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A Concise Study of Some Superhypergraph classes

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Abstract. In graph theory, the hypergraph [22] extends the traditional graph structure by allowing edges to connect multiple vertices, and this concept is further broadened by the superhypergraph [174,176]. Additionally, several types of uncertain graphs have been explored, including fuzzy graphs [136, 153], neutrosophic graphs [35,36], and plithogenic graphs [66,75,185].

This study explores the SuperHyperGraph, Single-Valued Neutrosophic Quasi SuperHyperGraph, and Plithogenic Quasi SuperHyperGraph, analyzing their relationships with other graph classes. Future work will define the Semi Superhypergraph, Multi Superhypergraph, Pseudo Superhypergraph, Mixed Superhypergraph, and Bidirected Superhypergraph and examine their connections to existing classes in hypergraphs and graphs.

Keywords: Neutrosophic graph, Fuzzy graph, Hypergraph, Superhypergraph

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1. Short Introduction

1.1. *Hypergraph and many applications*

Graphs are widely recognized as powerful tools for representing concepts and the relationships between them [57]. Graphs continue to play a significant role in computer science and societal applications, especially in areas such as graph neural networks [14, 39, 60, 76, 121, 123, 163, 211, 212, 214, 226]. In graph theory, various classes of graphs have been developed to address diverse applications and objectives [30].

In graph theory, the concept of a hypergraph [22], which extends the idea of a traditional graph, is well established and further generalized by the concept of a superhypergraph [174, 176]. The hypergraph concept has found applications in fields such as database theory [87, 100] and hypergraph neural networks [44, 94, 117, 122, 206, 213]. Examples in database theory include hypertree and hypergraph databases [13, 90], while in hypergraph neural networks, various applications have emerged, such as GroupNet, Metro, Hypergraph Neural Networks

(HyperNN), RAHGAR, and others. In graph parameters, hypertree-width is a notable metric that has been widely studied, among other research efforts [1, 61, 86, 113, 128, 218].

Directed hypergraphs, a variant where directionality is added to hypergraphs [85, 107, 108, 132, 135], have also been defined and studied in various research areas, including applications in pattern mining [149], Hypernetworks [205], traffic forecasting [124], Directed hypergraph neural network [200, 216], Personalized E-learning [191], and PageRank analysis [199]. Superhypergraphs further generalize hypergraphs by incorporating concepts from supergraph theory, expanding the framework for hypergraphs [68, 174, 176]. The author anticipates that superhypergraphs will inspire extensive research into both their mathematical structures and various applications, similar to the broad interest seen in hypergraphs.

1.2. *Various types of Uncertain Hypergraphs*

Additionally, various types of uncertain graphs have been explored, including fuzzy graphs [136, 153], neutrosophic graphs [35, 36], and plithogenic graphs [66, 75, 185], along with their corresponding hypergraph versions [23]. In these contexts, extensive research has also been conducted on applications such as decision-making [5, 10, 34, 106, 116, 164, 186] and uncertain neural networks [42, 89, 109, 120, 146, 208, 227].

Although some superhypergraph concepts, such as Fuzzy Superhypergraphs [91–93, 175], have been explored, their full mathematical properties and applications remain largely unexplored.

1.3. *Our Contribution in This Short Paper*

The discussion above highlights the mathematical value and significance of research on uncertain graphs, hypergraphs, and their generalizations to superhypergraphs. In this brief paper, we explore superhypergraphs in greater depth, specifically presenting key findings by defining the Intuitionistic Fuzzy Quasi SuperHyperGraph, the Single-Valued Neutrosophic Quasi SuperHyperGraph, and the Plithogenic Quasi SuperHyperGraph, as well as examining their relationships with other graph classes. Future work will involve defining the Semi Superhypergraph, Multi Superhypergraph, Pseudo Superhypergraph, Mixed Superhypergraph, and Bidirected Superhypergraph and investigating their connections to existing classes in hypergraphs and graphs.

2. **Preliminaries and Definitions**

This section provides an overview of the fundamental definitions and notations used throughout the paper. Basic concepts from set theory are also applied in parts of this work; for additional details, please refer to relevant references as needed [63, 95, 97, 102, 115].

2.1. *Basic Graph Concepts*

This subsection explains the basic concepts of graph theory. Below are some of the foundational ideas in this field. For more in-depth information on graph theory and its notations, please refer to [52, 55, 57, 210].

Definition 2.1 (Graph)**.** [57] A *graph* G is a mathematical structure that represents relationships between objects. It consists of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge connects a pair of vertices. Formally, a graph is represented as $G = (V, E)$, where V is the set of vertices and E is the set of edges.

Definition 2.2 (Degree)**.** [57] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is defined as the number of edges connected to v. For undirected graphs, the degree is given by:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

For directed graphs, the *in-degree* $\deg^-(v)$ refers to the number of edges directed towards v, while the *out-degree* $\deg^+(v)$ represents the number of edges directed away from v.

2.2. *Uncertain Graph*

This subsection explains the basic concepts of graph theory. Below are some of the foundational ideas in this field. For more in-depth information on graph theory and its notations, please refer to [52, 55, 57, 210]. As highlighted in the introduction, this paper centers on uncertain graph models, including Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Graphs. Addressing real-world uncertainties requires robust mathematical frameworks, such as Fuzzy Sets [219–225], Soft Sets [131], Picture Fuzzy Sets [45,77,105,159,168,188,207], Spherical Fuzzy Sets [16, 111, 112], Vague Sets [4, 38, 43], Hesitant Fuzzy Sets [197, 198], Rough Sets [143, 144], and Neutrosophic Sets [169–172, 184, 187]. These frameworks are invaluable for analyzing ambiguous or imprecise information. The concept of Fuzzy Graphs and related uncertain graph models can be viewed as extensions of these sets into graph theory, offering versatile tools for managing uncertainty in network structures.

The following sections define these graph concepts. For more on uncertain graphs, recent surveys provide comprehensive insights [67, 69–71, 192].

Definition 2.3 (Uncertain Graphs)**.** (cf. [69]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

(1) *Fuzzy Graph [153]:*

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0,1]$.
- Each edge $e = (u,v) \in E$ is assigned a membership degree $\mu(u,v) \in [0,1]$.

(2) *Intuitionistic Fuzzy Graph (IFG) [47, 134, 195]:*

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0,1]$ (degree of membership) and $\nu_A(v) \in [0,1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
- Each edge $e = (u,v) \in E$ is assigned two values: $\mu_B(u,v) \in [0,1]$ and $\nu_B(u,v) \in [0,1]$, with $\mu_B(u,v) + \nu_B(u,v) \leq 1$.

(3) *Neutrosophic Graph [11, 104]:*

- Each vertex $v \in V$ is assigned a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where $\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0,1]$ and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u,v) \in E$ is assigned a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.

2.3. *Plithogenic Graph*

This subsection provides an overview of the plithogenic graph. A plithogenic graph extends Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Graphs, representing the graphical counterpart of a plithogenic set [64, 167, 173].

A plithogenic set consists of elements with multiple attributes, where each attribute has specific values accompanied by degrees of membership and contradiction (fuzzy, intuitionistic, or neutrosophic) [173]. The formal definition of a plithogenic graph is outlined below [173,190].

Definition 2.4. [173,190] Let $G = (V,E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

(1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

- $M \subseteq V$ is the set of vertices.
- l is an attribute associated with the vertices.
- Ml is the range of possible attribute values.
- $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

(2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

- $N \subseteq E$ is the set of edges.
- m is an attribute associated with the edges.
- Nm is the range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.

- $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

(1) *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf\left((xy), (a, b)\right) \leq \min\{adf(x, a), adf(y, b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

(2) *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf\left((a, b), (c, d)\right) \leq \min\{aCf(a, c), aCf(b, d)\}$$

(3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a, a) &= 0, && \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\ bCf(a, a) &= 0, && \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Example 2.5. (cf. [67]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Graph*.
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Graph*.
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic Graph*.
- When $s = 7, t = 1$, PG is called a *Plithogenic heptapartitioned Neutrosophic Graph*.
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic Graph*.
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic Graph*.

2.4. *Hypergraph and Uncertain Hypergraph*

This subsection provides an overview of Hypergraphs and Uncertain Hypergraphs. A hypergraph is a generalized graph structure that extends traditional graph concepts by allowing hyperedges, which connect multiple vertices rather than just pairs, enabling more complex relationships between elements [19, 22, 87, 88]. Hypergraphs have a wide range of applications, notably in database systems [100]. For additional information, readers are encouraged to consult comprehensive surveys on hypergraphs, such as those in [32, 33, 204].

Definition 2.6 (Hypergraph)**.** [22] A *hypergraph* $H = (V, E)$ consists of a set V of vertices and a set E of hyperedges. Each hyperedge $e \in E$ is defined as a subset of V, thus $e \subseteq V$, and $E \subseteq \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the power set of V.

Extensions of the hypergraph concept, including Fuzzy Hypergraphs [8,23,133], incorporate elements from Fuzzy, Intuitionistic Fuzzy [2,28,51,141], and Neutrosophic theories [6,7,125, 126]. These graphs extend traditional hypergraphs by applying concepts from Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs to create more nuanced hypergraph structures. For more details on specific operations and further theoretical underpinnings, readers are encouraged to consult the cited papers.

Definitions of these extended hypergraph types are provided below. Note that the Single-Valued Neutrosophic Hypergraph generalizes the Intuitionistic Fuzzy Hypergraph, which in turn generalizes the Fuzzy Hypergraph, and the Fuzzy Hypergraph itself generalizes the Hypergraph.

Definition 2.7 (Fuzzy Hypergraph)**.** [8,23,133] A *fuzzy hypergraph* $G = (V, E, \psi, w)$ is a hypergraph where vertices have fuzzy membership degrees in hyperedges, and each hyperedge has an associated weight. The fuzzy hypergraph is defined as follows:

- V is the set of vertices.
- E is the set of hyperedges, where each hyperedge $e \in E$ is a subset of V.
- $\psi \in [0,1]^{|E|\times|V|}$ is a matrix where ψ_{ei} represents the degree of membership of vertex $i \in V$ in hyperedge $e \in E$, satisfying $\sum_{i\in V} \psi_{ei} = 1$ for each $e \in E$ and $\sum_{e\in E} \psi_{ei} > 0$ for each $i \in V$.
- $w : E \to \mathbb{R}_+$ assigns a positive weight $w(e)$ to each hyperedge $e \in E$.

