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Neutrosophic Circular-arc Graphs and Proper circular-arc graphs

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Abstract. Graph theory is a fundamental branch of mathematics that studies networks made up of nodes (vertices) and connections (edges). A key concept in graph theory is the intersection graph, where vertices represent sets, and edges are drawn between vertices if their corresponding sets intersect. A circular-arc graph specifically models the intersections of arcs on a circle, with vertices corresponding to the arcs and edges existing between intersecting arcs. This paper delves into the study of circular-arc graphs within the frameworks of fuzzy, intuitionistic fuzzy, neutrosophic, and Turiyam Neutrosophic graphs, all of which incorporate uncertainty into graph structures. Additionally, we examine the concept of proper circular-arc graphs.

Keywords: Neutrosophic graph, Fuzzy graph, Intersection graph, Circular-arc Graph

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1. **Introduction**

1.1. *Circular arc graphs*

Graph theory is a fundamental branch of mathematics that studies networks consisting of nodes (vertices) and connections (edges), which are crucial for analyzing the structure, paths, and properties of these networks [16]. One important example in graph theory is the intersection graph, where vertices correspond to sets, and edges are drawn between vertices if their corresponding sets intersect [32,61,73]. Many related graph classes have been extensively researched, such as interval graphs [28, 33], mixed interval graphs [36–38].

In this paper, we focus on circular-arc graphs [31, 35, 84], a specific type of intersection graph that has garnered significant attention due to its practical applications and importance in the study of various graph classes. A circular-arc graph represents the intersections of arcs on a circle, where vertices correspond to the arcs, and edges exist between arcs that intersect [31,35,84]. Circular-arc graphs are known for their linear-time recognition algorithms [44, 46, 48].

Additionally, several related graph classes have been studied, including: Circular Arc Bigraphs [7], Proper Circular-Arc Bigraphs [69], Chordal Proper Circular Arc Graphs [5], Unit Circular-Arc Graphs [18,47,56], Proper Circular-Arc Graphs [1,8,20,81,86], Helly Circular-Arc Graphs [9, 34, 54, 68], Proper Helly Circular-Arc Graphs [55], Weighted Circular-Arc Graphs [53, 59], Circular-Arc Overlap Graphs [49, 82, 83], Bipartite Co-Circular-Arc Graphs [19], and Circular-Arc Product Graphs [39].

1.2. *Fuzzy Graphs and Neutrosophic Graphs*

In this paper, we explore Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. These graph concepts were developed to handle uncertainty in practical applications.

A fuzzy graph assigns a membership value between 0 and 1 to each vertex and edge, indicating the degree of uncertainty or imprecision associated with them [67]. Fuzzy graphs essentially represent fuzzy sets [17, 52, 88, 89] and are widely used in fields such as social networks, decision-making, and transportation systems, where relationships can be uncertain or ambiguous [62, 67].

Intuitionistic Fuzzy Graphs extend fuzzy graphs by introducing both membership and non-membership degrees for vertices and edges, allowing for a more nuanced representation of uncertainty in relationships [30, 63, 64].

Neutrosophic Graphs [12,22,23,25–27,76–79], derived from neutrosophic set theory [4,75,80], add three components—truth, indeterminacy, and falsity—to classical and fuzzy logic. This approach offers greater flexibility in representing uncertainty .

Turiyam Neutrosophic Graphs, an extension of neutrosophic and fuzzy graphs, assign four attributes—truth, indeterminacy, falsity, and a liberal state—to each vertex and edge [29]. Note that Turiyam Neutrosophic Graphs are a subset of quadripartitioned neutrosophic graphs (cf. [74]).

In addition, various types of intersection graphs have been studied within these frameworks. Research has focused on fuzzy intersection graphs [14, 60, 65, 72], neutrosophic intersection graphs [11], fuzzy permutation graphs [66], and fuzzy interval graphs [14]. Many studies have also explored related topics [24].

1.3. *Our Contribution*

Our motivation and contribution are as follows. As previously mentioned, extensive research has been conducted on Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. Likewise, significant work has been done on Circular-arc graphs in classical graph theory. These studies hold both mathematical and practical significance due to their wide range of applications. However, while intersection graphs have been explored within the frameworks of fuzzy and neutrosophic theories, their full properties and characteristics remain largely unexplored.

In this paper, we define Circular-arc graphs within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs, and examine their properties and interrelationships. Through these studies, we aim to contribute to the advancement of graph theory, particularly in the areas of intersection graphs, Fuzzy Graphs, and Neutrosophic Graphs.

2. **Preliminaries and definitions**

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, and neutrosophic graphs.

