistics [3]. When data is crisp and deterministic, it becomes easy to draw a conclusion by giving single-valued performance measures that may define a real-life scenario. Classical estimators become inefficient when attempting to estimate such properties in practical settings. Estimation under neutrosophy is still a developing area where many researchers have shown that estimators based on neutrosophy are proven to be more efficient than classical estimators, especially when indeterminacy in the data is concerned.

In the field of sampling theory [4] made an attempt to estimate the population mean when data has some uncertainty and indeterminacy. He proposed some neutrosophic adaptations for the classical ratio estimator. His study shows that neutrosophic estimators are more efficient than classical estimators when indeterminacy in the data is concerned. [5], introduced a neutrosophic generalized exponential type estimators. [6], proposed a generalized neutrosophic strategy for elevated estimation of population mean. [7], proposed a robust neutrosophic ratio type estimator.

Generally, under the sampling estimation theory, we found that the incorporation of auxiliary variable increases the efficiency of an estimator. Several neutrosophic adaptations of classical estimation have been proposed when the study variable is positively correlated with the auxiliary variable [4-7]. By using single auxiliary variable and multiple auxiliary variables several estimators have been proposed in the classical frame work. Ratio estimator, product estimator and regression estimator performs almost equally in the optimum situations. Regression estimator performs better than ratio and regression estimators when regression line based on study and auxiliary variable deviates from the origin.[8], proposed a ratio-cum-product estimator. He showed that ratio-cum-product estimators perform better than usual estimator and ratio and product estimator under some defined conditions. [9], tried to improve the ratio-cum-product estimator by using a regression based unbiased estimator of auxiliary variables in place of usual mean estimator. His estimators showed improved efficiency over the estimator proposed by [8]. [10], proposed a ratio-cum-product type exponential estimator. His estimator revealed a better performance over the exponential ratio and regression type estimators.

[14-19] have proposed classical estimators incorporating two auxiliary variables that are correlated with the study variable, leads to greater precision. However, the estimation of population parameters using two correlated auxiliary variables under the neutrosophic framework remains unexplored. This article makes an attempt to provide an improved ratio-cum-product type exponential estimator by utilizing the idea of [9] and provide a neutrosophic adaptation of classical estimators. In this ongoing development of finite population mean estimation under neutrosophic framework, we have proposed a neutrosophic ratio-cum-product type exponential estimator that performs efficiently in situations where we use two neutrosophic auxiliary variables, one of which is positively correlated and the other is negatively correlated with the study variable. We have also given an extension of the flow chart created by [4] which guides the process of estimation under neutrosophic study.

Section 2 gives a flow chart which briefly demonstrates our study under neutrosophic framework. Section 3 describes symbolic representation of neutrosophic data and standards used. Section 4 encompasses a review study on some existed neutrosophic estimators for the estimation of finite population mean under neutrosophic framework. Section 5 consists of our proposed estimator and the derivations for the bias and mean square error of the given neutrosophic ratio-cum-product exponential type estimator for the first order of approximation. To confirm the accuracy of our suggested estimator, we conducted a numerical studyin section 6. Within this section, we performed an empirical investigation using agricultural data [11], concerning rice production. Additionally, we conducted a simulation study to further validate the theoretical properties of our suggested estimator. The Remaining parts include results,discussion and conclusions.

2. Flow chart suggesting the study for neutrosophic data

We have introduced an additional box in the flow chart constructed by [4] which shows that, If we have more than one neutrosophic auxiliary variables which are weakly/moderately correlated with neutrosophic study variable and out of which few are positively correlated and remaining are negatively correlated with the neutrosophic study variable then a fresh approach can be opted where ratio and product method along with exponential type estimation methods may be used simultaneously.

