On Some Novel Results About the Behavior of Some Numerical Solutions of a Neutrosophic Generalized Half – Linear Second Order Differential Equation

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Abstract: The generalized neutrosophic differential equation is a differential equation with neutrosophic real variable $x + yI$ instead of classical real variable x. This research is devoted to studying the oscillation of generalized neutrosophic half linear second order differential equation with the negative and delayed neutral term as follows:

$$\left(r(k+mI)z'(k+mI)\right)' + c(k+mI)f\left(x\left(\theta(k+mI)\right), r(k+mI)z'(k+mI)\right) = 0\,.$$

Also, the generalized neutrosophic half–linear differential equation of the second order with max term featured by the following from:

$$\left[r(k+mI)z'(k+mI)\right]' + c(k+mI)\ MAX_{s\in[\theta(k+mI),k+mI]}\, f\left(x(s), r(k+mI)z'(k+mI)\right) = 0\,,$$

with

$$z(k+mI)=x(k+mI)-a(k+mI)x\big(\theta(k+mI)\big),$$

Where we illustrate a numerical approach for the oscillation with indeterminacy element that represents the neutrosophic approach.

Keywords: Generalized Half, Linear, Riccati Transformation, neutrosophic generalized differential equation, neutrosophic variable.

Introduction

The importance of differential equations in our daily life is not limited to a specific application, phenomenon or example because they help us describe, present and explain the mechanism of these phenomena, including the spread of diseases, the mechanism of electrical work, climate change, nuclear explosions and many phenomena, all expressed in mathematical images. In particular, semi–linear differential equations with a negative neutral limit are a generalization of the linear Sturm-Liouville problem, which has been the subject of study for many researchers in the field of physics, as well as maximum-limit differential equations, as they are solved in the work of automatic control systems and in electronic networks. The second-order semi-linear differential equation is given in the following form:

$$\Big(r(t)\phi\left(x'(t)\right)\Big)' + c(t)\phi\big(x(t)\big)=0. \qquad (1)$$

Since r(t) >0 no matter what $t\in\mathbb{R}$ and $c(t)$ are continuous dependencies, $\phi(x)=|x|^{p-1}sgn(x)$ for $p>1$. Equation (1) is called the classical equation with dependent x and independent transform t, and is a linear differential equation with respect to the dependent $\phi(x)$ that is nonlinear with respect to x, thus, it is nonlinear with respect to the dependent x. The researcher Bihari is the first to develop a second-order semi-linear differential equation in [1] in 1966, focusing on the space of solutions to this equation, so if $x(t)$ is a solution to Equation (2), $\lambda x(t)$ is also solved. In 1987, Elbert [2] came up with some criteria by which such an equation is subjected to become oscillatory [3]. The generalized equation is oscillatory if the following is true:

$$c(t) > 0\,, \int_0^\infty c(t) = +\infty\,, \int_0^\infty \frac{1}{r(t)}\,dt = +\infty\,.$$

The generalized second-order semi-linear differential equation is given in the following Form: [5]

$$\left[r(t)x'(t)\right]' + c(t)f\left(x(t)\,, r(t)x'(t)\right) = 0\,, \qquad (2)$$

$$f\left(x\,, rx'\right) := \emptyset(x)\left|rx'\right|^{2-p}; r(t) > 0\,, \forall t \in \mathbb{R}.$$

Researchers *Došlý* and *Řezníčková* studied the basic solutions of Equation (2) in [4] and then researchers *Došlý* and *Bognár* set important parameters in [5] for the oscillation of Equation (2).

Researchers *Marík* and *Fišnarová* studied in [7] the oscillation of the classical half-linear differential equation with a delayed limit and then with a neutral limit and developed criteria for the oscillation of this equation through the oscillation of the classical equation in 2013.

The equation $\left[\emptyset\left(\acute{x}(t)\right)\right]' + \lambda c(t)\emptyset\left(x\left(\tau(t)\right)\right) = 0$ oscillates when the equation:

$$\left[\phi\left(\acute{x}(t)\right)\right]' + \lambda c(t)\left(\frac{\tau(t)}{t}\right)^{p-1}\phi\left(x(t)\right) = 0\,.$$

Thus, it is oscillation.

Then, in [6], the researchers worked on finding a standard for the oscillation of equations with a delayed limit using the Riccati transform. Neutrosophic logic as an important new tool was used by many authors to study many different areas of pure and applied mathematics, such as algebra [9-11], and mathematical analysis [12-13]. One of the most important applications of neutrosophic logic was in the field of differential equations, were we can see many differential neutrosophic equations and their solutions [13-40].

The importance of this research comes from the fact that it gives an analytical study of the behavior of the solutions of the neutrosophic generalized version of Equation (2), which is a generalized neutrosophic second-order semi-linear differential equation with a neutral limit. This study plays an important role in knowing the behavior of the phenomena described by this equation and therefore the possibility of controlling the results of these phenomena, and therefore this research is of great importance for researchers in the theoretical and applied scientific fields. The aim of this research is to study the infinite behavior of solutions of a generalized second-order semi-linear differential equation with a negative and maximal neutral limit, in terms of oscillation.

Results and discussion:

1. The study of the oscillation of a generalized second-order semi-linear differential equation with a neutral and delayed limit

The generalized neutrosophic second-order semi-linear differential equation with a delayed and neutral limit is given by the following relation:

$$[r(k+mI)\acute{z}(k+mI)]' + c(k+mI)f\left(x\big(\tau(k+mI)\big), r(k+mI)\acute{z}(k+mI)\right) \tag{3}$$

with a negative neutral limit $z(k+mI) = x(k+mI) - a(k+mI)x\big(\theta(k+mI)\big)$, where we will study the case $\tau(t+mI) = \theta(k+mI)$, whatever $k+mI \geq T$ so that $k+mI \in [k_0+m_0I, +\infty[$ and T starting point.