Here, the matrix ψ serves as the incidence matrix of the fuzzy hypergraph, where each hyperedge quantifies the participation of each vertex. The weight function w provides a quantitative measure for the importance or relevance of each hyperedge.

Definition 2.8 (Intuitionistic Fuzzy Hypergraph)**.** [140] An *intuitionistic fuzzy hypergraph* (IFHG) $H = (V, E)$ consists of:

- A finite set $V = \{x_1, x_2, \ldots, x_n\}$ of vertices.
- A family $E = \{E_1, E_2, \ldots, E_m\}$ of intuitionistic fuzzy subsets of V, where each hyperedge E_j is defined as:

  $$E_j = \{(x_i, \mu_j(x_i), \nu_j(x_i)) \mid x_i \in V\},$$

  with $\mu_j(x_i)$ denoting the degree of membership of vertex x_i to the hyperedge E_j, and $\nu_j(x_i)$ denoting the degree of non-membership. It is required that $0 \leq \mu_j(x_i)+\nu_j(x_i) \leq 1$ for all $x_i \in V$ and all $E_j \in E$.

Additionally, it holds that:

$$\bigcup_{j=1}^{m} \text{supp}(E_j) = V,$$

where $\text{supp}(E_j) = \{x_i \in V \mid \mu_j(x_i) > 0 \text{ and } \nu_j(x_i) > 0\}$ is the support of E_j.

Definition 2.9 (Single-Valued Neutrosophic Hypergraph)**.** [12] Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set of vertices, and let $E = \{E_1, E_2, \ldots, E_m\}$ be a collection of single-valued neutrosophic subsets of V, such that:

$$\bigcup_{i=1}^{m} \text{supp}(E_i) = V,$$

where each edge E_i is defined as:

$$E_i = \{(v_j, T_{E_i}(v_j), I_{E_i}(v_j), F_{E_i}(v_j)) \mid v_j \in V\},$$

with $T_{E_i}(v_j), I_{E_i}(v_j), F_{E_i}(v_j)$ representing the truth-membership, indeterminacy-membership, and falsity-membership of vertex v_j to edge E_i respectively. The pair $H = (V, E)$ is called a *single-valued neutrosophic hypergraph* (SVNH).

2.5. *Superhypergraph and Quasi-SuperHyperGraph*

This subsection explains the concepts of the Superhypergraph and Quasi-SuperHyperGraph. The SuperHyperGraph is a generalized graph concept that extends the Hypergraph by introducing supervertices and superedges [174, 175, 177, 178]. This framework includes related structures such as the quasi-SuperHyperGraph [92]. As briefly mentioned in the short introduction of this paper, these structures are further extended using fuzzy relations, leading to the concept known as the Fuzzy SuperHyperGraph [92]. The definitions are provided below.

Definition 2.10 (SuperHyperGraph)**.** [174] Let V be a finite set of vertices. A *superhypergraph* is an ordered pair $H = (V, E)$, where:

- $V \subseteq P(V)$ (the power set of V), meaning that each element of V can be either a single vertex or a subset of vertices (called a *supervertex*).
- $E \subseteq P(V)$ represents the set of edges, called *superedges*, where each $e \in E$ can connect multiple supervertices.

In this framework, a superhypergraph can accommodate complex relationships among groups of vertices, including single edges, hyperedges, superedges, and multi-edges. Superhypergraphs provide a flexible structure to represent high-order and hierarchical relationships.

The following relationship between a superhypergraph and a hypergraph is well-known and clearly holds.

Proposition 2.11. *Every superhypergraph can be transformed into a hypergraph.*

Proof. Let $H = (V, E)$ be a superhypergraph, where:

- $V \subseteq P(U)$ for some underlying set U, meaning each $v \in V$ is either a single element of U or a subset of U (termed a *supervertex*).
- $E \subseteq P(V)$, with each $e \in E$ linking subsets of elements in V (known as *superedges*).

Our objective is to construct a hypergraph $H' = (V', E')$ that is equivalent to H but adheres to the structure of a hypergraph.

Construction of H': 1. Define V' as the union of all elements in the supervertices of V, specifically:

$$V' = \bigcup_{v \in V} v.$$

Here, each element in V' is a unique element of U, effectively "flattening" the supervertices of V into single elements in V'.

2. Define the edge set E' as follows: for each superedge $e = \{v_1, v_2, \ldots, v_n\} \in E$ (where $v_i \in V$), construct an edge $e' \subseteq V'$ by taking the union of the elements within each supervertex v_i, i.e.,

$$e' = \bigcup_{v \in e} v.$$

This construction ensures that each $e' \subseteq V'$, thus meeting the definition of a hyperedge in H'.

Verification: The transformed structure $H' = (V', E')$ satisfies the following properties of a hypergraph:

- V' is a set of vertices, where each element corresponds to an element in the underlying set U (not subsets of U), ensuring $V' \subseteq U$.
- Each edge $e' \in E'$ is a subset of V' and does not contain supervertices, adhering to the standard hypergraph structure where each edge is a subset of the vertex set.

Thus, the transformation from $H = (V, E)$ to $H' = (V', E')$ is valid, and H' is a hypergraph as required. □

Definition 2.12 (Quasi-SuperHyperGraph). [92] A *quasi-superhypergraph* is a triple $H = (V, S, \Phi)$ where:

- V is a set of elements called *vertices*.
- $S = \{S_i\}_{i=1}^{k} \subset P(V)$ is a family of subsets of V called *supervertices*.
- $\Phi = \{\phi_{i,j} \mid i \neq j\}$ is a set of mappings $\phi_{i,j} : S_i \to S_j$, known as *superedges*, linking different supervertices.

The quasi-superhypergraph becomes a complete superhypergraph if, for each supervertex S_i, there exists at least one supervertex S_j such that S_i links to S_j.

The following relationship between a quasi-superhypergraph and a superhypergraph is well-known and clearly holds.

Proposition 2.13. *Every quasi-superhypergraph can be transformed into a superhypergraph.*

Proof. Let $H = (V, S, \Phi)$ be a quasi-superhypergraph where:

- V is a set of vertices.
- $S = \{S_i\}_{i=1}^k \subset P(V)$ is a collection of subsets of V (supervertices).
- $\Phi = \{\phi_{i,j} \mid i \neq j\}$ is a set of mappings $\phi_{i,j} : S_i \to S_j$, linking different supervertices S_i and S_j.

We aim to construct a superhypergraph $H' = (V', E')$ from H by defining an appropriate vertex set V' and edge set E' that satisfy the properties of a superhypergraph.

To define the vertex set V', let $V' = S$, where each supervertex S_i in S becomes a vertex in V'. Thus, each vertex in V' represents a subset of elements from V, satisfying the requirement that $V' \subseteq P(V)$.

Next, we define the edge set E' based on the mappings Φ. For each mapping $\phi_{i,j} \in \Phi$, where $\phi_{i,j} : S_i \to S_j$, we define a corresponding edge $e_{i,j}$ in E' that links S_i and S_j. Thus, we set $E' = \{e_{i,j} \mid \phi_{i,j} \in \Phi\}$, where each $e_{i,j} = \{S_i, S_j\}$. This construction ensures that each superedge in the quasi-superhypergraph H has a corresponding edge in H' connecting supervertices in V'.

Finally, we verify that $H' = (V', E')$ satisfies the properties of a superhypergraph. Each element in V' is a subset of V, fulfilling the requirement that vertices may represent groups of elements. Furthermore, each edge in E' connects subsets of V, with each edge $e_{i,j}$ linking two supervertices. Thus, the structure H' constructed from H meets the definition of a superhypergraph.

Since this transformation process can be applied to any quasi-superhypergraph $H = (V, S, \Phi)$, we conclude that every quasi-superhypergraph can indeed be transformed into a superhypergraph, as required. □

Next, the definitions for the Fuzzy Quasi-SuperHyperGraph and the q-Fuzzy Quasi-SuperHyperGraph are provided below. These are graph concepts that extend the Quasi-SuperHyperGraph through fuzzy relations.

Definition 2.14 (q-Fuzzy Quasi-SuperHyperGraph)**.** [92] Let $H^* = (V, S, \Phi)$ be a quasi-superhypergraph, and let $\sigma_i = \{(x, \sigma_i(x)) \mid x \in S_i, 0 \leq \sigma_i(x) \leq 1\}$ represent fuzzy supervertices, where $\sigma_i(x)$ denotes the degree of membership of x in S_i. Define $\mu_{i,j} : \phi_{i,j} \to [0, 1]$ as fuzzy superedges.

Then $H = (\{\sigma_i\}_{i=1}^{k}, \{\mu_{i,j}\}_{i,j})$ is called a *q-fuzzy quasi-superhypergraph* on H^*, where:

$$\mu_{i,j}(x, \phi_{i,j}(x)) \leq \frac{\sigma_i(x)}{q\, w(\sigma_i)} \wedge \frac{\sigma_j(\phi_{i,j}(x))}{q\, w(\sigma_j)},$$

for all $x \in V$, and where $w(\sigma_t) = \sum_{x \in S_t} \sigma_t(x)$ denotes the total weight of the fuzzy supervertex σ_t. Here, σ_i and σ_j represent fuzzy supervertices, and $\mu_{i,j}$ represents fuzzy superedges or fuzzy links from σ_i to σ_j.

If $q = 1$, we denote this as a *fuzzy quasi-superhypergraph*.

3. Result in this paper

In this section, we present the main results of this paper by defining the Intuitionistic Fuzzy Quasi SuperHyperGraph, Single-Valued Neutrosophic Quasi SuperHyperGraph, and Plithogenic Quasi SuperHyperGraph, and examining their relationships with other graph classes. While the ideas underlying these graph concepts have been suggested in sources such as [174], this paper reinterprets them through formal mathematical definitions.

3.1. *Intuitionistic Fuzzy Quasi SuperHyperGraph*

The definition of an Intuitionistic Fuzzy Quasi SuperHyperGraph and its relationships with other graph classes are presented below.