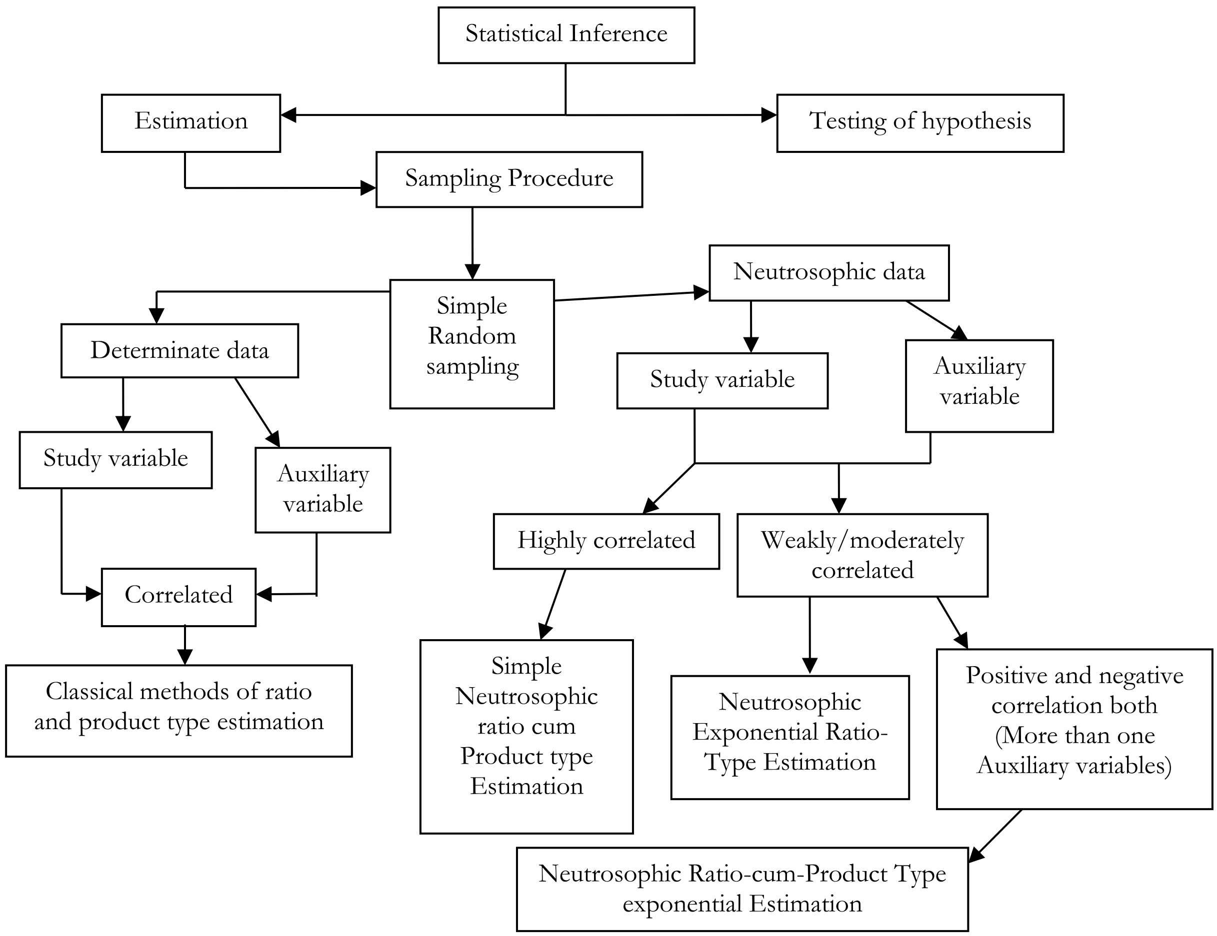

Figure 1 Flow chart of the study

3. Symbolic representations for neutrosophic data

Neutrosophic observations are represented in the following form $Z_N = Z_L + Z_U I_N$, where $I_N \in [I_L, I_U]$, $Z_N \in [Z_L, Z_U]$, where Z_L and Z_U are the lower and upper values of neutrosophic variable Z_N. I_N shows the level of indeterminacy in Z_N which takes values between 0 and 1. Here we see that Z_N takes values in the interval form which implies that any further calculation based on it will provide us a interval value instead of a point value.

Some of the notations mentioned here are taken from [4]. Consider a neutrosophic finite population $U_N = (U_{1N}, U_{2N}, \dots, U_{NN})$ of size N_{NN}. On the each unit $U_{iN} \in (i = 1,2,\dots,N)$ two neutrosophic auxiliary variables $X_{1N} \in [X_{1L}, X_{1U}]$ and $X_{2N} \in [X_{2L}, X_{2U}]$ are observed. $Y_N \in [Y_L, Y_U]$ be the study variable which is positively correlated with X_{1N} and negatively correlated with X_{2N}. A neutrosophic sample of size $n_N \in [n_L, n_U]$ is taken by using simple random sampling without replacement scheme.

Let $\left(y_N(i), x_{1N}(i), x_{2N}(i)\right)$ be the triplet corresponding to i^{th} sample observation on study variable $Y_N \in [Y_L, Y_U]$ and auxiliary variables $X_{1N} \in [X_{1L}, X_{1U}]$ and $X_{2N} \in [X_{2L}, X_{2U}]$ respectively. $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$, $\bar{X}_{1N} \in [\bar{X}_{1L}, \bar{X}_{1U}]$ and $\bar{X}_{2N} \in [\bar{X}_{2L}, \bar{X}_{2U}]$ are the population means corresponding to neutrosophic study variable Y_N and auxiliary variables X_{1N} and X_{2N} respectively. $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$, $\bar{x}_{1N} \in [\bar{x}_{1L}, \bar{x}_{1U}]$ and $\bar{x}_{2N} \in [\bar{x}_{2L}, \bar{x}_{2U}]$ are the neutrosophic sample means corresponding to i^{th} sample observation on study variable $Y_N \in [Y_L, Y_U]$ and auxiliary variables $X_{1N} \in [X_{1L}, X_{1N}]$ and $X_{2N} \in [X_{2L}, X_{2U}]$ respectively. Where, $y_N(i) \in [y_L(i), y_U(i)]$, $x_{1N}(i) \in [x_{1L}(i), x_{1U}(i)]$, $x_{2N}(i) \in [x_{2L}(i), x_{2U}(i)]$.