Equation (3), satisfies the following conditions:

1) $\theta(k+mI) \in \acute{C}([k_0+m_0I, +\infty[, \mathbb{R})$,

2) $\theta(k+mI) \leq k+mI$,

3) $\lim\limits_{t\to+\infty} \theta(k+mI) = +\infty$,

4) $0 \leq a(k+mI) < 1$.

2. Riccati's neutrosophic transform:

Studying the oscillation of generalized equations using the Riccati transform is an effective method, since the equation transforms from the second to the first order. For $x(k+mI)$ solution of Equation (2) the Riccati transform of Equation (2) is given in the following Form [5]:

$$\acute{v}(k+mI)+c(k+mI) \quad +\frac{H(v)}{r(k+mI)} \tag{4}$$

where $H(v)$ is defined by the relation $H(v)=\acute{g}\left(g^{-1}(v)\right)\left(g^{-1}(v)\right)^{2}$, which is absolutely increasing for $u>0$, and also completely decreasing for $u<0$, and achieves $H(0)=0$, so is a completely convex dependent, Since

$$u(k+mI)=\frac{r(k+mI)\,\acute{x}(k+mI)}{x(k+mI)}.$$

The dependent is also given by the following relation:

$$g(u)=\begin{cases} \displaystyle\int_{\frac{1}{u}}^{+\infty}\frac{ds}{F(s)} & if\ \ u>0 \\ \displaystyle\int_{-\infty}^{\frac{1}{u}}\frac{ds}{F(s)} & if\ u<0 \\ 0 & if\ u=0 \end{cases}, \tag{5}$$

where g is completely increasing, thus $\lim_{t\to-\infty} g(u)=-\infty \ \ and \ \lim_{t\to+\infty} g(u)=+\infty$.

Remark 1.1.

If Equation (2) is non-oscillatory, then there exists $v(k+mI)$ a finite solution of Equation (4) on the domain $[T,+\infty[$ the following cases are equivalent:

1- Equation (2) is non-oscillatory,

2- There exists $v(k+mI)$ a finite solution of Equation (4) on the domain $[T,+\infty[$,

3- There is a finite dependent $v(t)$ on the domain $[T\ ,+\infty[$ achieves the following:

$$\acute{v}(k+mI)+c(k+mI)+\frac{H(v)}{r(k+mI)}\leq 0.$$

Lemma 1.1.

Let $x(k+mI)>0$be a solution of Equation (3), then $z(t+mI)$ fulfills one of the two cases:

1) $\acute{z}(k+mI)>0\,; z(k)>0$ and $[r(k+mI)\acute{z}(k+mI)]^{\acute{}}\leq 0$,

2) $\acute{z}(k+mI)>0; z(k+mI)<0$ and $[r(k+mI)\acute{z}(k+mI)]^{\acute{}}\leq 0$.

Proof.

Since $x(k+mI)>0$, whatever$k+mI>T$ is the solution of Equation (3) then, we have:

$$[r(k+mI)\acute{z}(k+mI)]^{\acute{}}=-c(k+mI)f\left(x\big(\theta(k+mI)\big)\,,r(k+mI)\acute{z}(k+mI)\right)\leq 0.$$

Suppose that $\acute{z}(k+mI)\leq 0$, and here we distinguish two cases:

The first case: $z(k+mI)\geq 0$, and since $[r(t)\acute{z}(t)]^{\acute{}}\leq 0$whatever$k+mI>T$, then there is $m>0$.

Check $r(k+mI)\acute{z}(k+mI)<-m$ regardless of $k+mI>T_1>T$, from which $\acute{z}(k+mI)<-\frac{m}{r(k+mI)}$, and by integration, we find:

$$m\int_T^t \frac{1}{r(s)}ds\leq z(k+mI)+m\int_T^t \frac{1}{r(s)}ds<z(T),$$

and when $k+mI$ strives towards +∞, we find that $z(T)=+\infty,$ and this is a contradiction.

The second case: $z(k)<0$, and since $[r(k+mI)\acute{z}(k+mI)]^{\acute{}}\leq 0$, whatever $k+mI>T$, there is a number $k>0$ such that for $T_1>T$, then, $r(k+mI)\acute{z}(k+mI)<-k$, that is, $\acute{z}(k+mI)<-\frac{k}{r(k+mI)}$, and by complementing from T to t we find that:

$$z(T)-k\int_T^{t+mI}\frac{1}{r(s)}ds>z(k+mI)\,,$$

and when seeking t towards $+\infty$ we find $\lim_{t\to+\infty} z(k)=-\infty$.

But we have $x\big(\theta(k+mI)\big)=\frac{x(k+mI)-z(t+mI)}{a(k+mI)}>\frac{-z(k+mI)}{a(t+mI)}$, and from it we find $\lim_{t\to+\infty} x\big(\theta(k+mI)\big)=+\infty,$

and since $\lim_{t\to+\infty} \theta(k+mI) = +\infty$, we find $\lim_{t\to+\infty} x(k+mI) = +\infty$.