Definition 3.1 (Intuitionistic Fuzzy Quasi SuperHyperGraph)**.** Let $H^* = (V, S, \Phi)$ be a quasi-superhypergraph, where:

- V is a non-empty set of vertices.
- $S = \{S_i\}_{i=1}^{k}$ is a family of non-empty subsets of V called *supervertices*, i.e., $S_i \subseteq V$.
- $\Phi = \{\phi_{i,j}\}_{i,j}$ is a set of mappings $\phi_{i,j} : S_i \to S_j$ called *superedges*, for $i \neq j$.

An *intuitionistic fuzzy quasi superhypergraph* (IFQSHG)

$$H = (\{\langle S_i, \mu_{S_i}, \nu_{S_i}\rangle\}_{i=1}^{k}, \{\langle \phi_{i,j}, \mu_{i,j}, \nu_{i,j}\rangle\}_{i,j})$$

on H^* consists of:

- For each supervertex S_i, membership and non-membership functions $\mu_{S_i}, \nu_{S_i} : S_i \to [0, 1]$ satisfying:

$$0 \leq \mu_{S_i}(x) + \nu_{S_i}(x) \leq 1, \quad \forall x \in S_i.$$

- For each superedge $\phi_{i,j} : S_i \to S_j$, membership and non-membership functions $\mu_{i,j}, \nu_{i,j} : \phi_{i,j} \to [0, 1]$ satisfying:

$$0 \leq \mu_{i,j}(x, \phi_{i,j}(x)) + \nu_{i,j}(x, \phi_{i,j}(x)) \leq 1, \quad \forall x \in S_i.$$

Additionally, we define the hesitation degree for vertices and edges as:

$$\pi_{S_i}(x) = 1 - \mu_{S_i}(x) - \nu_{S_i}(x), \quad \pi_{i,j}(x, \phi_{i,j}(x)) = 1 - \mu_{i,j}(x, \phi_{i,j}(x)) - \nu_{i,j}(x, \phi_{i,j}(x)),$$

representing the indeterminacy or hesitation of x in S_i and the connection from x to $\phi_{i,j}(x)$, respectively.

Theorem 3.2. *Any intuitionistic fuzzy quasi superhypergraph H can be transformed into a fuzzy quasi superhypergraph H' by neglecting the non-membership functions and adjusting the membership functions accordingly.*

Proof. Given an intuitionistic fuzzy quasi superhypergraph H, we construct a fuzzy quasi superhypergraph H' as follows:

- For each supervertex S_i, define the membership function $\mu'_{S_i} : S_i \to [0, 1]$ by:

$$\mu'_{S_i}(x) = \mu_{S_i}(x), \quad \forall x \in S_i.$$

- For each superedge $\phi_{i,j}$, define the membership function $\mu'_{i,j} : \phi_{i,j} \to [0, 1]$ by:

$$\mu'_{i,j}(x, \phi_{i,j}(x)) = \mu_{i,j}(x, \phi_{i,j}(x)), \quad \forall x \in S_i.$$

By omitting the non-membership functions ν_{S_i} and $\nu_{i,j}$, we effectively reduce the intuitionistic fuzzy structure to a fuzzy one. The resulting fuzzy membership functions μ'_{S_i} and $\mu'_{i,j}$ retain the original membership degrees, ensuring that the essential relationships in H are preserved in H'. □

Theorem 3.3. *Any intuitionistic fuzzy quasi superhypergraph H can be transformed into an intuitionistic fuzzy hypergraph H'' by representing the supervertices and superedges as standard vertices and edges in a hypergraph.*

Proof. To transform H into an intuitionistic fuzzy hypergraph H'', we proceed as follows:

- Let the vertex set V' of H'' be the union of all supervertices:

$$V' = \bigcup_{i=1}^{k} S_i.$$

- For each element $x \in V'$, define the membership and non-membership functions $\mu_{V'}(x)$ and $\nu_{V'}(x)$ as:

$$\mu_{V'}(x) = \mu_{S_i}(x), \quad \nu_{V'}(x) = \nu_{S_i}(x), \quad \text{if } x \in S_i.$$

- For each mapping $\phi_{i,j} : S_i \to S_j$, create edges in H'' connecting $x \in S_i$ to $y = \phi_{i,j}(x) \in S_j$, with membership and non-membership functions $\mu_E(x, y)$ and $\nu_E(x, y)$ defined by:

$$\mu_E(x, y) = \mu_{i,j}(x, y), \quad \nu_E(x, y) = \nu_{i,j}(x, y).$$

By treating each element of the supervertices as individual vertices and each mapping as edges, we construct an intuitionistic fuzzy hypergraph H'' that encapsulates the same relationships as H. □

3.2. *Single-Valued Neutrosophic Quasi SuperHyperGraph*

The definition of a Single-Valued Neutrosophic Quasi SuperHyperGraph and its relationship with other graph classes are outlined below.

Definition 3.4 (Single-Valued Neutrosophic Quasi SuperHyperGraph)**.** A *Single-Valued Neutrosophic Quasi SuperHyperGraph* $H = (X, S, \Phi, T, I, F)$ is a quasi superhypergraph where:

- For each supervertex S_i:
  - Truth-membership function $T_{S_i} : S_i \to [0, 1]$.
  - Indeterminacy-membership function $I_{S_i} : S_i \to [0, 1]$.
  - Falsity-membership function $F_{S_i} : S_i \to [0, 1]$.
  - For all $x \in S_i$:

$$0 \leq T_{S_i}(x) + I_{S_i}(x) + F_{S_i}(x) \leq 1.$$

- For each superedge $\phi_{i,j}$:
  - Truth-membership function $T_{\phi_{i,j}} : \phi_{i,j} \to [0, 1]$.
  - Indeterminacy-membership function $I_{\phi_{i,j}} : \phi_{i,j} \to [0, 1]$.
  - Falsity-membership function $F_{\phi_{i,j}} : \phi_{i,j} \to [0, 1]$.
  - For all $(x, y) \in \phi_{i,j}$:

$$0 \leq T_{\phi_{i,j}}(x, y) + I_{\phi_{i,j}}(x, y) + F_{\phi_{i,j}}(x, y) \leq 1.$$

Theorem 3.5. *Any single-valued neutrosophic quasi superhypergraph H can be transformed into an intuitionistic fuzzy quasi superhypergraph H' by merging the indeterminacy and falsity memberships.*

Proof. Given H, we construct an intuitionistic fuzzy quasi superhypergraph H' as follows:

- For each supervertex S_i, define the membership and non-membership functions μ_{S_i} and ν_{S_i} by:

$$\mu_{S_i}(x) = T_{S_i}(x), \quad \nu_{S_i}(x) = F_{S_i}(x) + I_{S_i}(x), \quad \forall x \in S_i.$$

- Since $0 \leq T_{S_i}(x) + I_{S_i}(x) + F_{S_i}(x) \leq 1$, it follows that $\mu_{S_i}(x) + \nu_{S_i}(x) \leq 1$, satisfying the conditions of an intuitionistic fuzzy membership.
- For each superedge $\phi_{i,j}$, define $\mu_{i,j}$ and $\nu_{i,j}$ similarly:

$$\mu_{i,j}(x, \phi_{i,j}(x)) = T_{i,j}(x, \phi_{i,j}(x)), \quad \nu_{i,j}(x, \phi_{i,j}(x)) = F_{i,j}(x, \phi_{i,j}(x)) + I_{i,j}(x, \phi_{i,j}(x)).$$

By combining the indeterminacy and falsity degrees into a single non-membership function, we effectively reduce the neutrosophic structure to an intuitionistic fuzzy structure. □

Theorem 3.6. *Any single-valued neutrosophic quasi superhypergraph H can be transformed into a single-valued neutrosophic hypergraph H'' by representing supervertices and superedges as standard vertices and edges.*

Proof. We construct H'' as follows:

- The vertex set V' of H'' is $V' = \bigcup_{i=1}^{k} S_i$.
- For each vertex $x \in V'$, define the neutrosophic membership functions:

$$T_{V'}(x) = T_{S_i}(x), \quad I_{V'}(x) = I_{S_i}(x), \quad F_{V'}(x) = F_{S_i}(x), \quad \text{if } x \in S_i.$$

- For each mapping $\phi_{i,j}$, create edges between $x \in S_i$ and $y = \phi_{i,j}(x) \in S_j$, with neutrosophic membership functions:

$$T_E(x,y) = T_{i,j}(x,y), \quad I_E(x,y) = I_{i,j}(x,y), \quad F_E(x,y) = F_{i,j}(x,y).$$

This construction transforms the quasi superhypergraph H into a single-valued neutrosophic hypergraph H'' by flattening the supervertex and superedge structures into standard vertices and edges. □

Theorem 3.7. *The intersection of two Single-Valued Neutrosophic Quasi SuperHyperGraphs with the same vertex set X results in a Single-Valued Neutrosophic Quasi SuperHyperGraph.*

Proof. Let $H_1 = (X, S^1, \Phi^1, T^1, I^1, F^1)$ and $H_2 = (X, S^2, \Phi^2, T^2, I^2, F^2)$. Define $H = (X, S, \Phi, T, I, F)$ where:

$$S = S^1 \cap S^2,$$

$$\Phi = \Phi^1 \cap \Phi^2,$$

$$T_{S_i}(x) = \min\{T^1_{S_i}(x), T^2_{S_i}(x)\},$$

$$I_{S_i}(x) = \max\{I^1_{S_i}(x), I^2_{S_i}(x)\},$$

$$F_{S_i}(x) = \max\{F^1_{S_i}(x), F^2_{S_i}(x)\}.$$

Since $T + I + F \leq 1$, H satisfies the conditions of a Single-Valued Neutrosophic Quasi SuperHyperGraph. □

3.3. *Plithogenic Quasi SuperHyperGraph*

The definition of a Plithogenic Quasi SuperHyperGraph is outlined below.

