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Optimizing Network Structures Through Neutrosophic Graph Product Operations and its Coloring: A Comprehensive Approach for Enhanced Connectivity and Robustness

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Abstract

Optimal network analysis requires advanced techniques to handle the inherent complexity and uncertainty of real-world systems. We have used vertex order coloring on neutrosophic graphs to find the most effective approach to improve network reliability and performance. Neutrosophic graphs($\mathcal{NG}$) offer a comprehensive framework for modelling real-world networks with inherent uncertainties by incorporating degrees of truth, falsity, and indeterminacy. In this paper, we have investigated various graph product operations as a means of optimizing network structures. We further investigated the applications of vertex order coloring to identify α, β and γ strong vertices within various graph operations of $\mathcal{NG}$. We examined several $\mathcal{NG}$ products with the goal of determining the most optimal network based on particular important metrics including the total number of alpha-strong vertices, the weight of alpha-strong vertices, the chromatic number, and the weight of the graph's minimum spanning tree. The objective of our research is to identify the best solutions that strike a balance between robustness and association by rigorously studying and comparing various product operations. Our research advances the subject of network theory and provides useful information for a variety of applications, including social networks, transportation, and telecommunications.

Keywords: Neutrosophic graph, vertex order coloring, co-normal, tensor, residue

Introduction

Graph theory, a foundational field in mathematics, originated in 1736 with Leonhard Euler's solution to the Königsberg bridge problem, marking the birth of graph theory. This field of study investigates the characteristics and uses of graphs, which are mathematical structures that represent pairwise relationships between objects [8]. The concept of graph coloring, which holds significant importance in graph theory, originated in the 19th century when Francis Guthrie proposed the four-color theorem in 1852[25], which was subsequently validated by Kenneth Appel and Wolfgang Haken in 1976. It entails giving network vertices distinct colors in order to differentiate between neighboring vertices. This is a technique that has applications in resource allocation, scheduling, and map coloring. Lotfi A. Zadeh's 1965 invention of fuzzy set theory completely changed how mathematical models deal with imprecision and uncertainty [2]. This development led to the rise of fuzzy graph theory, introduced by Azriel Rosenfeld in 1975[22], It models networks with ambiguous or imprecise links by fusing the ideas of fuzzy sets with graph theory. Bhutani initially presented the idea of coloring in fuzzy graph theory in 1993[3, 4]. By extending the classic graph coloring problem to fuzzy graphs, Bhutani's work made it possible to investigate coloring techniques in graphs with edges that have different membership levels. Gong and zhang [11, 13] explored the concepts of adjacent vertex distinguishing proper edge coloring and adjacent vertex distinguishing total coloring within the context of fuzzy graphs. In addition to outlining the necessary and sufficient conditions for a vertex to be strong, Nagoor Gani et al. [12] established three different kinds of vertices namely α, β and γ strong vertices. Based on these principles, Krassimir Atanassov's intuitionistic fuzzy graph theory introduced in 1986 adds another level of complexity by taking into account both the degree of membership and the degree of non-membership. This method makes it possible to model uncertainty more thoroughly. Talebi et al. [26] introduced a new concept of coloring intuitionistic fuzzy graphs, which have good capabilities in handling vague and incompatible information, unlike traditional fuzzy graphs. This concept addresses problems where fuzzy graphs may not provide satisfactory results. Rosyida et al. presented an algorithm for coloring picture fuzzy graphs based on strong and weak adjacencies between vertices, which is an extension of the concepts from fuzzy graphs and intuitionistic fuzzy graphs ($\mathcal{IFGs}$)[23]. Rifayathali et al. [24] introduced the concept of chromatic excellence in intuitionistic fuzzy graphs, providing new insights into graph coloring. Florentin Smarandache, introduced the concept of neutrosophic fuzzy set ($\mathcal{NFS}$) by combining fuzzy set ($\mathcal{FS}$) with neutrosophic set ($\mathcal{NS}$), leading to the development of new concept and also proposed the single valued neutrosophic fuzzy set ($\mathcal{SVNFS}$) to address difficulties faced in dealing with real-life problems due to nonstandard intervals of neutrosophic components[13]. By permitting elements to have different degrees of membership, fuzzy set theory expands on conventional set theory and offers a more complex depiction of real-world situations. By adding a third parameter to indicate the degree of indeterminancy, Florentin Smarandache's 1998 development of $\mathcal{NG}$ expands on these ideas and provides an even more comprehensive framework for managing ambiguity and uncertainty. Meenakshi et al. [17,19,20] explored the use of neutrosophic graphs to model real-world problems containing inconsistent information, showcasing the versatility and applicability of this approach in various scenarios. They also developed an optimal algorithm to illustrate the applications of the proposed single-valued neutrosophic graph, providing a practical demonstration of its efficiency in information flow within networks Assigning colors to vertices under fuzzy constraints is known as coloring fuzzy graphs, which includes intuitionistic and neutrosophic graphs. This technique has applications in network design, resource allocation, and decision-making processes. These developments demonstrate how mathematical modelling is always changing when it comes to solving challenging real-world issues. Meenakshi et al.'s (17) goal is to build a single-valued neutrosophic graph using the max product of graphs, which should increase the efficiency of the maximized network. Aparna et al. introduced the concept of Single Valued Neutrosophic R-dynamic Vertex Coloring, combining Single Valued Neutrosophic Vertex Coloring and r-dynamic coloring to address graph coloring problems [1, 3]. Graph products are procedures that create a new graph by joining two graphs. Graph product operations play a significant role in network design and optimization. The residue product of two fuzzy graphs was defined by K. Radha et al. [23] and its features, including efficiency, connectedness, and completeness, were examined. The co-normal product of IFG was first proposed by K. Kalaiarasi et al. [14, 15], who also examined its importance in relation to the completeness, regularity, and pseudo-regularity of fuzzy graphs. Mohanta et al. [22] presented various operations on $\mathcal{SVNG}$, including rejection, symmetric difference, maximal product, and residue product. These operations are accompanied by appropriate examples, enhancing understanding and application

This introductory section is followed by a detailed analysis of pertinent literature, and the article proceeds as follows: We have included the preliminary information for a better grasp of $\mathcal{NG}$ concepts and coloring ideas in Section 2. Section 3 deals with the approach used in this investigation. Single-Valued Neutrosophic Vertex Order Coloring is a concept that incorporates the idea of neutrosophic graphs into traditional vertex coloring. This method minimizes coloring conflicts based on neutrosophic logic by assigning colors to each vertex in a graph based on a certain order prioritizing α strong vertices followed by β and γ strong vertices. Section 4 summarizes our research's conclusions, including theorems and an analysis of different operations. In Section 5, different operations on $\mathcal{NG}$ are detailed and its coloring using Neutrosophic Vertex Order Coloring(NVOC) is demonstrated. We have critically discussed these findings in Section 6, examining possible mechanisms and providing an algorithm to determine the optimal network. The article concludes with a summary of its most important findings. By using this systematic approach, we offered an extensive understanding of various operations and identified the optimal network depending on specific factors.

2. Preliminaries

2.1 Single valued Neutrosophic set [4]

Let $\mathcal{X}$ be a universe of discourse.

$\mathcal{N}$= {(x, $\mathcal{T}_{\mathcal{N}}(x)$, $\mathcal{I}_{\mathcal{N}}(x)$, $\mathcal{F}_{\mathcal{N}}$ (x): x ∈ X} is a $\mathcal{SVNFS}$ on $\mathcal{X}$. The degree of truth membership value is given by $\mathcal{T}_{\mathcal{N}}$ (x): $\mathcal{X}$ → [0, 1], degree of indeterminancy value, and degree of falsity membership value of x on $\mathcal{N}$ are denoted by $\mathcal{I}_{\mathcal{N}}$ (x): $\mathcal{X}$ → [0, 1] and $\mathcal{F}_{\mathcal{N}}$ (x): $\mathcal{X}$ → [0, 1], respectively, meeting the criteria $0 < \mathcal{T}_{\mathcal{N}}(x) + \mathcal{I}_{\mathcal{N}}(x) + \mathcal{F}_{\mathcal{N}}(x) \leq 3, \forall x \in \mathcal{X}$.

2.2 Single valued Neutrosophic Graph [5]

The $\mathcal{SVNG}$, $\widetilde{\mathcal{N}}_{\mathcal{G}}$ of $\mathcal{N}_{\mathcal{G}}$=($\mathcal{V}_{\mathcal{G}}$, $\mathcal{E}_{\mathcal{G}}$) is denoted by $\widetilde{\mathcal{N}}_{\mathcal{G}}$ =($\mathcal{V}_{\mathcal{G}}$,$\zeta$,$\omega$) where $\zeta = (\mathcal{T}_{\zeta}, \mathcal{I}_{\zeta}, \mathcal{F}_{\zeta})$ is a single-valued Neutrosophic set on $\mathcal{V}_{\mathcal{G}}$ and $\omega = (\mathcal{T}_{\omega}, \mathcal{I}_{\omega}, \mathcal{F}_{\omega})$ is a single-valued Neutrosophic symmetric relation on $\mathcal{E}_{\mathcal{G}} \subseteq \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$ is defined as follows:

i) $\mathcal{T}_{\mu}(x, y) \leq \mathcal{T}_{\zeta}(x) \wedge \mathcal{T}_{\zeta}(y)$, $\forall (x, y) \in \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$.

ii) $\mathcal{I}_{\mu}(x, y) \leq \mathcal{I}_{\zeta}(x) \wedge \mathcal{I}_{\zeta}(y)$, $\forall (x, y) \in \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$.

iii) $\mathcal{F}_{\mu}(x, y) \geq \mathcal{F}_{\zeta}(x) \vee \mathcal{F}_{\zeta}(y)$, $\forall (x, y) \in \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$.

2.3 Complete single valued Neutrosophic Graph [5]

The $\mathcal{SVNG}$ $\widetilde{\mathcal{N}}_{\mathcal{G}}$ is said to be complete if $\forall (x, y) \in \mathcal{E}_{\mathcal{G}}$,

$\mathcal{T}_{\omega_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x, y) = \mathcal{T}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x) \wedge \mathcal{T}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(y)$,

$\mathcal{I}_{\omega_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x, y) = \mathcal{I}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x) \wedge \mathcal{I}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(y)$,

$\mathcal{F}_{\omega_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x, y) = \mathcal{F}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x) \vee \mathcal{F}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(y)$,

2.4 Strong single valued Neutrosophic Graph [5]

The $\mathcal{SVNG}$ $\widetilde{\mathcal{N}}_{\mathcal{G}}$ is known as strong single valued Neutrosophic graph if $\forall (x, y) \in \mathcal{V}_{\mathcal{G}}$,

$\mathcal{T}_{\omega_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x, y) = \mathcal{T}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x) \wedge \mathcal{T}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(y)$,

$\mathcal{I}_{\omega_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x, y) = \mathcal{I}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x) \wedge \mathcal{I}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(y)$,

$\mathcal{F}_{\omega_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x, y) = \mathcal{F}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(x) \vee \mathcal{F}_{\zeta_{\widetilde{\mathcal{N}}_{\mathcal{G}}}}(y)$.