In addition $C_{yN} \in [C_{yL}, C_{yU}]$, $C_{x1N} \in [C_{x1L}, C_{x1U}]$, and $C_{x2N} \in [C_{x2L}, C_{x2U}]$ are neutrosophic coefficients of variation for neutrosophic variables Y_N, X_{1N} , andX_{2N} respectively. $\rho_{yx1N} \in [\rho_{yx1L}, \rho_{yx1U}]$ and $\rho_{yx2N} \in [\rho_{yx2L}, \rho_{yx2U}]$are the neutrosophic correlation coefficients between study variable $Y_N \in [Y_L, Y_U]$ and neutrosophic auxiliary variables $X_{1N} \in [X_{1L}, X_{1N}]$ and $X_{2N} \in [X_{2L}, X_{2N}]$ respectively. The performance measures used for the mean estimator are mean square error (MSE), and percentage relative efficiency (PRE) which are neutrosophically represented as $MSE_N(\bar{y}_N) \in [MSE_L(\bar{y}_N), MSE_U(\bar{y}_N)]$ and $PRE_N(\bar{y}_N) \in [PRE_L(\bar{y}_N), PRE_U(\bar{y}_N)]$ respectively.
$\varepsilon_{0N} = \frac{\bar{y}_N - \bar{Y}_N}{\bar{Y}_N}$, $\varepsilon_{1N} = \frac{\bar{x}_{1N} - \bar{X}_{1N}}{\bar{X}_{1N}}$ and $\varepsilon_{2N} = \frac{\bar{x}_{2N} - \bar{X}_{2N}}{\bar{X}_{2N}}$, are the neutrosophic relative errors corresponding to sample mean of neutrosophic study variable Y_N and auxiliary variables X_{1N} and X_{2N} respectively.

$$\varepsilon_{0N} \in [\varepsilon_{0L}, \varepsilon_{0U}],\ \varepsilon_{1N} \in [\varepsilon_{1L}, \varepsilon_{1U}] \text{ and } \varepsilon_{2N} \in [\varepsilon_{2L}, \varepsilon_{2U}]$$

$$E(\varepsilon_{0N}) = E(\varepsilon_{1N}) = E(\varepsilon_{2N}) = 0; E(\varepsilon_{0N}^2) = \theta_N C_{yN}^2; E(\varepsilon_{1N}^2) = \theta_N C_{x1N}^2; E(\varepsilon_{2N}^2) = \theta_N C_{x2N}^2$$

$$\theta_N = \frac{(1-f_N)}{n_N}; \theta_N \in [\theta_L, \theta_U]$$

$$E(\varepsilon_{0N}\varepsilon_{1N}) = \theta_N C_{x1N} C_{yN} \rho_{yx1N}; E(\varepsilon_{0N}\varepsilon_{2N}) = \theta_N C_{x2N} C_{yN} \rho_{yx2N}; E(\varepsilon_{1N}\varepsilon_{2N}) = \theta_N C_{x1N} C_{x2N} \rho_{x1x2N}$$

$$C_{yN}^2 = \frac{\sigma_{yN}^2}{\bar{Y}_N^2}; C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]; C_{x1N}^2 = \frac{\sigma_{x1N}^2}{\bar{X}_{1N}^2}; C_{x1N}^2 \in [C_{x1L}^2, C_{x1U}^2]; C_{x2N}^2 = \frac{\sigma_{x2N}^2}{\bar{X}_{2N}^2}; C_{x2N}^2 \in [C_{x2L}^2, C_{x2U}^2]; \sigma_{yN}^2 \in [\sigma_{yL}^2, \sigma_{yU}^2]; \sigma_{x1N}^2 \in [\sigma_{x1L}^2, \sigma_{x1U}^2]; \sigma_{x2N}^2 \in [\sigma_{x1L}^2, \sigma_{x2U}^2]$$

4. Review of some neutrosophic estimators

A Neutrosophic variant of usual sample mean estimator is proposed by [6].

$t_{0N} = \bar{y}_N$; $t_{0N} \in [t_{0L}, t_{0U}]$

Variance expression is given by:

$$Var(t_{0N}) = \theta_N \bar{Y}_N^2 C_{yN}^2 \tag{1}$$

Several adaptations of classical estimators within the framework of neutrosophic theory were proposed by [4], a subset of which are discussed below,

$$t_{1N} = \frac{\bar{y}_N}{\bar{x}_{1N}} \bar{X}_{1N}; t_{1N} \in [t_{1L}, t_{1U}] \tag{2}$$

$$Bias(t_{1N}) = \theta_N \bar{Y}_N [C_{x1N}^2 - C_{x1N} C_{yN} \rho_{yxN}] \tag{3}$$

$$MSE(t_{1N}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + C_{x1N}^2 - 2C_{x1N} C_{yN} \rho_{yx1N}] \tag{4}$$

$$t_{2N} = \bar{y}_N exp\left[\frac{\bar{X}_{1N} - \bar{x}_{1N}}{\bar{X}_{1N} + \bar{x}_{1N}}\right]; t_{2N} \in [t_{2L}, t_{2U}] \tag{5}$$

$$Bias(t_{2N}) = \theta_N \bar{Y}_N \left[\frac{3}{8} C_{x1N}^2 - \frac{1}{2} C_{x1N} C_{yN} \rho_{yx1N}\right] \tag{6}$$

$$MSE(t_{2N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4} C_{x1N}^2 - C_{x1N} C_{yN} \rho_{yx1N}\right] \quad (7)$$

$$t_{3N} = \bar{y}_N \left[\frac{\bar{X}_{1N} + C_{x1N}}{\bar{x}_{1N} + C_{x1N}}\right]; t_{3N} \in [t_{3L}, t_{3U}] \quad (8)$$

$$Bias(t_{3N}) = \theta_N \bar{Y}_N \left[\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N}+C_{xN}}\right)^2 C_{x1N}^2 - \left(\frac{\bar{X}_{1N}}{\bar{X}_{1N}+C_{xN}}\right) C_{x1N} C_{yN} \rho_{yx1N}\right] \quad (9)$$

$$MSE(t_{3N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \left(\frac{\bar{X}_{1N}}{\bar{X}_{1N}+C_{x1N}}\right)^2 C_{x1N}^2 - 2\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N}+C_{x1N}}\right) C_{x1N} C_{yN} \rho_{yx1N}\right] \quad (10)$$

