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Multiple-Attribute Decision Making Based on Novel Single Valued Neutrosophic Aczel-Alsina Power Muirhead Mean Operators

Peide Liu[1,2,*], Qaisar Khan[3,*], Rashid Ali[4], Hidayat ULLAH[4] , Rashid Jan[5,6]

[11]School of Business Administration, Shandong Women's University, Jinan 250300, Shandong, China

[2]Shandong Key Laboratory of New Smart Media, Jinan Shandong 250014, China Email: peide.liu@gmail.com

[3]Department of Mathematics and Statistics, University of Haripur KPK, Pakistan, Email:qaisar.khan@uoh.edu.pk

[4]Department of mathematics, University of Malakand KPK Pakistan, rashidalikhanbaidara@gmail.com, hidayatullak@yahoo.com

[5]Institute of Energy Infrastructure (IEI), Department of Civil Engineering, College of Engineering, Universiti Tenaga Nasional (UNITEN), Putrajaya Campus, Jalan IKRAM-UNITEN, 43000 Kajang, Selangor, Malaysia

[6]Mathematics Research Center, Near East University TRNC, Mersin 10, Nicosia 99138, Turkey, Email:rashid.jan@uniten.edu.my

Corresponding author: Peide Liu, Qaisar Khan (e-mail: peide.liu@gmail.com, qaisar.khan@uoh.edu.pk).

Abstract: Neutrosophic multiple-attribute decision making (MADM) is one of the major study areas in ambiguous and inconsistent decision-making problems. The single-valued neutrosophic number (S^vNN), which is an expansion of the fuzzy number, intuitionistic fuzzy number, can handle difficulties involving a huge quantity of erroneous, partial, and inconsistent information. To address the multi-criteria decision-making problems, novel aggregation operators (AGOs) are proposed in this article considering the benefits of the S^vNN. The new AGOs consider the interrelationship involving any number of input data and can remove the effect of unreliable data. This article adds the Aczel-Alsina (AA) operations to increase the flexibility of the new AGOs. This study proposes the single-valued neutrosophic Aczel-Alsina power Muirhead mean ($S^vNAAPMM$) operator, single-valued neutrosophic Aczel-Alsina power dual Muirhead mean ($S^vNAAPDMM$) operator which combines the Aczel-Asina operational rules with the power average/geometric operator and the Muirhead/ dual Muirhead mean operators. Some core characteristics and special cases with respect to the parameters are explored and it is found that various existing AGOs are special cases of the newly initiated AGOs. Further, weighted form these AGOs are introduced. Then, we set up the MADM technique using the two AGOs that are suggested to solve MADM problems. We then give a numerical example about strategic supplier's selection and compare it to other related MADM techniques already in use in the S^vNN information to demonstrate the effectiveness and appropriateness of the anticipated technique.

Keywords Single valued neutrosophic sets; Muirhead/dual Muirhead mean operators, power average/geometric operators, MADM.

1. Introduction

Multiple-attribute decision making (MADM) problems are more common than ever in today's sophisticated as well as complicated technological environments [1, 2]. For instance, an enterprise with confined production capacity will need to concurrently address the problem of appropriate product price and lot-size determinations [3]. The creation of energy from renewable sources and the restrictions on carbon emissions must be associated into a single electric network to create a hybrid renewable energy system [4]. Material handling concerns and pertinent expenditures should be considered simultaneously in a dynamic facility layout challenge [5].

Zadeh's theory is a well-known and extended version of the crisp set theory because of its range. In crisp set theory, we only have two options, such as zero or one, but in the case of Fuzzy set (FS) theory, we have a lot of opinions when evaluating any kind of problem. Zadeh developed the main idea of the FS to expand the scope of the classical set theory [6]. FSs, however, are unable to handle some situations in which it is tough to describe the membership degree operating a single distinctive number. Atanassov [7] presented intuitionistic FSs (IFSs), an expansion of Zadeh's FSs, to address the problem of non-membership degrees not being well understood. IFSs have so far been frequently used to solve MADM problems [8-14]. Like Zadeh FSs, IFSs are unable to describe ambiguous and inconsistent information, thus Smarandache [15, 16] developed the neutrosophic set (NS) from a philosophical perspective to do so. In a NS, its membership, indeterminacy, and falsity degrees are independently represented and are real standard or nonstandard subclasses of $\left]0^-,1^+\right[$. Consequently, the nonstandard interval $\left]0^-,1^+\right[$ might make practical applications more challenging. Thus, the idea of a single-valued neutrosophic set (S^vNS) [17] is created on the real standard interval $\left[0,1\right]$. After the introduction many scholars applied S^vNS in various fields [18-24].

The previous several years have seen a significant increase in researcher knowledge of information aggregation operators [25–29], who are now a highly important study area for MADM problems. The conventional AGOs suggested by Xu, Xu, and Yager [30, 31] are limited to aggregating a collection of real numbers into a single factual number. Now, many authors have expanded on these traditional AGOs. For instance, Peng et al. [32] proposed the S^vN power shapely Choquet weighted averaging operator for MADM and Liu et al. [33] created various prioritized weighted AGOs for S^vNNs and employed them to MADM. Additionally, certain decision-making techniques were created for MADM problems. For instance, Garg [34] created a MADM model based on TOPSIS and divergence measure to deal with S^vN information. Mishra et al [35] initiated an integrated decision support system for evaluating low carbon tourist strategies utilizing single-valued MEREC-MULTIMOORA. Rong et al. [36] pharmaceutical enterprise evaluation using a hybrid group decision method based on MARCOS and regret theory in a S^vN environment. These techniques can only provide a ranking result, but AGOs can also provide the comprehensive value of each option by averaging the values of each characteristic.

Various aggregation operators obviously serve various purposes, but a few of them, like Yager's suggested power average (PA) operator [37], may mitigate the effect of some challenging data provided by the DMs. The PA operator may perform by selecting a weight vector depending on the degree of support between the input arguments and aggregating the input data. Now, other studies have expanded the PA operator into further environments. For S^vNNs, Yang and Li [38] presented PA operators, which they then applied to MADM. The interrelationship between the aggregating parameters can therefore be included in some AGOs, such as the Bonferroni mean (BM) operators created by Bonferroni [39], the Heronian mean (HM) operator familiarized by Sykora [40], the Muirhead Mean (MM) operator [41], and the Maclaurin symmetric mean (MSM) [42] operators. To cope with ambiguous information, numerous authors have additionally expanded these AGOs to various fuzzy structures [43–50]. Some AGOs have been established by using various T-norms (TNs) and T-conorms (TCNs), such as algebraic, Einstein, and Hamacher, to aggregate SVNNs. Typically, the algebraic, Einstein, Hamacher, Frank, and Dombi TNs and TCNs are generalized to create the Archimedean TN and TCN. Recently, Senapati [51] initiated various core operational rules for S^vNNs based on Aczel-Alsina TN and TCN. Furthermore, they have developed some AA weighted aggregation operators and apply them to solve MADM problems under SVN environment. Motivated from the above discussion, it is clear that no one has yet tried to deal with S^vN data by combining Aczel-Alsina operational laws with the PA operator and the MM operator. The hybrid structure of these AOs can have the advantages of Aczel-Alsina operational laws, the PA operator, and the MM operator at the same time. That is, these types of AOs have Aczel-Alsina parameter, the capacity to remove the effect of uncomfortable data and can consider the interrelation amongst all input data. As a result, we suggest:

- The Aczel-Alsina operational laws are merged with the PA operators, MM and dual MM operators to launch a hybrid AOs $S^vNAAPMM$ and $S^vNAAPDMM$ operators and are much more flexible and superior to the prior AOs in terms of a general parameter.
- Unfortunately, there exist several MADM problems where the attributes are interconnected. Consequently, many current AOs are limited in their ability to mitigate these instances when the attributes take the form of real numbers or other fuzzy structures.
- As of right now, there are no AOs in place to handle MADM problems arising from S^vN data. These problems can be made more flexible in the decision-making process by having general parameters, considering interrelationships among any number of input arguments, and being able to remove the negative effects of awkward data. In response to this constraint, we integrated Aczel-Alsina operational laws with the PA operator, MM and DMM operators to handle MADM problems employing S^vN information.

Due to significant effects from previous research, the following are the goals and services this initiative is focusing on:

- To combine the Muirhead, mean (MM) operator, PA operator with Aczel-Alsina operational laws for S^vNNs, to initiate the theory of the $S^vNAAPMM$ operator.
- To use the theory of dual Muirhead, mean (DMM) operators, PG operator with Aczel-Alsina operational laws for S^vNNs, to initiate the theory of the S^vNPDHM operator.
- To investigate the basic properties and various special cases with respect to the parameters of the derived theory.
- To demonstrate the MADM technique based on the presented operators to deal with MADM problems under S^vN information.

- To compare the derived operator with many existing operators and discuss their impact and benefits.

This article is constructed in the shape: In Section 2, we discussed the S^vNSs, Muirhead Mean operators, and dual Muirhead Mean operator and AA operational for S^vNNs. In Section 3, a novel concept of the S^vNPMM operator, S^vNPDMM operator, and their major properties. In Section 4, the weighted form of the initiated AGOs are introduced. In section 5, the MADM technique is introduced based on the presented AGOs. In section 6, a numerical example about the selection of strategic suppliers is given to show the efficacy and practicality of the proposed MADM approach. At the end comparison with some existing MADM approaches and conclusion are given.

2. Preliminaries

In this portion, some core concepts about S^vNSs, score, accuracy and certainty functions, distances measure, PA operator, MM operators are given.

2.1. The single valued neutrosophic set and their operational laws

A philosophical perspective was used to present the neutrosophic sets (NSs). It was evidently challenging to apply in the actual applications. From a scientific or technical perspective, Wang et al. [17] further proposed the single-valued neutrosophic sets (S^vNSs).

Definition 1 [17]. Let U be a universal set, with F'' indicating a generic element in the fix set U. Then, the S^vNSs $\overline{O}$ in U are described as follows:

$$\overline{O} = \left\{ \left\langle F'', \overline{\overline{TR}}_{\overline{O}}(F''), \overline{\overline{IN}}_{\overline{O}}(F''), \overline{\overline{FL}}_{\overline{O}}(F'') \,|\, F'' \in U \right\rangle \right\}. \quad (1)$$

Where the truth-membership grade $\overline{\overline{TR}}_{\overline{O}}(F'')$, indeterminacy-grade $\overline{\overline{IN}}_{\overline{O}}(F'')$ non-membership-grade $\overline{\overline{FL}}_{\overline{O}}(F'')$ and are single values in the real standard closed interval [0,1]. That is, $\overline{\overline{TR}}_{\overline{O}}(F''): U \to [0,1], \overline{\overline{IN}}_{\overline{O}}(F''): U \to [0,1]$ and $\overline{\overline{FL}}_{\overline{O}}(F''): U \to [0,1]$ with the condition that sum of the three grades will be less or equal to 3.

For a S^vNS $\overline{O} = \left\{ \left\langle F'', \overline{\overline{TR}}_{\overline{O}}(F''), \overline{\overline{IN}}_{\overline{O}}(F''), \overline{\overline{FL}}_{\overline{O}}(F'') \,|\, F'' \in U \right\rangle \right\}$, the ordered triplet $\left\langle \Phi'', \overline{\overline{TR}}_{\overline{O}}(\Phi''), \overline{\overline{IN}}_{\overline{O}}(\Phi''), \overline{\overline{FL}}_{\overline{O}}(\Phi'') \right\rangle$ is said to be S^vNN and is denoted by $\underline{\&}$. That is, $\underline{\&} = \left\langle \overline{\overline{TR}}, \overline{\overline{IN}}, \overline{\overline{FL}} \right\rangle$, where $\overline{\overline{TR}}, \overline{\overline{IN}}, \overline{\overline{FL}} \in [0,1]$ and $0 \le \overline{\overline{TR}} + \overline{\overline{IN}} + \overline{\overline{FL}} \le 3$.