Definition 3.8 (Plithogenic Quasi SuperHyperGraph). A *Plithogenic Quasi SuperHyperGraph* $H = (X, S, \Phi, \Theta)$ extends the quasi superhypergraph by incorporating attributes and their associated degrees. It consists of:

- A non-empty set X of elements called *vertices*.
- A set $S = \{S_i\}_{i=1}^k$ of non-empty subsets of X, called *supervertices*.
- A set of mappings $\Phi = \{\phi_{i,j}\}_{i \neq j}$, where $\phi_{i,j} : S_i \to S_j$ are called *superedges*.
- For each supervertex S_i:
  - An attribute l_i with a domain of possible values M_{l_i}.
  - A *Degree of Appurtenance Function* (DAF) $\text{adf}_{S_i} : S_i \times M_{l_i} \to [0, 1]^s$.
- For each superedge $\phi_{i,j}$:
  - An attribute $m_{i,j}$ with a domain of possible values $M_{m_{i,j}}$.
  - A DAF $\text{adf}_{\phi_{i,j}} : \phi_{i,j} \times M_{m_{i,j}} \to [0, 1]^s$.
- *Degrees of Contradiction Functions* (DCFs):
  - For supervertices: $\text{aCf}_{S_i} : M_{l_i} \times M_{l_i} \to [0, 1]^t$.
  - For superedges: $\text{aCf}_{\phi_{i,j}} : M_{m_{i,j}} \times M_{m_{i,j}} \to [0, 1]^t$.

These functions satisfy:

(1) Reflexivity and symmetry of DCFs:

$$\text{aCf}_{S_i}(a, a) = 0, \quad \forall a \in M_{l_i},$$
$$\text{aCf}_{S_i}(a, b) = \text{aCf}_{S_i}(b, a), \quad \forall a, b \in M_{l_i},$$
$$\text{aCf}_{\phi_{i,j}}(c, c) = 0, \quad \forall c \in M_{m_{i,j}},$$
$$\text{aCf}_{\phi_{i,j}}(c, d) = \text{aCf}_{\phi_{i,j}}(d, c), \quad \forall c, d \in M_{m_{i,j}}.$$

(2) The overall membership degree of an element $x \in S_i$ with attribute value a is adjusted by the contradiction between a and other attribute values.

Theorem 3.9. *Every Plithogenic Quasi SuperHyperGraph can be transformed into a Single-Valued Neutrosophic Quasi SuperHyperGraph.*

Proof. Given a Plithogenic Quasi SuperHyperGraph $H = (X, S, \Phi, \Theta)$, we can define the neutrosophic membership functions for each supervertex and superedge based on the Degrees of Appurtenance and Contradiction.

For each supervertex S_i and $x \in S_i$ with attribute value $a \in M_{l_i}$:

$$T_{S_i}(x) = \text{adf}_{S_i}(x, a) \cdot (1 - \text{aCf}_{S_i}(a, a')),$$
$$I_{S_i}(x) = \text{aCf}_{S_i}(a, a'),$$
$$F_{S_i}(x) = 1 - T_{S_i}(x) - I_{S_i}(x),$$

where a' is a reference attribute value.

Similarly, for each superedge $\phi_{i,j}$ and $(x, y) \in \phi_{i,j}$ with attribute value $c \in M_{m_{i,j}}$:

$$T_{\phi_{i,j}}(x, y) = \text{adf}_{\phi_{i,j}}((x, y), c) \cdot (1 - \text{aCf}_{\phi_{i,j}}(c, c')),$$
$$I_{\phi_{i,j}}(x, y) = \text{aCf}_{\phi_{i,j}}(c, c'),$$
$$F_{\phi_{i,j}}(x, y) = 1 - T_{\phi_{i,j}}(x, y) - I_{\phi_{i,j}}(x, y),$$

with c' as a reference attribute value for edges.

Since the sum $T + I + F \leq 1$ holds by construction, the transformed structure satisfies the conditions of a Single-Valued Neutrosophic Quasi SuperHyperGraph. □

Theorem 3.10. *In a Plithogenic Quasi SuperHyperGraph, the degree of contradiction function aCf is symmetric and reflexive.*

Proof. By the definition of a Plithogenic Quasi SuperHyperGraph, the degree of contradiction functions satisfy:

$$\text{aCf}(a, a) = 0, \quad \text{aCf}(a, b) = \text{aCf}(b, a).$$

Reflexivity follows from $\text{aCf}(a, a) = 0$, and symmetry from $\text{aCf}(a, b) = \text{aCf}(b, a)$. □

4. **Future tasks and Discussions**

The following outlines future prospects for this study.

4.1. *Semi Superhypergraph*

As noted in the introduction, numerous graph classes have been proposed within graph theory, alongside ongoing efforts to generalize these classes [30]. The semigraph is recognized as a generalized graph class extending the standard concept of a graph [46,161,165]. Recently, an even broader generalization, the semihypergraph, has been introduced, further expanding on this structure [101]. Similarly, it is reasonable to expect the existence of a semi superhypergraph, which would build on the structure of the superhypergraph. In future work, we aim to explore the mathematical structure of the semi superhypergraph, as well as potential extensions incorporating Fuzzy and Neutrosophic properties and concepts such as the semi n-superhypergraph [174].

The definitions of the already established concepts of the semigraph and semihypergraph, along with the conceptual framework for the semi superhypergraph, are presented below.

Definition 4.1 (Semigraph)**.** [161] A *semigraph* $G = (V, E)$ consists of:

- A non-empty set V of elements called *vertices*.
- A set E of ordered n-tuples of distinct vertices, called *edges*, where $n \geq 2$ for each edge. Each edge connects a specific sequence of vertices.

The semigraph G must satisfy the following conditions:

(1) *Intersection Condition*: Any two edges in E have at most one vertex in common.
(2) *Edge Equivalence*: Two edges $E_1 = (u_1, u_2, \dots, u_m)$ and $E_2 = (v_1, v_2, \dots, v_n)$ are considered equal if and only if:
  - $m = n$ (they have the same length).
  - Either $u_i = v_i$ for all $i = 1, \dots, n$ (identical ordering), or $u_i = v_{n-i+1}$ for all $i = 1, \dots, n$ (reverse ordering).

Definition 4.2 (Semihypergraph)**.** [101] A *semihypergraph* $H_s = (V, E_h)$ consists of:

- A finite set $V = \{v_1, v_2, \dots, v_n\}$ of vertices.
- A set $E_h = \{E_h^1, E_h^2, \dots, E_h^p\}$ of ordered hyperedges, where each hyperedge E_h^j is an ordered k_j-tuple of distinct vertices from V, with $k_j \geq 2$.

The semihypergraph H_s must satisfy the following conditions:

(1) *Intersection Condition*: Any two hyperedges in E_h have at most one vertex in common.
(2) *Hyperedge Equivalence*: Two hyperedges $E_h^m = (u_1, u_2, \dots, u_m)$ and $E_h^n = (v_1, v_2, \dots, v_n)$ are considered equal if and only if:
  - $m = n$ (they have the same length).
  - Either $u_i = v_i$ for all $i = 1, \dots, n$ (identical ordering), or $u_i = v_{n-i+1}$ for all $i = 1, \dots, n$ (reverse ordering).

Proposition 4.3. *Every semihypergraph $H_s = (V, E_h)$ can be transformed into a semigraph $G = (V, E)$.*

Proof. Let $H_s = (V, E_h)$ be a semihypergraph with a vertex set V and a set E_h of hyperedges, where each hyperedge $E_h^j \in E_h$ is an ordered k_j-tuple of distinct vertices from V with $k_j \geq 2$.

We construct a corresponding semigraph $G = (V, E)$ as follows:

(1) Set V as the vertex set of G, identical to the vertex set of H_s.
(2) For each hyperedge $E_h^j = (v_{j_1}, v_{j_2}, \dots, v_{j_{k_j}})$ in H_s, define a corresponding edge $E_j = (v_{j_1}, v_{j_2}, \dots, v_{j_{k_j}})$ in E of G. Add E_j to E.

This construction ensures that each hyperedge in H_s corresponds directly to an edge in G with identical ordering and vertex sequence. Next, we verify that $G = (V, E)$ satisfies the conditions of a semigraph.

The set V in G is identical to that in H_s, so V is non-empty, meeting the requirement for a semigraph. Additionally, each edge $E_j \in E$ is an ordered k_j-tuple of distinct vertices from V, with $k_j \geq 2$ by construction, which aligns with the edge structure required for semigraphs.

By the intersection condition of the semihypergraph H_s, any two hyperedges E_h^a and E_h^b share at most one vertex. Since each edge in G corresponds to a hyperedge in H_s, the intersection condition is preserved in G; any two edges in E have at most one vertex in common.

The hyperedge equivalence condition in H_s specifies that two hyperedges are equal if they have the same length and either identical or reverse ordering of vertices. Since each edge in G corresponds to a hyperedge in H_s, this equivalence criterion holds for the edges in G as well.

Thus, the constructed graph $G = (V, E)$ satisfies all conditions of a semigraph. Therefore, any semihypergraph H_s can indeed be transformed into a semigraph G, as required. □

Definition 4.4 (Semi SuperHyperGraph)**.** A *semi superhypergraph* $H = (V, S, E)$ consists of:

- A finite set V of elements called *vertices*.
- A collection $S = \{S_1, S_2, \ldots, S_k\}$ of non-empty subsets of V, called *supervertices*.
- A set E of ordered n-tuples of supervertices, called *superedges*, where $n \geq 2$ for each superedge.

The semi superhypergraph H must satisfy the following conditions:

(1) *Intersection Condition*: Any two superedges in E have at most one supervertex in common.

(2) *Superedge Equivalence*: Two superedges $E_1 = (S_{i_1}, S_{i_2}, \ldots, S_{i_m})$ and $E_2 = (S_{j_1}, S_{j_2}, \ldots, S_{j_n})$ are considered equal if and only if:
   - $m = n$ (they have the same length).
   - Either $S_{i_k} = S_{j_k}$ for all $k = 1, \ldots, n$ (identical ordering), or $S_{i_k} = S_{j_{n-k+1}}$ for all $k = 1, \ldots, n$ (reverse ordering).

We can show that both superhypergraphs and semihypergraphs can be represented as semi superhypergraphs.

Theorem 4.5. *Every superhypergraph and every semihypergraph can be represented as a semi superhypergraph.*

Proof. We will prove this theorem in two parts.

Part 1: Superhypergraphs as Semi SuperHyperGraphs

Let $G = (V, E)$ be a superhypergraph, where:

- V is a set of supervertices, each being a subset of some underlying set U.
- $E \subseteq \mathcal{P}(V)$ is a set of superedges, where each superedge $e \in E$ is a subset of V.