A neutrosophic ratio estimator based on correlation coefficient were suggested by [6],

$$t_{4N} = \bar{y}_N \left[\frac{\bar{X}_{1N} + \rho_{yx1N}}{\bar{x}_{1N} + \rho_{yx1N}}\right]; t_{4N} \in [t_{4L}, t_{4U}] \quad (11)$$

$$Bias(t_{4N}) = \theta_N \bar{Y}_N \left[\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N}+\rho_{yx1N}}\right)^2 C_{x1N}^2 - \left(\frac{\bar{X}_{1N}}{\bar{X}_{1N}+\rho_{yx1N}}\right) C_{x1N} C_{yN} \rho_{yx1N}\right] \quad (12)$$

$$MSE(t_{4N}) = \theta_N \bar{Y}_N^2 \left[\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N}+\rho_{yx1N}}\right)^2 C_{yN}^2 + C_{x1N}^2 - 2\left(\frac{\bar{X}_{1N}}{\bar{X}_{1N}+\rho_{yx1N}}\right) C_{x1N} C_{yN} \rho_{yx1N}\right] \quad (13)$$

Motivated by [12], a neutrosophic product type exponential estimator may be given as,

$$t_{5N} = \bar{y}_N exp\left[\frac{\bar{x}_{2N} - \bar{X}_{2N}}{\bar{x}_{2N} + \bar{X}_{2N}}\right]; t_{5N} \in [t_{5L}, t_{5U}] \quad (14)$$

$$MSE(t_{5N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \{C_{x2N}^2 (1 - 4K_{02N})/4\}\right]; \quad (15)$$

$$K_{02N} = \rho_{yx2N \frac{C_{yN}}{C_{x2N}}}; K_{02N} \in [K_{02L}, K_{02U}]$$

Motivated by [13], neutrosophic product type estimator may be given as,

$$t_{6N} = \frac{\bar{y}_N \bar{x}_{2N}}{\bar{X}_{2N}}; t_{6N} \in [t_{6L}, t_{6U}] \quad (16)$$

$$Bias(t_{6N}) = \theta_N \bar{Y}_N \rho_{yx2N} C_{x2N} C_{yN} \quad (17)$$

$$MSE(t_{6N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + C_{x1N}^2 (1 + 2K_{02N})\right]; \quad (18)$$

$$K_{02N} = \rho_{yx2N \frac{C_{yN}}{C_{x2N}}}; K_{02N} \in [K_{02L}, K_{02U}]$$

Motivated by [8], a neutrosophic ratio-cum-product estimator may be given as

$$t_{7N} = \bar{y}_N \frac{\bar{X}_{1N}}{\bar{x}_{1N}} \frac{\bar{x}_{2N}}{\bar{X}_{2N}} ; t_{7N} \in [t_{7L}, t_{7U}]$$

(19)

$$MSE(t_{7N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \{C_{x1N}^2(1 - 2K_{01N})\} + \{C_{x2N}^2\left(1 + 2(K_{02} - K_{12})\right)\}\right]$$

(20)

$$K_{01N} = \rho_{yx1N \frac{C_{yN}}{C_{x1N}}} ; K_{01N} \in [K_{01L}, K_{01U}] ; \; K_{02N} = \rho_{yx2N \frac{C_{yN}}{C_{x2N}}} ; K_{02N} \in [K_{02L}, K_{02U}];$$

$$K_{12N} = \rho_{x1x2N \frac{C_{x1N}}{C_{x2N}}} ; K_{12N} \in [K_{12L}, K_{12U}]$$

Motivated by [10], neutrosophic ratio-cum-product type exponential estimator may be given as,

$$t_{8N} = \bar{y}_N exp \left[\frac{\bar{X}_{1N} - \bar{x}_{1N}}{\bar{X}_{1N} + \bar{x}_{1N}}\right] exp \left[\frac{\bar{x}_{2N} - \bar{X}_{2N}}{\bar{x}_{2N} + \bar{X}_{2N}}\right] ; t_{8N} \in [t_{8L}, t_{8U}]$$

(P*)

$$Bias(t_{8N}) = \left(\theta_N/2\right) \bar{Y}_N \left[\frac{3}{4} C_{x1N}^2 - C_{x1N} C_{yN} \rho_{yx1N} + C_{x2N} C_{yN} \rho_{yx2N} - C_{x1N} C_{x2N} \rho_{x1x2N} - \frac{1}{4} C_{x2N}^2\right]$$

(21)

$$MSE(t_{8N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \{C_{x1N}^2 (1 - 4K_{01N})/4\} + \{C_{x2N}^2(1 + 4K_{02} - 2K_{12})/4\}\right] ;$$