2.1. *Basic Graph Concepts*

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [15, 16, 87].

Definition 2.1 (Graph)**.** [16] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2.2 (Degree)**.** [16] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 2.3 (Subgraph)**.** [16] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Definition 2.4 (Induced subgraph)**.** [42, 51] Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For a subset $V' \subseteq V$, the *induced subgraph* $G[V']$ is the graph whose vertex set is V' and whose edge set consists of all edges from E that have both endpoints in V'. Formally, the induced subgraph $G[V'] = (V', E')$ is defined as follows:

$$E' = \{(u, v) \in E \mid u \in V', v \in V'\}.$$

In other words, $G[V']$ is the subgraph of G that contains all vertices in V' and all edges from G whose endpoints are both in V'.

Definition 2.5 (Complete Graph)**.** A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.
- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Each vertex has degree $\deg(v) = n - 1$ for all $v \in V$.

2.2. *Intersection graph*

Intersection graphs have been extensively studied. The definition is provided below [32, 61, 73].

Definition 2.6 (Intersection graph)**.** [32,61,73] A *intersection graph* is a graph that represents the intersection relationships between sets. Formally, let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a collection of sets. The *intersection graph* $G = (V, E)$ associated with $\mathcal{S}$ is a graph where:

- The vertex set V corresponds to the sets in $\mathcal{S}$, i.e., $V = \{v_1, v_2, \ldots, v_n\}$, where each vertex v_i represents the set $S_i \in \mathcal{S}$.
- There is an edge $(v_i, v_j) \in E$ if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$.

2.3. *Fuzzy, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs*

In this subsection, we examine Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs.

Fuzzy graphs are frequently discussed in comparison with crisp graphs, which represent the classical form of graphs [67].

Definition 2.7. (cf. [50, 67]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- V is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form (v, v)) and edges represent binary relationships between distinct vertices.

Taking the above into consideration, we define Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs as follows. Please note that the definitions have been consolidated for simplicity.

Definition 2.8 (Unified Graphs Framework: Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs). (cf. [23]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, and falsity.

(1) *Fuzzy Graph [67]:*
- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$, representing the degree of participation of v in the fuzzy graph.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$, representing the strength of the connection between u and v.

(2) *Intuitionistic Fuzzy Graph (IFG) [2]:*
- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $v_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + v_A(v) \leq 1$.
- Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ (degree of membership) and $v_B(u, v) \in [0, 1]$ (degree of non-membership), such that $\mu_B(u, v) + v_B(u, v) \leq 1$.

(3) *Neutrosophic Graph [3, 43, 45, 70]:*
- Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
  - $\sigma_T(v) \in [0, 1]$ is the truth-membership degree,
  - $\sigma_I(v) \in [0, 1]$ is the indeterminacy-membership degree,
  - $\sigma_F(v) \in [0, 1]$ is the falsity-membership degree,
  - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between u and v.

(4) *Turiyam Neutrosophic Graph [29]:*
- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
  - $t(v) \in [0, 1]$ is the truth value,
  - $iv(v) \in [0, 1]$ is the indeterminacy value,
  - $fv(v) \in [0, 1]$ is the falsity value,
  - $lv(v) \in [0, 1]$ is the liberal state value,
  - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.

- Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between u and v.

3. Result: Circular-arc graph

In this section, we extend the concept of Circular-arc graphs to Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The definition of a Circular-arc graph is provided below. As mentioned in the introduction, Circular-arc graphs are a well-studied topic in graph theory, with numerous ongoing research efforts [10,35,57,71, 81,85].

Definition 3.1 (Circular-arc graph)**.** [85] A *circular-arc graph* is the intersection graph of a set of arcs on a circle. Formally, let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1, where each I_i represents an arc of the circle. The corresponding circular-arc graph $G = (V, E)$ is defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$ consists of one vertex for each arc in the set.
- The edge set E is defined such that there is an edge between two vertices I_α and I_β if and only if their corresponding arcs I_α and I_β intersect, i.e.,

$$\{I_\alpha, I_\beta\} \in E \iff I_\alpha \cap I_\beta \neq \emptyset.$$

A family of arcs corresponding to the graph G is referred to as an *arc model* of the circular-arc graph.

The following theorem is clearly valid.

Theorem 3.2. *A Circular-Arc Graph satisfies the definition of an Intersection Graph.*

Proof. In a circular-arc graph $G = (V, E)$, the vertex set $V = \{I_1, I_2, \ldots, I_n\}$ corresponds to a collection of arcs $\{I_1, I_2, \ldots, I_n\}$ on a circle C_1. Each vertex v_i in the graph represents an arc I_i on the circle. This directly satisfies the definition of an intersection graph, where each vertex corresponds to a set (in this case, an arc).