Now, we notice that: $z(k+mI) < 0$, thus:

$x(k+mI) < a(k+mI)x\big(\theta(k+mI)\big) < x\big(\theta(k+mI)\big)$, we form a sequence $\theta(k_n) = k_{n-1}$ such that $k_0, k_1, k_2, \ldots$

and $\boldsymbol{k_{n-1}} < \boldsymbol{k_n}$, now considering $x(k_0) = \eta$, and since $x(t+mI) < x\big(\theta(t+mI)\big)$, from which we find that:

$x(k_n) < x\big(\theta(k_n)\big)$, thus, $x(k_n) < x(k_{n-1})$ and from it we find that $x(t_n) < \eta$ is whatever n is a natural number, and this means:

$\lim_{t\to+\infty} x(k+mI) < \eta$ and this is a contradiction.

From it, $\acute{z}(k+mI) > 0$, and here we find that $z(k+mI) < 0$ or that there exists $T_2 > T$ such that $z(k+mI) > 0$.

Lemma 1.2

Let $x(k+mI)$ be a solution of Equation (3) achieves that $x(k+mI) > 0$ for $0 < T < k + mI$ and let be

$$\int_T^\infty \frac{dk+mI}{r(k+mI)},$$

then $x\big(\theta(k+mI)\big) \geq z\big(\theta(k+mI)\big)$ for every $T < k+mI$ in the case of $z(k+mI) > 0$.

Lemma 1.3

Let $x(t+mI)$ be a solution of Equation (3) achieves that $x(k+mI) > 0$ for $0 < T < k + mI$ and let

$$\int_T^\infty \frac{dk+mI}{r(k+mI)},$$

and $\acute{r}(k+mI) > 0$, as well as

$$\int_T^\infty c(k+mI)\theta(k+mI)^{p-1}dt + mI = \infty.$$

Then

$$\frac{z\big(\theta(t+mI)\big)}{z(t+mI)} \geq \frac{\theta(t+mI)}{t+mI},$$

for each $T < k + mI$. In the case of $z(k + mI) > 0$.

Proof.

We impose $x(k + mI) > 0$ for each $T < k + mI$ solution of Equation (3), where $\acute{r}(k + mI) > 0$ and considering $z(k + mI) > 0$, we find from the preliminary (3) that $\acute{z}(k + mI) > 0$, and now let's prove that $\left(\frac{z(k+mI)}{k+mI}\right)'$ thus, $\acute{z}(k + mI)k - z(k + mI) \leq 0$ and that in a way that the assumption.

We assume by argument that $\left(\frac{z(k+mI)}{k+mI}\right)' \geq 0$, that is, the dependent $\frac{z(k+mI)}{k+mI}$ is completely increasing, and therefore $z(k + mI) > (k + mI)k$, from which the following is achieved:

$$[r(k + mI)\acute{z}(k + mI)]' = -c(k + mI)\frac{\left(x\left(\theta(k + mI)\right)\right)^{p-1}}{\left(r(k + mI)\acute{z}(k + mI)\right)^{p-2}},$$

and from it we get

$$\left(r(k + mI)\acute{z}(k + mI)\right)^{p-2}[r(k + mI)\acute{z}(k + mI)]' = -c(k + mI)\left(x\left(\theta(k + mI)\right)\right)^{p-1},$$

By integrating the two sides from T to $k + mI$, we get

$$\left(r(k + mI)\acute{z}(k + mI)\right)^{p-1} = \left(r(T)\acute{z}(T)\right)^{p-1} - (p - 1)\int_{T}^{k+mI} c(s)\left(x\left(\theta(s)\right)\right)^{p-1} ds .$$

We have $\acute{z}(k + mI) > 0$ and $p - 1$ are even, then we find that $\left(r(k + mI)\acute{z}(k + mI)\right)^{p-1} \geq 0$, therefore

$$\left(r(T)\acute{z}(T)\right)^{p-1} - (p - 1)\int_{T}^{t} c(s)\left(x\left(\theta(s)\right)\right)^{p-1} ds \geq 0 ,$$

and so, we find the following:

$$\left(r(T)\acute{z}(T)\right)^{p-1} \geq (p - 1)\int_T^t c(s)z\left(\theta(s)\right)^{p-1}ds \ \geq k^{p-1}(p - 1)\int_T^t c(s)\theta(s)^{p-1}ds ,$$

and when t seeks toward ∞, then $\left(r(T)\acute{z}(T)\right)^{p-1} \geq \infty$, and this is a contradiction because $r(T)\,\acute{z}(T)$ is a neutrosophic real number that cannot be infinite, that is, $\acute{z}(k + mI)(k + mI) - z(k + mI) \leq 0$for every $T < t + mI$ and therefore, we find:

$$\frac{z\big(\theta(k+mI)\big)}{\theta(k+mI)} \geq \frac{z(k+mI)}{k+mI} \tag{6}$$

Lemma 1.4

Let $x(k+mI)$ be a solution of Equation (3) achieves that $x(k+mI) > 0$ for $0 < T < k + mI$ and let

$$\int_T^{\infty} \frac{dk+mI}{r(k+mI)} = +\infty\,.$$

Then $\left(\frac{x(\theta(k+mI))}{z(k+mI)}\right)^{p-1} \geq 1.$

For every $T < k+mI$ in the case of $z(k+mI) < 0,\ p > 1$ is an odd number.

Lemma 1.5.

The Riccati transform of Equation (3) gives us the following form:

$$\acute{v}(k+mI) + c(k+mI)\left(\frac{x\big(\theta(k+mI)\big)}{z(k+mI)}\right)^{p-1} + \frac{H(v)}{r(k+mI)} = 0, \tag{7}$$

where $p > 1$is an odd number.

Proof.