(22)

$$K_{01N} = \rho_{yx1N} \frac{C_{yN}}{C_{x1N}} ; K_{01N} \in [K_{01L}, K_{01U}] ; \; K_{02N} = \rho_{yx2N} \frac{C_{yN}}{C_{x2N}} ; K_{02N} \in [K_{02L}, K_{02U}];$$

$$K_{12N} = \rho_{x1x2N} \frac{C_{x1N}}{C_{x2N}} ; K_{12N} \in [K_{12L}, K_{12U}]$$

5. Proposed Estimator

Motivated by [10] and [9], we have proposed a neutrosophic ratio-cum-product exponential type estimator, using two neutrosophic auxiliary variables X_{1N} and X_{2N}. The proposed estimator have been obtained by replacing neutrosophic sample means, $\bar{x}_{iN} (i = 1,2)$ of auxiliary variables in (P*) with the unbiased estimators $\hat{t}_{iN} = \bar{x}_{iN} + \alpha_{iN}(\bar{X}_{iN} - \bar{x}_{iN})$, where$\alpha_{iN}(i = 1,2)$ are real constants to be suitably chosen. Hence the proposed estimator can be given as,

$$t_{RP_eN} = \bar{y}_N exp \left[\frac{\bar{X}_{1N} - \bar{t}_{1N}}{\bar{X}_{1N} + \bar{t}_{1N}}\right] exp \left[\frac{\bar{t}_{2N} - \bar{X}_{2N}}{\bar{t}_{2N} + \bar{X}_{2N}}\right] ; t_{RPeN} \in [t_{RPeL}, t_{RPeU}]$$

(23)

Expressing the proposed estimator in error terms we get

$$t_{RPeN} = \bar{Y}_N (1 + \varepsilon_0) exp \left[\frac{\bar{X}_{1N} - \left(\bar{X}_{1N}\left(1 + \varepsilon_{1N}(1 - \alpha_{1N})\right)\right)}{\bar{X}_{1N} + \left(\bar{X}_{1N}\left(1 + \varepsilon_{1N}(1 - \alpha_{1N})\right)\right)}\right] exp \left[\frac{\bar{X}_{2N}\left(1 + \varepsilon_{2N}(1 - \alpha_{2N})\right) - \bar{X}_{2N}}{\bar{X}_{2N}\left(1 + \varepsilon_{2N}(1 - \alpha_{2N})\right) + \bar{X}_{2N}}\right]$$

$$= \bar{Y}_N (1 + \varepsilon_0) exp \left[\frac{-\varepsilon_{1N}(1 - \alpha_{1N})}{2\left(1 + \frac{\varepsilon_{1N}}{2}(1 - \alpha_{1N})\right)}\right] exp \left[\frac{\varepsilon_{2N}(1 - \alpha_{2N})}{2\left(1 + \frac{\varepsilon_{2N}}{2}(1 - \alpha_{2N})\right)}\right]$$

(24)

After expanding the terms on the right-hand side and algebraic simplifying and retaining the terms for the first degree of approximation, we get

$$t_{RPeN} = \bar{Y}_N\left[1-\frac{\varepsilon_{1N}}{2}(1-\alpha_{1N})+\frac{3}{8}\varepsilon_{1N}^2(1-\alpha_{1N})^2+\frac{\varepsilon_{2N}}{2}(1-\alpha_{2N})-\frac{\varepsilon_{1N}\varepsilon_{2N}}{4}(1-\alpha_{1N})(1-\alpha_{2N})-\frac{\varepsilon_{2N}^2(1-\alpha_{2N})^2}{8}+\varepsilon_{0N}-\frac{\varepsilon_{0N}\varepsilon_{1N}}{2}(1-\alpha_{1N})+\frac{\varepsilon_{0N}\varepsilon_{2N}}{2}(1-\alpha_{2N})\right] \quad (25)$$

Subtracting $\bar{Y}_N$ in the both sides of the above equation, we get

$$t_{RPeN}-\bar{Y}_N = \bar{Y}_N\left[-\frac{\varepsilon_{1N}}{2}(1-\alpha_{1N})+\frac{3}{8}\varepsilon_{1N}^2(1-\alpha_{1N})^2+\frac{\varepsilon_{2N}}{2}(1-\alpha_{2N})-\frac{\varepsilon_{1N}\varepsilon_{2N}}{4}(1-\alpha_{1N})(1-\alpha_{2N})-\frac{\varepsilon_{2N}^2(1-\alpha_{2N})^2}{8}+\varepsilon_{0N}-\frac{\varepsilon_{0N}\varepsilon_{1N}}{2}(1-\alpha_{1N})+\frac{\varepsilon_{0N}\varepsilon_{2N}}{2}(1-\alpha_{2N})\right] \quad (26)$$

Taking expectation in the both side of equation (26) and putting expectation of terms, we get the bias express as,

$$Bias(t_{RPeN}) = \bar{Y}_N\theta_N\left[\frac{3}{8}(1-\alpha_{1N})^2C_{x1N}^2-\frac{1}{8}(1-\alpha_{2N})^2C_{x2N}^2-\frac{(1-\alpha_{1N})(1-\alpha_{2N})}{4}\rho_{x1x2N}C_{x1N}C_{x2N}-\frac{(1-\alpha_{1N})}{2}\rho_{yx1N}C_{x1N}C_{yN}+\frac{(1-\alpha_{2N})}{2}\rho_{yx2N}C_{x2N}C_{yN}\right] \quad (27)$$