2.5 Degree, neighborhood and Cardinality of a vertex in $\mathcal{SVNG}$ [5]

Let $\widetilde{\mathcal{N}_{\mathcal{G}}}$ =($\mathcal{V}_{\mathcal{G}}$,$\zeta$,$\omega$) be a $\mathcal{SVNG}$ and $\mathfrak{a}_i$ be a vertex in a $\widetilde{\mathcal{N}_{\mathcal{G}}}$. A vertex's degree, represented by $d(\mathfrak{a}_i)$, is the total weight of the strong arcs that incident at it.

The neighborhood of $\mathfrak{a}_i$ is represented by the strong arc $\widetilde{\mathcal{N}_{\mathcal{G}}}$ ={$\omega_i \in \mathcal{V}_{\mathcal{G}}$ / ($\mathfrak{a}_i$, ω_i)}.

The minimum degree of $\widetilde{\mathcal{N}_{\mathcal{G}}}$ is given by $\delta(\widetilde{\mathcal{N}_{\mathcal{G}}}) = \min\{d_{\widetilde{\mathcal{N}_{\mathcal{G}}}}(\mathfrak{a}_i) / \ \mathfrak{a}_i \in \mathcal{V}_{\mathcal{G}}\}$.

The maximum degree of $\widetilde{\mathcal{N}_{\mathcal{G}}}$ is given by $\Delta(\widetilde{\mathcal{N}_{\mathcal{G}}}) = \max\{d_{\widetilde{\mathcal{N}_{\mathcal{G}}}}(\mathfrak{a}_i) / \ \mathfrak{a}_i \in \mathcal{V}_{\mathcal{G}}\}$.

The cardinality of a vertex $\mathfrak{a}_i \in \mathcal{V}$ in a $\mathcal{SVNG}$, $\mathcal{N}_{\mathcal{G}}$=($\mathcal{V}_{\mathcal{G}}$, $\mathcal{E}_{\mathcal{G}}$) is defined by

$$|\mathfrak{a}_i| = \mathcal{T}_\zeta(\mathfrak{a}_i) + \mathcal{I}_\zeta(\mathfrak{a}_i) + \mathcal{F}_\zeta(\mathfrak{a}_i).$$

The cardinality of an edge $\mathfrak{a}_i\mathfrak{a}_j \in \mathcal{E}$ in a $\mathcal{SVNG}$, $\mathcal{N}_{\mathcal{G}}$=($\mathcal{V}_{\mathcal{G}}$, $\mathcal{E}_{\mathcal{G}}$) is defined by

$$|\mathfrak{a}_i\mathfrak{a}_j| = \mathcal{T}_\omega(\mathfrak{a}_i\mathfrak{a}_j) + \mathcal{I}_\omega(\mathfrak{a}_i\mathfrak{a}_j) + \mathcal{F}_\omega(\mathfrak{a}_i\mathfrak{a}_j).$$

2.7 Adjacent Vertex of a fuzzy graph [10]

Let $\mathfrak{a}_i$ be the vertex of a fuzzy graph $\mathcal{F}_{\mathcal{G}} = (\mathbb{V}_{\mathcal{G}}, \mathbb{E}_{\mathcal{G}})$ then the adjacent vertices to the vertex $\mathfrak{a}_i$ is

$$\mathbb{A}\ (\mathfrak{a}_i) = \{\mathfrak{a}_i\ \epsilon\ \mathbb{V}_{\mathcal{G}} \ni \tfrac{1}{2}[\sigma(i) \wedge \sigma(j)]\ \leq \mu(ij)\}.$$

$$\mathbb{A}_s(\mathfrak{a}_i) = \{\mathfrak{a}_i\ \epsilon\ \mathbb{A}(\mathbb{V}_{\mathcal{G}}) | d(j) \geq d(i)\}$$

$$\mathbb{A}_w(\mathfrak{a}_i) = \{\mathfrak{a}_i\ \epsilon\ \mathbb{A}(\mathbb{V}_{\mathcal{G}}) | d(j) < d(i)\}.$$

3. Neutrosophic vertex order coloring

A sophisticated but effective development of conventional vertex coloring is NVOC. It is appropriate for situations where ambiguity and vagueness are crucial since it enables a more nuanced representation of these components. This approach offers a more advanced and adaptable solution to the vertex coloring problem by ensuring that the coloring complies with the rules of neutrosophic logic.

3.1 $\mathcal{K}$-vertex coloring of $\mathcal{SVNG}$

A family $\mathfrak{E} = \{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_{\mathcal{K}}\}$ of Neutrosophic sets on a set $\mathcal{V}$ is called a $\mathcal{K}$-vertex coloring of $\widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$ if

a) $\cup\ \mathfrak{E} = \zeta$

b) $\mathrm{e}_i \wedge \mathrm{e}_j = 0$

c) For every strong edge xy of $\mathcal{N}_{\mathcal{G}}$, min{ $\mathrm{e}_i(\mu_1(x))$, $\mathrm{e}_i(\mu_1(y))$}= 0 and max {$\mathrm{e}_i(\gamma_1(x))$, $\mathrm{e}_i(\gamma_1(x))$}= 1,$(1 \leq j \leq k)$

The least value of $\mathcal{K}$ for which the $\mathcal{N}_{\mathcal{G}}$ has a $\mathcal{K}$-vertex coloring denoted by $\chi^{\tau_0}(\widetilde{\mathcal{N}_{\mathcal{G}}})$, is called the chromatic number of the $\mathcal{NG}$ $\widetilde{\mathcal{N}_{\mathcal{G}}}$.

3.2 Adjacent Vertex of $\mathcal{SVNG}$

Let $\mathfrak{a}_i$ be the vertex of a $\mathcal{SVNG}$, $\widetilde{\mathcal{N}_{\mathcal{G}}}$ =($\mathcal{V}_{\mathcal{G}}$,$\zeta$,$\omega$) then the adjacent vertices to the vertex $\mathfrak{a}_i$ is

$\mathbb{A}\ (\mathfrak{a}_i) = \{x\ \epsilon\ \mathcal{V}_{\mathcal{G}} \ni \frac{1}{2}[\sigma(x) \wedge \sigma(y)]\ \leq \mu(xy)\}$.

$\mathbb{A}_s(\mathfrak{a}_i)$ = $\{x\ \epsilon\ \mathbb{A}(\mathfrak{a}_i) \ni \{d(y) \geq d(x)\}$ where $\mathbb{A}_s(\mathfrak{a}_i)$ denotes the strong adjacent vertices of a vertex $\mathfrak{a}_i$

$\mathbb{A}_w(\mathfrak{a}_i)$ = $\{x\ \epsilon\ \mathbb{A}(\mathfrak{a}_i) \ni \{d(y) < d(x)\}$ where $\mathbb{A}_w(\mathfrak{a}_i)$denotes the weak adjacent vertices of a vertex $\mathfrak{a}_i$

$\mathbb{A}_{P_{S_T}}(\mathfrak{a}_i)$ = $\{x\ \epsilon\ \mathbb{A}(\mathfrak{a}_i) \ni \{d(y) \geq d(x)\}$ and $\{x\ \epsilon\ \mathbb{A}\ (\mathfrak{a}_i) \ni \{d(y) < d(x)\}$ where $\mathbb{A}_{P_{S_T}}(\mathfrak{a}_i)$ denotes the partially strong adjacent vertices of a vertex $\mathfrak{a}_i$.

3.3 α –Strong Vertex of $\mathcal{SVNG}$

A vertex $\mathfrak{a}_i\ \epsilon\ \zeta$ of an $\mathcal{SVNG}$, $\widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$ is said to be $\alpha -$ strong vertex,, if for every $\mathfrak{a}_i\ \epsilon\ \mathbb{A}(\zeta)$ satisfies $d_s(\mathfrak{a}_i) \geq d(\mathbb{A}_s(\mathfrak{a}_i))$. It is denoted by $\alpha_{sv}(\zeta)$.

3.4 γ –Strong Vertex of $\mathcal{SVNG}$

A vertex $\mathfrak{a}_i \in \zeta$ of an $\mathcal{SVNG}, \widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$ is said to be a $\gamma -$ strong or weak vertex, if for every $\mathfrak{a}_i \epsilon\, \mathbb{A}(\zeta)$ satisfies $d_w(\mathfrak{a}_i) \leq d(\mathbb{A}_w(\mathfrak{a}_i))$. It is denoted by $\gamma_{wv}(\zeta)$.

3.5 $\beta -$ Strong Vertex of $\mathcal{SVNG}$

A vertex that satisfies both $\alpha - strong\ and\ \gamma - strong$ conditions is said to be $a\ \beta - strong\ Vertex$.

It is denoted by $\beta(\zeta)$.

3.5.1 All of Alpha's strong vertices are also weak ones, and vice versa.
3.5.2 All gamma strong vertices are strong silent .

3.6 Neutrosophic vertex order coloring of $\mathcal{SVNG}$

Neutrosophic vertex order colouring of $\widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$ is defined as a family ψ={$\mathfrak{S}_\alpha(\mathcal{V})$, β($\mathcal{V}$), $\mathfrak{W}_\gamma$ ($\mathcal{V}$)} of strong vertices on a set $\mathcal{V}_{\mathcal{G}}$ if

i)∪ ψ = ζ

ii)∩ ψ = ϕ

iii)If u, v∈ $\mathfrak{S}_\alpha(\mathcal{V})$ (or) u, v ∈ β($\mathcal{V}$)for each (μ(u, v) > 0) of G, then c(u)≠c(v), where c(u)is the colour of u

iv)For every (μ(u, v) > 0) of G, if u, v ∈ $\mathfrak{W}_\gamma$ ($\mathcal{V}$), then c(u)=c(v) and μ(u, v) = 0

v)If u, v ∈ ξ, then c(u)=c(v) and for every (μ(u, v) = 0) of G.

The chromatic number of neutrosophic graph $\widetilde{\mathcal{N}_{\mathcal{G}}}$, χ^{τ^*} ($\widetilde{\mathcal{N}_{\mathcal{G}}}$) is the least number k for which a k -fuzzy vertex order colouring exists.

3.7 Illustration

Figure 1 is an example of a neutrosophic graph $\widetilde{\mathcal{N}_{\mathcal{G}}}$, where $\widetilde{\mathcal{N}_{\mathcal{G}}}$=($\mathcal{V}_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}}$) =(8,13) in the below example.