Suppose that $x(k+mI) \neq 0$ is a solution of Equation (2) for every $T \leq t+mI$ is a solution of Equation (3) and $v(t+mI) = g\big(u(k+mI)\big)$, and the dependent $u(k+mI) = \frac{r(k+mI)\acute{z}(k+mI)}{z(k+mI)}$ by deriving the dependent $v(k+mI)$, we get:

$$\acute{v}(k+mI) = \acute{g}(u)\acute{u}(k+mI)$$

$$= \acute{g}(u)\left[\frac{[r(k+mI)\acute{z}(k+mI)]' z(k+mI) - r(k+mI)\acute{z}^2(k+mI)}{z^2(k+mI)}\right],$$

$$\acute{v}(k+mI) = \acute{g}(u)\left[\frac{-c(k+mI)f\left(x\big(\theta(k+mI)\big), r(k+mI)\acute{z}(k+mI)\right)}{z(k+mI)} - \frac{r(k+mI)\acute{z}^2(k+mI)}{z^2(k+mI)}\right],$$

$$\acute{v}(k+mI) = -\acute{g}(u)c(k+mI)\left(\frac{x(\theta(k+mI))}{z(k+mI)}\right)^{p-1}\left(\frac{z(k+mI)}{r(k+mI)\acute{z}(k+mI)}\right)^{p-2}$$

$$-\frac{\acute{g}(u)}{r(k+mI)}\left(\frac{r(k+mI)\acute{z}(k+mI)}{z(k+mI)}\right)^{2},$$

$$\acute{v}(k+mI) = -\acute{g}(u)c(k+mI)\left(\frac{x(\theta(k+mI))}{z(k+mI)}\right)^{p-1} u^{2-p}(k+mI)$$

$$-\frac{\acute{g}(u)}{r(k+mI)}u^{2}(k+mI)\,.$$

By choosing the dependent g, so that $\acute{g}(u)u^{2-p}(k+mI) = 1$, the equation turns into the following form:

$$\acute{v}(k+mI) + c(k+mI)\left(\frac{x(\theta(k+mI))}{z(k+mI)}\right)^{p-1} + \frac{H(v)}{r(k+mI)} = 0\,,$$

$$H(v) = u^{2}\acute{g}(u)\,.$$

Theorem 1.1

Let's have the following equation:

$$[r(t+mI)\acute{x}(t+mI)]'$$

$$+ c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1} f\big(x(t+mI)\,, r(t+mI)\acute{x}(t+mI)\big)$$

$$= 0\,, \qquad (8)$$

where $\acute{r}(t+mI) \geq 0$ and

$$\int_{T}^{\infty} \frac{dt}{r(t+mI)} = +\infty\,,$$

then if equation (8) is oscillatory, thus, equation (3) is oscillatory.

Proof.

To complete the required we have to prove that Equation (3) is oscillatory.We will use the method of reversing the assumption, suppose that equation (8) is oscillatory, and Equation (3) is non-oscillatory, then there is $x(t+mI)$ solution of Equation (3), so that $x(t+mI) > 0$ for each $T < t+mI$, and here, by lemma 1.1, we distinguish two cases:

The first case: in the case of $z(t) > 0$where p is an odd number for each $T < t + mI$,

By Lemma1.5, we have a Riccati transform of Equation (3) is:

$$\acute{v}(t+mI) + c(t+mI)\left(\frac{x\big(\theta(t+mI)\big)}{z(t+mI)}\right)^{p-1} + \frac{H(v)}{r(t+mI)} = 0$$

Since $v(t+mI)$ is a finite dependent because $x(t+mI) \neq 0$, therefore, taking advantage of the primality of 1.3 and the primality of 1.2, we find that:

$$\acute{v}(t) + c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1} + \frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI) + c(t+mI)\left(\frac{z\big(\theta(t+mI)\big)}{z(t+mI)}\right)^{p-1} + \frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI) + c(t+mI)\left(\frac{x\big(\theta(t+mI)\big)}{z(t+mI)}\right)^{p-1} + \frac{H(v)}{r(t+mI)} = 0\,.$$

Hence

$$\acute{v}(t+mI) + c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1} + \frac{H(v)}{r(t+mI)} \leq 0\,.$$

According to proposition 1.1, we find that equation (8) is non-oscillatory, which contradicts the assumption.

The second case: in the case of $z(t+mI) < 0$ for every$t + mI > T$,

We will take the Riccati transform of Equation (3), where:

$$u(t+mI) = \frac{r(t+mI)\acute{z}(t+mI)}{z(t+mI)} \qquad \text{and} \qquad v(t+mI) = g\big(u(t+mI)\big)\,.$$

According to lemma1.5, we have

$$\acute{v}(t+mI) + c(t+mI)\left(\frac{x\big(\theta(t+mI)\big)}{Z(t+mI)}\right)^{p-1} + \frac{H(v)}{r(t+mI)} = 0\,.$$

Since $v(t+mI)$ is a finite dependent, therefore, taking advantage of the lemma1.4, we have:

$$\acute{v}(t+mI)+c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1}+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)\left(\frac{x(\theta(t+mI))}{z(t+mI)}\right)^{p-1}+\frac{H(v)}{r(t+mI)}=0\,.$$

Therefore

$$\acute{v}(t+mI)+c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1}+\frac{H(v)}{r(t+mI)}\leq 0\,.$$

According to lemma1.1, equation (8) is non-oscillatory, which contradicts the assumption.