Squaring both sides of the equation (26) and retaining terms for the first order of approximation, we get

$$(t_{RPeN}-\bar{Y}_N)^2 = \bar{Y}^2\left[\varepsilon_{0N}^2+\frac{\varepsilon_{2N}^2}{4}(1-\alpha_{2N})^2+\frac{\varepsilon_{1N}^2}{4}(1-\alpha_{1N})^2-\frac{\varepsilon_{1N}\varepsilon_{2N}}{2}(1-\alpha_{1N})(1-\alpha_{2N})+\varepsilon_{0N}\varepsilon_{2N}(1-\alpha_{2N})-\varepsilon_{0N}\varepsilon_{1N}(1-\alpha_{1N})\right] \quad (28)$$

Taking expectation both sides of the equation (28), expression for the mean square error can be obtained as,

$$MSE(t_{RPeN}) = \bar{Y}_N^2\theta_N\left[C_{yN}^2+\frac{(1-\alpha_{2N})^2}{4}C_{x2N}^2+\frac{(1-\alpha_{1N})^2}{4}C_{x1N}^2-\frac{(1-\alpha_{1N})(1-\alpha_{2N})}{2}\rho_{x1x2N}C_{x1N}C_{x2N}+(1-\alpha_{2N})\rho_{yx2N}C_{x2N}C_{yN}-(1-\alpha_{1N})\rho_{yx1N}C_{x1N}C_{yN}\right] \quad (29)$$

Optimum values of the coefficients $\alpha_{iN} (i = 1, 2)$ which minimizes the MSE_N can be obtained by taking partial derivatives of equation (13) with respect to α_{1N} and α_{2N} respectively.

$$\alpha_{1N}^* = 1-\left[\frac{2C_{YN}\rho_{x1x2N}}{C_{x1N(1-\rho_{x1x2N}^2)}}\left(\rho_{yx1N}-\rho_{yx2N}\right)+2\rho_{yx1N}\frac{C_{yN}}{C_{x1N}}\right]$$

$$\alpha_{2N}^* = 1-\left[\frac{2C_{YN}}{C_{x2N(1-\rho_{x1x2N}^2)}}\left(\rho_{yx1N}-\rho_{yx2N}\right)\right]$$

$$MSE(t_{RPeN})_{opt} = \bar{Y}_N^2\theta_N\left[C_{yN}^2+\frac{(1-\alpha_{2N}^*)^2}{4}C_{x2N}^2+\frac{(1-\alpha_{1N}^*)^2}{4}C_{x1N}^2-\frac{(1-\alpha_{1N}^*)(1-\alpha_{2N}^*)}{2}\rho_{x1x2N}C_{x1N}C_{x2N}+(1-\alpha_{2N}^*)\rho_{yx2N}C_{x2N}C_{yN}-(1-\alpha_{1N}^*)\rho_{yx1N}C_{x1N}C_{yN}\right] \quad (30)$$

6. Numerical Study

6.1. Empirical Study

To show the utility of our proposed estimator usability of our proposed estimator t_{RPeN}, we have used data provided by [11]. We have considered rice yield as our neutrosophic study variable ($\bar{Y}_N$) and rain sowing ($\bar{X}_{1N}$) and rain ripening ($\bar{X}_{2N}$) as our two neutrosophic auxiliary variables.

Table 1 depicts the population descriptive statistics and Table 2 illustrates the MSEs and PREs corresponding to reviewed estimators and proposed estimators.

Table 1 Descriptive statistics for population under study

μ_{yN}	[3.9222, 18.4333]	C_{yN}	[0.5664, 0.1192]
μ_{x1N}	[11.5889, 35.6556]	C_{x1N}	[0.9258, 0.7679]
μ_{x2N}	[40.5889, 93.7333]	C_{x2N}	[0.7927, 0.8014]
σ^2_{yN}	[4.9351, 4.8289]	ρ_{yx1N}	[0.5973, -0.5332]
σ^2_{x1N}	[115.1054, 749.6602]	ρ_{yx2N}	[-0.0647, -0.7512]
σ^2_{x2N}	[1035.0988, 5643.2067]	ρ_{x1x2N}	[0.4817, 0.8284]

Table 2 Neutrosophic performance measures

SN	Estimators	MSE	I (Indeterminacy)	PRE
1	t_{0N}	[0.2720, 0.2662]	[0, 0.0215]	[100, 100]
2	t_{1N}	[0.4157, 11.6799]	[0, 0.9644]	[65.4387, 2.2791]
3	t_{2N}	[0.1673, 3.5038]	[0, 0.9523]	[162.6514, 7.5972]
4	t_{3N}	[0.3586, 11.2360]	[0, 0.9681]	[75.8625, 2.3691]
5	t_{4N}	[0.3771, 12.0049]	[0, 0.9686]	[72.1436, 2.2174]
6	t_{5N}	[0.3821, 4.1051]	[0, 0.9069]	[71.1925, 6.4844]
7	t_{6N}	[0.7592, 8.5407]	[0, 0.9111]	[35.8311, 3.1168]
8	t_{7N}	[0.3127, 3.0081]	[0, 0.8961]	[87.0123, 8.8492]
9	t_{8N}	[0.1305, 0.7384]	[0, 0.8232]	[208.4060, 36.0514]
10	t_{RPeN}	**[0.1466, 0.0047]**	[0, 0.9681]	**[185.5373, 5684.8530]**

The data presented in Table 2 strongly indicates that our suggested estimator t_{RPeN} is more effective compared to other. The table clearly illustrates that our estimator has lowest MSE compared to all other estimators listed. In addition to having the lowest MSE, our proposed estimator t_{RPeN} also exhibits the highest PRE among all the metrics considered.