Definition 2 [50]. The score function $\overline{\overline{SOE}}(\underline{\&})$, accuracy function $\overline{\overline{AR}}(\underline{\&})$ and certainty function $\overline{\overline{CR}}(\underline{\&})$ of a S^vNN $\underline{\&} = \left\langle \overline{\overline{TR}}, \overline{\overline{IN}}, \overline{\overline{FL}} \right\rangle$ can be emitted as

$$\begin{aligned}
&\overline{\overline{SOE}}(\underline{\&}) = \frac{2 + \overline{\overline{TR}}_{\underline{\&}} - \overline{\overline{IN}}_{\underline{\&}} - \overline{\overline{FL}}_{\underline{\&}}}{3}, \ Where \ \mathcal{O}(\underline{\&}) \in [0,1]; \\
&\overline{\overline{AR}}(\underline{\&}) = \overline{\overline{TR}}_{\underline{\&}} - \overline{\overline{FL}}_{\underline{\&}}, \ \ Where \ \overline{\overline{AR}}(\underline{\&}) \in [-1,1]; \\
&\overline{\overline{CR}}(\underline{\&}) = \overline{\overline{TR}}_{\underline{\&}} \ \ Where \ \overline{\overline{CR}}(\underline{\&}) \in [0,1]
\end{aligned} \quad (2)$$

For rating the S^vNNs, Smarandache [50] devised a comparative approach based on the score and exactness function, which is expressed as:

Definition 3 [50]. Suppose that the two S^v^NNs are $\tilde{\wp}_1=\left\langle \overline{\overline{TR}}_1,\overline{\overline{IN}}_1,\overline{\overline{FL}}_1 \right\rangle$ and $\tilde{\wp}_2=\left\langle \overline{\overline{TR}}_2,\overline{\overline{IN}}_2,\overline{\overline{FL}}_2 \right\rangle$. Let $\overline{\overline{SOE}}(\tilde{\wp}_1)$, $\overline{\overline{SOE}}(\tilde{\wp}_2)$ be the score functions and $\overline{\overline{AR}}(\tilde{\wp}_1),\overline{\overline{AR}}(\tilde{\wp}_2)$ be the accuracy functions of $\tilde{\wp}_i(i=1,2)$ respectively. If

(i) $\overline{\overline{SOE}}(\tilde{\wp}_1)\prec \overline{\overline{SOE}}(\tilde{\wp}_2)$, then $\tilde{\wp}_1\prec\tilde{\wp}_2$;

(ii) $\overline{\overline{SOE}}(\tilde{\wp}_1)\succ \overline{\overline{SOE}}(\tilde{\wp}_2)$, then $\tilde{\wp}_1\succ\tilde{\wp}_2$;

(iii) $\overline{\overline{SOE}}(\tilde{\wp}_1)= \overline{\overline{SOE}}(\tilde{\wp}_2)$, then;

(i) $\overline{\overline{AR}}(\tilde{\wp}_1)\prec \overline{\overline{AR}}(\tilde{\wp}_2)$, then $\tilde{\wp}_1\prec\tilde{\wp}_2$;

(ii) $\overline{\overline{AR}}(\tilde{\wp}_1)\succ \overline{\overline{AR}}(\tilde{\wp}_2)$, then $\tilde{\wp}_1\succ\tilde{\wp}_2$;

(iii) $\overline{\overline{AR}}(\tilde{\wp}_1)= \overline{\overline{AR}}(\tilde{\wp}_2)$, then;

(i) $\overline{\overline{CR}}(\tilde{\wp}_1)\prec \overline{\overline{CR}}(\tilde{\wp}_2)$, then $\tilde{\wp}_1\prec\tilde{\wp}_2$;

(ii) $\overline{\overline{CR}}(\tilde{\wp}_1)\succ \overline{\overline{CR}}(\tilde{\wp}_2)$, then $\tilde{\wp}_1\succ\tilde{\wp}_2$;

(iii) $\overline{\overline{CR}}(\tilde{\wp}_1)= \overline{\overline{CR}}(\tilde{\wp}_2)$, then $\tilde{\wp}_1=\tilde{\wp}_2$.

Definition 4 [51]. Let $\tilde{\wp}=\left\langle \overline{\overline{TR}},\overline{\overline{IN}},\overline{\overline{FL}} \right\rangle,\tilde{\wp}_1=\left\langle \overline{\overline{TR}}_1,\overline{\overline{IN}}_1,\overline{\overline{FL}}_1 \right\rangle$ and $\tilde{\wp}_2=\left\langle \overline{\overline{TR}}_2,\overline{\overline{IN}}_2,\overline{\overline{FL}}_2 \right\rangle$ be the three S^v^NNs and $\kappa \succ 0$. Then the AA operational laws for S^v^NNs are explained as follows:

(1) $$\tilde{\wp}_1\oplus\tilde{\wp}_2=\left\langle 1-e^{-\left(\left(-\ln\left(1-\overline{\overline{TR}}_1\right)\right)^{\phi}+\left(-\ln\left(1-\overline{\overline{TR}}_2\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}, e^{-\left(\left(-\ln\overline{\overline{IN}}_1\right)^{\phi}+\left(-\ln\overline{\overline{IN}}_2\right)^{\phi}\right)^{\frac{1}{\phi}}}, e^{-\left(\left(-\ln\overline{\overline{FL}}_1\right)^{\phi}+\left(-\ln\overline{\overline{FL}}_2\right)^{\phi}\right)^{\frac{1}{\phi}}} \right\rangle; \tag{3}$$

(2) $$\tilde{\wp}_1\otimes\tilde{\wp}_2=\left\langle e^{-\left(\left(-\ln\overline{\overline{TR}}_1\right)^{\phi}+\left(-\ln\overline{\overline{TR}}_2\right)^{\phi}\right)^{\frac{1}{\phi}}}, 1-e^{-\left(\left(-\ln\left(1-\overline{\overline{IN}}_1\right)\right)^{\phi}+\left(-\ln\left(1-\overline{\overline{IN}}_2\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}, 1-e^{-\left(\left(-\ln\left(1-\overline{\overline{FL}}_1\right)\right)^{\phi}+\left(-\ln\left(1-\overline{\overline{FL}}_2\right)\right)^{\phi}\right)^{\frac{1}{\phi}}} \right\rangle; \tag{4}$$

(3) $$\kappa\tilde{\wp}=\left\langle 1-e^{-\left(\kappa\left(-\ln\left(1-\overline{\overline{TR}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}, e^{-\left(\kappa\left(-\ln\overline{\overline{IN}}\right)^{\phi}\right)^{\frac{1}{\phi}}}, e^{-\left(\kappa\left(-\ln\overline{\overline{FL}}\right)^{\phi}\right)^{\frac{1}{\phi}}} \right\rangle; \tag{5}$$

(4) $$\tilde{\wp}^{\kappa}=\left\langle e^{-\left(\kappa\left(-\ln\overline{\overline{TR}}\right)^{\phi}\right)^{\frac{1}{\phi}}}, 1-e^{-\left(\kappa\left(-\ln\left(1-\overline{\overline{IN}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}, 1-e^{-\left(\kappa\left(-\ln\left(1-\overline{\overline{FL}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}} \right\rangle. \tag{6}$$

Definition 5 [20]. Let $\tilde{\wp}_1=\left\langle \overline{\overline{TR}}_1,\overline{\overline{IN}}_1,\overline{\overline{FL}}_1 \right\rangle$ and $\tilde{\wp}_2=\left\langle \overline{\overline{TR}}_2,\overline{\overline{IN}}_2,\overline{\overline{FL}}_2 \right\rangle$ be the three S^v^NNs. Then, the hamming distance measure among $\tilde{\wp}_1$ and $\tilde{\wp}_2$ is demarcated as:

$$\tilde{D}(\tilde{\wp}_1,\tilde{\wp}_2)=\frac{\left|\overline{\overline{TR}}_1-\overline{\overline{TR}}_2\right|+\left|\overline{\overline{IN}}_1-\overline{\overline{IN}}_2\right|+\left|\overline{\overline{FL}}_1-\overline{\overline{FL}}_2\right|}{3}. \tag{7}$$

2.2. PA operator

For aggregation, the PA [37] operator is another important AO that may be used to reduce the negative effects of uncomfortable data on the outcomes of the final ranking process. It is defined as:

Definition 6 [37]. Let $\overline{\overline{\Omega}}_l\,(l=1,2,...,g)$ be a group of positive crisp numbers. Then, the PA operator is quantified as

$$PA\left(\overline{\overline{\Omega}}_1,\overline{\overline{\Omega}}_2,....,\overline{\overline{\Omega}}_g\right)=\frac{\bigoplus_{l=1}^{g}\left(1+T\left(\overline{\overline{\Omega}}_l\right)\right)\overline{\overline{\Omega}}_l}{\bigoplus_{l=1}^{g}\left(1+T\left(\overline{\overline{\Omega}}_l\right)\right)}; \tag{8}$$

where $T\left(\overline{\overline{\Omega}}_l\right)=\bigoplus_{m=1,l\neq m}^{g} Sprt\left(\overline{\overline{\Omega}}_l,\overline{\overline{\Omega}}_m\right)$, $Sprt\left(\overline{\overline{\Omega}}_l,\overline{\overline{\Omega}}_m\right)$ indicates the support for $\overline{\overline{\Omega}}_l$ from $\overline{\overline{\Omega}}_m$ executing the axioms:

1) $0\leq Sprt\left(\overline{\overline{\Omega}}_l,\overline{\overline{\Omega}}_m\right)\leq 1$, 2) $Sprt\left(\overline{\overline{\Omega}}_l,\overline{\overline{\Omega}}_m\right)=Sprt\left(\overline{\overline{\Omega}}_m,\overline{\overline{\Omega}}_l\right)$ and

3) $Sprt\left(\overline{\overline{\Omega}}_j,\overline{\overline{\Omega}}_h\right)\leq Sprt\left(\overline{\overline{\Omega}}_l,\overline{\overline{\Omega}}_m\right)$, if $\left|\overline{\overline{\Omega}}_l,\overline{\overline{\Omega}}_m\right|\geq\left|\overline{\overline{\Omega}}_j,\overline{\overline{\Omega}}_h\right|$.

2.3. Muirhead Mean operator.

For classical numbers, Muirhead [41] was the first to introduce the MM operator. The benefit of MM operator is that it considers the interrelationship of each aggregated argument is related to each other.

Definition 7 [41]. Let $J_{\text{i}}\,(\text{i}=1,2...,\phi)$ be a set of classical numbers and $Œ''=\left(Œ''_1,Œ''_2,...,Œ''_\phi\right)\in R^{\phi}$ be a vector of parameters. Then the MM operator is suggested as:

$$\text{MM}^{Œ''}\left(J_1,J_2,...,J_\phi\right)=\left(\frac{1}{\phi!}\sum_{\xi\in A_\phi}\prod_{\text{i}=1}^{\phi}R_{\xi(\text{i})}^{Œ''_{\text{i}}}\right)^{\frac{1}{\sum_{\text{i}=1}^{\phi}Œ''_{\text{i}}}}. \tag{9}$$

Where A_ϕ is a set of permutation of $(1,2,...,\phi)$ and $\xi(\text{i})$ is any arrangement of $(1,2,...,\phi)$.

3. The Single valued neutrosophic Aczel-Alsina Power Muirhead Mean Operator

In this part, the SVNAAPMM and SVNAAPDMM operators are suggested.

Definition 8. Let $\underset{\&}{\Xi}_z=\left\langle \overline{TR}_z,\overline{IN}_z,\overline{FL}_z\right\rangle (z=1,2,...,K)$ be series of SVNNs, and $Œ''=\left(Œ''_1,Œ''_2,...,Œ''_K\right)\in R^K$ be a set of parameters. Then, the SVNAAPMM operator is put forward as:

$$S^V NAAPMM^{Œ''}\left(\underset{\&}{\Xi}_1,\underset{\&}{\Xi}_2,...,\underset{\&}{\Xi}_K\right)=\left(\frac{1}{K!}\sum_{\zeta\in H_K}\prod_{z=1}^{K}\left(K\frac{\left(1+T\left(\underset{\&}{\Xi}_{\zeta(z)}\right)\right)}{\sum_{z=1}^{K}\left(1+T\left(\underset{\&}{\Xi}_z\right)\right)}\underset{\&}{\Xi}_{\zeta(z)}\right)^{Œ''_z}\right)^{\frac{1}{\sum_{z=1}^{K}Œ''_z}}, \tag{10}$$

where,

$$T\left(\underset{\&}{\Xi}_j\right)=\sum_{z=1}^{K}Spt\left(\underset{\&}{\Xi}_i,\underset{\&}{\Xi}_z\right), \tag{11}$$

and

$$Spt\left(\tilde{\Xi}_i,\tilde{\Xi}_z\right)=1-\tilde{D}\left(\tilde{\Xi}_i,\tilde{\Xi}_z\right), \tag{12}$$

$\zeta(z)\left(z=1,2,...,K\right)$ epitomizes any arrangement of $\left(1,2,...,K\right)$. $\tilde{H}_K$ signifies all possible combinations of $\left(1,2,...,K\right)$, and K is the balancing coefficient $\tilde{D}\left(\tilde{\Xi}_i,\tilde{\Xi}_z\right)$ denotes the Hamming distance among two SVNNs $\tilde{\Xi}_i$ and $\tilde{\Xi}_z$, and $Spt\left(\tilde{\Xi}_i,\tilde{\Xi}_z\right)$ is the support for $\tilde{\Xi}_i$ from $\tilde{\Xi}_z$ sustaining the axioms given in Definition 6.

To write Equation (10), in simplified form, let

$$\tilde{H}_z=\frac{\left(1+\tilde{T}\left(\tilde{\Xi}_j\right)\right)}{\sum_{z=1}^{K}\left(1+\tilde{T}\left(\tilde{\Xi}_z\right)\right)}, \tag{13}$$

then, Equation (10) can be written in simplified form as,

$$S^V NAAPMM^{\text{Œ}''}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_K\right)=\left(\frac{1}{K!}\sum_{\zeta\in \tilde{H}_K}\prod_{z=1}^{K}\left(K\tilde{H}_{\zeta(z)}\tilde{\Xi}_{\zeta(z)}\right)^{\text{Œ}''_z}\right)^{\frac{1}{\sum_{z=1}^{K}\text{Œ}''_z}}. \tag{14}$$

Where $\sum_{z=1}^{K}\tilde{H}_z=1$ and $0\le \tilde{H}_z\le 1$.