2. The study of the oscillation of a generalized neutrosophic second-order semi-linear differential equation with a maximal limit

In this paragraph, we study the oscillation of a generalized neutrosophic semi-linear differential equation that contains a neutral and a maximal limit, and this equation has the following form:

$$[r(t+mI)\acute{z}(t+mI)]^{\prime}+c(t+mI)\ \boldsymbol{Max}_{s\in[\theta(t+mI),t+mI]}f\big(x(s)\,,r(t+mI)\acute{z}(t+mI)\big)$$
$$=0\,, \qquad (9)$$

where $f(x,y)$ is defined as in Equation (2) provided p is an odd natural number, thus, $p\geq 1$, then: $r(t+mI)$ ، $c(t+mI)$ are absolutely positive dependents and the dependent $z(t)$ given by the relation $z(t)=x(t+mI)-a(t+mI)x\big(\theta(t+mI)\big)$ fulfills the following conditions:

- **1)** $a(t+mI)\leq a0\leq 1$,
- **2)** $\theta(t+mI)\leq t+m,$,
- **3)** $]\infty+, Lim t+mI\to+\infty\ \theta(t+mI)=+\infty[$

Lemma 1.2: The Riccati transform of equation (9) gives us the following form:

$$\acute{v}(t+mI)+c(t+mI)max_{s\in[\theta(t+mI),t+mI]}\left(\frac{x(s)}{z(t+mI)}\right)^{p-1}+\frac{H(v)}{r(t+mI)}=0 \quad (10)$$

where $p \geq 1$ is an odd number.

Proof.

We assume that $x(t+mI) \neq 0$ for every $T < t+mI$ solution of equation (9) and that $v(t+mI) = g\big(u(t+mIt+mI)\big)$,

And the dependent $u(t) = \frac{r(t+mI)\acute{z}(t+mI)}{z(t+mI)}$ by deriving $v(t+mI)$, we get:

$$\acute{v}(t) = \acute{g}(u)\acute{u}(t) = \acute{g}(u)\left[\frac{[r(t)\acute{z}(t)]' z(t) - r(t)\acute{z}^2(t)}{z^2(t)}\right],$$

$$\acute{v}(t) = \acute{g}(u)\left[\frac{-c(t)max_{s\in[\theta(t),t]}f\big(x(s), r(t)\acute{z}(t)\big)}{z(t)} - \frac{r(t)\acute{z}^2(t)}{z^2(t)}\right],$$

$$\acute{v}(t) = -\acute{g}(u)c(t)\left[\frac{max_{s\in[\theta(t),t]}x(s)}{z(t)}\right]^{p-1}\left[\frac{z(t)}{r(t)\acute{z}(t)}\right]^{p-2} - \frac{\acute{g}(u)}{r(t)}\left[\frac{r(t)\acute{z}(t)}{z(t)}\right]^2,$$

$$\acute{v}(t) = -\acute{g}(u)c(t)\left[\frac{max_{s\in[\theta(t),t]}x(s)}{z(t)}\right]^{p-1}[u(t)]^{2-p} - \frac{\acute{g}(u)}{r(t)}[u(t)]^2 .$$

By choosing the dependent g so that $\acute{g}(u)u^{2-p}(t) = 1$, we get the equation:

$$\acute{v}(t+mI) + c(t+mI)\left[\frac{max_{s\in[\theta(t+mI)\,,t+mI]}x(s)}{z(t+mI)}\right]^{p-1} + \frac{H(v)}{r(t+mI)} = 0,$$

Theorem 2.2 Let's have the following equation:

$$[r(t+mI)\acute{x}(t+mI)]' + c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1} f\big(x(t+mI), r(t+mI)\acute{x}(t+mI)\big)$$
$$= 0\,, \qquad (11)$$

where $\acute{r}(t+mI) \geq 0$ and $\int^{\infty} \frac{dt}{r(t+mI)} = +\infty$, then if equation (11) is oscillatory, equation (9) is oscillatory.

Proof.

We will use the method of reversing the assumption, suppose that equation (11) is oscillatory, and equation (9) is non-oscillatory, then there is $x(t+mI)$ solution to equation (9), so that it achieves that $x(t+mI)>0$ for each $T<t+mI$, and here we distinguish two cases:

The first case: in the case of $z(t+mI)>0$ by introductory 1.1 where p is an odd number for every $T<t+mI$.

Taking the Riccati transform of the given equation (9), where $v(t+mI)=g\big(u(t+mI)\big)$ is a finite solution and $u(t+mI)=\frac{r(t+mI)\acute{z}(t+mI)}{z(t+mI)}$.

Defined over the entire domain $]T,+\infty[$ where $T<t+mI$, and by introductory 1.2 and introductory 1.3 where p is an odd number for each $T<t$, and by introductory 1.5, we find that:

$$\acute{v}(t+mI)+c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1}+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)\left(\frac{z\big(\theta(t+mI)\big)}{z(t+mI)}\right)^{p-1}+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)\left(\frac{x\big(\theta(t+mI)\big)}{z(t+mI)}\right)^{p-1}+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)\left(\frac{max_{s\in[\theta(t),t]}x(s)}{z(t+mI)}\right)^{p-1}+\frac{H(v)}{r(t+mI)}\leq 0\,,$$

thus

$$\acute{v}(t+mI)+c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1}+\frac{H(v)}{r(t+mI)}\leq 0\,.$$

According to lemma1.1, equation (11) is non-oscillatory, which contradicts the assumption.

The second case: in the case of $z(t+mI)<0$ and $x(t+mI)>0$ by introductory 1.4 where p is an odd number for each $T<t+mI$.