To represent G as a semi superhypergraph $H = (V', S, E')$, we proceed as follows:

- Set $V' = V$. That is, the vertices of the semi superhypergraph are the supervertices of the superhypergraph.
- Set $S = V'$. Each supervertex in V' is a supervertex in H.
- For each superedge $e = \{S_{i_1}, S_{i_2}, \ldots, S_{i_m}\} \in E$, define an ordered superedge $E' = (S_{i_1}, S_{i_2}, \ldots, S_{i_m})$ and include it in E'.

The Intersection Condition and Superedge Equivalence in the semi superhypergraph are satisfied because they correspond directly to those in the superhypergraph, with the addition of ordering, which is accounted for in the equivalence condition.

Part 2: Semihypergraphs as Semi SuperHyperGraphs

Let $H_s = (V, E_h)$ be a semihypergraph, where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is a set of vertices.
- E_h is a set of ordered hyperedges, each hyperedge being an ordered tuple of vertices from V.

To represent H_s as a semi superhypergraph $H = (V, S, E')$, we proceed as follows:

- Define the supervertex set $S = \{\{v\} \mid v \in V\}$, where each supervertex is a singleton set containing a single vertex from V.
- The vertex set V remains the same.
- For each hyperedge $E_h^j = (v_{j_1}, v_{j_2}, \ldots, v_{j_m}) \in E_h$, define a superedge $E' = (\{v_{j_1}\}, \{v_{j_2}\}, \ldots, \{v_{j_m}\})$ and include it in E'.

Again, the Intersection Condition and Superedge Equivalence in the semi superhypergraph are satisfied because they correspond to those in the semihypergraph, with vertices replaced by singleton supervertices. □

4.2. *Soft Semihypergraph*

In this paper, we also introduce the concept of a Soft Semihypergraph, a hybrid graph structure that merges elements from both Soft Hypergraphs [9, 82, 162] and Soft Semigraphs [79–81]. A soft graph, in general, is a mathematical framework that extends soft set theory to graph structures, assigning parameters to vertices and edges to effectively manage uncertainty. Below, we provide definitions of Soft Hypergraphs and Soft Semigraphs, followed by an initial definition of a Soft Semihypergraph as a concept still under development.

Definition 4.6. [131] Let U be a non-empty finite set, called the *universe* of discourse, and let E be a non-empty set of parameters. A *soft set* over U is defined as follows:

$$F = (F, A) \text{ over } U \text{ is an ordered pair, where } A \subseteq E \text{ and } F : A \to P(U),$$

where $F(a) \subseteq U$ for each $a \in A$ and $P(U)$ denotes the power set of U. The set of all soft sets over U is denoted by $S(U)$.

(1) *Soft Subset:* Let $F = (F, A)$ and $G = (G, B)$ be two soft sets over the common universe U. We say that F is a soft subset of G, denoted $F \subseteq G$, if:
- $A \subseteq B$
- $F(a) \subseteq G(a)$ for all $a \in A$.

(2) *Union of Soft Sets:* The union of two soft sets $F = (F, A)$ and $G = (G, B)$ over U is defined as $H = (H, C)$ where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e), & e \in A - B \\ G(e), & e \in B - A \\ F(e) \cup G(e), & e \in A \cap B \end{cases}$$

(3) *Intersection of Soft Sets:* The intersection of two soft sets $F = (F, A)$ and $G = (G, B)$ with disjoint parameter sets $A \cap B = \emptyset$ is defined as $H = (H, C)$, where $C = A \cap B$ and

$$H(e) = F(e) \cap G(e), \quad \forall e \in C.$$

Definition 4.7. [82] Let $H^* = (V, E)$ be a hypergraph with a vertex set V and a hyperedge set E. A *soft hypergraph* is defined by a 4-tuple $H = (H^*, A, B, C)$, where:

(1) $H^* = (V, E)$ is a simple hypergraph.
(2) C is a nonempty set of parameters.
(3) (A, C) is a soft set over V, where $A : C \to \mathcal{P}(V)$ is a mapping such that $A(c) \subseteq V$ for each $c \in C$.
(4) (B, C) is a soft set over E_s, the set of all subhyperedges of H^*, where $B : C \to \mathcal{P}(E_s)$ maps each $c \in C$ to a subset of the maximal subhyperedges of H^*.
(5) For each $c \in C$, the pair $(A(c), B(c))$ forms a semisubhypergraph of H^*, meaning each element in $B(c)$ is a subhyperedge of a hyperedge in H^* and corresponds to the vertices in $A(c)$.

If we denote $(A(c), B(c))$ by $F(c)$, then the soft hypergraph H is represented by $\{F(c) : c \in C\}$, where each $F(c)$ is termed as a *hyperpart* (or simply *h-part*) of the soft hypergraph H.

Definition 4.8 (Soft Semigraph)**.** [79,81] Let $S^* = (T, D)$ be a semigraph with vertex set T and edge set D. A *soft semigraph* is defined by a 4-tuple $S = (S^*, I, J, K)$, where:

(1) $S^* = (T, D)$ is a semigraph with a set T of vertices and a set D of edges.

(2) K is a nonempty set of parameters.

(3) (I, K) is a soft set over T, where $I : K \to \mathcal{P}(T)$ is a mapping such that $I(k) \subseteq T$ for each $k \in K$. For each $k \in K$, $I(k)$ denotes the subset of vertices associated with parameter k.

(4) (J, K) is a soft set over D_p, the set of all maximal partial edges (or *mp edges*) in S^*. Here, $J : K \to \mathcal{P}(D_p)$ maps each $k \in K$ to a subset $J(k)$ of D_p, where each element in $J(k)$ is an mp edge formed by some or all vertices in $I(k)$.

The soft semigraph $S = (S^*, I, J, K)$ represents a collection of partial semigraphs of S^* defined by different parameters in K. Specifically, for each $k \in K$, the pair $L(k) = (I(k), J(k))$ is called a *partial semigraph* (or *p-part*) of S^*.

Definition 4.9 (Soft Semihypergraph)**.** Let $H^* = (V, E_h)$ be a semihypergraph, where:

- V is a finite set of vertices.
- $E_h = \{E_h^1, E_h^2, \ldots, E_h^p\}$ is a set of hyperedges, where each hyperedge E_h^j is an ordered n_j-tuple of distinct vertices from V, with $n_j \geq 2$.

Let C be a non-empty set of parameters. A *soft semihypergraph* is a quadruple $H = (H^*, A, B, C)$, where:

(1) (A, C) is a soft set over V, i.e., $A : C \to \mathcal{P}(V)$ is a mapping such that $A(c) \subseteq V$ for each $c \in C$.

(2) (B, C) is a soft set over E_s, where E_s is the set of all subhyperedges of H^*. The mapping $B : C \to \mathcal{P}(E_s)$ assigns to each $c \in C$ a set $B(c)$ of maximal subhyperedges corresponding to $A(c)$.

(3) For each $c \in C$, the pair $F(c) = (A(c), B(c))$ forms a *partial semihypergraph* of H^*, meaning:
   - Each hyperedge in $B(c)$ is a subhyperedge of some hyperedge in E_h.
   - The vertices of each hyperedge in $B(c)$ are elements of $A(c)$.

The collection $H = \{F(c) : c \in C\}$ represents the soft semihypergraph, where each $F(c)$ is called a *partial semihypergraph* (or *p-part*) of H.

Theorem 4.10. *The concept of a soft semihypergraph generalizes both semihypergraphs and soft semigraphs. Specifically:*

(1) *Every semihypergraph can be considered as a soft semihypergraph.*

(2) *Every soft semigraph can be represented as a soft semihypergraph.*

Proof. Let $H^* = (V, E_h)$ be a semihypergraph. We can construct a soft semihypergraph $H = (H^*, A, B, C)$ by choosing:

- $C = \{c\}$ is a singleton parameter set.
- Define $A : C \to \mathcal{P}(V)$ by $A(c) = V$.
- Define $B : C \to \mathcal{P}(E_h)$ by $B(c) = E_h$.

Then, $(A(c), B(c)) = (V, E_h)$ is the original semihypergraph H^*. Since H consists of a single partial semihypergraph identical to H^*, the semihypergraph is represented as a soft semihypergraph.

Let $S = (S^*, I, J, K)$ be a soft semigraph as defined in [82], where:

- $S^* = (T, D)$ is a semigraph with vertex set T and edge set D.
- K is a non-empty set of parameters.
- $I : K \to \mathcal{P}(T)$ is a mapping representing the soft set over T.
- $J : K \to \mathcal{P}(D_p)$, where D_p is the set of partial edges of S^*.

We can construct a semihypergraph $H^* = (V, E_h)$ corresponding to S^* by setting:

- $V = T$.
- $E_h = D_p$, treating each partial edge as an ordered hyperedge.

Now, define the soft semihypergraph $H = (H^*, A, B, K)$ by:

- $A = I$.
- $B = J$.

For each $k \in K$, the partial semihypergraph $F(k) = (A(k), B(k))$ corresponds to the partial semigraph $L(k) = (I(k), J(k))$ in the soft semigraph S.

Therefore, the soft semigraph $S = (S^*, I, J, K)$ is represented as a soft semihypergraph $H = (H^*, A, B, K)$. □

Question 4.11. Is it possible to extend a Soft Semihypergraph to a Soft Semisuperhypergraph? What potential applications might such an extension have?

4.3. *Multi-Superhypergraph*

One well-known graph class is the multigraph, which allows for multiple (parallel) edges between the same pair of vertices [40,62,110,127]. This concept extends to the multi-hypergraph, a generalization to hypergraphs where parallel hyperedges are permitted. Both multigraphs and multi-hypergraphs [145, 209] have numerous applications, particularly in areas such as neural network theory [25, 114, 119, 189]. Here, we aim to explore an even broader extension with the definition of the Multi-Superhypergraph, provided below.

Definition 4.12 (Multiset). [26, 27, 166] A multiset M over a set S is a collection where each element $x \in S$ can appear more than once. Formally, a multiset is defined as a pair $M = (S, m)$, where:

- S is the underlying set of distinct elements, called the *support* of M.
- $m : S \to \mathbb{Z}_{\geq 0}$ is a function called the *multiplicity function*, which assigns to each element $x \in S$ a non-negative integer $m(x)$, representing the number of occurrences of x in M.