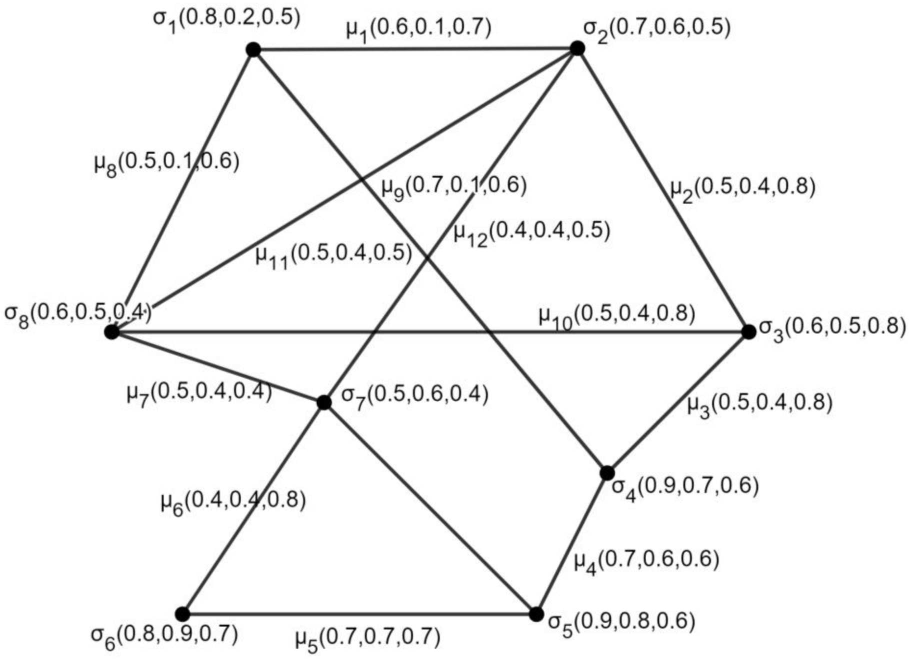

Figure 1: $\widetilde{\mathcal{N}}_{\mathcal{G}}$

Determine the degree of each vertex by definition 2.8. Sort vertices into α, β, and γ strong categories based on the criteria listed in definitions 3.2, 3.3, and 3.4. The degree and category of vertices are displayed in table 1. Use the fuzzy vertex order coloring approach to color the vertices so that no two neighboring vertices have the same color applied to them. Section 6 provides an algorithm for coloring $\mathcal{NG}$'s.

Table 1: Degree and category of vertices

Vertices	Degree of Vertices	Category
$d_{T,I,F}(\sigma_1)$	1.87	β Strong
$d_{T,I,F}(\sigma_2)$	2.06	α Strong
$d_{T,I,F}(\sigma_3)$	1.29	γ Strong
$d_{T,I,F}(\sigma_4)$	1.6	β Strong
$d_{T,I,F}(\sigma_5)$	1.4	β Strong
$d_{T,I,F}(\sigma_6)$	0.83	γ Strong
$d_{T,I,F}(\sigma_7)$	1.87	β Strong

$d_{T,I,F}(\sigma_8)$	2.06	α $Strong$

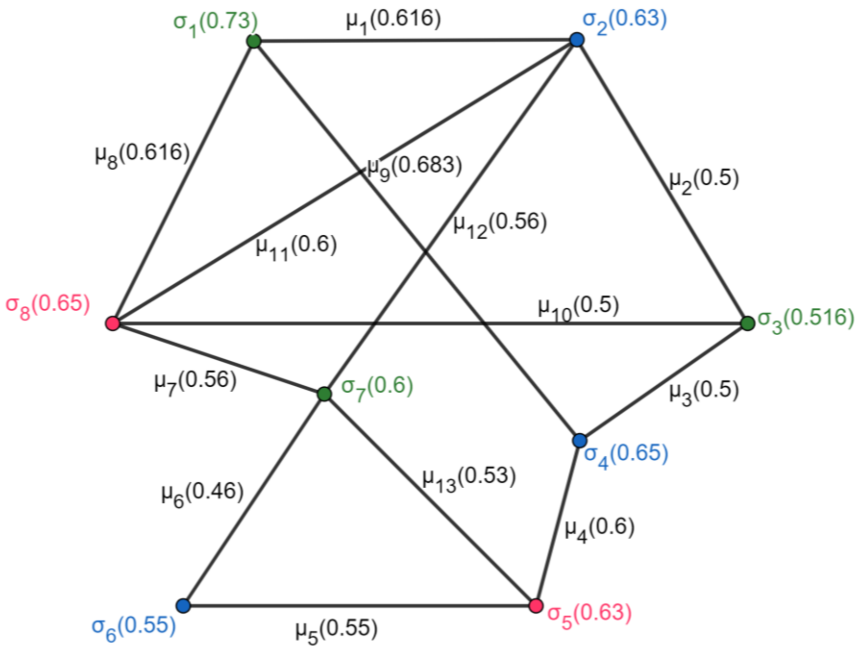

Figure 2: Vertex order coloring of $\widetilde{\mathcal{N}_{\mathcal{G}}}$

Vertex Category	Identification of color
α strong vertex	$\sigma_2 - 1,\ \sigma_8 - 3$
β strong vertex	$\sigma_1 - 2, \sigma_4 - 1, \sigma_5 - 3, \sigma_7 - 2$
γ strong vertex	$\sigma_3 - 2, \sigma_6 - 1$

For Figure 2, the chromatic number (χ^{τ^*}) is observed to be 3.

4. Results and discussion

The most recent developments in Neutrosophic vertex order coloring have been discussed in this section, along with several new theorems that broaden the scope of our research. These new theorems improve our theoretical knowledge of Neutrosophic vertex order coloring and present us with fresh avenues for practical applications in a variety of fields.

Theorem 4.1

Let $\widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$ be a regular neutrosophic graph. Then All the vertices of $\mathcal{NG}$ are α-strong vertices

Proof:

A neutrosophic graph $\widetilde{\mathcal{N}}_{\mathcal{G}}:(\mathcal{V}_{\mathcal{G}},\zeta,\omega)$ is regular if every vertex $\mathfrak{a}_i \in \mathcal{V}$ has the same degree. The degree $d(\mathfrak{a}_i)$ is the sum of the truth membership values of the edges incident to $\mathfrak{a}_i$

$$d(\mathfrak{a}_i)=\sum_{\mathfrak{a}_i \in \mathcal{V}}[\sigma((v,u))]$$

Since $\widetilde{\mathcal{N}}_{\mathcal{G}}$ is regular, for every vertex $\mathfrak{a}_i \in \mathcal{V}_{\mathcal{G}}$, we have $d(\mathfrak{a}_i) = d$.

As a result of their regularity, the edges incident to each vertex's total truth membership values must be equal to d. Each vertex $\mathfrak{a}_i$ has a uniformly high total of truth membership values for its incident edges, according to this consistent degree.

The truth membership values of the edges incident to σ_i must be sufficiently high for a vertex $\mathfrak{a}_i$ to maintain a degree d. All vertices $\mathfrak{a}_i$ in$\mathcal{V}_{\mathcal{G}}$ are α-strong since every vertex in a regular $\mathcal{NG}$ has low indeterminacy and falsity values and high truth membership values.

Therefore, All the vertices of $\mathcal{NG}$ are α-strong vertices if it is a regular neutrosophic graph.

Theorem 4.2

A complete $\mathcal{NG}$, $\widetilde{\mathcal{N}}_{\mathcal{G}}:(\mathcal{V}_{\mathcal{G}},\zeta,\omega)$must have at least one pair of α-strong vertices whose degrees are same.

Proof: Every vertex in a complete $\mathcal{NG}$ with n vertices is connected to every other vertex. As a result, every vertex $\mathfrak{a}_i$ has $n-1$ edges incident with it.

The degree $d(\mathfrak{a}_i)$ is the sum of the truth membership values of the edges incident to $\mathfrak{a}_i$

$$d(\mathfrak{a}_i)=\sum_{\mathfrak{a}_i \in \mathcal{V}}[\sigma((v,u))].$$

The degrees are constrained within a specified range due to the total connectivity of the graph and the limited range of truth membership values. Therefore, if we distribute the degrees across n vertices, there must be at least one pair of vertices with the same degree, according to the pigeonhole principle.

Therefore, A complete $\mathcal{NG}$, $\widetilde{\mathcal{N}}_{\mathcal{G}}:(\mathcal{V}_{\mathcal{G}},\zeta,\omega)$have at least one pair of α-strong vertices whose degrees are same.

Theorem 4.3

A vertex u is γ-strong in a neutrosophic graph $\widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$ if there is at least one distinct weakest arc among the edges incident to u.

Proof:

Consider of a vertex $\mathfrak{a}_i$ in the neutrosophic network $\widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$.

The set $\mathcal{E}_u = \{ e_i = (u, v_i) | v_i \in V \}$ represents the edges incident to u. The truth membership values of these edges are provided by $\sigma_{\mathcal{T}}(e_i)$.

In order for u to be γ-strong, there needs to be an edge $e_{min} = (u, v_{min}) \in \mathcal{E}_u$

such that $\sigma(e_{min})$ is the smallest truth membership value among all the edges in $\mathcal{E}_u$, and this value is unique, (i.e) no other edge in $\mathcal{E}_u$ shares this minimum value. This condition can be mathematically represented as follows: $\sigma(e_{min}) < \sigma(e) \ \forall\ e \in \mathcal{E}_u,\ e \neq e_{min}$.

The definition of a γ-strong vertex is satisfied by u if it has at least one unique weakest arc among its incident edges, which is indicated by the existence of the edge e_{min}. In contrast to other vertices that might not have such a unique weakest arc, this condition guarantees that the vertex u is distinct in having a uniquely minimal truth membership value for one of its incident edges. Therefore, the neutrosophic graph $\widetilde{\mathcal{N}_{\mathcal{G}}}'s$ unique weakest arc verifies that u is a γ-strong vertex.

Theorem 4.4

The degree of every γ-strong node in a $\mathcal{NG}, \widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$ with a crisp graph underlying $\mathcal{C}_{\mathcal{G}}$ that is regular is smaller than the minimum degree of any α-strong node $\mathfrak{a}_i$

$[d_{\gamma}(\mathfrak{v}) < \min\ d_{\alpha}(\mathfrak{v}_i)]$, where $i \in Z^+$.