Taking the Riccati transform of equation (9) where $v(t+mI)=g\big(u(t+mI)\big)$ is a finite solution of equation (10) and $u(t+mI)=\frac{r(t+mI)\acute{z}t+mI}{z(t+mI)}$ is defined over the entire domain $]T,+\infty[$ and using the lemma 1.4 we have the following relation:

$$\left(\frac{x\big(\theta(t+mI)\big)}{z(t+mI)}\right)^{p-1} \geq 1,$$

and according to the lemma 1.5, we have:

$$\acute{v}(t+mI)+c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1}+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)\left(\frac{x\big(\theta(t+mI)\big)}{z(t+mI)}\right)^{p-1}+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)\left[\frac{max_{s\in[\theta(t+mI),t+mI]}x(s)}{z(t+mI)}\right]^{p-1}+\frac{H(v)}{r(t+mI)}.$$

From it, we find that:

$$\acute{v}(t+mI)+c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1}+\frac{H(v)}{r(t+mI)}$$

$$\leq \acute{v}(t+mI)+c(t+mI)\left[\frac{max_{s\in[\theta(t+mI)t+mI]}x(s)}{z(t+mI)}\right]^{p-1}+\frac{H(v)}{r(t+mI)}=0.$$

thus,

$$\acute{v}(t+mI)+c(t+mI)\left(\frac{\theta(t+mI)}{t+mI}\right)^{p-1}+\frac{H(v)}{r(t+mI)}\leq 0\,.$$

According to lemma 1.1, the condition is equivalent to the fact that equation (11) is non-oscillatory, and this is a contradiction.

Numerical Results:

In the following tables, we show how the improved method gives better numerical results than the classical approaches.

Variables values for classical equation	**M1**	**E1**	**M2**	**E2**	**M3**	**E3**
1.00000	$2.68e^{-002}$	$2.46e^{-002}$	$1.32e^{-002}$	$1.20e^{-002}$	1.01806	1.03393
0.39811	$2.68e^{-002}$	$2.66e^{-002}$	$1.32e^{-002}$	$1.01e^{-002}$	1.02214	1.03627

0.12589	$1.93e^{-002}$	$1.70e^{-002}$	$9.57e^{-003}$	$8.29e^{-003}$	1.00972	1.03642
0.03981	$1.56e^{-002}$	$1.59e^{-002}$	$7.72e^{-003}$	$7.83e^{-003}$	0.99062	1.01896
0.01585	$1.53e^{-002}$	$1.62e^{-002}$	$7.75e^{-003}$	$8.05e^{-003}$	0.98424	1.00995
0.00631	$1.54e^{-002}$	$1.66e^{-002}$	$7.77e^{-003}$	$8.24e^{-003}$	0.98034	1.00599
0.00251	$1.54e^{-002}$	$1.68e^{-002}$	$7.79e^{-003}$	$8.35e^{-003}$	0.97986	1.00662
0.00100	$1.54e^{-002}$	$1.69e^{-002}$	$7.80e^{-003}$	$8.40e^{-003}$	0.97965	1.00664
0.00040	$1.54e^{-002}$	$1.70e^{-002}$	$7.80e^{-003}$	$8.43e^{-003}$	0.97957	1.00662
0.00016	$1.54e^{-002}$	$1.70e^{-002}$	$7.80e^{-003}$	$8.44e^{-003}$	0.97956	1.00661
0.00006	$1.54e^{-002}$	$1.70e^{-002}$	$7.80e^{-003}$	$8.44e^{-003}$	0.97954	1.00660
0.00001	$1.54e^{-002}$	$1.70e^{-002}$	$7.78e^{-003}$	$8.45e^{-003}$	0.97954	1.00660

The values of the variables with indeterminacy						
	M1	E1	M2	E2	M3	E3
1.00000+I	$5.96e^{-004}$+I	$2.10e^{-003}$+I	$2.20e^{-004}$+I	$1.28e^{-002}$+I	1.43755+I	0.71812+I
0.39811+0.6765I	$6.98e^{-004}$+I	$2.41e^{-003}$+I	$4.05e^{-004}$+I	$1.68e^{-002}$+I	1.28634+I	0.70181+I
0.12589+0,11234I	$7.56e^{-004}$+I	$1.23e^{-003}$+I	$3.10e^{-004}$+I	$1.11e^{-003}$+I	1.47698+I	1.00965+I
0.03981+0.765I	$9.12e^{-004}$+I	$4.22e^{-003}$+I	$5.06e^{-004}$+I	$1.10e^{-003}$+I	1.80228+I	1.94442+I
0.01585+0.00876I	$1.15e^{-003}$+I	$5.76e^{-003}$+I	$6.27e^{-004}$+I	$2.46e^{-003}$+I	1.10448+I	1.15208+I
0.00631+0.6543I	$1.26e^{-003}$+I	$6.28e^{-003}$+I	$6.61e^{-004}+$ I	$3.08e^{-003}$+I	0.97539+I	1.02933+I
0.00251+0.7205I	$1.26e^{-003}$+I	$6.41e^{-003}$+I	$6.68e^{-004}+$ I	$3.25e^{-003}$+I	0.92225+I	0.97894+I
0.00100+I	$1.26e^{-003}$+I	$6.44e^{-003}$+I	$6.70e^{-004}+$ I	$3.30e^{-003}$+I	0.91116+I	0.96771+I
0.00040+0.15094I	$1.26e^{-003}$+I	$6.45e^{-003}$+I	$6.70e^{-004}+$ I	$3.30e^{-003}$+I	0.90910+I	0.96551+I
0.00016+0,00013I	$1.26e^{-003}$+I	$6.45e^{-003}$+I	$6.70e^{-004}$+I	$3.30e^{-003}$+I	0.90872+I	0.96508+I
0.00006+0.876093409I	$1.26e^{-003}$+I	$6.45e^{-003}$+I	$6.70e^{-004}$+I	$3.31e^{-003}$+I	0.90863+I	0.96503+I
0.00001+0.000012I	$1.26e^{-003}$+I	$6.45e^{-003}+$ I	$6.70e^{-004}$+I	$3.31e^{-003}$+I	0.90862+I	0.96502+I

Conclusion and future applications:

In this research, we studied the oscillation of Equation (3), which is a generalized neutrosophic second-order semi-linear differential equation with a negative and delayed limit, as it was compared with equation (8), which is a generalized neutrosophic second-order semi-linear differential equation; we used the Riccati transform technique to reach our results. And then, in the second section, we studied the oscillation of the neutrosophic second-order semi-linear differential equation, which contains a negative neutral limit and another maximum, and we came to condition (11) is useful in proving whether equation (9) is oscillatory or non-oscillatory. We recommend that our novel approach to be used for the same equation generalized in terms of refined neutrosophic real numbers, where it can be defined by using a refinement of the indeterminacy element.