Corresponding to Definition 8, the resulting theorem can be obtained.

Theorem 1. Let $\tilde{\Xi}_z=\left\langle \overline{TR}_z,\overline{IN}_z,\overline{FL}_z\right\rangle\left(z=1,2,...,K\right)$ be a group of SVNNs, the aggregated value by the SVNAAPMM operator is still a SVNN and

$$S^V NAAPMM^{\text{Œ}''}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_z\right)=\left\langle \left(e^{-\left(\frac{1}{\sum_{z=1}^{K}\text{Œ}''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in\tilde{H}_K}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(1-\overline{TR}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$
$$\left.\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}\text{Œ}''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in\tilde{H}_K}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(\overline{IN}_{\zeta(j)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}\text{Œ}''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in\tilde{H}_K}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(\overline{FL}_{\zeta(j)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle. \tag{15}$$

Proof. Corresponding to Aczel-Alsina operational rules suggested in Definition 4, we can have.

$$K\tilde{H}_z\tilde{\Xi}_{\zeta(z)}=\left\langle\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(1-\overline{TR}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(K\tilde{H}_z\left(-\ln\left(\overline{IN}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\tilde{H}_z\left(-\ln\left(\overline{FL}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right\rangle;$$

and

$$\left(K\tilde{H}_z\tilde{\Xi}_{\zeta(j)}\right)^{\text{Œ}''_z}=\left\langle\left(e^{-\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(1-\overline{TR}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(\overline{IN}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(\overline{FL}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)^{\frac{1}{q}}\right\rangle.$$

Therefore,

$$\prod_{z=1}^{K}\left(K\tilde{H}_z\tilde{\Xi}_{\zeta(z)}\right)^{\text{Œ}''_z}$$
$$=\left\langle\left(e^{-\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(1-\overline{TR}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\sum_{z=1}^{K}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(\overline{IN}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\sum_{z=1}^{K}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(K\tilde{H}_z\left(-\ln\left(\overline{FL}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle;$$

Let

$$\hbar = 1 - e^{-\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}}$$

$$\Rightarrow 1-\hbar = e^{-\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}} \quad \Rightarrow \ln(1-\hbar) = -\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}.$$

Utilizing this, we have

$$\sum_{\zeta\in H_{\text{Ķ}}}\prod_{z=1}^{\text{Ķ}}\left(\text{ĶH}_z\Xi_{\zeta(z)}\right)^{\text{Œ}''_z} = \left\langle \left(1-e^{-\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(1-\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$
$$\left.\left(e^{-\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle.$$

Furthermore,

$$\frac{1}{\text{Ķ}!}\sum_{\zeta\in H_z}\prod_{z=1}^{\text{Ķ}}\left(\text{ĶH}_z\Xi_{\zeta(z)}\right)^{\text{Œ}''_z} = \left\langle \left(1-e^{-\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(1-\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$
$$\left.\left(e^{-\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_n}\left(\sum_{z=1}^{\text{Ķ}}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(\text{ĶH}_z\left(-\ln\left(\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle;$$

Thus,

$$\left(\frac{1}{\text{Ķ}!}\sum_{\zeta\in H_z}\prod_{z=1}^{\text{Ķ}}\left(z\text{Ķ}_z\Xi_{\zeta(z)}\right)^{\text{Œ}''_z}\right)^{\frac{1}{\sum_{z=1}^{\text{Ķ}}\text{Œ}''_j}} = \left\langle \left(e^{-\left(\frac{1}{\sum_{z=1}^{\text{Ķ}}\text{Œ}''_z}\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(z\text{Ķ}_z\left(-\ln\left(1-\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)^{\frac{1}{q}},\right.$$
$$\left.\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{\text{Ķ}}\text{Œ}''_z}\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(z\text{Ķ}_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{\text{Ķ}}\text{Œ}''_z}\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(z\text{Ķ}_z\left(-\ln\left(\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)^{\frac{1}{q}}\right\rangle$$

Hence,

$$S^{V}NAAPMM^{(\text{Œ}'')}\left(\Xi_1,\Xi_2,\ldots,\Xi_z\right) = \left\langle \left(e^{-\left(\frac{1}{\sum_{z=1}^{\text{Ķ}}\text{Œ}''_z}\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(z\text{Ķ}_z\left(-\ln\left(1-\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)^{\frac{1}{q}},\right.$$
$$\left.\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{\text{Ķ}}\text{Œ}''_z}\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(z\text{Ķ}_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{\text{Ķ}}\text{Œ}''_z}\left(\frac{1}{\text{Ķ}!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(z\text{Ķ}_z\left(-\ln\left(\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)^{\frac{1}{q}}\right\rangle.$$

Theorem 2. (Idempotency) Let $\Xi_z = \left\langle \overline{\overline{TR}}_z, \overline{\overline{IN}}_z, \overline{\overline{FL}}_z \right\rangle (z = 1,2,\ldots,\text{Ķ})$ be a series of SVNNs, if $\Xi_z = \Xi = \left\langle \overline{\overline{TR}}, \overline{\overline{IN}}, \overline{\overline{FL}} \right\rangle (z = 1,2,\ldots,\text{Ķ})$ holds for all z, then

$$S^{V}NAAPMM^{(Œ'')}\left(\mathfrak{Z}_1,\mathfrak{Z}_2,...,\mathfrak{Z}_z\right)=\mathfrak{Z}. \tag{16}$$

Proof. Since $\mathfrak{Z}_z=\mathfrak{Z}=\langle\overline{\overline{TR}},\overline{\overline{IN}},\overline{\overline{FL}}\rangle(z=1,2,...,K)$, we can have $Spt(\mathfrak{Z}_ì,\mathfrak{Z}_z)=\frac{1}{K}$ for all $ì,z=1,2,..,K$ and $ì\neq z$.

Thus, we can derive $H_{\mathfrak{Z}_z}=\frac{1}{K}(z=1,2,..,K)$. According to Theorem 1, we have

$$S^{V}NAAPMM^{(Œ'')}\left(\mathfrak{Z}_1,\mathfrak{Z}_2,...,\mathfrak{Z}_z\right)=\left\langle\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H\mathfrak{Z}_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(1-\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)^{\frac{1}{q}},\right.$$

$$\left.\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H\mathfrak{Z}_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H\mathfrak{Z}_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)^{\frac{1}{q}}\right\rangle.$$

$$=\left\langle\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H\mathfrak{Z}_z}\left(\sum_{z=1}^{K}\left(Œ''_z\left(-\ln\left(\overline{\overline{TR}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H\mathfrak{Z}_z}\left(\sum_{z=1}^{K}\left(Œ''_j\left(-\ln\left(1-\left(\overline{\overline{IN}}\right)\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$

$$\left.\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H\mathfrak{Z}_z}\left(\sum_{z=1}^{K}\left(Œ''_j\left(-\ln\left(1-\left(\overline{\overline{FL}}\right)\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle;$$

$$=\left\langle\left(e^{-\left(\left(-\ln\left(\overline{\overline{TR}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\left(-\ln\left(1-\left(\overline{\overline{IN}}\right)\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\left(-\ln\left(1-\overline{\overline{FL}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)^{\frac{1}{q}}\right\rangle$$

$$=\langle\overline{\overline{TR}},\overline{\overline{IN}},\overline{\overline{FL}}\rangle=\mathfrak{Z}.$$

Theorem 3. Let $\mathfrak{Z}_z=\langle\overline{\overline{TR}}_z,\overline{\overline{IN}}_z,\overline{\overline{FL}}_z\rangle(z=1,2,...,K)$ be a series of SVNNs. Then

$$\mathfrak{Z}^{-}\leq S^{V}NAAPMM^{(Œ'')}\left(\mathfrak{Z}_1,\mathfrak{Z}_2,...,\mathfrak{Z}_n\right)\leq\mathfrak{Z}^{+}, \tag{17}$$

where,

$\mathfrak{Z}^{-}=\left\langle\min_{z=1}^{K}\overline{\overline{TR}}_z,\max_{z=1}^{K}\overline{\overline{IN}}_z,\max_{z=1}^{K}\overline{\overline{FL}}_z\right\rangle$ and $\mathfrak{Z}^{+}=\left\langle\max_{z=1}^{K}\overline{\overline{TR}}_z,\min_{z=1}^{K}\overline{\overline{IN}}_z,\min_{z=1}^{K}\overline{\overline{FL}}_z\right\rangle$.

Proof. Corresponding to Theorem 1 and Theorem 2, it is straightforward to acquire that.

$$\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_j}\left(\frac{1}{K!}\left(\sum_{\zeta\in H\mathfrak{Z}_K}\left(\sum_{z=1}^{K}\left(Œ''_j\left(-\ln\left(1-e^{-\left(KH_{\mathfrak{Z}_z}\left(-\ln\left(1-\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)$$

$$\geq\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_j}\left(\frac{1}{K!}\left(\sum_{\zeta\in H\mathfrak{Z}_K}\left(\sum_{z=1}^{K}\left(Œ''_j\left(-\ln\left(1-e^{-\left(KH_{\mathfrak{Z}_z}\left(-\ln\left(1-\min_{z=1}^{K}\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)=\min_{z=1}^{K}\overline{\overline{TR}}_z$$

;

$$\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_j}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_K}\left(\sum_{z=1}^{K}\left(Œ''_j\left(-\ln\left(1-e^{-\left(K\hbar_{\zeta(z)}\left(-\ln\left(\overline{IN}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)$$

$$\leq\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_j}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_K}\left(\sum_{z=1}^{K}\left(Œ''_j\left(-\ln\left(1-e^{-\left(K\hbar_{\zeta(z)}\left(-\ln\left(\max_{z=1}^{K}\overline{IN}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)=\max_{z=1}^{K}\overline{\overline{IN}}_z;$$

;

$$\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_j}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_K}\left(\sum_{z=1}^{K}\left(Œ''_j\left(-\ln\left(1-e^{-\left(K\hbar_{\zeta(z)}\left(-\ln\left(\overline{FL}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)$$

$$\leq\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_j}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_K}\left(\sum_{z=1}^{K}\left(Œ''_j\left(-\ln\left(1-e^{-\left(K\hbar_{\zeta(z)}\left(-\ln\left(\max_{z=1}^{K}\overline{FL}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)=\max_{z=1}^{K}\overline{\overline{FL}}_z.$$

.

According to Definition 2, we can get $\tilde{\Xi}^{-}\leq S^{V}NAAPMM^{(Œ'')}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_K\right)$. Similarly, we have

$S^{V}NAAPMM^{(Œ'')}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_K\right)\leq\tilde{\Xi}^{+}$. Therefore, $\tilde{\Xi}^{-}\leq S^{V}NAAPMM^{(Œ'')}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_n\right)\leq\tilde{\Xi}^{+}$.

Subsequently, we discuss some special cases of SVNAAPMM with respect to the parameters $Œ''$ *and* ϕ.

Case 1. If $Œ''=(1,0,0,...,0)$, then the SVNAAPMM operator deteriorates to the SVNAAPA operator, that is

$$S^{V}NAAPMM^{(1,0,0,...,0)}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_K\right)=\left\langle 1-e^{-\left(\sum_{z=1}^{K}\hbar_z\left(-\ln\left(1-\overline{TR}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}},e^{-\left(\sum_{z=1}^{K}\hbar_z\left(-\ln\overline{IN}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}},e^{-\left(\sum_{z=1}^{K}\hbar_z\left(-\ln\overline{FL}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right\rangle =\sum_{z=1}^{K}\hbar_z\tilde{\Xi}_z \tag{18}$$

In this case, if $Spt\left(\tilde{\Xi}_i,\tilde{\Xi}_z\right)=g$ for all $i\neq z$, then the SVNAAPMM operator deteriorates to the SVN average operator. i.e.,

$$S^{V}NAAPMM^{(1,0,0,...,0)}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_K\right)=\left\langle 1-e^{-\left(\sum_{z=1}^{k}\frac{1}{K}\left(-\ln\left(1-\overline{TR}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}},e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln\overline{IN}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}},e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln\overline{FL}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right\rangle =\frac{1}{K}\sum_{z=1}^{K}\tilde{\Xi}_z. \tag{19}$$