6.2. Monte-Carlo Simulation study

To support the results obtained through the empirical study and prove the validity of our proposed estimator as an efficient estimator we have performed a simulation study on a neutrosophic simulated data.

Table 3 shows the descriptive statistics for simulation under neutrosophic data. We have simulated a neutrosophic data by using R-studio software using 'ntsDists' package. In our study neutrosophic variables follows neutrosophic normal densities [4], i.e., $X_{1N} \sim NN(\mu_{x1N}, \sigma^2_{x1N})$; $X_{1N} \in [X_{1L}, X_{1U}]$, $\mu_{1N} \in [\mu_{x1L}, \mu_{x1U}]$, $\sigma^2_{x1N} \in [\sigma^2_{x1L}, \sigma^2_{x1U}]$; $X_{2N} \sim NN(\mu_{x2N}, \sigma^2_{x2N})$; $X_{2N} \in [X_{2L}, X_{2U}]$, $\mu_{2N} \in [\mu_{x2L}, \mu_{x2U}]$, $\sigma^2_{x2N} \in [\sigma^2_{x2L}, \sigma^2_{x2U}]$ and $Y_N \sim NN\left(\mu_{yN}, \sigma^2_{yN}\right)$; $Y_N \in [Y_L, Y_U]$, $\mu_{yN} \in [\mu_{yL}, \mu_{yU}]$, $\sigma^2_{yN} \in [\sigma^2_{yL}, \sigma^2_{yU}]$
We took $X_{1N} \sim NN((80.69, 110.86), (5.43^2, 37.86^2))$;$X_{2N} \sim NN((200.89, 250.50), (18.3^2, 25.5^2))$;
$Y_N \sim NN((30.8, 50.5), (2.3^2, 10.4^2))$ for generating 1000 neutrosophic normal variates.

Table 4 illustrates the descriptive statistics for simulation under classical statistics. Here, we have generated a simulated data under classical study from normal population, i.e., $X_{1N} \sim N(\mu_{x1}, \sigma^2_{x1})$; $X_{1N} \in [X_{1L}, X_{1U}]$, $\mu_{1N} \in [\mu_{x1L}, \mu_{x1U}]$, $\sigma^2_{x1N} \in [\sigma^2_{x1L}, \sigma^2_{x1U}]$; $X_{2N} \sim N(\mu_{x2}, \sigma^2_{x2})$; $X_{2N} \in [X_{2L}, X_{2U}]$, $\mu_{2N} \in [\mu_{x2L}, \mu_{x2U}]$; $\sigma^2_{x2N} \in [\sigma^2_{x2L}, \sigma^2_{x2U}]$ and $Y_N \sim N\left(\mu_y, \sigma^2_y\right)$; $Y_N \in [Y_L, Y_U]$, $\mu_{yN} \in [\mu_{yL}, \mu_{yU}]$, $\sigma^2_{yN} \in [\sigma^2_{yL}, \sigma^2_{yU}]$. We took $X_{1N} \sim N(90.56, 665.64)$; $X_{2N} \sim N(220.34, 424.36)$; $Y_N \sim N(40.2, 39.69)$ for generating 1000 normal variates under classical study.

Table 3 Descriptive statistics for simulation under neutrosophic data

$\boldsymbol{N_N}$	[1000, 1000]	$\boldsymbol{\sigma^2_{x2N}}$	[(18.3)², (25.5)²]
$\boldsymbol{n_N}$	[20, 20]	$\boldsymbol{C_{x1N}}$	[0.0673, 0.3415]
$\boldsymbol{\mu_{x1N}}$	[80.69, 110.86]	$\boldsymbol{C_{x2N}}$	[0.0911, 0.1018]
$\boldsymbol{\mu_{x2N}}$	[200.89, 250.50]	$\boldsymbol{C_{yN}}$	[0.0747, 0.2059]
$\boldsymbol{\mu_{yN}}$	[30.8, 50.5]	$\boldsymbol{\rho_{yx1N}}$	[0.0413, 0.0547]
$\boldsymbol{\sigma^2_{yN}}$	[(2.3)², (10.4)²]	$\boldsymbol{\rho_{yx2N}}$	[-0.0423, -0.0438]
$\boldsymbol{\sigma^2_{x1N}}$	[(5.43)², (37.86)²]	$\boldsymbol{\rho_{x1x2N}}$	[-0.0683, -0.0469]