Proof: Let $\mathcal{NG}$, $\widetilde{\mathcal{N}_{\mathcal{G}}}: (\mathcal{V}_{\mathcal{G}}, \zeta, \omega)$ be a neutrosophic graph. The underlying crisp graph $\mathcal{C}_{\mathcal{G}}$, is regular, (i.e) each vertex in $\mathcal{C}_{\mathcal{G}}$ has an equal number of edges. A vertex $\mathfrak{a}_i$ is considered γ-strong in $\mathcal{NG}$ if it is connected to at least one edge with a significantly lower truth membership value (i.e) there must be at least one unique weakest arc among the edges incident to σ_i. The total degree $d_{\gamma}(\mathfrak{a}_i)$ of $\mathfrak{a}_i$ is decreased by this particular weakest arc because a vertex's degree is determined by the total truth membership values of its incident edges. Conversely, a vertex $\mathfrak{a}_i$ with high truth membership values ($\mathcal{T}$) and low indeterminacy ($\mathcal{I}$) and falsity ($\mathcal{F}$) is said to be α-strong because of constant and comparatively high truth membership values of its incident edges, resulting in a higher degree d_α ($\mathfrak{a}_i$).The degrees are uniform since $\mathcal{C}_\mathcal{G}$ is regular. However, the γ-strong condition lowers the truth membership value of γ-strong vertices, decreasing their degree of comparison in α-strong vertices. Because the α-strong vertices consistently maintain higher truth membership values across their incident edges, for any γ-strong vertex $\mathfrak{a}_i$, its degree $d_\gamma(\mathfrak{a}_i)$ is less than the minimum degree $\min(d_\alpha(\mathfrak{a}_i))$ of any α-strong vertex $\mathfrak{a}_i$. This is because α-strong vertices consistently maintain higher truth membership values across their incident edges.

Integrating other advanced techniques, such as machine learning models, could help improve scalability and efficiency, particularly in large networks. Some managerial implications include network optimization, resource allocation and in healthcare systems. This approach can significantly enhance patient referral systems by optimizing the flow between different healthcare providers, ultimately reducing waiting times and improving service efficiency.

5. Study of α, β and γ strong vertices in Neutrosophic graph Operations

Using vertex and edge cardinality (definitions 2.6), we can find the membership values for the $\mathcal{NG}$ represented in Figure 2. Determine each vertex's degree by definition 2.5, Sort vertices into α, β, and γ strong categories based on the criteria listed in definitions 3.3, 3.4, and 3.5. Use the neutrosophic vertex order coloring approach to color the vertices so that no two neighboring vertices have the same color applied to them. This section deals with an innovative technique for coloring $\mathcal{NG}$ networks.

5.1 α, β and γ strong vertices of Co-normal product

Consider two single-valued Neutrosophic graphs $\mathbb{N}_{\mathbb{G}_1}' = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathbb{N}_{\mathbb{G}_2}' = (\mathcal{V}_2, \mathcal{E}_2)$ whose $\mathbb{G}_{\mathcal{C}_{\mathcal{N}_1}} = (\mathcal{X}_1, \mathcal{Y}_1)$ and $\mathbb{G}_{\mathcal{C}_{\mathcal{N}_2}} = (\mathcal{X}_2, \mathcal{Y}_2)$ respectively.

Then the Co-normal product $\mathbb{N}_{\mathbb{G}_1}' \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2}' = (\mathcal{X}_1 \boxplus_{\mathcal{CN}} \mathcal{X}_2, \mathcal{Y}_1 \boxplus_{\mathcal{CN}} \mathcal{Y}_2)$ is defined as

i. $\forall\ (x, y) \in \mathcal{V}_1 \times \mathcal{V}_2$,

$\mathcal{T}_{x_1} \boxplus_{\mathcal{CN}} \mathcal{T}_{x_2} (u_1, u_2) = \mathcal{T}_{x_1} (u_1) \wedge \mathcal{T}_{x_2} (u_2)$;

$\mathcal{I}_{x_1} \boxplus_{\mathcal{CN}} \mathcal{I}_{x_2} (u_1, u_2) = \mathcal{I}_{x_1} (u_1) \wedge \mathcal{I}_{x_2} (u_2)$ and

$\mathcal{F}_{x_1} \boxplus_{\mathcal{CN}} \mathcal{F}_{x_2}(u_1, u_2) = \mathcal{F}_{x_1} (u_1) \vee \mathcal{F}_{x_2} (u_2)$

ii. $\mathcal{T}_{y}((u_1, u_2)(v_1, v_2)) = (\mathcal{T}_{y_1}(u_1 v_1) \wedge \mathcal{T}_{x_2} (v_1, v_2))$;

$\mathcal{I}_{y}((u_1, u_2)(v_1, v_2)) = (\mathcal{I}_{y_1}(u_1 v_1) \wedge \mathcal{I}_{x_2} (v_1, v_2))$;

$\mathcal{F}_{y}((u_1, u_2)(v_1, v_2)) = (\mathcal{F}_{x_1}(u_1 v_1) \vee \mathcal{F}_{y_2} (v_1, v_2))$; $\forall\ u_1 = u_2$ and $v_1, v_2 \in \mathcal{X}_2$ where

$u_1, u_2 \in \mathcal{X}_1$ *and* $v_1, v_2 \in \mathcal{X}_2$

iii. $\mathcal{T}_{y}((u_1, u_2)(v_1, v_2)) = (\mathcal{T}_{y_1}(u_1 v_1) \wedge \mathcal{T}_{x_2} (v_1, v_2))$;

$\mathcal{I}_{y}((u_1, u_2)(v_1, v_2)) = (\mathcal{I}_{y_1}(u_1 v_1) \wedge \mathcal{I}_{x_2} (v_1, v_2))$;

$\mathcal{F}_{y}((u_1, u_2)(v_1, v_2)) = (\mathcal{F}_{x_1}(u_1 v_1) \vee \mathcal{F}_{y_2} (v_1, v_2))$; $\forall\ u_1, u_2 \in \mathcal{X}_1$ and $v_1 = v_2$

where $u_1, u_2 \in \mathcal{X}_1$ *and* $v_1, v_2 \in \mathcal{X}_2$

The construction of the Co-normal product network $\mathbb{N}_{\mathbb{G}_1'} \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2'}$ is shown in the following steps.

Step 1: Construct the co-normal $\mathcal{SVNG}$ product network, $\mathbb{N}_{\mathbb{G}_1'} \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2'}$(figure 5) of two $\mathcal{NG}$ (Figure 3,4).

Step 2: The truth, indeterminancy and falsity membership values, makes up each vertex. The values are assigned in accordance with Table 2 using Vertex cardinality (definition 2.6).

Step 3: The values are assigned in accordance with Table 3 by using edge cardinality (definition 2.6).

Step 4: Determine each vertex's adjacent vertices and degree for each vertex. Table 4 provides the degree of each vertex in Figure 5.

Step 5: Sort the α, β, and γ strong vertices according to the conditions given in definition 3.3,3.4 and 3.5.

Figure 3 and 4 are the examples of neutrosophic graphs where $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_1}'}$=(3,3) and $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_2}'}$=(4,5) respectively.

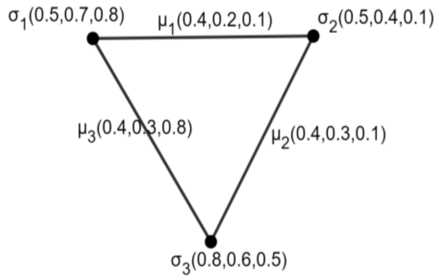

Figure 3: $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_1}}'$

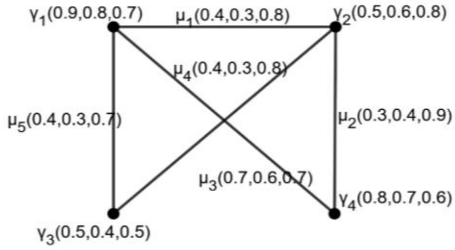

Figure 4: $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_2}}'$

Figure 5 is the neutrosophic network, $\mathbb{N}_{\mathbb{G}_1}' \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2}'$ which is driven by applying co-normal product between $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_1}}'$ and $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_2}}'$.

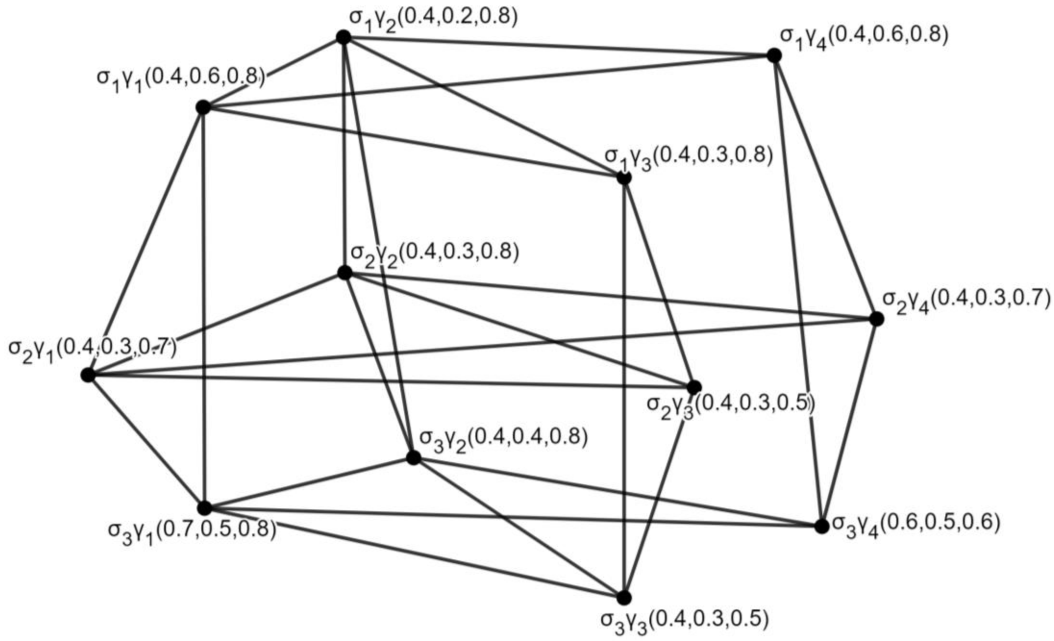

Figure 5: $\mathbb{N}_{\mathbb{G}_1}' \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2}'$

Table 2: Membership values of the vertices of $\mathbb{N}_{\mathbb{G}_1}' \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2}'$

Vertices	Membership Values	Vertex Cardinality $\lvert\mathbb{E}\rvert$

$\sigma(\sigma_1\gamma_1)$	(0.4,0.6,0.8)	0.33
$\sigma(\sigma_1\gamma_2)$	(0.4.0.2,0.8)	0.46
$\sigma(\sigma_1\gamma_3)$	(0.4,0.3,0.8)	0.43
$\sigma(\sigma_1\gamma_4)$	(0.4.0.6,0.8)	0.33
$\sigma(\sigma_2\gamma_1)$	(0.4,0.3,0.7)	0.46
$\sigma(\sigma_2\gamma_2)$	(0.4,0.3,0.8)	0.43
$\sigma(\sigma_2\gamma_3)$	(0.4,0.3,0.5)	0.53
$\sigma(\sigma_2\gamma_4)$	(0.4,0.3,0.7)	0.46
$\sigma(\sigma_3\gamma_1)$	(0.7,0.5,0.8)	0.46
$\sigma(\sigma_3\gamma_2)$	(0.4,0.4,0.8)	0.4
$\sigma(\sigma_3\gamma_3)$	(0.4,0.3,0.5)	0.53
$\sigma(\sigma_3\gamma_4)$	(0.6,0.5,0.6)	0.5