Case 2. If $Œ''=(1,1,0,...,0)$, then the SVNAAPMM operator deteriorates to the SVN power Bonferroni mean operator. i.e.,

$$\begin{aligned}S^{V}NAAPMM^{(1,1,0,...,0)}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_K\right)=\Bigg\langle&\left(e^{-\left(\frac{1}{2}\left(\frac{1}{K(K-1)}\left(\sum_{\substack{i=1,z=1\\i\neq z}}^{K}\left(1*\left(-\ln\left(1-e^{-\left(K\hbar_i\left(-\ln\left(1-\overline{TR}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)+\left(1*\left(-\ln\left(1-e^{-\left(K\hbar_z\left(-\ln\left(1-\overline{TR}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\\&\left(1-e^{-\left(\frac{1}{2}\left(\frac{1}{K(K-1)}\left(\sum_{\substack{i=1,z=1\\i\neq z}}^{K}\left(1*\left(-\ln\left(1-e^{-\left(K\hbar_i\left(-\ln\left(\overline{IN}_i\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)+\left(1*\left(-\ln\left(1-e^{-\left(K\hbar_z\left(-\ln\left(\overline{IN}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\\&\left(1-e^{-\left(\frac{1}{2}\left(\frac{1}{K(K-1)}\left(\sum_{\substack{i=1,z=1\\i\neq z}}^{K}\left(1*\left(-\ln\left(1-e^{-\left(K\hbar_i\left(-\ln\left(\overline{FL}_i\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)+\left(1*\left(-\ln\left(1-e^{-\left(K\hbar_z\left(-\ln\left(\overline{FL}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right)\Bigg\rangle=\left(\frac{1}{K(K-1)}\sum_{\substack{i,z=1\\i\neq z}}^{K}\left(K\hbar_i\tilde{\Xi}_i\otimes K\hbar_z\tilde{\Xi}_z\right)\right)^{\frac{1}{2}}.\end{aligned} \tag{20}$$

In this case, if $Spt\left(\Xi_i, \Xi_z\right) = g \succ 0$ for all $i \neq z$, then the SVNAAMM operator deteriorates to the SVNAA Bonferroni mean (SVNAABM) operator. i.e.,

$$S^V NAAPMM^{(1,1,0,\ldots,0)}\left(\Xi_1, \Xi_2, \ldots, \Xi_K\right) = \left\langle \left(e^{-\left(\frac{1}{2}\left(\frac{1}{K(K-1)}\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(\overline{TR}_i\right)\right)^{\Phi} + \left(1*\left(-\ln\left(\overline{TR}_z\right)\right)^{\Phi}\right)\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right), \right.$$
$$\left(1-e^{-\left(\frac{1}{2}\left(\frac{1}{K(K-1)}\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(1-\overline{IN}_i\right)\right)^{\Phi} + \left(1*\left(-\ln\left(1-\overline{IN}_z\right)\right)^{\Phi}\right)\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right), \left. \left(1-e^{-\left(\frac{1}{2}\left(\frac{1}{K(K-1)}\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(1-\overline{FL}_i\right)\right)^{\Phi} + \left(1*\left(-\ln\left(1-\overline{FL}_z\right)\right)^{\Phi}\right)\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right) \right\rangle \quad (21)$$
$$= \left(\frac{1}{K(K-1)} \sum_{\substack{i=1,z=1\\ i\neq z}}^{K} \Xi_i \otimes \Xi_z \right)^{\frac{1}{2}}.$$

Case 3. If $Œ'' = (\overbrace{1,1,1,1,\ldots,1}^{k}, \overbrace{0,0,0,\ldots,0}^{K-k})$, then the SVNAAPMM operator deteriorates to the SVNAAPMSM operator, i.e.,

$$S^V NAAPMM^{(\overbrace{1,1,1,\ldots,1}^{k},\overbrace{0,0,0,\ldots,0}^{K-k})}\left(\Xi_1, \Xi_2, \ldots, \Xi_K\right) = \left\langle \left(e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\le i_1 < i_2 < \ldots < i_K}\left(\sum_{z=1}^{k}\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(1-\overline{TR}_{i_z}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right), \right.$$
$$\left(1-e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\le i_1 < i_2 < \ldots < i_K}\left(\sum_{z=1}^{k}\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(\overline{IN}_{i_z}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right), \left. \left(1-e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\le i_1 < i_2 < \ldots < i_K}\left(\sum_{z=1}^{k}\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(\overline{FL}_{i_z}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right) \right\rangle \quad (22)$$
$$= \left(\frac{1}{C_K^k} \sum_{1\le i_1 < i_2 < \ldots < i_K} \prod_{z=1}^{k} \left(KH_z \Xi_{i_z} \right) \right)^{\frac{1}{k}}.$$

In this case, if $Spt\left(\Xi_i, \Xi_z\right) = g \succ 0$ for all $i \neq z$, then the SVNAAMM operator deteriorates to the SVNAAMSM operator. i.e.,

$$S^V NAAPMM^{(\overbrace{1,1,1,\ldots,1}^{k},\overbrace{0,0,0,\ldots,0}^{K-k})}\left(\Xi_1, \Xi_2, \ldots, \Xi_K\right)$$
$$= \left\langle \left(e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\le i_1 < i_2 < \ldots < i_K}\left(\sum_{z=1}^{k}\left(-\ln \overline{TR}_{i_z}\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right), \left(1-e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\le i_1 < i_2 < \ldots < i_K}\left(\sum_{z=1}^{n}\left(-\ln\left(1-\overline{IN}_{i_z}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right), \left(1-e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\le i_1 < i_2 < \ldots < i_K}\left(\sum_{z=1}^{n}\left(-\ln\left(1-\overline{FL}_{i_z}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}} \right) \right\rangle \quad (23)$$
$$= \left(\frac{1}{C_K^k} \sum_{1\le i_1 < i_2 < \ldots < i_k} \prod_{z=1}^{k} \left(\Xi_{i_z} \right) \right)^{\frac{1}{k}}$$

Case 4. If $Œ'' = (1,1,1,\ldots,1)$ *or* $\left(\frac{1}{Y}, \frac{1}{Y}, \frac{1}{Y}, \ldots, \frac{1}{Y}\right)$, then the SVNAAPMM operator deteriorates to the following form:

$$S^V NAAPMM^{Œ''=(1,1,1,\ldots,1) \text{ or } \left(\frac{1}{Y},\frac{1}{Y},\frac{1}{Y},\ldots,\frac{1}{Y}\right)}\left(\Xi_1, \Xi_2, \ldots, \Xi_K\right) = \left\langle \left(e^{-\left(\sum_{z=1}^{K} Œ''_z\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(1-\overline{TR}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}} \right), \right.$$
$$\left(1-e^{-\left(\sum_{z=1}^{K} Œ''_z\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(\overline{IN}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}} \right), \left. \left(1-e^{-\left(\sum_{z=1}^{K} Œ''_z\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(\overline{FL}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}} \right) \right\rangle = \prod_{z=1}^{K} \left(KH_z \Xi_z \right)^{\frac{1}{K}}. \quad (24)$$

In this case, if $Spt\left(\Xi_i, \Xi_z\right) = g \succ 0$ for all $i \neq z$, then the SVNAAMM operator deteriorates to the SVNAA geometric operator. i.e.,

$$S^V NAAPMM^{Œ''=(1,1,1,\ldots,1)\, or \left(\frac{1}{Y},\frac{1}{Y},\frac{1}{Y},\ldots,\frac{1}{Y}\right)}\left(\tilde{\mathcal{Z}}_1,\tilde{\mathcal{Z}}_2,\ldots,\tilde{\mathcal{Z}}_K\right)$$
$$=\left\langle \left(e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln \overline{TR}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right),\left(1-e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln\left(1-\overline{IN}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right),\left(1-e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln\left(1-\overline{FL}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right\rangle=\prod_{z=1}^{K}\left(\tilde{\mathcal{Z}}_z\right)^{\frac{1}{K}}. \tag{25}$$

Definition 9. Let $\tilde{\mathcal{Z}}_z=\langle \overline{TR}_z,\overline{IN}_z,\overline{FL}_z\rangle (z=1,2,\ldots,K)$ be series of SVNNs, and $Œ''=\left(Œ''_1,Œ''_2,\ldots,Œ''_K\right)\in R^K$ be a series of parameters. The SVNAAPDMM operator is delivered as

$$S^V NAAPDMM^{Œ''}\left(\tilde{\mathcal{Z}}_1,\tilde{\mathcal{Z}}_2,\ldots,\tilde{\mathcal{Z}}_K\right)=\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\prod_{\zeta\in H_K}\sum_{z=1}^{K}\left(Œ''_z\tilde{\mathcal{Z}}_{\zeta(z)}^{K\frac{\left(1+T\left(\tilde{\mathcal{Z}}_{\zeta(z)}\right)\right)}{\sum_{z=1}^{K}\left(1+T\left(\tilde{\mathcal{Z}}_z\right)\right)}}\right)\right)^{\frac{1}{K!}}, \tag{26}$$

where

$$T\left(\tilde{\mathcal{Z}}_z\right)=\sum_{z=1}^{K}Spt\left(\tilde{\mathcal{Z}}_i,\tilde{\mathcal{Z}}_z\right) \tag{27}$$

and

$$Spt\left(\tilde{\mathcal{Z}}_i,\tilde{\mathcal{Z}}_z\right)=1-D\left(\tilde{\mathcal{Z}}_i,\tilde{\mathcal{Z}}_z\right), \tag{28}$$

$\zeta(z)(z=1,2,\ldots,K)$ signifies any combination of $(1,2,\ldots,K)$. H_n indicates all possible combinations of $(1,2,\ldots,K)$, and K is the balancing coefficient $D\left(\tilde{\mathcal{Z}}_i,\tilde{\mathcal{Z}}_z\right)$ denotes the Hamming distance among two SVNNs $\tilde{\mathcal{Z}}_i$ and $\tilde{\mathcal{Z}}_z$, and $Spt\left(\tilde{\mathcal{Z}}_i,\tilde{\mathcal{Z}}_z\right)$ is the support for $\tilde{\mathcal{Z}}_i$ from $\tilde{\mathcal{Z}}_z$ sustaining the axioms given in Definition 6.

To write Equation (26), in simplified form, let

$$H_z=\frac{\left(1+T\left(\tilde{\mathcal{Z}}_z\right)\right)}{\sum_{z=1}^{K}\left(1+T\left(\tilde{\mathcal{Z}}_z\right)\right)}, \tag{29}$$

then, Equation (26) can be written in simplified form as,

$$S^V NAAPDMM^{Œ''}\left(\tilde{\mathcal{Z}}_1,\tilde{\mathcal{Z}}_2,\ldots,\tilde{\mathcal{Z}}_n\right)=\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\prod_{\zeta\in H_K}\sum_{z=1}^{K}\left(Œ''_z\tilde{\mathcal{Z}}_{\zeta(z)}^{KH_z}\right)\right)^{\frac{1}{K!}}. \tag{30}$$

Where $\sum_{z=1}^{K}H_z=1$ and $0\le H_z\le 1$.

Conferring to Definition 9, the subsequent Theorem 4, can be attained.

Theorem 4. Let $\tilde{\mathcal{Z}}_z=\langle \overline{TR}_z,\overline{IN}_z,\overline{FL}_z\rangle (z=1,2,\ldots,K)$ be a group of SVNNs, the aggregated value by the SVNAAPDMM operator is still a SVNN and

$$S^{V}NAAPDMM^{\text{Œ}''}\left(\Xi_1,\Xi_2,...,\Xi_z\right)=\left\langle\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}\text{Œ}''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_K}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(j)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$

$$\left.\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}\text{Œ}''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_K}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}\text{Œ}''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_K}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle. \quad (31)$$

Proof. Conferring to the Aczel-Alsina operational rules, we can have.

$$\Xi_{\zeta(z)}^{KHb_z}=\left\langle\left(e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right\rangle$$

and

$$\text{Œ}''_{\zeta(z)}\left(\Xi_{\zeta(z)}^{KHb_z}\right)=\left\langle\left(1-e^{-\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right\rangle.$$

Therefore,

$$\sum_{z=1}^{K}\text{Œ}''_z\left(\Xi_{\zeta(z)}^{KHb_z}\right)$$

$$=\left\langle\left(1-e^{-\left(\sum_{z=1}^{K}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle;$$

Let

$$\hbar=1-e^{-\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}}$$

$$\Rightarrow e^{-\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}} \qquad \Rightarrow \ln(1-\hbar)=-\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)^{\frac{1}{\phi}}.$$

Utilizing this, we have

$$\prod_{\zeta\in H_K}\sum_{z=1}^{K}\text{Œ}''_z\left(\Xi_{\zeta(z)}^{KHb_z}\right)=\left\langle\left(e^{-\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{K}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$

$$\left.\left(1-e^{-\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{K}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{K}\left(\text{Œ}''\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle.$$

Furthermore,

$$\left(\prod_{\zeta\in H_K}\sum_{z=1}^{K}\text{Œ}''_z\left(\Xi_{\zeta(z)}^{KHb_z}\right)\right)^{\frac{1}{K!}}=\left\langle\left(e^{-\left(\frac{1}{K!}\left(\sum_{\zeta\in H_n}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$

$$\left.\left(1-e^{-\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(1-e^{-\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{K}\left(\text{Œ}''_z\left(-\ln\left(1-e^{-\left(KHb_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle;$$

Thus,

$$\frac{1}{\sum_{z=1}^{K}Œ''_j}\left(\prod_{\zeta\in H_K}\sum_{z=1}^{K}Œ''_z\left(\Xi_{\zeta(z)}^{KH_z}\right)\right)^{\frac{1}{K!}}=\left\langle\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$

$$\left.\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle;$$

Hence,

$$S^{V}NAAPDMM^{(Œ'')}\left(\Xi_1,\Xi_2,\ldots,\Xi_z\right)=\left\langle\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\right.$$

$$\left.\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{j=1}^{n}\left(Œ''_z\left(-\ln\left(1-e^{-\left(zK_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)^{\phi}\right)\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)\right\rangle.$$

Theorem 5. (Idempotency) Let $\Xi_z=\langle\overline{\overline{TR}}_z,\overline{\overline{IN}}_z,\overline{\overline{FL}}_z\rangle(z=1,2,\ldots,K)$ be a set of SVNNs, if $\Xi_z=\Xi=\langle\overline{\overline{TR}},\overline{\overline{IN}},\overline{\overline{FL}}\rangle(z=1,2,\ldots,K)$ holds for all z, then

$$S^{V}NAAPDMM^{(Œ'')}\left(\Xi_1,\Xi_2,\ldots,\Xi_z\right)=\Xi. \tag{32}$$

Theorem 6. Let $\Xi_z=\langle\overline{\overline{TR}}_z,\overline{\overline{IN}}_z,\overline{\overline{FL}}_z\rangle(z=1,2,\ldots,K)$ be a series of SVNNs. Then

$$\Xi^{-}\leq SVNAAPDMM^{(Œ'')}\left(\Xi_1,\Xi_2,\ldots,\Xi_n\right)\leq\Xi^{+}, \tag{33}$$

where

$\Xi^{-}=\left\langle\min_{z=1}^{K}\overline{\overline{TR}}_z,\max_{z=1}^{K}\overline{\overline{IN}}_z,\max_{z=1}^{K}\overline{\overline{FL}}_z\right\rangle$ and $\Xi^{+}=\left\langle\max_{z=1}^{K}\overline{\overline{TR}}_z,\min_{z=1}^{K}\overline{\overline{IN}}_z,\min_{z=1}^{K}\overline{\overline{FL}}_z\right\rangle$.