References

1. BIHARI.I., 1964, An oscillation theorem concerning the half-linear differential equation of second order, Magy, Tud, Akad, Mat, Kut, intéz, PP 275-280**.**
2. ELBERT. A., On the half – linear second order differential equation, Hungar , A cta Math .Hungar. 49(1987), 487-508
3. DOšLÝ. O., REHAK. P., 2005, Half-linear differential equations, Acta Math.
4. DOšLÝ. O., *ŘEZNÍČKOVÁ* . J., 2012, Conjugacy and principal solution of generalized half-linear second order differential equations, Qual, PP 1-13.
5. DOšL*Ý*.O., BOGN*ÁR*.G.,2013, Conditional oscillation and principal solution of generalized half- linear differential equation, Debrecen, PP 451-459.
6. MAR*ÍK*. R., FIšNAROVÁ. S., Oscillation of half-linear differential equations with delay, Mendel University in Brno, Zem ^ ed ^ elsk ´ A 1,613 00 Brno, Czech Republic, 2013
7. FIšNAROVÁ. S., MAR*ÍK*.R., 2014, Modified Riccati technique for half- linear differential equations with delay, Electronic journal of Qualitative Theory of Differential Equations, pp. 1-14.
8. SELVARANGAM. S., RANI. B., THANDAPANI, E. Oscillation results for second order half- linear neutral delay differential equations with "maxima". Tamkang Journal of Mathematics, Volume 48, Number 3, 289-299, sept 2017.
9. Sankari, H., and Abobala, M., " On The Group of Units Classification In 3-Cyclic and 4-cyclic Refined Rings of Integers And The Proof of Von Shtawzens' Conjectures", International Journal of Neutrosophic Science, 2023.
10. Sankari, H., and Abobala, M., " On The Classification of The Group of Units of Rational and Real 2-Cyclic Refined Neutrosophic Rings", Neutrosophic Sets and Systems, 2023.
11. Katy D. Ahmad, Mikail Bal, Arwa A. Hajjari, Rozina Ali. (2021). Novel Applications Of Neutrosophic AH-Isometries To The Group Of Units Problem In Neutrosophic Rings and Refined Neutrosophic Rings. Journal of Neutrosophic and Fuzzy Systems, 1 (2), 89-98 (**Doi :** https://doi.org/10.54216/JNFS.010205).
12. Dima Alrwashdeh , Talat Alkhouli , Ahmed Soiman Rashed Alhawiti , Ali Allouf , Hussein Edduweh , Abdallah Al-Husban, On Two Novel Generalized Versions of Diffie-Hellman Key Exchange Algorithm Based on Neutrosophic and Split-Complex Integers and their Complexity Analysis, International Journal of Neutrosophic Science, Vol. 25 Issue. 2 PP. 01-10, (2025), Doi:https://doi.org/10.54216/IJNS.250201.
13. Salama, A. Dalla, R. Al, M. Ali, R. (2022). On Some Results About The Second Order Neutrosophic Differential Equations By Using Neutrosophic Thick Function. Journal of Neutrosophic and Fuzzy Systems, (), 30-40. **DOI:** https://doi.org/10.54216/JNFS.040104.