Table 4 Descriptive statistics for simulation under classical data

$\boldsymbol{N}$	1000	$\boldsymbol{\sigma^2_{x2}}$	424.36
n	20	$\boldsymbol{C_{x1}}$	0.2849
$\boldsymbol{\mu_{x1}}$	90.56	$\boldsymbol{C_{x2}}$	0.0935
$\boldsymbol{\mu_{x2}}$	220.34	$\boldsymbol{C_y}$	0.1567
$\boldsymbol{\mu_y}$	40.2	$\boldsymbol{\rho_{yx1}}$	-0.0025
$\boldsymbol{\sigma^2_y}$	39.69	$\boldsymbol{\rho_{yx2}}$	0.0036
$\boldsymbol{\sigma^2_{x1}}$	665.64	$\boldsymbol{\rho_{x1x2}}$	0.0157

6.2.1. Steps followed for simulation study

1. Generate a 1000 neutrosophic normal variates by using assumed characteristics of the population.

2. Draw a sample of size 20 from the generated population by using simple random sampling without replacement scheme.
3. Obtain estimates of different estimators considered in our article.
4. Repeat step 2 and step 3 thousand times and calculate neutrosophic MSEs and PREs.

Similarly, we can obtain MSEs and PREs for the estimators under classical simulation studies, with the distinction that data is generated using classical normal distribution.

Table 5 Neutrosophic versus Classical estimates of MSEs

SN	Estimator	MSE Neutrosophic	I (Indeterminacy)	MSE Classical
1	t_{0N}	[0.2726, 5.0344]	[0, 0.9458]	2.0568
2	t_{1N}	[1.7284, 15.8633]	[0, 0.8910]	8.6187
3	t_{2N}	[0.6367, 7.7064]	[0, 0.9174]	3.6910
4	t_{3N}	[1.7202, 15.8054]	[0, 0.8912]	17.1468
5	t_{4N}	[1.7293, 15.8691]	[0, 0.8910]	8.6231
6	t_{5N}	[0.3771, 5.3905]	[0, 0.9300]	2.2434
7	t_{6N}	[0.6889 6.4545]	[0, 0.8933]	2.8016
8	t_{7N}	[2.1423, 17.2567]	[0, 0.8759]	9.3782
9	t_{8N}	[0.7398 8.0544]	[0, 0.9082]	3.8805
10	$\boldsymbol{t_{RPeN}}$	**[0.2614 1.7610]**	[0, 0.8516]	**1.9698**

Table 6 PREs under simulation study

SN	Estimator	PRE Neutrosophic	PRE Classical
1	t_{0N}	[100, 100]	100
2	t_{1N}	[15.7733, 31.7359]	23.8642
3	t_{2N}	[12.7253, 29.1734]	55.7239
4	t_{3N}	[15.8488, 31.8523]	11.9951
5	t_{4N}	[15.7651, 31.7244]	23.8520
6	t_{5N}	[72.2912, 93.3940]	91.6818
7	t_{6N}	[15.7734, 31.7359]	73.4144
8	t_{7N}	[39.5764, 77.9982]	21.9313
9	t_{8N}	[36.8534, 62.5045]	53.0029
10	$\boldsymbol{t_{RPeN}}$	**[104.3013, 285.8807]**	**104.4135**

Simulation results presented in the Table 5 demonstrate that our proposed neutrosophic estimator outperforms all other reviewed estimators in terms of MSE. Table 6 indicates that proposed neutrosophic estimator t_{RPeN} has the maximum PRE among all the reviewed estimators. When data is imprecise and vague it becomes quite difficult to stand with single value estimate of MSEs.Table 5 and Table 6 clearly illustrate that neutrosophic MSEs and PREs reveal more reliable results than classical MSEs and PREs respectively.

7. Results and Discussion

Table 2 provides neutrosophic MSE under empirical study which shows that usual neutrosophic mean estimator $\bar{Y}_N$ $(= t_{0N})$ has variance [0.2720, 0.2662] which is better than some of the reviewed neutrosophic estimators. This might be because of the poor correlation between the study variable and auxiliary variables. Our proposed estimator has MSE **[0.1466, 0.0047]**, showing that this estimator performs better than all reviewed estimators.Under simulation study Table 5 shows that usual estimator has the variance [0.2726, 5.0344] and our proposed estimator has the MSE [.2614, 1.7610]. Simulation under classical study also shows the similar pattern of the performance of the estimators. Table 6 shows that under empirical study, neutrosophic PRE of the proposed estimator is **[185.5373, 5684.8530]** and under simulation study it is **[104.3013, 285.8806].**

It is very well known that the quality of data is a very important factor in empirical studies. Though we have considered a very small population, we still have gotten a similar trend in the MSE and PRE as in the simulation results, which support the better performance of our study.

8. Conclusions

This study presents a ratio-cum-product type exponential estimator using two correlated auxiliary variables for estimating population mean when dealing with imprecise data. To show the practical application, we have used agricultural data. We found that our proposed estimator performs outstandingly, and using two auxiliary variables for mean estimation under neutrosophic framework reduces precision remarkably. We have also found that neutrosophic estimators perform better than classical estimators when there is indeterminacy in the data. So, it is reliable to use neutrosophic estimators for indeterminate data.

Though we have used only two auxiliary estimators in our study on agriculture based secondary data, there is an open space to further improve the estimator by using more than two auxiliary variables. Moreover, application is not bounded to only agriculture-based data, however, researchers can further make an empirical study on different fields provided estimation of population mean is concerned.

Conflicts of Interest

None

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