Subsequently, we will deliberate some special cases of SVNAAPDMM with respect to the parameters $Œ''$ and Φ.

Case 1. If $Œ''=(1,0,0,\ldots,0)$, then the SVNAAPDMM operator relegates to the SVNAAPG operator, that is

$$S^{V}NAAPDMM^{(1,0,0,\ldots,0)}\left(\Xi_1,\Xi_2,\ldots,\Xi_K\right)=\left\langle e^{-\left(\sum_{z=1}^{K}H_z\left(-\ln\overline{\overline{TR}}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}},1-e^{-\left(\sum_{z=1}^{K}H_z\left(-\ln\left(1-\overline{\overline{IN}}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}},1-e^{-\left(\sum_{z=1}^{K}H_z\left(-\ln\left(1-\overline{\overline{FL}}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right\rangle$$
$$=\prod_{z=1}^{K}\Xi_z^{H_z}. \tag{34}$$

In this case, if $Spt\left(\Xi_i,\Xi_z\right)=g$ for all $i\neq z$, then the SVNAAPDMM operator relegates to the SVN geometric operator. i.e.,

$$S^V NAAPDMM^{(1,0,0,\dots,0)}\left(\ddot{\Xi}_1,\ddot{\Xi}_2,\dots,\ddot{\Xi}_K\right)=\left\langle e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln \overline{TR}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}},1-e^{-\left(\sum_{z=1}^{k}\frac{1}{K}\left(-\ln\left(1-\overline{IN}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}},1-e^{-\left(\sum_{z=1}^{k}\frac{1}{K}\left(-\ln\left(1-\overline{FL}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right\rangle$$
$$=\left(\prod_{z=1}^{K}\ddot{\Xi}_z\right)^{\frac{1}{K}}. \tag{35}$$

Case 2. If $Œ''=(1,1,0,\dots,0)$, then the SVNAAPDMM operator relegates to the SVN power geometric Bonferroni mean (SVNPGBM) operator. i.e.,

$$S^V NAAPDMM^{(1,1,0,\dots,0)}\left(\ddot{\Xi}_1,\ddot{\Xi}_2,\dots,\ddot{\Xi}_K\right)$$
$$=\left\langle\left(1-e^{-\left(\frac{1}{2}\left(\left(\frac{1}{K(K-1)}\right)\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(1-e^{-\left(KH\xi_i\left(-\ln\left(\overline{TR}_i\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)+\left(1*\left(-\ln\left(1-e^{-\left(KH\xi_z\left(-\ln\left(\overline{TR}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\right.$$
$$\left(e^{-\left(\frac{1}{2}\left(\left(\frac{1}{K(K-1)}\right)\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(1-e^{-\left(KH\xi_i\left(-\ln\left(1-\overline{IN}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)+\left(1*\left(-\ln\left(1-e^{-\left(KH\xi_z\left(-\ln\left(1-\overline{IN}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right), \tag{36}$$
$$\left.\left(e^{-\left(\frac{1}{2}\left(\left(\frac{1}{K(K-1)}\right)\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(1-e^{-\left(KH\xi_i\left(-\ln\left(1-\overline{FL}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)+\left(1*\left(-\ln\left(1-e^{-\left(KH\xi_z\left(-\ln\left(1-\overline{FL}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right)\right\rangle$$
$$=\left(\frac{1}{2}\prod_{\substack{i,z=1\\ i\neq z}}^{K}\left(\ddot{\Xi}_i^{KH\xi_i}\oplus\ddot{\Xi}_z^{KH\xi_z}\right)\right)^{\frac{1}{K(K-1)}}.$$

In this case, if $Spt\left(\ddot{\Xi}_i,\ddot{\Xi}_z\right)=g \succ 0$ for all $i\neq z$, then the SVNAAPDMM operator relegates to the SVNAA geometric Bonferroni mean (SVNAAGBM) operator. i.e.,

$$S^V NAAPDMM^{(1,1,0,\dots,0)}\left(\ddot{\Xi}_1,\ddot{\Xi}_2,\dots,\ddot{\Xi}_K\right)=\left\langle\left(1-e^{-\left(\frac{1}{2}\left(\left(\frac{1}{K(K-1)}\right)\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(1-\overline{TR}_i\right)\right)^{\Phi}\right)+\left(1*\left(-\ln\left(1-\overline{TR}_z\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\right.$$
$$\left.\left(e^{-\left(\frac{1}{2}\left(\left(\frac{1}{K(K-1)}\right)\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(\overline{IN}_i\right)\right)^{\Phi}+\left(1*\left(-\ln\left(\overline{IN}_z\right)\right)^{\Phi}\right)\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\left(e^{-\left(\frac{1}{2}\left(\left(\frac{1}{K(K-1)}\right)\left(\sum_{\substack{i=1,z=1\\ i\neq z}}^{K}\left(1*\left(-\ln\left(\overline{FL}_i\right)\right)^{\Phi}+\left(1*\left(-\ln\left(\overline{FL}_z\right)\right)^{\Phi}\right)\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right)\right\rangle \tag{37}$$
$$=\left(\frac{1}{2}\prod_{z=1}^{K}\ddot{\Xi}_i\oplus\ddot{\Xi}_z\right)^{\frac{1}{K(K-1)}}.$$

Case 3. If $Œ''=\left(\overbrace{1,1,1,1,\dots,1}^{k},\overbrace{0,0,0,\dots,0}^{K-k}\right)$, then the SVNAAPDMM operator deteriorates to the SVNAAP dual MSM (SVNAAPDMSM) operator, i.e,

$$S^V NAAPDMM^{\left(\overbrace{1,1,1,\dots,1}^{k},\overbrace{0,0,0,\dots,0}^{K-k}\right)}\left(\ddot{\Xi}_1,\ddot{\Xi}_2,\dots,\ddot{\Xi}_K\right)=\left\langle\left(1-e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\preceq i_1\prec i_2\prec\dots\prec i_K}\left(\sum_{z=1}^{k}\left(-\ln\left(1-e^{-\left(KH\xi_z\left(-\ln\left(\overline{TR}_{i_z}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\right.$$
$$\left.\left(e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\preceq i_1\prec i_2\prec\dots\prec i_K}\left(\sum_{z=1}^{k}\left(-\ln\left(1-e^{-\left(KH\xi_z\left(-\ln\left(1-\overline{IN}_{i_z}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\left(e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\preceq i_1\prec i_2\prec\dots\prec i_K}\left(\sum_{z=1}^{k}\left(-\ln\left(1-e^{-\left(KH\xi_z\left(-\ln\left(1-\overline{FL}_{i_z}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right)\right\rangle \tag{38}$$
$$=\frac{1}{k}\left(\prod_{1\le i_1\prec i_2\prec\dots\prec i_K}\sum_{z=1}^{K}\left(\ddot{\Xi}_{i_z}^{KH\xi_z}\right)\right)^{\frac{1}{C_K^k}}.$$

In this case, if $Spt(\Xi_i,\Xi_z)=g \succ 0$ for all $i\neq z$,, then the SVNAADMM operator deteriorates to the SVNAADMSM operator. i.e.,

$$S^V NAAPDMM^{(\overbrace{1,1,1,\ldots,1}^{k},\overbrace{0,0,0,\ldots,0}^{K-k})}(\Xi_1,\Xi_2,\ldots,\Xi_K)=\left\langle\left(1-e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\prec i_1\prec i_2\prec\ldots\prec i_K}\left(\sum_{z=1}^{n}\left(-\ln\left(1-\overline{TR}_{i_z}\right)\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\right.$$
$$\left.\left(e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\prec i_1\prec i_2\prec\ldots\prec i_K}\left(\sum_{z=1}^{k}\left(-\ln\overline{IN}_{i_z}\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\left(e^{-\left(\frac{1}{k}\left(\frac{1}{C_K^k}\left(\sum_{1\prec i_1\prec i_2\prec\ldots\prec i_K}\left(\sum_{z=1}^{k}\left(-\ln\overline{FL}_{i_z}\right)^{\Phi}\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right)\right\rangle=\left(\frac{1}{k}\left(\prod_{1\le i_1<i_2<\ldots<i_K}\sum_{z=1}^{K}(\Xi_{i_z})\right)^{\frac{1}{C_K^k}}\right) \tag{39}$$

Case 4. If $Œ''=(1,1,1,\ldots,1)$ *or* $\left(\frac{1}{K},\frac{1}{K},\frac{1}{K},\ldots,\frac{1}{K}\right)$, then the SVNAAPDMM operator deteriorates to the following form:

$$S^V NAAPDMM^{Œ''=(1,1,1,\ldots,1)\ or\left(\frac{1}{K},\frac{1}{K},\frac{1}{K},\ldots,\frac{1}{K}\right)}(\Xi_1,\Xi_2,\ldots,\Xi_K)=\left\langle\left(1-e^{-\left(\sum_{z=1}^{K}\left(Œ''_z\left(-\ln\left(1-e^{-\left(K\hbar_z\left(-\ln\left(\overline{TR}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\right.$$
$$\left.\left(e^{-\left(\sum_{z=1}^{K}\left(Œ''_z\left(-\ln\left(1-e^{-\left(K\hbar_z\left(-\ln\left(1-\overline{IN}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\sum_{z=1}^{K}\left(Œ''_z\left(-\ln\left(1-e^{-\left(K\hbar_z\left(-\ln\left(1-\overline{FL}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right\rangle=\left(\sum_{z=1}^{K}\frac{1}{K}\Xi_z^{K\hbar_z}\right). \tag{40}$$

In this case, if $Spt(\Xi_i,\Xi_z)=g \succ 0$ for all $i\neq z$, then the SVNAAMM operator deteriorates to the SVNAA average operator. i.e.,

$$S^V NAAPDMM^{Œ''=(1,1,1,\ldots,1)\ or\left(\frac{1}{K},\frac{1}{K},\frac{1}{K},\ldots,\frac{1}{K}\right)}(\Xi_1,\Xi_2,\ldots,\Xi_K)$$
$$=\left\langle\left(1-e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln\left(1-\overline{TR}_z\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right),\left(e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln\overline{IN}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right),\left(e^{-\left(\sum_{z=1}^{K}\frac{1}{K}\left(-\ln\overline{FL}_z\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right\rangle=\sum_{z=1}^{K}\frac{1}{K}(\Xi_z). \tag{41}$$

4. The Single Valued Neutrosophic Aczel-Alsina Power Weighted Muirhead Mean and Power Weighted Dual Muir-Head Mean Operators

In this part, the SVNAAPWMM and SVNAAPWDMM operators are suggested.