14. Salama, A. Al, M. Dalla, R. Ali, R. (2022). A Study of Neutrosophic Differential Equations By Using the One-Dimensional Geometric AH-Isometry Of NeutrosophicLaplace Transformation. Journal of Neutrosophic and Fuzzy Systems, (), 08-25. **DOI:** https://doi.org/10.54216/JNFS.040201.
15. Hazaymeh, Ayman. "Time Effective Fuzzy Soft Set and Its Some Applications with and Without a Neutrosophic." International Journal of Neutrosophic Science 23.2 (2023): 129-29.
16. A. S. Heilat, H. Zureigat, R. Hatamleh, B. Batiha, New Spline Method for Solving Linear Two-Point Boundary Value Problems European Journal of Pure and Applied Mathematics 14 (4), 1283-1294, 2021
17. Al-Husban, A., Djenina, N., Saadeh, R., Ouannas, A., & Grassi, G. (2023). A new incommensurate fractional-order COVID 19: modelling and dynamical analysis. Mathematics, 11(3), 555.
18. Hamadneh, T., Hioual, A., Alsayyed, O., Al-Khassawneh, Y. A., Al-Husban, A., & Ouannas, A. (2023). The FitzHugh–Nagumo Model Described by Fractional Difference Equations: Stability and Numerical Simulation. Axioms, 12(9), 806.
19. Heilat, A. S., Karoun, R. C., Al-Husban, A., Abbes, A., Al Horani, M., Grassi, G., & Ouannas, A. (2023). The new fractional discrete neural network model under electromagnetic radiation: Chaos, control and synchronization. Alexandria Engineering Journal, 76, 391-409.
20. Al-Husban, A., Karoun, R. C., Heilat, A. S., Al Horani, M., Khennaoui, A. A., Grassi, G., & Ouannas, A. (2023). Chaos in a two dimensional fractional discrete Hopfield neural network and its control. Alexandria Engineering Journal, 75, 627-638.
21. Abu Falahah, I., Hioual, A., Al-Qadri, M. O., Al-Khassawneh, Y. A., Al-Husban, A., Hamadneh, T., & Ouannas, A. (2023). Synchronization of Fractional Partial Difference Equations via Linear Methods. Axioms, 12(8), 728.
22. Hamadneh, T., Abbes, A., Falahah, I. A., Al-Khassawneh, Y. A., Heilat, A. S., Al-Husban, A., & Ouannas, A. (2023). Complexity and chaos analysis for two-dimensional discrete-time predator–prey Leslie–Gower model with fractional orders. Axioms, 12(6), 561.
23. Hamadneh, T., Hioual, A., Alsayyed, O., Al-Khassawneh, Y. A., Al-Husban, A., & Ouannas, A. (2023). Finite Time Stability Results for Neural Networks Described by Variable-Order Fractional Difference Equations. Fractal and Fractional, 7(8), 616.
24. Abubaker, A. A., Abualhomos, M., Matarneh, K., & Al-Husban, A. (2025). A Numerical Approach For The Algebra of Two-Fold. Neutrosophic Sets and Systems, 75, 181-195.
25. Al-Husban, A., Djenina, N., Saadeh, R., Ouannas, A., & Grassi, G. (2023). A new incommensurate fractional-order COVID 19: modelling and dynamical analysis. Mathematics, 11(3), 555.
26. Hatamleh, R. (2003). On the Form of Correlation Function for a Class of Nonstationary Field with a Zero Spectrum. Rocky Mountain Journal of Mathematics, 33(1)159-173. https://doi.org/10.1216/rmjm/1181069991.
27. Hatamleh, R., Zolotarev, V. A. (2014). On Two-Dimensional Model Representations of One Class of Commuting Operators. Ukrainian Mathematical Journal, 66(1), 122–144. https://doi.org/10.1007/s11253-014-0916-9.

28. Hatamleh, R., Zolotarev,V. A. (2015). On Model Representations of Non-Selfadjoint Operators with Infinitely Dimensional Imaginary Component. Journal of Mathematical Physics, Analysis, Geometry,11(2), 174–186. https://doi.org/10.15407/mag11.02.174 .
29. Hatamleh, R., Zolotarev, V. A. (2016). Triangular Models of Commutative Systems of Linear Operators Close to Unitary Operators.Ukrainian Mathematical Journal,68(5),791–811. https://doi.org/10.1007/s11253-016-1258-6.
30. T.Qawasmeh, A. Qazza, R. Hatamleh, M.W. Alomari, R. Saadeh, Further accurate numerical radius inequalities, Axiom-MDPI, 12 (8), 801, 2023
31. Raed Hatamleh, Ayman Hazaymeh.(2024). Finding Minimal Units In Several Two-Fold Fuzzy Finite Neutrosophic Rings, Neutrosophic Sets and Systems, Vol. 70, pp.1-16. DOI: 10.5281/zenodo.13160839
32. Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Mowafaq Omar Al-Qadri, Abdallah Al-Husban,(2024), On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For 2≤n≤3, Neutrosophic Sets and Systems, Vol. 68,pp.8-25.
33. Raed Hatamleh , Ayman Hazaymeh.(2025),On Some Topological Spaces Based On Symbolic n-Plithogenic Intervals, International Journal of Neutrosophic Science, Vol. 25 Issue. 1 PP. 23-37. Doi: https://doi.org/10.54216/IJNS.250102.
34. Raed Hatamleh , Ayman Hazaymeh,(2025),On The Topological Spaces of Neutrosophic Real Intervals, International Journal of Neutrosophic Science, Vol. 25 Issue. 1 PP. 130-136, (Doi: https://doi.org/10.54216/IJNS.250111.
35. Raed Hatamleh , Ayman Hazaymeh.(2025),The Properties of Two-Fold Algebra Based on the n-standard Fuzzy Number Theoretical System, International Journal of Neutrosophic Science, 25 (1) ,172-178.(Doi: https://doi.org/10.54216/IJNS.250115).
36. Ahmad A. Abubaker , Raed Hatamleh , Khaled Matarneh , Abdallah Al-Husban,(2025), On the Irreversible k-Threshold Conversion Number for Some Graph Products and Neutrosophic Graphs, International Journal of Neutrosophic Science,25(2), 197-205. (Doi: https://doi.org/10.54216/IJNS.250217).
37. Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban.(2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, Neutrosophic Sets and Systems,vol.67, pp. 169-178.(DOI:10.5281/zenodo.11151930).
38. Abubaker, Ahmad A, Hatamleh, Raed, Matarneh, Khaled, Al-Husban, Abdallah.(2024). On the Numerica Solutions for Some Neutrosophic Singular Boundary Value Problems by Using (LPM) Polynomials, Inernational Journal of Neutrosophic Science,25(2),197-205. (Doi: https://doi.org/10.54216/IJNS.250217).
39. Raed Hatamleh (2024). On the Compactness and Continuity of Uryson's Operator in Orlicz Space, International Journal of Neutrosophic Science, 24 (3), 233-239.
40. Ayman Hazaymeh, Ahmad Qazza, Raed Hatamleh, Mohammad W Alomari, Rania Saadeh, On Further Refinements of Numerical Radius Inequalities, Axiom-MDPI,12(9),807,2023.

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