If $m(x) > 0$ for some $x \in S$, then x is said to be an element of M with multiplicity $m(x)$. A multiset with all multiplicities equal to 1 is equivalent to a set.

Definition 4.13 (Multigraph)**.** [40] Let $G = (V, E)$ be a multigraph, where:

- V is a non-empty set of vertices.
- E is a multiset of unordered pairs of vertices, meaning that E allows multiple edges (parallel edges) between the same pair of vertices.

In a multigraph, loops (edges that connect a vertex to itself) may or may not be permitted, depending on the specific multigraph definition being used.

Definition 4.14 (Multi-Hypergraph)**.** [209] Let $H = (V, E)$ be a multi-hypergraph, where:

- V is a non-empty set of vertices.
- E is a multiset of subsets of V, called hyperedges, where each hyperedge can contain any number of vertices. The multiset property allows for multiple occurrences of the same hyperedge (parallel hyperedges).

In a multi-hypergraph, hyper-loops (hyperedges that contain repeated vertices) may be allowed, providing further generalization.

Definition 4.15 (Multi-Superhypergraph)**.** A *Multi-Superhypergraph* is a triple $H = (V, S, E)$, where:

- V is a non-empty finite set of elements called *vertices*.
- S is a multiset of non-empty subsets of V, called *supervertices*. Each supervertex $s \in S$ satisfies $s \subseteq V$, and the multiset allows for multiple occurrences of the same supervertex.
- E is a multiset of non-empty subsets of S, called *superedges*. Each superedge $e \in E$ is a subset of supervertices from S, and multiple occurrences of the same superedge are permitted.

Theorem 4.16. *Every Multigraph and every Multi-Hypergraph can be represented as a Multi-Superhypergraph. Conversely, the Multi-Superhypergraph generalizes both the Multigraph and the Multi-Hypergraph.*

Proof. Let $G = (V, E_G)$ be a multigraph, where:

- V is a non-empty set of vertices.

- E_G is a multiset of unordered pairs of vertices, allowing multiple edges between the same pair of vertices.

Construction:

Define a Multi-Superhypergraph $H = (V', S, E)$ as follows:

- Set $V' = V$.
- Let $S = \{\{v\} \mid v \in V\}$. Each vertex v becomes a singleton supervertex $\{v\}$. Since S is a multiset, multiplicities are considered if necessary.
- For each edge $e = \{u, v\} \in E_G$, include the superedge $e' = \{\{u\}, \{v\}\}$ in the multiset E. The multiplicity of e' in E matches the multiplicity of e in E_G.

Verification:

- Supervertices in S correspond to individual vertices in V.
- Superedges in E connect singleton supervertices, reflecting the edges in G.
- Multiplicities of edges are preserved due to the multiset nature of E.

And let $H_G = (V, E_H)$ be a multi-hypergraph, where:

- V is a non-empty set of vertices.
- E_H is a multiset of subsets of V, called hyperedges, allowing multiple occurrences of the same hyperedge.

Construction:

Define a Multi-Superhypergraph $H = (V', S, E)$ as follows:

- Set $V' = V$.
- Let $S = \{\{v\} \mid v \in V\}$. Each vertex becomes a singleton supervertex.
- For each hyperedge $e_H \in E_H$, include the superedge $e = \{\{v\} \mid v \in e_H\}$ in the multiset E. The multiplicity of e in E matches that of e_H in E_H.

Verification:

- Supervertices represent the original vertices as singleton sets.
- Superedges correspond to hyperedges by connecting the supervertices representing the vertices in each hyperedge.
- Multiplicities of hyperedges are preserved in E.

The Multi-Superhypergraph accommodates:

- Supervertices that are subsets of V, not limited to singletons.
- Superedges connecting any subsets of supervertices.
- Multiplicities of supervertices and superedges through the use of multisets.

Thus, the Multi-Superhypergraph generalizes both the Multigraph and the Multi-Hypergraph by allowing more complex relationships and multiplicities. □

4.4. *Pseudo-SuperHypergraph*

A pseudograph is a type of graph that allows parallel edges and self-loops (edges that return to the originating vertex) [21, 29, 110]. This structure enables the representation of more complex relationships and intricate networks compared to standard graphs [203, 217]. A pseudo-hypergraph generalizes the pseudograph concept by incorporating hypergraphs, extending its capabilities for modeling even more sophisticated connections [17, 31, 118]. Moving forward, We aim to elucidate the mathematical structure and potential applications of the pseudo-superhypergraph. Below, we present the established definitions of pseudograph and pseudo-hypergraph, followed by an overview of the pseudo-superhypergraph.

Definition 4.17 (Pseudograph)**.** [202] A *pseudograph* $G = (V, E)$ is a graph that consists of:

- A non-empty set V of elements called *vertices*.
- A set E of unordered pairs of vertices, where each edge $e \in E$ is of the form $\{u, v\}$ with $u, v \in V$. The following are permitted in E:
  - *Parallel edges*: Multiple instances of the same pair $\{u, v\}$ are allowed in E.
  - *Loops*: An edge of the form $\{v, v\}$, which connects a vertex to itself, is allowed.

Definition 4.18 (Pseudo-Hypergraph)**.** [17] A *pseudo-hypergraph* $H = (V, E)$ generalizes the concept of a hypergraph and consists of:

- A non-empty set V of elements called *vertices*.
- A set E of multisets of vertices, where each hyperedge $e \in E$ is a multiset over V. This allows:
  - *Multiple occurrences of vertices within a hyperedge*: A vertex $v \in V$ may appear more than once in a single hyperedge e, resembling loops in pseudographs.
  - *Multiple identical hyperedges*: The same multiset of vertices may appear multiple times in E, resembling parallel edges in pseudographs.

Remark 4.19 (Pseudo-Hypergraph)**.** In a pseudo-hypergraph, each hyperedge can connect any subset of vertices from V, with repetitions of vertices within hyperedges permitted. This allows a single hyperedge to act both as a multi-edge and as a ”loop” by including repeated occurrences of vertices.

Definition 4.20 (Pseudo-Superhypergraph)**.** A *pseudo-superhypergraph* $H = (V, S, E)$ consists of:

- *Vertices*: A non-empty finite set V of elements called *vertices*.
- *Supervertices*: A multiset S of multisets over V, called *supervertices*. Each supervertex $s \in S$ is a multiset of vertices from V, allowing:

- Multiple occurrences of the same vertex within a supervertex (vertices can repeat within s).
- Multiple occurrences of the same supervertex in S (supervertices can repeat in S).

- *Superedges*: A multiset E of multisets over S, called *superedges*. Each superedge $e \in E$ is a multiset of supervertices from S, allowing:
  - Multiple occurrences of the same supervertex within a superedge (supervertices can repeat within e).
  - Multiple occurrences of the same superedge in E (superedges can repeat in E).

Remark 4.21 (Pseudo-Superhypergraph)**.** In a pseudo-superhypergraph:

- Supervertices can have repeated vertices from V.
- Superedges can have repeated supervertices from S.
- Superedges themselves can appear multiple times in E.

Theorem 4.22. *Every pseudograph and every pseudo-hypergraph can be represented as a pseudo-superhypergraph. Conversely, the pseudo-superhypergraph generalizes both the pseudograph and the pseudo-hypergraph.*

Proof. We prove this theorem in two parts.
Part 1: Pseudographs as Pseudo-Superhypergraphs. Let $G = (V, E_G)$ be a pseudograph, where:

- V is a non-empty set of vertices.
- E_G is a multiset of unordered pairs $\{u, v\}$ with $u, v \in V$, allowing for loops and multiple edges.

Construction:

Define a pseudo-superhypergraph $H = (V', S, E)$ as follows:

- Set $V' = V$.
- Let $S = \{\{v\} \mid v \in V\}$. Each vertex v becomes a supervertex $s_v = \{v\}$ in S.
- For each edge $e = \{u, v\} \in E_G$:
  - If $u \neq v$:
    - Define $e' = \{s_u, s_v\}$, a multiset containing s_u and s_v.
    - Include e' in E with multiplicity equal to the number of edges between u and v in E_G.
  - If $u = v$ (a loop at v):
    - Define $e' = \{s_v, s_v\}$, a multiset containing s_v twice.
    - Include e' in E with multiplicity equal to the number of loops at v in E_G.

Verification:

- Supervertices represent the vertices in V.
- Superedges correspond to edges in E_G, including loops and multiple edges.
- Repeated supervertices within a superedge represent loops.
- Multiplicities in E reflect the multiplicities in E_G.

Part 2: Pseudo-Hypergraphs as Pseudo-Superhypergraphs. Let $H_G = (V, E_H)$ be a pseudo-hypergraph, where:

- V is a non-empty set of vertices.
- E_H is a multiset of multisets over V, allowing for repeated vertices within hyperedges and multiple hyperedges.

Construction:

Define a pseudo-superhypergraph $H = (V', S, E)$ as follows:

- Set $V' = V$.
- Let $S = \{\{v\} \mid v \in V\}$.
- For each hyperedge $e_H \in E_H$:
  - Construct a superedge e by replacing each occurrence of v in e_H with s_v.
  - Thus, if v appears m times in e_H, s_v appears m times in e.
  - Include e in E with multiplicity equal to the multiplicity of e_H in E_H.

Verification:

- Supervertices correspond to vertices in V.
- Superedges represent hyperedges, preserving multiplicities and repeated vertices.
- Repeated supervertices within a superedge represent repeated vertices within a hyperedge.
- Multiplicities of hyperedges are preserved in E.

Since both pseudographs and pseudo-hypergraphs can be represented as pseudo-superhypergraphs, and the pseudo-superhypergraph allows for more general structures, it follows that the pseudo-superhypergraph generalizes both concepts. □

4.5. *Mixed Superhypergraph*

A mixed graph integrates undirected and directed edges, allowing both bidirectional and directional connections between vertices [64, 154]. This concept has been expanded to mixed hypergraphs [193], extending the structure to hypergraphs, whose mathematical properties have been examined in depth. In this paper, we further introduce the framework of mixed superhypergraphs. Future work will explore fuzzy mixed hypergraphs and neutrosophic mixed hypergraphs, combining elements of fuzzy and neutrosophic graphs within this extended structure. Although detailed definitions are beyond this paper's scope, readers needing background on directed graphs [24, 37, 103], directed hypergraphs [85, 108, 132, 135], or directed superhypergraphs [65] are encouraged to consult relevant literature as needed.