Table 3: Membership values of the edges of

$\mathbb{N}_{\mathbb{G}_1'} \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2'}$

Edges	Membership Values	Edge Cardinality $\lvert\mathbb{E}\rvert$	Edges	Membership Values	Edge Cardinality $\lvert\mathbb{E}\rvert$
$\mu(\sigma_1\gamma_1\sigma_2\gamma_1)$	(0.3,0.1,0.7)	0.5	$\mu(\sigma_1\gamma_4\sigma_2\gamma_4)$	(0.3,0.1,0.6)	0.53
$\mu(\sigma_1\gamma_1\sigma_3\gamma_1)$	(0.3,0.2,0.7)	0.43	$\mu(\sigma_1\gamma_4\sigma_3\gamma_4)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_1\sigma_1\gamma_2)$	(0.3,0.1,0.7)	0.43	$\mu(\sigma_2\gamma_1\sigma_3\gamma_1)$	(0.3,0.2,0.7)	0.43
$\mu(\sigma_1\gamma_1\sigma_1\gamma_3)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_1\sigma_2\gamma_2)$	(0.3,0.2,0.8)	0.43

$\mu(\sigma_1\gamma_1\sigma_1\gamma_4)$	(0.4,0.5,0.8)	0.36	$\mu(\sigma_2\gamma_1\sigma_2\gamma_3)$	(0.3,0.2,0.7)	0.46
$\mu(\sigma_1\gamma_2\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_1\sigma_2\gamma_4)$	(0.4,0.3,0.7)	0.46
$\mu(\sigma_1\gamma_2\sigma_1\gamma_3)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_2\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_2\sigma_1\gamma_4)$	(0.2,0.3,0.9)	0.33	$\mu(\sigma_2\gamma_2\sigma_2\gamma_3)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_3\sigma_2\gamma_3)$	(0.3,0.1,0.5)	0.56	$\mu(\sigma_2\gamma_2\sigma_2\gamma_4)$	(0.2,0.3,0.9)	0.33
$\mu(\sigma_1\gamma_3\sigma_3\gamma_3)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_3\sigma_3\gamma_3)$	(0.3,0.2,0.5)	0.53
$\mu(\sigma_1\gamma_2\sigma_2\gamma_2)$	(0.3,0.1,0.8)	0.46	$\mu(\sigma_2\gamma_4\sigma_3\gamma_4)$	(0.3,0.2,0.6)	0.5
$\mu(\sigma_3\gamma_1\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.53	$\mu(\sigma_3\gamma_1\sigma_3\gamma_3)$	(0.3,0.2,0.7)	0.46
$\mu(\sigma_3\gamma_2\sigma_3\gamma_3)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_3\gamma_1\sigma_3\gamma_4)$	(0.5,0.4,0.7)	0.46
$\mu(\sigma_3\gamma_2\sigma_3\gamma_4)$	(0.2,0.3,0.9)	0.33			

Table 4: List of α strong, β strong, and γ strong vertices of $\mathbb{N}_{\mathbb{G}_1'} \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2'}$

Vertex	Adjacent vertices	Degree	α strong	β strong	γ strong
$\sigma_1\gamma_1$	$\sigma_2\gamma_1,\sigma_3\gamma_1,\sigma_1\gamma_2,\sigma_1\gamma_3,\sigma_1\gamma_4$	2.15	-	$\sigma_1\gamma_1$	-
$\sigma_1\gamma_2$	$\sigma_1\gamma_1,\sigma_2\gamma_2,\sigma_3\gamma_2,\sigma_1\gamma_3,\sigma_1\gamma_4$	2.08	-	$\sigma_1\gamma_2$	-
$\sigma_1\gamma_3$	$\sigma_1\gamma_1,\sigma_1\gamma_2,\sigma_2\gamma_3,\sigma_3\gamma_3$	1.85	-	$\sigma_1\gamma_3$	-
$\sigma_1\gamma_4$	$\sigma_1\gamma_1,\sigma_1\gamma_2,\sigma_2\gamma_4,\sigma_3\gamma_4$	1.65	-	-	$\sigma_1\gamma_4$
$\sigma_2\gamma_1$	$\sigma_1\gamma_1,\sigma_3\gamma_1,\sigma_2\gamma_2,\sigma_2\gamma_3,\sigma_2\gamma_4$	2.31	$\sigma_2\gamma_1$	-	-
$\sigma_2\gamma_2$	$\sigma_1\gamma_2,\sigma_3\gamma_2,\sigma_2\gamma_1,\sigma_2\gamma_3,\sigma_2\gamma_4$	2.08	-	$\sigma_2\gamma_2$	-
$\sigma_2\gamma_3$	$\sigma_2\gamma_1,\sigma_2\gamma_2,\sigma_1\gamma_3,\sigma_3\gamma_3$	1.98	-	$\sigma_2\gamma_3$	-
$\sigma_2\gamma_4$	$\sigma_2\gamma_1,\sigma_2\gamma_2,\sigma_1\gamma_4,\sigma_3\gamma_4$	1.82	-	$\sigma_2\gamma_4$	-

$\sigma_3\gamma_1$	$\sigma_1\gamma_1, \sigma_2\gamma_1, \sigma_3\gamma_2, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.24	-	$\sigma_3\gamma_1$	-
$\sigma_3\gamma_2$	$\sigma_1\gamma_2, \sigma_2\gamma_2, \sigma_3\gamma_1, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.15	-	$\sigma_3\gamma_2$	-
$\sigma_3\gamma_3$	$\sigma_1\gamma_3, \sigma_2\gamma_3, \sigma_3\gamma_1, \sigma_3\gamma_2$	1.85	-	$\sigma_3\gamma_3$	-
$\sigma_3\gamma_4$	$\sigma_1\gamma_4, \sigma_2\gamma_4, \sigma_3\gamma_1, \sigma_3\gamma_2$	1.72	-	$\sigma_3\gamma_4$	-

Table 5: Color identification of α, β, and γ strong vertices

Vertex	Identification of Color
α strong vertex	$\sigma_2\gamma_1 - 2$
β strong vertex	$\sigma_1\gamma_1 - 1, \sigma_1\gamma_2 - 2, \sigma_2\gamma_2 - 1, \sigma_3\gamma_2 - 3, \sigma_3\gamma_1$- 4, $\sigma_1\gamma_3 - 4, \sigma_2\gamma_3 - 3,$ $\sigma_3\gamma_3 - 2, \sigma_2\gamma_4 - 4, \sigma_3\gamma_4 - 2$
γ strong vertex	$\sigma_1\gamma_4 - 3$

The minimum spanning tree of $\mathbb{N}_{\mathbb{G}_1}' \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2}'$ is computed using kruskal's algorithm and its weights is observed as $\mathcal{MST}_{\mathcal{W}_0}(\mathbb{N}_{\mathbb{G}_1}' \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2}') = 4.36$

From Table 7, The chromatic number for $\mathbb{N}_{\mathbb{G}_1}' \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2}'$ is given by, $\chi^{\tau^*}(\mathbb{N}_{\mathbb{G}_1}' \boxplus_{\mathcal{CN}} \mathbb{N}_{\mathbb{G}_2}') = 4$

5.2 α, β *and* γ strong vertices of Tensor product

Assume that the single-valued neutrosophic graphs of $\mathbb{G}_{\mathcal{T}_1} = (\mathcal{X}_1, \mathcal{Y}_1)$ and $\mathbb{G}_{\mathcal{T}_2} = (\mathcal{X}_2, \mathcal{Y}_2)$ are $\mathbb{N}_{\mathbb{G}_1}' = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathbb{N}_{\mathbb{G}_2}' = (\mathcal{V}_2, \mathcal{E}_2)$ are respectively. A pair $(\mathcal{X}, \mathcal{Y})$ is considered to be the tensor product $\mathbb{N}_{\mathbb{G}_1}' \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2}'$ if:

i. $\mathcal{T}_{\mathcal{X}}(u_1, u_2) = \mathcal{T}_{x_1}(u_1) \wedge \mathcal{T}_{x_2}(u_2)$;

$\mathcal{I}_{\mathcal{X}}(u_1, u_2) = \mathcal{T}_{x_1}(u_1) \wedge \mathcal{T}_{x_2}(u_2)$;

$\mathcal{F}_{\mathcal{X}}(u_1, u_2) = \mathcal{T}_{x_1}(u_1) \vee \mathcal{T}_{x_2}(u_2) \ \forall \ (u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$

ii. $\mathcal{T}_{\mathcal{Y}}((u_1, u_2)(v_1, v_2)) = \mathcal{T}_{\mathcal{Y}_1}(u_1 v_1) \wedge \mathcal{T}_{x_2}(v_1, v_2)$;

$\mathcal{I}_{\mathcal{Y}}((u_1, u_2)(v_1, v_2)) = \mathcal{I}_{\mathcal{Y}_1}(u_1 v_1) \wedge \mathcal{I}_{\mathcal{X}_2}(v_1, v_2);$

$\mathcal{F}_{\mathcal{Y}}((u_1, u_2)(v_1, v_2)) = \mathcal{F}_{\mathcal{X}_1}(u_1 v_1) \vee \mathcal{F}_{\mathcal{Y}_2}(v_1, v_2); \ \forall\ u_1 u_2 \in \mathcal{X}_1$ and $v_1, v_2 \in \mathcal{X}_2$

The construction of the tensor product network $\mathbb{N}_{\mathbb{G}_1}' \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2}'$ is shown in the following steps.

Step 1: Construct the tensor $\mathcal{SVNG}$ product network, $\mathbb{N}_{\mathbb{G}_1}' \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2}'$(figure 6) of two $\mathcal{NG}$ (Figure 3,4).

Step 2: Three truth, indeterminancy and falsity membership values, make up each vertex. The values are assigned in accordance with Table 6 using Vertex cardinality (definition 2.6).

Step 3: The values are assigned in accordance with Table 7 by using edge cardinality (definition 2.7).

Step 4: Determine each vertex's adjacent vertices and degree for each vertex. Table 8 provides the degree of each vertex in Figure 6.

Step 5: Sort the α, β, and γ strong vertices according to the conditions given in definition 3.3,3.4 and 3.5.

Figure 6 is the neutrosophic network, $\mathbb{N}_{\mathbb{G}_1}' \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2}'$ which is driven by applying tensor product between $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_1}}'$ and $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_2}}'$.