Definition 10. Let $\Xi_z=\langle\overline{TR}_z,\overline{IN}_z,\overline{FL}_z\rangle(z=1,2,\ldots,K)$ be series of SVNNs, and $Œ''=(Œ''_1,Œ''_2,\ldots,Œ''_K)\in R^K$ be a group of parameters. Let $\hbar=(\hbar_1,\hbar_2,\ldots,\hbar_n)^T$ be the importance degree, such that $0\le\hbar_z\le 1$ and $\sum_{z=1}^{K}\hbar_z=1$. The single valued neutrosophic Aczel-Alsina power weighted Muirhead mean (SVNAAPWMM) operator is conveyed as

$$S^V NAAPMM^{Œ''_z}(\Xi_1,\Xi_2,\ldots,\Xi_K)=\left(\frac{1}{K!}\sum_{\zeta\in H_k}\prod_{z=1}^{K}\left(K\frac{\hbar_{\zeta(z)}\left(1+T(\Xi_{\zeta(z)})\right)}{\sum_{z=1}^{K}\hbar_{\zeta(z)}\left(1+T(\Xi_z)\right)}\Xi_{\zeta(z)}\right)^{Œ''_z}\right)^{\frac{1}{\sum_{z=1}^{K}Œ''_z}}, \tag{42}$$

where

$$T(\Xi_j)=\sum_{z=1}^{K}Spt(\Xi_i,\Xi_z) \tag{43}$$

and

$$Spt\left(\tilde{\Xi}_i,\tilde{\Xi}_z\right)=1-\tilde{D}\left(\tilde{\Xi}_i,\tilde{\Xi}_z\right), \tag{44}$$

$\zeta(z)\left(z=1,2,...,\c{K}\right)$ signifies any arrangement of $\left(1,2,...,\c{K}\right)$. H_n signifies all possible arrangements of $\left(1,2,...,\c{K}\right)$, and $\c{K}$ is the balancing coefficient $\tilde{D}\left(\tilde{\Xi}_i,\tilde{\Xi}_j\right)$ denotes the Hamming distance among two SVNNs $\tilde{\Xi}_i$ and $\tilde{\Xi}_z$, and $Spt\left(\tilde{\Xi}_i,\tilde{\Xi}_z\right)$ is the support for $\tilde{\Xi}_i$ from $\tilde{\Xi}_z$ sustaining the axioms given in Definition 6.

To write Equation (42), in simplified form, let

$$\tilde{\hbar}_j=\frac{\tilde{\mathcal{R}}_j\left(1+\tilde{T}\left(\tilde{\Xi}_z\right)\right)}{\sum_{z=1}^{\c{K}}\tilde{\mathcal{R}}_j\left(1+\tilde{T}\left(\tilde{\Xi}_z\right)\right)}, \tag{45}$$

then, Equation (42) can be written in simplified form as,

$$S^V NAAPWMM^{Œ''}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_{\c{K}}\right)=\left(\frac{1}{\c{K}!}\sum_{\zeta\in H_{\c{K}}}\prod_{z=1}^{\c{K}}\left(\c{K}\tilde{\hbar}_z\tilde{\Xi}_{\zeta(z)}\right)^{Œ''_z}\right)^{\frac{1}{\sum_{z=1}^{\c{K}}Œ''_z}}, \tag{46}$$

Where $\sum_{z=1}^{\c{K}}\tilde{\hbar}_z=1$ and $0\le\tilde{\hbar}_z\le 1$.

According to Definition 10, the following Theorem 7, can be obtained.

Theorem 7. Let $\tilde{\Xi}_z=\left\langle\overline{\overline{TR}}_z,\overline{\overline{IN}}_z,\overline{\overline{FL}}_z\right\rangle(z=1,2,...,\c{K})$ be a group of SVNNs, the aggregated value by the SVNAAPWMM operator is still a SVNN and

$$S^V NAAPWMM^{(Œ'')}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_{\c{K}}\right)=\left\langle\left(e^{-\left(\frac{1}{\sum_{z=1}^{\c{K}}Œ''_z}\left(\frac{1}{\c{K}!}\left(\sum_{\zeta\in H_{\c{K}}}\left(\sum_{z=1}^{\c{K}}\left(Œ''_j\left(-\ln\left(1-e^{-\left(\c{K}\tilde{\hbar}_z\left(-\ln\left(1-\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\right.$$
$$\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{\c{K}}Œ''_z}\left(\frac{1}{\c{K}!}\left(\sum_{\zeta\in H_n}\left(\sum_{z=1}^{\c{K}}\left(Œ''_z\left(-\ln\left(1-e^{-\left(\c{K}\tilde{\hbar}_z\left(-\ln\left(\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right),\left.\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{\c{K}}Œ''_z}\left(\frac{1}{\c{K}!}\left(\sum_{\zeta\in H_n}\left(\sum_{z=1}^{\c{K}}\left(Œ''_z\left(-\ln\left(1-e^{-\left(\c{K}\tilde{\hbar}_z\left(-\ln\left(\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\Phi}\right)^{\frac{1}{\Phi}}}\right)\right)^{\Phi}\right)\right)\right)\right)\right)^{\frac{1}{\Phi}}}\right)\right\rangle. \tag{47}$$

Definition 11. Let $\tilde{\Xi}_z=\left\langle\overline{\overline{TR}}_z,\overline{\overline{IN}}_z,\overline{\overline{FL}}_z\right\rangle(z=1,2,...,\c{K})$ be series of SVNNs, and $Œ''=(Œ''_1,Œ''_2,...,Œ''_n)\in R^n$ be a group of parameters. Let $\tilde{\mathcal{R}}=\left(\tilde{\mathcal{R}}_1,\tilde{\mathcal{R}}_2,...,\tilde{\mathcal{R}}_{\c{K}}\right)^T$ be the importance degree, such that $0\le\tilde{\mathcal{R}}_{\c{K}}\le 1$ and $\sum_{z=1}^{\c{K}}\tilde{\mathcal{R}}_z=1$. The single valued neutrosophic Aczel-Alsina power weighted dual Muirhead mean (SVNAAPWDMM) operator is conveyed as

$$S^V NAAPWDMM^{Œ''}\left(\tilde{\Xi}_1,\tilde{\Xi}_2,...,\tilde{\Xi}_{\c{K}}\right)=\frac{1}{\sum_{z=1}^{\c{K}}Œ''_z}\left(\prod_{\zeta\in H_{\c{K}}}\sum_{z=1}^{\c{K}}\left(Œ''_z\Xi_{\zeta(z)}^{\c{K}\frac{\tilde{\mathcal{R}}_z\left(1+\tilde{T}\left(\tilde{\Xi}_{\zeta(z)}\right)\right)}{\sum_{z=1}^{\c{K}}\tilde{\mathcal{R}}_j\left(1+\tilde{T}\left(\tilde{\Xi}_z\right)\right)}}\right)\right)^{\frac{1}{\c{K}!}}, \tag{48}$$

where

$$T\left(\Xi_j\right)=\sum_{z=1}^{K} Spt\left(\Xi_i,\Xi_z\right) \tag{49}$$

and

$$Spt\left(\Xi_i,\Xi_z\right)=1-D\left(\Xi_i,\Xi_z\right), \tag{50}$$

$\zeta(z)(z=1,2,...,K)$ signifies any arrangement of $(1,2,...,K)$. H_n signifies all possible arrangements of $(1,2,...,K)$, and K is the balancing coefficient $D\left(\Xi_i,\Xi_z\right)$ denotes the Hamming distance among two SVNNs Ξ_i and Ξ_z, and $Spt\left(\Xi_i,\Xi_z\right)$ is the support for Ξ_i from Ξ_z satisfying the axioms given in Definition 6.

To write Equation (48), in simplified form, let

$$H_z=\frac{\mathcal{H}_z\left(1+T\left(\Xi_z\right)\right)}{\sum_{z=1}^{K}\mathcal{H}_z\left(1+T\left(\Xi_z\right)\right)}; \tag{51}$$

then, Equation (48) can be written in simplified form as,

$$S^V NAAPWDMM^{Œ''}\left(\Xi_1,\Xi_2,...,\Xi_K\right)=\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\prod_{\zeta\in H_K}\sum_{z=1}^{K}\left(Œ''_z\Xi_{\zeta(z)}^{KH_z}\right)\right)^{\frac{1}{K!}}, \tag{52}$$

Where $\sum_{z=1}^{K}H_z=1$ and $0\leq H_z\leq 1$.

According to Definition 11, the following Theorem 8, can be obtained.

Theorem 8. Let $\Xi_z=\left\langle \overline{\overline{TR}}_z,\overline{\overline{IN}}_z,\overline{\overline{FL}}_z\right\rangle (z=1,2,...,K)$ be a group of SVNNs, the aggregated value by the SVNAAPWDMM operator is still a SVNN and

$$S^V NAAPWDMM^{(Œ'')}\left(\Xi_1,\Xi_2,...,\Xi_K\right)=\left\langle\begin{array}{l}\left(1-e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_n}\left(\sum_{z=1}^{K}\left(Œ''_z\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(\overline{\overline{TR}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\\ \left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{K}\left(Œ''_z\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(1-\overline{\overline{IN}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right),\left(e^{-\left(\frac{1}{\sum_{z=1}^{K}Œ''_z}\left(\frac{1}{K!}\left(\sum_{\zeta\in H_z}\left(\sum_{z=1}^{K}\left(Œ''_z\left(-\ln\left(1-e^{-\left(KH_z\left(-\ln\left(1-\overline{\overline{FL}}_{\zeta(z)}\right)\right)^{\phi}\right)^{\frac{1}{\phi}}}\right)\right)\right)^{\phi}\right)\right)\right)\right)^{\frac{1}{\phi}}}\right)\end{array}\right\rangle. \tag{53}$$

5. A Methodology to Multiple Attribute Decision Making Based on The Single Valued Neutrosophic Aczel-Alsina Power Weighted Muirhead Mean Operators

The applications of these operators in the MADM will be covered in this section.

For a MADM situation with SVNs that includes z workable alternatives $\overline{\overline{C}}_i=\left(\overline{\overline{C}}_1,\overline{\overline{C}}_2,...,\overline{\overline{C}}_z\right)(i=1,2,...,z)$ and s attributes $\overline{\overline{T}}_j=\left(\overline{\overline{T}}_1,\overline{\overline{T}}_2,,,\overline{\overline{T}}_s\right)(j=1,2,...,s)$, and $F=\{F_1,F_2,F_3,...,F_z\}$ is the importance degree of the attributes for which the weight vectors of attributes are defined in $F_s(j=1,2,3,...,s)$, where $F_j\geq 0, j=1,...,s,\sum_{j=1}^{s}F_j=1$

Assume that $Y = \left[\overline{\overline{B}}_{ij}\right]_{m\times n}$ is the decision matrix, where $\overline{\overline{B}}_{ij} = \left\langle \overline{\overline{TR}}_{ij}, \overline{\overline{IN}}_{ij}, \overline{\overline{FL}}_{ij} \right\rangle$ acquires the structure of SVNN for the alternative $\overline{\overline{C}}_i$ with respect to the attribute $\overline{\overline{T}}_j (i = 1,...,z; j = 1,....,s)$. The rating of alternative is then necessary.

Afterwards, we address the MADM problems using the SVNAAPWMM and SVNAAPWDMM operators. The specific decision-making processes are listed below.

Step 1. Confirm that the attribute values are comparable. In real-life situations, attributes often come in two distinguishes: cost type and benefit type. The following formula must be used to convert attribute values from cost type to benefit type to obtain the desired result:

$$\overline{\overline{B}}_{ij} = \begin{cases} \overline{\overline{B}}_{ij} & \text{for benefit type;} \\ \left(\overline{\overline{B}}_{ij}\right)^c & \text{for cost type.} \end{cases} \tag{54}$$

Step 2. Determine the supports utilizing the following formula:

$$\text{spt}\left(\overline{\overline{B}}_{ij}, \overline{\overline{B}}_{il}\right) = 1 - \tilde{D}\left(\overline{\overline{B}}_{ij}, \overline{\overline{B}}_{il}\right)(i = 1,2,...,z;\ j,l = 1,2,...,s). \tag{55}$$

Where $\tilde{D}\left(\overline{\overline{B}}_{ij}, \overline{\overline{B}}_{il}\right)$ is hamming distance among two SVNNs.

Step 3. Determine $\tilde{T}\left(\overline{\overline{B}}_{ij}\right)$ utilizing the following formula:

$$\tilde{T}\left(\overline{\overline{B}}_{ij}\right) = \sum_{\substack{l=1 \\ j\neq l}}^{s} \text{spt}\left(\overline{\overline{B}}_{ij}, \overline{\overline{B}}_{il}\right)(i = 1,2,...,z;\ j,l = 1,2,...,s) . \tag{56}$$

Step 4. Utilizing the SVNAAPWMM operator or SVNAAPWDMM operator, combine all the attribute values $\overline{\overline{B}}_{ij}\ (j = 1,2,....,s)$ to the comprehensive value $\overline{\overline{C}}_i\ (i = 1,...,z)$ shown as follows:

$$\overline{\overline{C}}_i = \text{S}^{\text{V}}\text{NAAPWMM}\left(\overline{\overline{B}}_{i1}, \overline{\overline{B}}_{i2},....,\overline{\overline{B}}_{is}\right) \tag{57}$$

Or

$$\overline{\overline{C}}_i = \text{S}^{\text{V}}\text{NAAPWDMM}\left(\overline{\overline{B}}_{i1}, \overline{\overline{B}}_{i2},....,\overline{\overline{B}}_{is}\right) \tag{58}$$

Step 5. In this step, we determine the score value $\overline{\overline{\overline{\text{SOE}}}}\left(\overline{\overline{B}}_i\right)$ of all aggregated information for the assessment of alternative $\overline{\overline{B}}_i$ based on the score function of SVNNs described in Definition 2.