Definition 4.23 (Mixed Graph)**.** [152] A *mixed graph* $G = (V, E \cup A)$ is a graph structure with:

- V, a non-empty set of vertices.
- E, a set of undirected edges, where each edge $e \in E$ is an unordered pair $\{u, v\}$ of distinct vertices $u, v \in V$.
- A, a set of directed edges (also called arcs), where each arc $a \in A$ is an ordered pair (u, v) of distinct vertices $u, v \in V$.

In a mixed graph, an edge can either connect two vertices bidirectionally (undirected) or directionally (from one vertex to another).

Remark 4.24. A mixed graph combines features of both undirected and directed graphs. While undirected edges $e = \{u, v\}$ imply symmetric relationships between vertices u and v, directed edges $a = (u, v)$ define an asymmetric connection from u to v.

Definition 4.25 (Mixed Hypergraph)**.** [193] A *mixed hypergraph* $H = (V, E \cup A)$ is a hypergraph structure with:

- V, a non-empty set of vertices.
- E, a set of undirected hyperedges, where each hyperedge $e \in E$ is a non-empty subset of V containing at least two vertices.
- A, a set of directed hyperedges (also known as *dyperedges*), where each directed hyperedge $a = (Z, z) \in A$ is an ordered pair consisting of:
  - Z, a non-empty subset of $V \setminus \{z\}$, called the *tail set* of a.
  - $z \in V$, a single vertex called the *head* of a.

In a mixed hypergraph, undirected hyperedges connect subsets of vertices without directionality, whereas directed hyperedges represent a directed relationship from the vertices in the tail set Z to the head z.

Remark 4.26. Mixed hypergraphs generalize mixed graphs by allowing hyperedges, which can involve more than two vertices, and by permitting directed connections (dyperedges) between sets of vertices and individual vertices. The structure enables modeling of more complex relationships in network theory, such as group-to-individual influence dynamics.

Theorem 4.27. *Any mixed hypergraph can be transformed into a mixed graph.*

Proof. Given a mixed hypergraph $H = (V, E \cup A)$, we construct a mixed graph $G = (V', E' \cup A')$ as follows:

(1) *Vertex Set V':* Define $V' = V \cup \{e \mid e \in E\}$. For each undirected hyperedge $e \in E$, introduce a new vertex in V' to represent it.
(2) *Edge Set E':* For each undirected hyperedge $e \in E$ and each $v \in e$, create an undirected edge $\{v, e\}$ in E'.
(3) *Arc Set A':* For each directed hyperedge $a = (Z, z) \in A$, choose a vertex $y \in Z$ and create a directed edge (y, z) in A'.

Verification:

- The structure of the undirected hyperedges E is represented in G by the connections between the original vertices in V and the new vertices representing hyperedges in V'.
- Directed hyperedges in A are represented by directed edges connecting a vertex in the tail set to the head vertex, thereby preserving directionality in the transformed graph.

The resulting G is a mixed graph that represents the original mixed hypergraph H. □

Definition 4.28 (Mixed Superhypergraph)**.** A *mixed superhypergraph* $H = (V, S, E, A)$ consists of:

- A non-empty set V of elements called *vertices*.
- A set S of non-empty subsets of V, called *supervertices*, where each supervertex $s \in S$ satisfies $s \subseteq V$.
- A set E of undirected *superedges*, where each superedge $e \in E$ is a non-empty subset of S.
- A set A of directed *superedges* (also known as *super-dyperedges*), where each directed superedge $a = (Z, z) \in A$ consists of:
  - A non-empty subset $Z \subseteq S \setminus \{z\}$, called the *tail set*.
  - A supervertex $z \in S$, called the *head*.

In a mixed superhypergraph, undirected superedges connect subsets of supervertices without directionality, while directed superedges represent a directed relationship from the tail set Z to the head z.

Theorem 4.29. *Any mixed superhypergraph can be transformed into a mixed hypergraph.*

Proof. Let $H = (V, S, E, A)$ be a mixed superhypergraph. We construct a mixed hypergraph $H' = (V', E', A')$ as follows:

(1) *Vertex Set V':* Introduce a new vertex v_s in V' for each supervertex $s \in S$. Thus,

$$V' = \{v_s \mid s \in S\}.$$

(2) *Undirected Hyperedges E':* For each undirected superedge $e \in E$, create an undirected hyperedge e' in E' defined by:

$$e' = \{v_s \mid s \in e\}.$$

(3) *Directed Hyperedges A':* For each directed superedge $a = (Z, z) \in A$, create a directed hyperedge a' in A' defined by:

$$a' = (\{v_s \mid s \in Z\}, v_z),$$

where v_z is the vertex corresponding to the head supervertex z.

Verification:

- The vertex set V' represents the supervertices of H.
- Undirected superedges in E are transformed into undirected hyperedges in E' by mapping the supervertices in e to their corresponding vertices in V'.
- Directed superedges in A are transformed into directed hyperedges in A', preserving the direction from the tail set Z to the head z.

The mixed hypergraph $H' = (V', E', A')$ represents the original mixed superhypergraph H, with supervertices replaced by vertices and superedges appropriately transformed. □

Theorem 4.30. *Any mixed superhypergraph can be transformed into an undirected superhypergraph by removing the directed superedges.*

Proof. Given a mixed superhypergraph $H = (V, S, E, A)$, we construct an undirected superhypergraph $H' = (V, S, E')$ as follows:

(1) *Supervertex Set S:* Retain the same supervertex set from H.

(2) *Undirected Superedges E':* Form E' by combining the undirected superedges E with the directed superedges A converted to undirected superedges:

$$E' = E \cup \{e_a \mid a = (Z, z) \in A\},$$

where for each $a = (Z, z) \in A$, the undirected superedge e_a is defined as:

$$e_a = Z \cup \{z\}.$$

Verification:

- The undirected superedges in E remain unchanged.
- Each directed superedge $a = (Z, z)$ is converted into an undirected superedge e_a connecting all supervertices in $Z \cup \{z\}$, thus removing directionality while preserving the connectivity.

The undirected superhypergraph $H' = (V, S, E')$ maintains the structure of H without directionality, representing all relationships among supervertices with undirected superedges.

□

4.6. *Bidirected Superhypergraph*

The concept of the bidirected graph [15, 59, 78] has been introduced in recent years. Below, we provide definitions for the bidirected hypergraph and bidirected superhypergraph, which generalize bidirected graphs.

Definition 4.31 (Bidirected Graph). [15, 59, 78] A *bidirected graph* (also known as a *bigraph*) is a pair $B = (G, \tau)$, where:

- $G = (V, E)$ is a simple undirected graph, where V is a non-empty set of vertices and E is a set of edges (without parallel edges or loops).
- $\tau : V \times E \to \{-1, 0, 1\}$ is a function called the *bidirection function*, which assigns a *local orientation* to each vertex-edge pair (v, e) as follows:
  - $\tau(v, e) = 1$: Edge e is directed *towards* vertex v.
  - $\tau(v, e) = -1$: Edge e is directed *away from* vertex v.
  - $\tau(v, e) = 0$: Vertex v is not incident to edge e.

The graph G is referred to as the *underlying graph* of B, and the function τ provides the bidirected structure on G by assigning a direction at each endpoint of every edge in E.

Definition 4.32 (Bidirected Hypergraph). A *bidirected hypergraph* is a triple $H = (V, E, \tau)$, where:

- V is a non-empty set of vertices.
- E is a set of hyperedges, where each hyperedge $e \in E$ is a non-empty subset of V.
- $\tau : V \times E \to \{-1, 0, 1\}$ is a function called the *bidirection function*, assigning a *local orientation* to each vertex-hyperedge pair (v, e) such that:
  - $\tau(v, e) = 1$ if hyperedge e is directed *towards* vertex v.
  - $\tau(v, e) = -1$ if hyperedge e is directed *away from* vertex v.
  - $\tau(v, e) = 0$ if $v \notin e$.

Additionally, for each hyperedge $e \in E$, we require that:

$$\sum_{v \in e} \tau(v, e) = 0.$$

This condition ensures that the hyperedge is balanced in terms of its local orientations.

Definition 4.33 (Bidirected Superhypergraph). A *bidirected superhypergraph* is a quadruple $H = (V, S, E, \tau)$, where:

- V is a non-empty set of vertices.
- S is a set of non-empty subsets of V, called *supervertices*.
- E is a set of superedges, where each superedge $e \in E$ is a non-empty subset of S.
- $\tau : S \times E \to \{-1, 0, 1\}$ is a function called the *bidirection function*, assigning a *local orientation* to each supervertex-superedge pair (s, e) such that:
  - $\tau(s, e) = 1$ if superedge e is directed *towards* supervertex s.
  - $\tau(s, e) = -1$ if superedge e is directed *away from* supervertex s.
  - $\tau(s, e) = 0$ if $s \notin e$.

Similarly to bidirected hypergraphs, for each superedge $e \in E$, we require that:

$$\sum_{s \in e} \tau(s, e) = 0.$$

This condition ensures that the superedge is balanced in terms of its local orientations.

Theorem 4.34. *Any bidirected hypergraph can be transformed into a bidirected graph.*

Proof. Given a bidirected hypergraph $H = (V, E, \tau)$, we construct a bidirected graph $G = (V', E', \tau')$ as follows:

(1) *Vertex Set V':* Let $V' = V \cup \{e \mid e \in E\}$. That is, for each hyperedge $e \in E$, introduce a new vertex representing e.

(2) *Edge Set E':* For each hyperedge $e \in E$ and each vertex $v \in e$, create an edge $\{v, e\}$ connecting vertex v to the hyperedge vertex e.