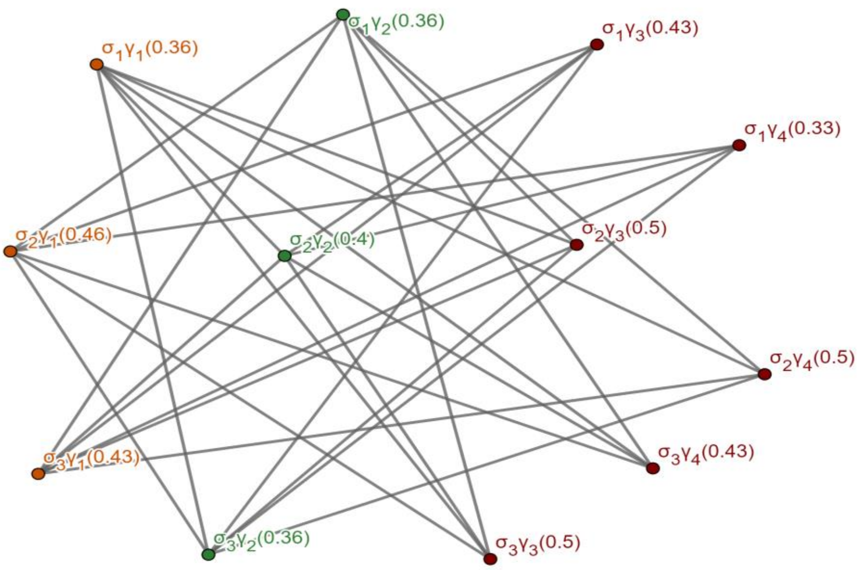

Figure 6: $\mathbb{N}_{\mathbb{G}_1}' \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2}'$

Table 6: Membership values of the vertices of $\mathbb{N}_{\mathbb{G}_1}' \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2}'$

Vertices	Membership Values	Vertex Cardinality $\lvert\mathbb{E}\rvert$

$\sigma(\sigma_1\gamma_1)$	(0.4,0.5,0.8)	0.36
$\sigma(\sigma_1\gamma_2)$	(0.3.0.4,0.8)	0.36
$\sigma(\sigma_1\gamma_3)$	(0.3,0.2,0.8)	0.43
$\sigma(\sigma_1\gamma_4)$	(0.3.0.5,0.8	0.33
$\sigma(\sigma_2\gamma_1)$	(0.3,0.2,0.7)	0.46
$\sigma(\sigma_2\gamma_2)$	(0.3,0.3,0.8)	0.4
$\sigma(\sigma_2\gamma_3)$	(0.3,0.3,0.5)	0.5
$\sigma(\sigma_2\gamma_4)$	(0.3,0.2.0.6)	0.5
$\sigma(\sigma_3\gamma_1)$	(0.5,0.5,0.7)	0.43
$\sigma(\sigma_3\gamma_2)$	(0.3,0.4,0.8)	0.36
$\sigma(\sigma_3\gamma_3)$	(0.3,0.3,0.5)	0.5
$\sigma(\sigma_3\gamma_4)$	(0.4,0.5,0.6)	0.43

Table 7: Membership values of the edges of

$\mathbb{N}_{\mathbb{G}_1'} \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2'}$

Edges	Membership Values	Edge Cardinality $\lvert\mathbb{E}\rvert$	Edges	Membership Values	Edge Cardinality $\lvert\mathbb{E}\rvert$
$\mu(\sigma_1\gamma_1\sigma_2\gamma_2)$	(0.3,0.1,0.8)	0.46	$\mu(\sigma_1\gamma_4\sigma_3\gamma_1)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_1\sigma_2\gamma_3)$	(0.2,0.1,0.7)	0.46	$\mu(\sigma_1\gamma_4\sigma_3\gamma_2)$	(0.2,0.2,0.9)	0.36
$\mu(\sigma_1\gamma_1\sigma_2\gamma_4)$	(0.3,0.1,0.7)	0.5	$\mu(\sigma_2\gamma_1\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43

$\mu(\sigma_1\gamma_1\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_1\sigma_3\gamma_3)$	(0.3,0.2,0.7)	0.46
$\mu(\sigma_1\gamma_1\sigma_3\gamma_3)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_1\sigma_3\gamma_4)$	(0.3,0.2,0.7)	0.46
$\mu(\sigma_1\gamma_1\sigma_3\gamma_4)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_2\sigma_3\gamma_1)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_2\sigma_2\gamma_1)$	(0.3,0.1,0.8)	0.46	$\mu(\sigma_2\gamma_2\sigma_3\gamma_3)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_2\sigma_2\gamma_3)$	(0.3,0.1,0.8)	0.46	$\mu(\sigma_2\gamma_2\sigma_3\gamma_4)$	(0.2,0.2,0.9)	0.36
$\mu(\sigma_1\gamma_2\sigma_2\gamma_4)$	(0.2,0.1,0.9)	0.4	$\mu(\sigma_2\gamma_3\sigma_3\gamma_1)$	(0.3,0.2,0.7)	0.46
$\mu(\sigma_1\gamma_2\sigma_3\gamma_1)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_3\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_2\sigma_3\gamma_3)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_4\sigma_3\gamma_1)$	(0.3,0.2,0.7)	0.43
$\mu(\sigma_1\gamma_2\sigma_3\gamma_4)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_4\sigma_3\gamma_2)$	(0.2,0.2,0.9)	0.46
$\mu(\sigma_1\gamma_3\sigma_2\gamma_1)$	(0.3,0.1,0.8)	0.5	$\mu(\sigma_1\gamma_3\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_3\sigma_2\gamma_2)$	(0.3,0.1,0.8)	0.46	$\mu(\sigma_1\gamma_4\sigma_2\gamma_1)$	(0.3,0.1,0.7)	0.43
$\mu(\sigma_1\gamma_3\sigma_3\gamma_1)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_1\gamma_4\sigma_2\gamma_2)$	(0.2,0.1,0.9)	0.4

Table 8: List of α strong, β strong, and γ strong vertices of $\mathbb{N}_{\mathbb{G}_1}' \star_{\mathcal{J}} \mathbb{N}_{\mathbb{G}_2}'$

Vertex	Adjacent vertices	Degree	α strong	β strong	γ strong
$\sigma_1\gamma_1$	$\sigma_2\gamma_2,\sigma_2\gamma_3,\sigma_2\gamma_4,\sigma_3\gamma_2,\sigma_3\gamma_3,\sigma_3\gamma_4$	2.71	$\sigma_1\gamma_1$	-	-
$\sigma_1\gamma_2$	$\sigma_2\gamma_1,\sigma_2\gamma_3,\sigma_2\gamma_4,\sigma_3\gamma_1,\sigma_3\gamma_3,\sigma_3\gamma_4$	2.61	-	$\sigma_1\gamma_2$	-
$\sigma_1\gamma_3$	$\sigma_2\gamma_2,\sigma_2\gamma_1,\sigma_3\gamma_2,\sigma_3\gamma_1$	1.82	-	-	$\sigma_1\gamma_3$

$\sigma_1\gamma_4$	$\sigma_2\gamma_1, \sigma_2\gamma_2, \sigma_3\gamma_1, \sigma_3\gamma_2$	1.62	-	-	$\sigma_1\gamma_4$
$\sigma_2\gamma_1$	$\sigma_1\gamma_2, \sigma_1\gamma_3, \sigma_1\gamma_4, \sigma_3\gamma_2, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.81	$\sigma_2\gamma_1$	-	-
$\sigma_2\gamma_2$	$\sigma_1\gamma_1, \sigma_1\gamma_3, \sigma_1\gamma_4, \sigma_3\gamma_1, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.54	-	$\sigma_2\gamma_2$	-
$\sigma_2\gamma_3$	$\sigma_1\gamma_1, \sigma_1\gamma_2, \sigma_3\gamma_2, \sigma_3\gamma_1$	1.81	-	-	$\sigma_2\gamma_3$
$\sigma_2\gamma_4$	$\sigma_1\gamma_1, \sigma_1\gamma_2, \sigma_3\gamma_2, \sigma_3\gamma_1$	1.72	-	-	$\sigma_2\gamma_4$
$\sigma_3\gamma_1$	$\sigma_1\gamma_2, \sigma_1\gamma_3, \sigma_1\gamma_4, \sigma_2\gamma_2, \sigma_2\gamma_3, \sigma_2\gamma_4$	2.64	$\sigma_3\gamma_1$	-	-
$\sigma_3\gamma_2$	$\sigma_1\gamma_1, \sigma_1\gamma_3, \sigma_1\gamma_4, \sigma_3\gamma_1, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.44	-	$\sigma_3\gamma_2$	-
$\sigma_3\gamma_3$	$\sigma_2\gamma_2, \sigma_2\gamma_1, \sigma_1\gamma_1, \sigma_1\gamma_2$	1.75	-	-	$\sigma_3\gamma_3$
$\sigma_3\gamma_4$	$\sigma_2\gamma_2, \sigma_2\gamma_1, \sigma_1\gamma_1, \sigma_1\gamma_2$	1.68	-	-	$\sigma_3\gamma_4$

Table 9: Color identification of α, β, and γ strong vertices

Vertex	Identification of Color
α strong vertex	$\sigma_1\gamma_1 - 1, \sigma_2\gamma_1 - 1, \sigma_3\gamma_1$-1
β strong vertex	$\sigma_1\gamma_2 - 2, \sigma_2\gamma_2 - 2, \sigma_3\gamma_2 - 2$
γ strong vertex	$\sigma_1\gamma_3 - 3, \sigma_2\gamma_3 - 3, \sigma_3\gamma_3 - 3, \sigma_1\gamma_4 - 3, \sigma_2\gamma_4 - 3, \sigma_3\gamma_4 - 3$

The minimum spanning tree of $\mathbb{N}_{\mathbb{G}_1'} \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2'}$ is computed using kruskal's algorithm and its weights is observed as $\mathcal{MST}_{\mathcal{W}_0}(\mathbb{N}_{\mathbb{G}_1'} \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2'}) = 4.53$

From Table 7, The chromatic number for $\mathbb{N}_{\mathbb{G}_1'} \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2'}$ is given by, $\chi^{\tau^*}(\mathbb{N}_{\mathbb{G}_1'} \star_{\mathcal{T}} \mathbb{N}_{\mathbb{G}_2'}) = 3$

5.3 α, β *and* γ strong vertices of Residue product [15]

Consider two single-valued Neutrosophic networks of the graphs $\mathbb{N}_{\mathbb{G}_1'} = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathbb{N}_{\mathbb{G}_2'} = (\mathcal{V}_2, \mathcal{E}_2)$ with $\mathbb{G}_{\mathcal{R}_{\mathcal{P}_1}} = (\mathcal{X}_1, \mathcal{Y}_1)$ and $\mathbb{G}_{\mathcal{R}_{\mathcal{P}_2}} = (\mathcal{X}_2, \mathcal{Y}_2)$ respectively. Then the Residue product $\mathbb{N}_{\mathbb{G}_1'} \diamond_{\mathcal{R}_{\mathcal{P}}} \mathbb{N}_{\mathbb{G}_2'} = (\mathcal{X}_1 \diamond_{\mathcal{R}_{\mathcal{P}}} \mathcal{X}_2, \mathcal{Y}_1 \diamond_{\mathcal{R}_{\mathcal{P}}} \mathcal{Y}_2)$ is defined as

i) $\forall\ (x, y) \in \mathcal{V}_1 \times \mathcal{V}_2$,

$\mathcal{T}_{x_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathcal{T}_{x_2}\ (u_1,\ u_2) = (\mathcal{T}_{x_1}\ (u_1) \wedge\ \mathcal{T}_{x_2}\ (u_2)$;