Step 6. In this step, using the method of ranking and ordering technique to choose the best desirable alternative after ranking each possible choice $\overline{\overline{C}}_i = \left(\overline{\overline{C}}_1, \overline{\overline{C}}_2,...,\overline{\overline{C}}_z\right)(i = 1,2,...,z)$

Step 7. End

6. Illustrative Example

In this part, to demonstrate the approach suggested in this research, we will offer a numerical example adapted from [31] to choose strategic suppliers under supply chain risk with SVNNs.

A panel has five potential strategic suppliers $\overline{\overline{C}}_{\mathrm{i}}\ (\mathrm{i}=1,\ldots,5)$ from which to choose. Four attributes are chosen by the experts to assess the five potential strategic suppliers: $\overline{\overline{T}}_1$ stands for "technology level," $\overline{\overline{T}}_2$ for "service level," $\overline{\overline{T}}_3$ for "risk managing ability," and $\overline{\overline{T}}_4$ for "enterprise environment risk." The four aforementioned attributes (their weighting vectors are $F_{\mathrm{j}}=(0.2,\ 0.1,\ 0.3, 0.4)^{\mathrm{T}}$) are to be used by the decision maker to assess the five potential strategic suppliers $\overline{\overline{C}}_{\mathrm{i}}\ (\mathrm{i}=1,\ldots.,5)$ and provide the assessment information in the form of SVNN as given in Table 1 .

Table 1. The SVN decision matrix

Alternatives/Attributes	$\overline{\overline{T}}_1$	$\overline{\overline{T}}_2$	$\overline{\overline{T}}_3$	$\overline{\overline{T}}_4$
$\overline{\overline{C}}_1$	$\langle 0.5, 0.8, 0.1\rangle$	$\langle 0.6, 0.3, 0.3\rangle$	$\langle 0.3, 0.6, 0.1\rangle$	$\langle 0.5, 0.7, 0.2\rangle$
$\overline{\overline{C}}_2$	$\langle 0.7, 0.2, 0.1\rangle$	$\langle 0.7, 0.2, 0.2\rangle$	$\langle 0.7, 0.2, 0.4\rangle$	$\langle 0.8, 0.2, 0.1\rangle$
$\overline{\overline{C}}_3$	$\langle 0.6, 0.7, 0.2\rangle$	$\langle 0.5, 0.7, 0.3\rangle$	$\langle 0.5, 0.3, 0.1\rangle$	$\langle 0.6, 0.3, 0.2\rangle$
$\overline{\overline{C}}_4$	$\langle 0.8, 0.1, 0.3\rangle$	$\langle 0.6, 0.3, 0.4\rangle$	$\langle 0.3, 0.4, 0.2\rangle$	$\langle 0.5, 0.6, 0.1\rangle$
$\overline{\overline{C}}_5$	$\langle 0.6, 0.4, 0.4\rangle$	$\langle 0.4, 0.8, 0.1\rangle$	$\langle 0.7, 0.6, 0.1\rangle$	$\langle 0.5, 0.8, 0.2\rangle$

Subsequently, we apply the methodology established to choose the best strategic suppliers under supply chain risk.

Step 1. Since all the attributes are of the same type, there is no need to normalize it.

Step 2. Utilizing formula (55) to generate the support. For simplicity, we shall denote S_{ij} instead of $Spt\left(\overline{\overline{B}}^{i}_{ij}, \overline{\overline{B}}^{i}_{il}\right)(i=1,2,3,4,5;\ j,l=1,2,3,4)$. Which are listed below:

$S^{1}_{12}=0.2667, S^{1}_{13}=0.1333, S^{1}_{14}=0.0667, S^{1}_{23}=0.2667, S^{1}_{24}=0.2000, S^{1}_{34}\ 0.1333, S^{2}_{12}=0.0333, S^{2}_{13}=0.1000,$
$S^{2}_{14}=0.0333; S^{2}_{23}=0.0667, S^{2}_{24}=0.0667, S^{2}_{34}=0.1333, S^{3}_{12}=0.0667, S^{3}_{13}=0.2000, S^{3}_{14}=0.1333, S^{3}_{23}=0.2000,$
$S^{3}_{24}=0.2000, S^{3}_{34}=0.0667; S^{4}_{12}=0.1667, S^{4}_{13}=0.3000, S^{4}_{14}=0.3333, S^{4}_{23}=0.2000, S^{4}_{24}=0.2333, S^{4}_{34}=0.1667,$
$S^{5}_{12}=0.3000, S^{5}_{13}=0.2000, S^{5}_{14}=0.2333; S^{5}_{23}=0.1667, S^{5}_{24}=0.0667, S^{5}_{34}=0.1667.$

Step 3. Utilizing formula (56), to generate $T\left(\overline{\overline{B}}_{ij}\right)$. We shall denote T_{ij} instead of $T\left(\overline{\overline{B}}_{ij}\right)$. Which are listed below:

$T_{11}=0.4667, T_{12}=0.7333, T_{13}=0.5333, T_{14}=0.4000, T_{21}=0.1667, T_{22}=0.1667, T_{23}=0.3000, T_{24}=0.2333;$
$T_{31}=0.4000, T_{32}=0.4667, T_{33}=0.4667, T_{34}=0.4000, T_{41}=0.8000, T_{42}=0.6000, T_{43}=0.6667, T_{44}=0.7333;$
$T_{51}=0.7333, T_{52}=0.5333, T_{53}=0.5333, T_{54}=0.4667.$

Step 4. Utilizing formulas (57) or (58), to aggregate the comprehensive evaluation information of each alternative. For suitability, we firstly determine the $u_{ij} = \frac{sF_{ij}\left(1+T(\overline{\overline{B}}_{ij})\right)}{\sum_{z=1}^{s} F_z(1+T(\overline{\overline{B}}_z))}$ .Which are given below:

(Assume $\phi = 2, Œ'' = (0.1, 0.2, 0.3, 0.4)^{\mathrm{T}}$)

$u_{11} = 0.7892, u_{12} = 0.4664, u_{13} = 1.2377, u_{14} = 1.5067, u_{21} = 0.7568, u_{22} = 0.3784, u_{23} = 1.2649, u_{24} = 1.6000;$
$u_{31} = 0.7850, u_{32} = 0.4112, u_{33} = 1.2336, u_{34} = 1.5701, u_{41} = 0.8405, u_{42} = 0.3735, u_{43} = 1.1673, u_{44} = 1.6187;$
$u_{51} = 0.8966, u_{52} = 0.3966, u_{53} = 1.1897, u_{54} = 1.5172.$

$\overline{\overline{B}}_1 = \langle 0.4367, 0.6750, 0.2661 \rangle, \overline{\overline{B}}_2 = \langle 0.6552, 0.2553, 0.2885 \rangle,$
$\overline{\overline{B}}_3 = \langle 0.5006, 0.6488, 0.2945 \rangle; \overline{\overline{B}}_4 = \langle 0.4631, 0.4221, 0.3809 \rangle,$
$\overline{\overline{B}}_5 = \langle 0.4632, 0.7381, 0.2703 \rangle$

Or

$\overline{\overline{B}}_1 = \langle 0.5465, 0.4405, 0.1436 \rangle, \overline{\overline{B}}_2 = \langle 0.7475, 0.1840, 0.1427 \rangle,$
$\overline{\overline{B}}_3 = \langle 0.5806, 0.4347, 0.1751 \rangle; \overline{\overline{B}}_4 = \langle 0.6718, 0.2191, 0.2050 \rangle,$
$\overline{\overline{B}}_5 = \langle 0.5905, 0.5588, 0.1404 \rangle$

Step 5. Utilizing the score function suggested in Definition 2, to find the score values comprehensive evaluation information of each alternative.

$\overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_1) = 0.4985, \overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_2) = 0.7038, \overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_3) = 0.5191, \overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_4) = 0.5534, \overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_5) = 0.4849.$
and
$\overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_1) = 0.6541, \overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_2) = 0.8069, \overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_3) = 0.6569, \overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_4) = 0.7493, \overline{\overline{\mathrm{SOE}}}(\overline{\overline{B}}_5) = 0.6304.$

Step 6. According to score values the ranking of the alternatives are $\overline{\overline{B}}_2 \succ \overline{\overline{B}}_4 \succ \overline{\overline{B}}_3 \succ \overline{\overline{B}}_1 \succ \overline{\overline{B}}_5$ and $\overline{\overline{B}}_2 \succ \overline{\overline{B}}_4 \succ \overline{\overline{B}}_3 \succ \overline{\overline{B}}_1 \succ \overline{\overline{B}}_5$

So, the best alternative is $\overline{\overline{B}}_2$, while the worst one is $\overline{\overline{B}}_5$.

6.1. Impact of the parameters ϕ and Œ″ final ranking order

In this part, the impact of the parameters ϕ and Œ″ are investigated utilizing both the initiated AGOs.

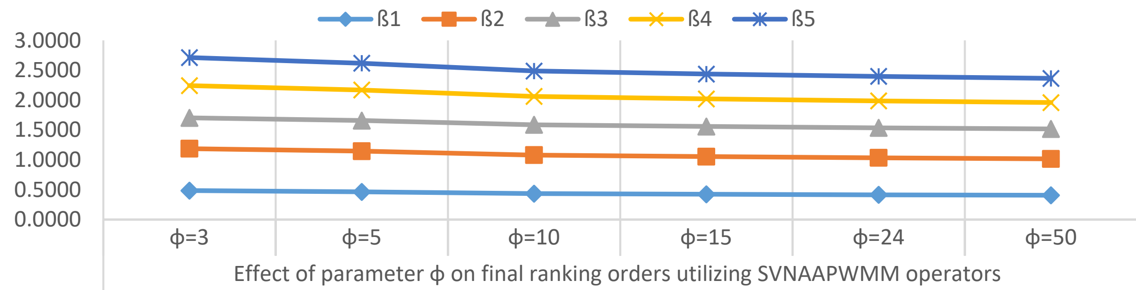
For different values of the parameter ϕ the values of the parameter $Œ'' = (1,1,1,1)^{T}$ are fixed for utilizing both SVNAAPWMM and SVNAAPWDMM operators and the score values are given in Figure 1 and 2.

From Figure 1 one can see that for different values of the parameters ϕ, utilizing the SVNAAPWMM operators different score values of the alternatives are obtained, but the ranking order remain the same. That is, $\overline{\overline{B}}_2$ is the best alternative and $\overline{\overline{B}}_5$ is worst alternative. one can also noticed from Figure 1, when the values of the parameter ϕ increases the score of the alternatives decreases. From Figure

2, we can see that for different values of the parameters ϕ , utilizing the SVNAAPWDMM operators different score values and different ranking order of the alternatives are obtained. When the values of the parameter $\phi \leq 9$ the best alternative is $\overline{\overline{B}}_2$ and $\overline{\overline{B}}_5$ is worst alternative.

Similarly, when the values of the parameter $\phi \geq 10$, the best alternative is $\overline{\overline{B}}_4$ and $\overline{\overline{B}}_5$ is worst alternative. From Figure 2, we can observe that when the values of the parameter ϕ increases the score of the alternatives increases.

From Figure 3, we can observe that for different values of parameter $Œ''$ different score values of the alternatives are obtained, and the ranking order remain the same. That is, $\overline{\overline{B}}_2$ is the best alternative and $\overline{\overline{B}}_5$ is worst alternative. By assigning some different values to parameter $Œ''$, we will get different existing aggregation operators for SVN information. From Figure 4, for different values of the parameter $Œ''$ utilizing SVNAAPWDMM operator different score values are obtained. Although, $\overline{\overline{B}}_2$ is the best alternative and $\overline{\overline{B}}_5$ is worst alternative.