(3) *Bidirection Function τ':* For each edge $\{v, e\} \in E'$, define:

$$\tau'(v, \{v, e\}) = \tau(v, e), \quad \tau'(e, \{v, e\}) = -\tau(v, e).$$

Verification:

- The sum of the bidirections at the endpoints of each edge $\{v, e\}$ is zero:

$$\tau'(v, \{v, e\}) + \tau'(e, \{v, e\}) = \tau(v, e) - \tau(v, e) = 0.$$

- This construction preserves the bidirectional relationships of the original hypergraph H in the graph G.

The resulting G is a bidirected graph that represents the original bidirected hypergraph H. Therefore, any bidirected hypergraph can be transformed into a bidirected graph. □

Theorem 4.35. *Any bidirected superhypergraph can be transformed into a bidirected hypergraph.*

Proof. Given a bidirected superhypergraph $H = (V, S, E, \tau)$, we construct a bidirected hypergraph $H' = (V', E', \tau')$ as follows:

(1) *Vertex Set V':* Let $V' = S$. Each supervertex $s \in S$ becomes a vertex in V'.
(2) *Hyperedge Set E':* For each superedge $e \in E$, define a hyperedge $e' = e$, consisting of the supervertices in e.
(3) *Bidirection Function τ':* For each $s \in V'$ and $e' \in E'$, define:

$$\tau'(s, e') = \tau(s, e).$$

Verification:

- The bidirected hypergraph H' has vertices corresponding to the supervertices of H, and hyperedges corresponding to the superedges.
- The local orientations in H' are preserved from H via τ'.

Thus, any bidirected superhypergraph can be transformed into a bidirected hypergraph by considering supervertices as vertices and superedges as hyperedges. □

Theorem 4.36. *Any bidirected superhypergraph can be transformed into an undirected superhypergraph by removing the bidirections.*

Proof. Given a bidirected superhypergraph $H = (V, S, E, \tau)$, we construct an undirected superhypergraph $H' = (V, S, E)$ by:

- Retaining the same vertex set V, supervertex set S, and superedge set E.
- Omitting the bidirection function τ.

Verification:

- The connectivity among supervertices via superedges remains unchanged.
- The removal of τ eliminates all local orientations, resulting in an undirected structure.

The resulting H' is an undirected superhypergraph that retains the original connectivity of H without bidirections. Therefore, any bidirected superhypergraph can be transformed into an undirected superhypergraph. □

We also aim to explore extensions of these concepts, such as fuzzy bidirected graphs and neutrosophic bidirected graphs, by integrating elements from fuzzy and neutrosophic graph theory. While these ideas are still in the conceptual stage, we propose the following definitions as a foundation. Further research into the mathematical structures of these graphs is expected to advance in the future.

Definition 4.37 (Bidirected Fuzzy Graph)**.** A *Bidirected Fuzzy Graph* is a tuple $BFG = (G, \tau, \sigma, \mu)$, where:

- $G = (V, E)$ is a simple undirected graph, where V is the set of vertices and E is the set of edges.

- $\tau : V \times E \to \{-1, 0, 1\}$ is the *bidirection function*, which assigns a direction to each vertex-edge pair as follows:

$$\tau(v, e) = \begin{cases} 1 & \text{if } e \text{ is directed towards } v, \\ -1 & \text{if } e \text{ is directed away from } v, \\ 0 & \text{if } v \text{ is not incident to } e. \end{cases}$$

- $\sigma : V \to [0, 1]$ is the vertex membership function, assigning a membership degree to each vertex $v \in V$.
- $\mu : E \to [0, 1]$ is the edge membership function, assigning a membership degree to each edge $e \in E$.

The membership functions must satisfy the condition:

$$\mu(e) \leq \min\{\sigma(u), \sigma(v)\}, \quad \text{for each edge } e = (u, v) \in E.$$

Definition 4.38 (Bidirected Neutrosophic Graph)**.** A *Bidirected Neutrosophic Graph* is a tuple $BNG = (G, \tau, \sigma, \mu)$, where:

- $G = (V, E)$ is a simple undirected graph.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function as defined above.
- $\sigma : V \to [0, 1]^3$ assigns a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ to each vertex $v \in V$, representing the truth, indeterminacy, and falsity degrees, respectively.
- $\mu : E \to [0, 1]^3$ assigns a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ to each edge $e \in E$.

The membership functions must satisfy the following conditions:

$$\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3, \quad \forall v \in V,$$
$$\mu_T(e) + \mu_I(e) + \mu_F(e) \leq 3, \quad \forall e \in E.$$

Definition 4.39 (Bidirected Plithogenic Graph)**.** A *Bidirected Plithogenic Graph* is a tuple $BPG = (G, \tau, PM, PN)$, where:

- $G = (V, E)$ is a simple undirected graph.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function as defined above.
- $PM = (M, l, Ml, adf, aCf)$ is the Plithogenic Vertex Set:
  - $M \subseteq V$ is the set of vertices.
  - l is the vertex attribute.
  - Ml is the range of possible vertex attribute values.
  - $adf : M \times Ml \to [0, 1]^s$ is the Degree of Appurtenance Function.
  - $aCf : Ml \times Ml \to [0, 1]^t$ is the Degree of Contradiction Function for vertices.
- $PN = (N, m, Nm, bdf, bCf)$ is the Plithogenic Edge Set:
  - $N \subseteq E$ is the set of edges.

– m is the edge attribute.
– Nm is the range of possible edge attribute values.
– $bdf : N \times Nm \to [0,1]^s$ is the Degree of Appurtenance Function.
– $bCf : Nm \times Nm \to [0,1]^t$ is the Degree of Contradiction Function for edges.

The functions adf and bdf must satisfy the Appurtenance Constraint, and aCf and bCf must satisfy the Contradiction Function Constraints as defined in Plithogenic Graphs.

Theorem 4.40. *Bidirected Fuzzy Graphs, Bidirected Neutrosophic Graphs, and Bidirected Plithogenic Graphs are generalizations of Bidirected Graphs.*

Proof. Let $B = (G, \tau)$ be a Bidirected Graph, where:

- $G = (V, E)$ is a simple undirected graph, with V as the set of vertices and E as the set of edges.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function, assigning a local direction to each vertex-edge pair.

A Bidirected Fuzzy Graph $BFG = (G, \tau, \sigma, \mu)$ extends B by introducing:

- $\sigma : V \to [0,1]$, the vertex membership function.
- $\mu : E \to [0,1]$, the edge membership function.

When $\sigma(v) = 1$ for all $v \in V$ and $\mu(e) = 1$ for all $e \in E$, the fuzzy graph reduces to the classical bidirected graph B. Hence, BFG generalizes B.

A Bidirected Neutrosophic Graph $BNG = (G, \tau, \sigma, \mu)$ extends B by introducing:

- $\sigma : V \to [0,1]^3$, assigning truth, indeterminacy, and falsity degrees to each vertex.
- $\mu : E \to [0,1]^3$, assigning truth, indeterminacy, and falsity degrees to each edge.

When $\sigma(v) = (1,0,0)$ and $\mu(e) = (1,0,0)$ for all $v \in V$ and $e \in E$, the neutrosophic graph reduces to B. Thus, BNG generalizes B.

A Bidirected Plithogenic Graph $BPG = (G, \tau, PM, PN)$ extends B by incorporating:

- Plithogenic Vertex Set $PM = (M, l, Ml, adf, aCf)$, which assigns attributes and appurtenance/contradiction values to vertices.
- Plithogenic Edge Set $PN = (N, m, Nm, bdf, bCf)$, which assigns attributes and appurtenance/contradiction values to edges.

When $adf(x, a) = 1$, $bdf(e, b) = 1$, and contradiction functions are zero for all $x \in M, e \in N$, and attributes a, b, the plithogenic graph reduces to B. Hence, BPG generalizes B.

Since each of these graph types reduces to the classical Bidirected Graph under specific conditions, it follows that Bidirected Fuzzy Graphs, Bidirected Neutrosophic Graphs, and Bidirected Plithogenic Graphs are generalizations of Bidirected Graphs. □

Theorem 4.41. *A Bidirected Plithogenic Graph is a generalization of a Plithogenic Graph.*

Proof. Let $PG = (G, PM, PN)$ be a Plithogenic Graph. Now, let $BPG = (G, \tau, PM, PN)$ be a Bidirected Plithogenic Graph, where:

- $G = (V, E)$ and PM, PN are as defined in PG.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function, which assigns a local orientation to each vertex-edge pair.

To prove the generalization, consider the following:

- If $\tau(v, e) = 0$ for all $v \in V$ and $e \in E$, the bidirection function does not influence the structure of the graph. In this case, BPG reduces to PG, as the bidirection function becomes irrelevant.
- All other components of BPG, including the vertex and edge attribute structures (DAF and DCF), remain identical to those in PG.

Since BPG retains all the characteristics of PG and extends it with the bidirection function τ, it follows that every Plithogenic Graph PG can be represented as a Bidirected Plithogenic Graph BPG under appropriate conditions ($\tau(v, e) = 0$ for all v and e).

Thus, Bidirected Plithogenic Graphs generalize Plithogenic Graphs. □

4.7. *Other future tasks: More Graph Classes*

Additionally, we aim to generalize above structures using Fuzzy and Neutrosophic principles [169,170], examining their mathematical frameworks and applications, particularly in decision-making contexts. Furthermore, we plan to extend the concept of the Soft Semihypergraph by integrating ideas from Rough Graphs [41, 49, 96, 137, 196], Soft Expert Graphs [18, 58, 151, 194, 201], Hypersoft Graphs [73, 74, 148, 156–158], Superhypersoft Sets [83, 130, 179, 182], TreeSoft Sets [50,139,180,181,183], Intersection Graphs [69,129], Vague Graphs [3,4,150,160], Single-Valued Pentapartitioned Neutrosophic Graphs [48,72,98,99,147], Hesitant Fuzzy Graphs [20, 84, 138, 142, 215], Infinite Graphs [53, 54, 56], Directed Graphs [24, 37, 103], Mixed Graphs [64, 154, 155], and Superhypergraphs. This approach aims to facilitate a deeper investigation and broader applicability of these advanced structures.

As an additional note, applications of the aforementioned graph classes extend beyond decision-making, which is often explored in uncertain graphs, to promising advancements in areas like superhypergraph neural networks, fuzzy hypergraph neural networks, neutrosophic graph networks, and more. The authors hopes that research in these areas will gain momentum, fostering further developments across these fields.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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