$\mathcal{I}_{x_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathcal{I}_{x_2}\ (u_1,\ u_2) = \mathcal{I}_{x_1}\ (u_1) \wedge \mathcal{I}_{x_2}\ (u_2)$ and

$\mathcal{F}_{x_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathcal{F}_{x_2}\ (u_1,\ u_2) = \mathcal{F}_{x_1}\ (u_1) \vee \mathcal{F}_{x_2}\ (u_2)$,

ii) $\forall\ (x, y) \in \mathcal{E}_1$ and $z \neq w \in \mathcal{V}_2$,

$(\mathcal{T}_{y_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathcal{T}_{y_2})\ ((x,\ z), (y,\ w)) = \mathcal{T}_{y_1}\ (x, y)$;

$(\mathcal{I}_{y_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathcal{I}_{y_2}\ )\ ((x,\ z), (y,\ w)) = \mathcal{I}_{y_1}\ (x, y)$ and

$(\mathcal{F}_{y_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathcal{F}_{y_2})\ ((x,\ z), (y,\ w)) = \mathcal{F}_{y_1}\ (x, y)$.

The construction of the residue product network $\mathbb{N}_{\mathbb{G}'_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathbb{N}_{\mathbb{G}'_2}$ is shown in the following steps.

Step 1: Construct the residue $\mathcal{SVNG}$ product network, $\mathbb{N}_{\mathbb{G}'_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathbb{N}_{\mathbb{G}'_2}$(figure 7) of two $\mathcal{NG}$ (Figure 3,4).

Step 2: The truth, indeterminancy and falsity membership values, make up each vertex. Step 3: The values are assigned in accordance with Table 10 by using edge cardinality (definition 2.7).

Step 4: Determine each vertex's adjacent vertices and degree for each vertex. Table 8 provides the degree of each vertex in Figure 7.

Step 5: Sort the α, β, and γ strong vertices according to the conditions given in definition 3.3,3.4 and 3.5.

Figure 7 is the neutrosophic network, $\mathbb{N}_{\mathbb{G}'_1} \diamond_{\mathcal{R}_\mathcal{P}} \mathbb{N}_{\mathbb{G}'_2}$ which is driven by applying residue product between $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_1}'}$ and $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_2}'}$.

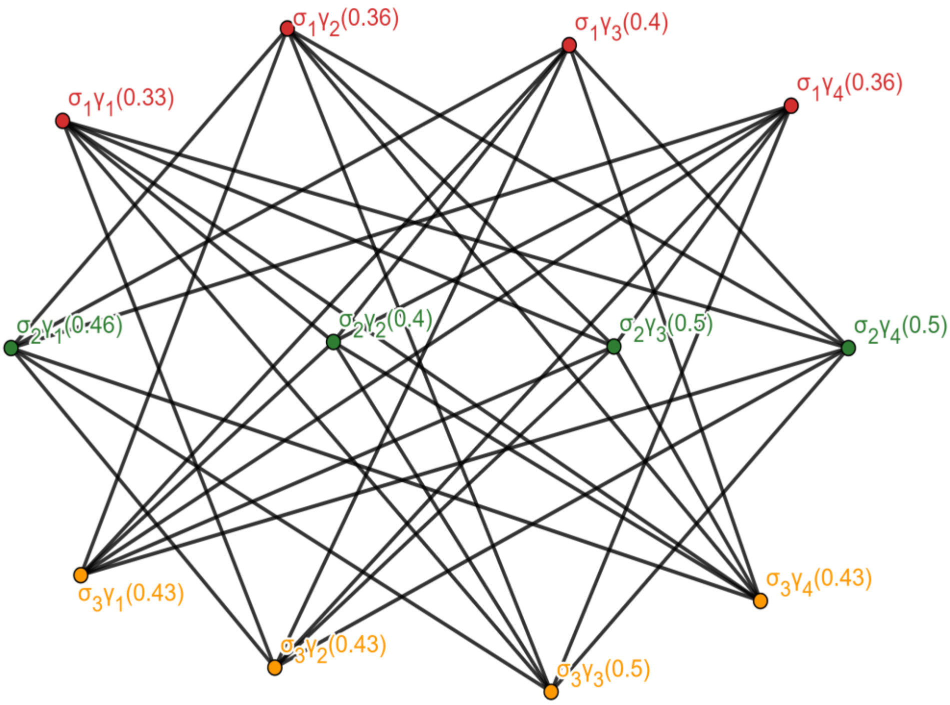

Figure 7: $\mathbb{N}_{\mathbb{G}_1}' \diamond_{\mathcal{R}_{\mathcal{P}}} \mathbb{N}_{\mathbb{G}_2}'$

Table 10: Membership values of the edges of $\mathbb{N}_{\mathbb{G}_1}' \diamond_{\mathcal{R}_{\mathcal{P}}} \mathbb{N}_{\mathbb{G}_2}'$

Edges	Membership Values	Edge Cardinality $\lvert\mathbb{E}\rvert$	Edges	Membership Values	Edge Cardinality $\lvert\mathbb{E}\rvert$
$\mu(\sigma_1\gamma_1\sigma_2\gamma_2)$	(0.3,0.1,0.8)	0.43	$\mu(\sigma_1\gamma_4\sigma_3\gamma_1)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_1\sigma_2\gamma_3)$	(0.3,0.1,0.7)	0.5	$\mu(\sigma_1\gamma_4\sigma_3\gamma_2)$	(0.2,0.2,0.9)	0.36
$\mu(\sigma_1\gamma_1\sigma_2\gamma_4)$	(0.3,0.1,0.7)	0.5	$\mu(\sigma_2\gamma_1\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_1\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_1\sigma_3\gamma_3)$	(0.3,0.2,0.7)	0.46
$\mu(\sigma_1\gamma_1\sigma_3\gamma_3)$	(0.3,0.2,0.8)	0.46	$\mu(\sigma_2\gamma_1\sigma_3\gamma_4)$	(0.3,0.2,0.7)	0.46
$\mu(\sigma_1\gamma_1\sigma_3\gamma_4)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_2\sigma_3\gamma_1)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_2\sigma_2\gamma_1)$	(0.3,0.1,0.8)	0.46	$\mu(\sigma_2\gamma_2\sigma_3\gamma_3)$	(0.3,0.2,0.8)	0.43

$\mu(\sigma_1\gamma_2\sigma_2\gamma_3)$	(0.3,0.1,0.8)	0.46	$\mu(\sigma_2\gamma_2\sigma_3\gamma_4)$	(0.2,0.2,0.9)	0.36
$\mu(\sigma_1\gamma_2\sigma_2\gamma_4)$	(0.2,0.1,0.9)	0.4	$\mu(\sigma_2\gamma_3\sigma_3\gamma_1)$	(0.3,0.2,0.7)	0.46
$\mu(\sigma_1\gamma_2\sigma_3\gamma_1)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_3\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_2\sigma_3\gamma_3)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_2\gamma_4\sigma_3\gamma_1)$	(0.3,0.2,0.7)	0.43
$\mu(\sigma_1\gamma_2\sigma_3\gamma_4)$	(0.3,0.1,0.8)	0.46	$\mu(\sigma_2\gamma_4\sigma_3\gamma_2)$	(0.2,0.2,0.9)	0.46
$\mu(\sigma_1\gamma_3\sigma_2\gamma_1)$	(0.3,0.1,0.7)	0.5	$\mu(\sigma_1\gamma_3\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43
$\mu(\sigma_1\gamma_3\sigma_2\gamma_2)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_1\gamma_4\sigma_2\gamma_1)$	(0.3,0.1,0.7)	0.43
$\mu(\sigma_1\gamma_3\sigma_3\gamma_1)$	(0.3,0.2,0.8)	0.46	$\mu(\sigma_1\gamma_4\sigma_2\gamma_2)$	(0.2,0.1,0.9)	0.4
$\mu(\sigma_1\gamma_3\sigma_2\gamma_4)$	(0.4,0.2,0.1)	0.7	$\mu(\sigma_1\gamma_3\sigma_3\gamma_4)$	(0.4,0.3,0.8)	0.43
$\mu(\sigma_2\gamma_3\sigma_3\gamma_2)$	(0.3,0.2,0.8)	0.43	$\mu(\sigma_1\gamma_4\sigma_2\gamma_3)$	(0.4,0.2,0.1)	0.7
$\mu(\sigma_2\gamma_4\sigma_3\gamma_3)$	(0.4,0.3,0.1)	0.66	$\mu(\sigma_2\gamma_3\sigma_3\gamma_4)$	(0.4,0.3,0.1)	0.66

Table 11: List of α strong, β strong, and γ strong vertices of $\mathbb{N}_{\mathbb{G}_1}' \diamond_{\mathcal{R}_\mathcal{P}} \mathbb{N}_{\mathbb{G}_2}'$

Vertex	Adjacent vertices	Degree	α strong	β strong	γ strong
$\sigma_1\gamma_1$	$\sigma_2\gamma_2, \sigma_2\gamma_3, \sigma_2\gamma_4, \sigma_3\gamma_2, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.75	-	$\sigma_1\gamma_1$	-
$\sigma_1\gamma_2$	$\sigma_2\gamma_1, \sigma_2\gamma_3, \sigma_2\gamma_4, \sigma_3\gamma_1, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.64	-	-	$\sigma_1\gamma_2$
$\sigma_1\gamma_3$	$\sigma_2\gamma_1, \sigma_2\gamma_2, \sigma_2\gamma_4, \sigma_3\gamma_1, \sigma_3\gamma_2, \sigma_3\gamma_4$	2.95	-	$\sigma_1\gamma_3$	-
$\sigma_1\gamma_4$	$\sigma_2\gamma_1, \sigma_2\gamma_2, \sigma_2\gamma_3, \sigma_3\gamma_1, \sigma_3\gamma_2, \sigma_3\gamma_3$	2.82	-	$\sigma_1\gamma_4$	-
$\sigma_2\gamma_1$	$\sigma_1\gamma_2, \sigma_1\gamma_3, \sigma_1\gamma_4, \sigma_3\gamma_2, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.81	-	$\sigma_2\gamma_1$	-
$\sigma_2\gamma_2$	$\sigma_1\gamma_1, \sigma_1\gamma_3, \sigma_1\gamma_4, \sigma_3\gamma_1, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.48	-	-	$\sigma_2\gamma_2$
$\sigma_2\gamma_3$	$\sigma_1\gamma_1, \sigma_1\gamma_2, \sigma_1\gamma_4, \sigma_3\gamma_1, \sigma_3\gamma_2, \sigma_3\gamma_4$	2.91	$\sigma_2\gamma_3$	-	-