	ϕ=3	ϕ=5	ϕ=10	ϕ=15	ϕ=24	ϕ=50
ß5	0.4698	0.4477	0.4252	0.4170	0.4106	0.4051
ß4	0.5398	0.5120	0.4755	0.4615	0.4509	0.4417
ß3	0.5161	0.5121	0.5070	0.5047	0.5029	0.5014
ß2	0.7011	0.6825	0.6453	0.6303	0.6188	0.6090
ß1	0.4854	0.4631	0.4346	0.4232	0.4145	0.4069

SCORE VALUES FOR DIFFERENT VALUES OF PARAMETER

Fig 1. Impact of the parameter ϕ on final ranking orders utilizing SVNAAPWMM operators

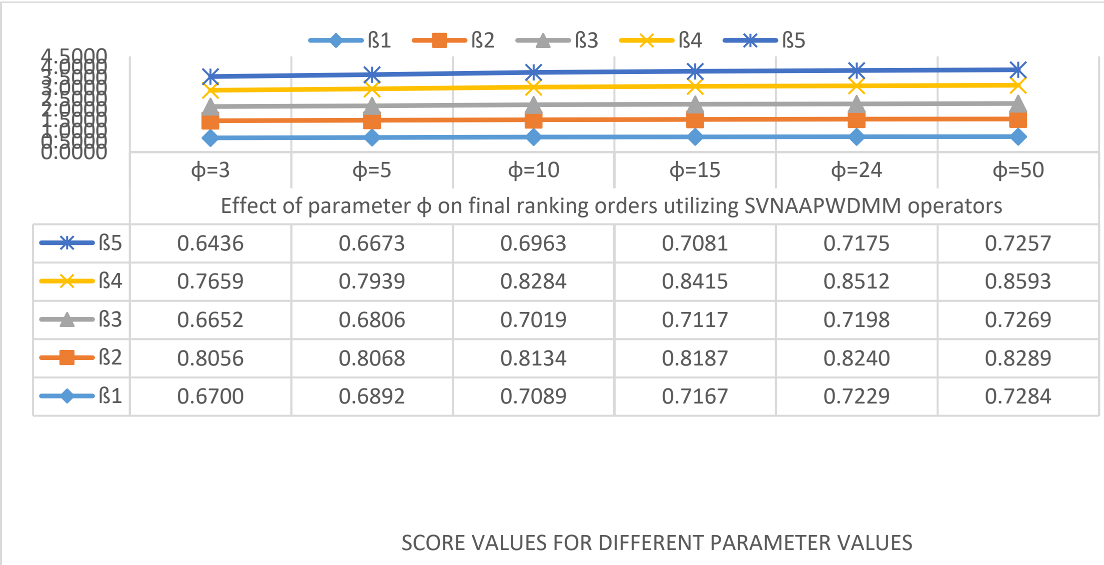
6.2. Comparative analysis advantages and disadvantages of the initiated MADM approach

In this subpart, a comparison between the initiated approach and the existing MADM approaches, advantages and disadvantages are investigated.

We solve the same example using five existing MADM methods, including SVN weighted averaging operator, SVN weighted geometric operator [49], SVNAAWA [51], SVN MSM operators developed by Khan et al. [18], and SVN power aggregation operator developed by Yang and [38], and Wei and

Zhang [47] SVNBPM operators to demonstrate the benefits and efficacy of the method developed in this article.

Figure 5, we can see that the ranking orders of the alternatives obtained from different approaches and the initiated approach are slightly different. That is, from the initiated MADM approach $\overline{\overline{B}}_2$ is the best alternative and $\overline{\overline{B}}_5$ is worst alternative, while utilizing the existing MADM approaches $\overline{\overline{B}}_2$ is the best alternative and $\overline{\overline{B}}_1$ is worst alternative.



	ϕ=3	ϕ=5	ϕ=10	ϕ=15	ϕ=24	ϕ=50
ß5	0.6436	0.6673	0.6963	0.7081	0.7175	0.7257
ß4	0.7659	0.7939	0.8284	0.8415	0.8512	0.8593
ß3	0.6652	0.6806	0.7019	0.7117	0.7198	0.7269
ß2	0.8056	0.8068	0.8134	0.8187	0.8240	0.8289
ß1	0.6700	0.6892	0.7089	0.7167	0.7229	0.7284

Fig 2. Impact of the parameter ϕ on final ranking orders utilizing SVNAAPWDMM

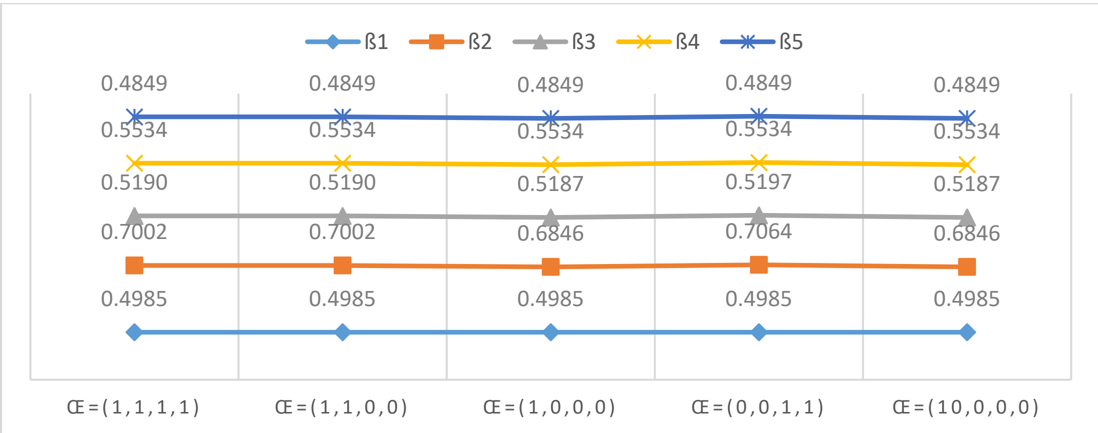


Fig 3. Impact of the parameter $Œ''$ on final ranking orders utilizing SVNAAPWMM operators

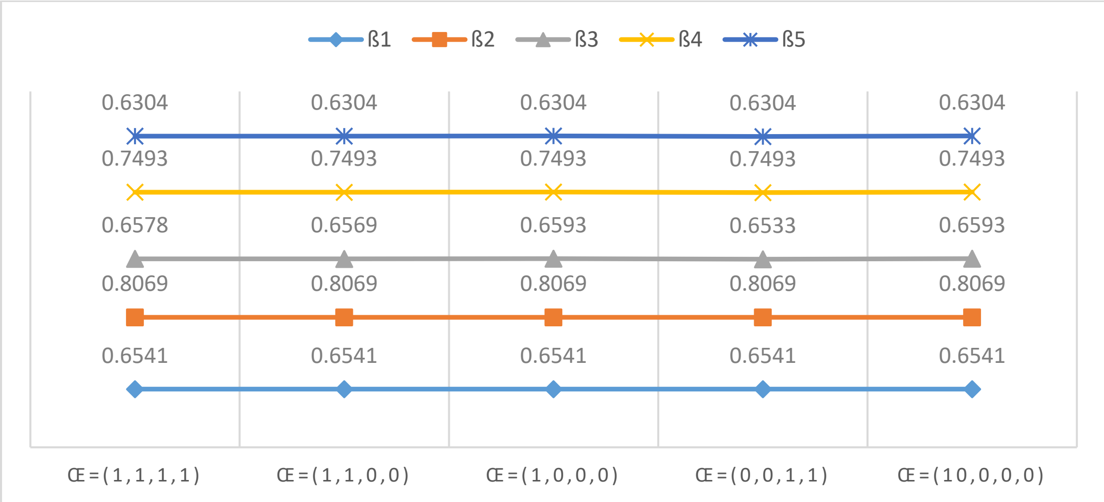
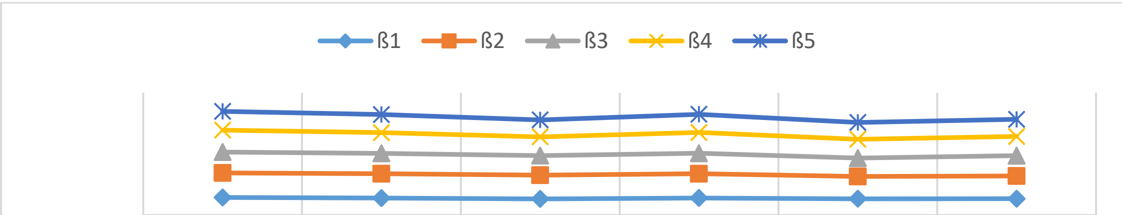


Fig 4. Impact of the parameter *Œ″* on final ranking orders utilizing SVNAAPWMM operators



	SVNAAWA OPERATOR	SVNWA OPERATOR	SVNWG OPERATOR	SVNPWA OPERATOR	SVNPMSM OPERATOR	SVNWBPM OPERATOR
ß5	0.6134	0.5898	0.5531	0.5926	0.549503687	0.5601
ß4	0.7160	0.6757	0.6117	0.6787	0.61274229	0.6269
ß3	0.6814	0.6689	0.6398	0.6689	0.600881976	0.6625
ß2	0.8007	0.7942	0.7753	0.7931	0.728815841	0.7436
ß1	0.5817	0.5604	0.5341	0.5637	0.538525781	0.5394

Fig 5. Comparison with different MADM approaches

As a result, it is confirmed that the approach we suggested is sensible and efficient.

There are several advantages of the proposed AGOs over the existing AGOs. Firstly, the existing AGOs are special cases with admiration to the parameter of the initiated AGOs. Secondly, the initiated AGOs are based on AA operational rules. The initiated AGOs have the capacity of removing the effect of awkward data and can take interrelationship among any number of input arguments at the same time. While the existing AGOs can only remove the effect of awkward data or can take interrelationship among input arguments. Thirdly, the initiated MADM approach is flexible due to the containment of several parameters which can deal with complex MADM problems more efficiently than the existing MADM approaches.

7. CONCLUSION

The PA operator, which can diminish the influence of elusive data provided by biased experts, and the MM operator, which can deliberate the interrelationship connecting any number input arguments, are advantages that the PMM operator may utilize. The advantage of considerable flexibility with general parameters is one of the advantages of Aczel-Alsina operational laws for SVNNs. Initially, we combined PMM with the AA operational laws and suggested various aggregation operators for SVNNs. Secondly, some of the basic characteristics and special cases with respect to the parameters are investigated and found that some of the existing AGOs are special cases of these initiated aggregation operators. The advantage of the intended aggregation operators is that they are more flexible in the aggregation process because they can simultaneously reduce the influence of uncomfortable data by PA operator and take the relationship between any number of input data by utilizing MM operator. Thirdly, weighted form of these AGOs are also initiated. Fourthly, A novel MADM technique is created based on the proposed aggregation operators to handle SVN information. To demonstrate the efficacy and applicability of the suggested MADM approach, an example is utilized, and comparisons are made with the current methods. The suggested aggregation operations are quite helpful in solving MADM problems due to the defined characteristic.

In future study, we will construct specific aggregation operators based on AA operational laws for SVN hesitant FSs, interval NHFSs, double valued neutrosophic sets and so on, and apply them to solve MAGDM and MADM problems in different fields.

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- Competing Interests: No
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- Data availability and access: The data utilized in this manuscript are hypothetical and artificial, and one can use these data before prior permission by just citing this manuscript.

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Peide Liu, received the B.S. and M.S. degrees in signal and information processing from Southeast University, Nanjing, China, in 1988 and 1991, respectively, and the Ph.D. degree in information management from Beijng Jiaotong University, Beijing, China, in 2010.

He is currently a Professor with the School of Management Science and Engineering, Shandong University of Finance and Economics, Shandong, China. He is an Associate Editor of the IEEE Transactions on Fuzzy Systems, Information Sciences, International Journal of Fuzzy Systems, Journal of Intelligent and Fuzzy Systems, the editorial board of the journal Technological and Economic Development of Economy, and the members of editorial board of the other 12 journals. He has authored or coauthored more than 400 publications. His research interests include aggregation operators, fuzzy logic, fuzzy decision making, and their applications.

Qaisar Khan did his Ph.D. from International Islamic University Islamabad Pakistan in 2019. He is currently working as an Assistant Professor in the Department of Mathematics and Statistics, University of Haripur, Pakistan. He has published more than 60 articles in well reputed journals. His area of research is fuzzy decision making and fuzzy algebraic structures.

Rashid Ali has completed M.sc from Hazara university Mansehra,in 2009 and M.S from Hazara university in 2015 in history of mathematics (history of exponential and logarithmic functions). Currently, he is Ph.D. student at the University of Malakand Chak Dara. His research field is decision making, fuzzy algebra.

Dr. Hidayat Ullah is currently serving as an Assistant Professor in the Department of Mathematics at the University of Malakand since 2018, he brings a wealth of experience to his role.His journey in academia began as a Lecturer in Mathematics at the University of Malakand in 2005, where he continued to inspire and educate students until 2018. Prior to this, Dr. Khan served as a Research Associate at Baluchistan University of IT and Emerging Sciences (BUITMS), Quetta, from 2004 to 2005.

Dr. Khan's dedication to education extends beyond his tenure at the University of Malakand. He also contributed as a Lecturer at Government Degree College Agra, Malakand Agency, Khyber Pakhtunkhwa, from 2000 to 2001, and at Government Jehanzeb Post Graduate College, Saidu Sharif, Swat, Khyber Pakhtunkhwa. His academic achievements include earning a Master of Science (MSc) in Mathematics from the University of Peshawar in 2000 and a Doctorate in Mathematics from Universiti Teknologi Malaysia, Malaysia, in 2015. Dr. Khan's research is primarily focused on Fuzzy Algebra, a field in which he has made significant contributions.

Throughout his career, Dr. Hidayat Ullah Khan has published more than 50 research articles in esteemed national and international journals, further solidifying his reputation as a distinguished mathematician and educator.

Dr. Rashid Jan Postdoctoral researcher at the Institute of Energy Infrastructure (IEI), Department of Civil Engineering, College of Engineering, Universiti Tenaga Nasional (UNITEN), Putrajaya Campus, Jalan IKRAM-UNITEN, 43000 Kajang, Selangor, Malaysia. He obtained PhD degree in applied mathematics from Xian Jiaotong University, Xian China and worked as an Assistant Professor in the Department of Mathematics, University of Swabi, KPK Pakistan. He has published more than 150 research articles in high ranked journals.

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