$\sigma_2\gamma_4$	$\sigma_1\gamma_1, \sigma_1\gamma_2, \sigma_1\gamma_3, \sigma_3\gamma_1, \sigma_3\gamma_2, \sigma_3\gamma_3$	3.08	$\sigma_2\gamma_4$	-	-
$\sigma_3\gamma_1$	$\sigma_1\gamma_2, \sigma_1\gamma_3, \sigma_1\gamma_4, \sigma_2\gamma_2, \sigma_2\gamma_3, \sigma_2\gamma_4$	2.67		$\sigma_3\gamma_1$	-
$\sigma_3\gamma_2$	$\sigma_1\gamma_1, \sigma_1\gamma_3, \sigma_1\gamma_4, \sigma_3\gamma_1, \sigma_3\gamma_3, \sigma_3\gamma_4$	2.44	-	-	$\sigma_3\gamma_2$
$\sigma_3\gamma_3$	$\sigma_1\gamma_1, \sigma_1\gamma_2, \sigma_1\gamma_4, \sigma_2\gamma_1, \sigma_2\gamma_2, \sigma_2\gamma_4$	2.87	-	$\sigma_3\gamma_3$	-
$\sigma_3\gamma_4$	$\sigma_1\gamma_1, \sigma_1\gamma_2, \sigma_1\gamma_3, \sigma_2\gamma_1, \sigma_2\gamma_2, \sigma_2\gamma_3$	2.8	-	$\sigma_3\gamma_4$	-

Table 12: Color identification of α, β, and γ strong vertices

Vertex	Identification of Color
α strong vertex	$\sigma_2\gamma_3 - 2$, $\sigma_2\gamma_4 - 2$
β strong vertex	$\sigma_1\gamma_1 - 1, \sigma_1\gamma_3 - 1, \sigma_1\gamma_4 - 1, \sigma_2\gamma_1 - 2, \sigma_3\gamma_1 - 3, \sigma_3\gamma_3 - 3, \sigma_3\gamma_4 - 3$
γ strong vertex	$\sigma_1\gamma_2 - 1, \sigma_2\gamma_2 - 2, \sigma_3\gamma_2 - 3$

The minimum spanning tree of $\mathbb{N}_{\mathbb{G}_1}' \diamond_{\mathcal{R}_{\mathcal{P}}} \mathbb{N}_{\mathbb{G}_2}'$ is computed using kruskal's algorithm and its weights is observed as $\mathcal{MST}_{\mathcal{W}_0}(\mathbb{N}_{\mathbb{G}_1}' \diamond_{\mathcal{R}_{\mathcal{P}}} \mathbb{N}_{\mathbb{G}_2}')$ = 4.56

From Table 7, The chromatic number for $\mathbb{N}_{\mathbb{G}_1}' \diamond_{\mathcal{R}_{\mathcal{P}}} \mathbb{N}_{\mathbb{G}_2}'$ is given by, $\chi^{\tau^*}(\mathbb{N}_{\mathbb{G}_1}' \diamond_{\mathcal{R}_{\mathcal{P}}} \mathbb{N}_{\mathbb{G}_2}')$ = 3

6. Applications of Neutrosophic product network- vertex order coloring

The fuzzy graph theory discipline has made great strides with the introduction of the NVOC method, which provides an advanced tool for network structure optimization under uncertainty. In this section, we have presented the algorithm for NVOC, a technique used to assign colors to vertices while accommodating uncertainty and vagueness. The NVOC algorithm makes use of the special qualities of neutrosophic graphs to offer practical recommendations for a variety of real-world applications, enhancing the effectiveness, dependability, and resilience of network systems.

Input: Neutrosophic graphs $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_1}}'$ and $\mathbb{N}_{\mathbb{F}_{\mathbb{G}_2}}'$

Output: Optimal Neutrosophic network $\mathcal{O}^*_{\mathcal{N}_{\mathcal{F}_i}}$.

Step 1: using several kinds of network operations, construct a connected network $\widetilde{\mathcal{N}_{\mathcal{G}}}$, with the vertex set $\mathcal{V}_1 \times \mathcal{V}_2$

Step 2: Select the relevant operations and ascertain the effective edges in the $\mathcal{NG}$ $\widetilde{\mathcal{N}_{\mathcal{G}}}$ with n vertices.

Step 3: Determine the adjacent edges and make sure they are effective edges in order to construct an effective network. The condition to verify the effectiveness of the edge is given as follows.

$$\mu(uv) = \frac{1}{2}[\sigma(u) \wedge \sigma(v)] \leq \mu(uv).$$

Construct a new network with effective edges and eliminate the non-effective ones.

Step 4: Next, get the degree of each vertex using the formula $d(\sigma_i)= \sum_{i=1}^{n} d(\mathcal{A}(\sigma_{i.}))$

Step 5: Based on the criteria listed in definitions 3.2, 3.3, and 3.4, evaluate the properties of vertices and categorize them as α strong, β strong, or γ strong vertices.

Step 6: Determine an m-coloring of the vertices in the $\mathcal{NG}$. Let $\{\sigma_1, \sigma_2, \ldots \sigma_n\}$ represent the vertices of $\widetilde{\mathcal{N}_{\mathcal{G}}}$: ($\mathcal{V}_{\mathcal{G}}$,$\zeta$,$\omega$), and let {1, 2,..., m} represent the vertex colors in Z+.

Step 7: From $\mathbb{Z}^+$ to $\alpha_{str}(\mathcal{V})$, assign colors 1 to j.

Step 8: Assign each strong vertex a unique color that corresponds to $\alpha_{str}(\mathcal{V})$ if they are adjacent. If they are not adjacent, then assign the same color to each vertex.

Step 9: Assign color values j+1 to k for each $\beta(\mathcal{V})$, and then repeat step 7 for all $\beta(\mathcal{V})$.

Step 10: Give each vertex of γ_{str} $(\mathcal{V})$ the same value from [$\mathcal{K}$+1,m]. An edge that connects two γ strong vertices should be removed. If not, apply the same color to each vertex that is a member of $\gamma_{str}(\mathcal{V})$.

Step 11: For every product network, the weight of the minimum spanning tree ($\mathcal{MST}_{\mathcal{W}_0}$) has been calculated in order to evaluate the effective product graph. The chromatic number for each selected network can be obtained using χ^{τ^*}.

Step 12: ($|\alpha_{str}|$) provides the cardinality of α strong vertices, and ($\mathcal{W}_0$ (α_{str})) provides the weight of α strong vertices. The analysis of the most optimal network among the many networks will be done using the summation ($\mathcal{S}^*_{\mathcal{ON}_i}$) of all these parameters.

Step 13: To determine the optimal Neutrosophic product network ($\mathcal{O}^*_{\mathcal{N}_{\mathcal{F}_i}}$), the smallest value of $\mathcal{S}^*_{\mathcal{ON}_i}$, where i=1,2,...,n. Therefore, $\mathcal{O}^*_{\mathcal{N}_{\mathcal{F}_i}} = \min\{\mathcal{S}^*_{\mathcal{ON}_i}\}$

From Section 5.1, 5.2, 5.3, we arrived at the following,

Let $\mathbb{G}_{\mathcal{C}_\mathcal{N}}$ be the Co-normal product of Neutrosophic network and its sum of the values of χ^{τ^*}, $|\alpha_{str}|$, $\mathcal{W}_0$ (α_{str}) and $\mathcal{W}(\mathcal{MST}_{\mathcal{W}_0})$ say $\mathcal{S}^*_{\mathcal{ON}_1}$=11.67.

Let $\mathbb{G}_{\mathcal{T}}$ be the Tensor product of Neutrosophic network and its sum of the values of χ^{τ^*}, $|\alpha_{str}|$, $\mathcal{W}_0$ (α_{str}) and $\mathcal{W}(\mathcal{MST}_{\mathcal{W}_0})$ say $\mathcal{S}^*_{\mathcal{ON}_2}$=18.69.

Let $\mathbb{G}_{\mathcal{R}_\mathcal{P}}$ be the Residue product of Neutrosophic network and its sum of the values of χ^{τ^*}, $|\alpha_{str}|$, $\mathcal{W}_0$ (α_{str}) and $\mathcal{W}(\mathcal{MST}_{\mathcal{W}_0})$ say $\mathcal{S}^*_{\mathcal{ON}_3}$=15.55.

Table 13: Comparison of optimal values of neutrosophic product networks

Operations on $\mathcal{NG}$'s	$\lvert\alpha_{str}\rvert$	$\mathcal{W}_0$ (α_{str})	$\mathcal{W}(\mathcal{MST}_{\mathcal{W}_0})$	χ^{τ^*}	$\mathcal{S}^*_{\mathcal{ON}_i}$
Co- normal product	1	2.31	4.36	4	11.67
Tensor product	3	8.16	4.53	3	18.69
Residue product	2	5.99	4.56	3	15.55

The four elements that make up the ideal network are the identification of various node types, examination of the network's effective nodes, assessment of the effectiveness level, and calculation of the chromatic number. The Co-normal Neutrosophic product network is the most optimal network, according to Table 13. The proposed method effectively handles network structures using neutrosophic graph product operations and vertex order coloring, particularly in dealing with uncertainty and indeterminacy. However, the limitations may include scalability issues and applications scope. The complexity of neutrosophic graphs can increase significantly with the size of the network, which may lead to computational challenges in larger datasets. While neutrosophic graphs are well-suited for uncertain data, the method may not be optimal for networks where certainty and crisp values dominate.

7. Conclusion

Network optimization can be effectively achieved by examining several graph product operations, such as co-normal, tensor, and residue products, through vertex order coloring on neutrosophic graphs. With this approach, we could analyze network stability, performance, and relationship while taking into consideration the inherent uncertainties of real-world systems. This research improves the ability to design more resilient, dependable, and efficient networks by discovering ideal configurations. The healthcare industry is one of the areas where this idea has the greatest influence. Neutrosophic graph theory optimization of patient referral networks can result in notable enhancements in the provision of healthcare services. Through efficient management of patient movement across several healthcare providers and institutions, this methodology guarantees prompt access to medical care. This strategy guarantees prompt access to medical care, lowers patient wait times, and maximizes the use of medical resources by efficiently regulating the flow of patients between different healthcare providers and facilities. Such developments show the significant societal benefits of this research by improving patient outcomes as well as the general sustainability and efficiency of healthcare systems. Future studies could explore the application of neutrosophic graph coloring in diverse areas such as financial networks, transportation logistics, and supply chain management. Development of more efficient algorithms to handle large-scale neutrosophic graphs and to improve computational performance is another important future direction.

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