And in a circular-arc graph, there is an edge between two vertices I_α and I_β if and only if the corresponding arcs I_α and I_β intersect, i.e., $I_\alpha \cap I_\beta \neq \emptyset$. This is consistent with the edge definition of an intersection graph, where an edge exists between two vertices if their corresponding sets (arcs) have a non-empty intersection.

By these definitions, we can see that the structure of a circular-arc graph satisfies the criteria of an intersection graph. Each vertex in a circular-arc graph represents a set (an arc), and there is an edge between two vertices if and only if their corresponding arcs (sets) intersect. Therefore, a *Circular-Arc Graph* is an instance of an *Intersection Graph.* □

We extend the above concept to Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The definitions are provided below.

Definition 3.3 (Fuzzy Circular-Arc Graph)**.** (cf. [41, 65]) Let C_1 be a circle, and let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on C_1. A *Fuzzy Circular-Arc Graph* is a fuzzy graph $\psi = (V, \sigma, \mu)$ defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$, where each vertex corresponds to an arc I_i.
- The vertex membership function $\sigma : V \to [0, 1]$ assigns to each vertex I_i a membership degree $\sigma(I_i)$, representing the degree of membership of the arc I_i in the fuzzy graph.
- The edge membership function $\mu : V \times V \to [0, 1]$ is defined by:

$$\mu(I_\alpha, I_\beta) = \begin{cases} \min\{\sigma(I_\alpha), \sigma(I_\beta)\}, & \text{if } I_\alpha \cap I_\beta \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

These functions satisfy the following conditions:

(1) $\mu(I_\alpha, I_\beta) \leq \min\{\sigma(I_\alpha), \sigma(I_\beta)\} \quad \text{for all } I_\alpha, I_\beta \in V$,

(2) $\mu(I_\alpha, I_\beta) = \mu(I_\beta, I_\alpha) \quad \text{(symmetry)}$,

(3) $\mu(I_\alpha, I_\alpha) = 0 \quad \text{(no self-loops)}$.

The resulting fuzzy graph represents the degrees of connection between arcs based on their intersections and membership degrees.

Definition 3.4 (Intuitionistic Fuzzy Circular-Arc Graph)**.** Let C_1 be a circle, and let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on C_1. An *Intuitionistic Fuzzy Circular-Arc Graph* is an intuitionistic fuzzy graph $G_{IF} = (A, B)$ defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$.
- The intuitionistic fuzzy vertex set $A = \{\langle I_i, \mu_A(I_i), \nu_A(I_i)\rangle : I_i \in V\}$, where:
  - $\mu_A(I_i) \in [0, 1]$ is the degree of membership of I_i,
  - $\nu_A(I_i) \in [0, 1]$ is the degree of non-membership of I_i,
  - $\mu_A(I_i) + \nu_A(I_i) \leq 1$.
- The intuitionistic fuzzy edge set $B = \{\langle (I_\alpha, I_\beta), \mu_B(I_\alpha, I_\beta), \nu_B(I_\alpha, I_\beta)\rangle : I_\alpha, I_\beta \in V\}$, where:
  - $\mu_B(I_\alpha, I_\beta) = \begin{cases} \min\{\mu_A(I_\alpha), \mu_A(I_\beta)\}, & \text{if } I_\alpha \cap I_\beta \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$
  - $\nu_B(I_\alpha, I_\beta) = \begin{cases} \max\{\nu_A(I_\alpha), \nu_A(I_\beta)\}, & \text{if } I_\alpha \cap I_\beta \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$

Definition 3.5 (Neutrosophic Circular-Arc Graph)**.** Let C_1 be a circle, and let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on C_1. A *Neutrosophic Circular-Arc Graph* is a neutrosophic graph $G = (V, E, \sigma, \mu)$ defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$.
- The neutrosophic vertex membership function $\sigma : V \to [0,1]^3$, assigning to each vertex I_i a triple $\sigma(I_i) = (\sigma_T(I_i), \sigma_I(I_i), \sigma_F(I_i))$, where:
  - $\sigma_T(I_i)$ is the truth-membership degree,
  - $\sigma_I(I_i)$ is the indeterminacy-membership degree,
  - $\sigma_F(I_i)$ is the falsity-membership degree,
  - $0 \leq \sigma_T(I_i) + \sigma_I(I_i) + \sigma_F(I_i) \leq 3$.
- The edge set $E = \{(I_\alpha, I_\beta) : I_\alpha, I_\beta \in V, I_\alpha \cap I_\beta \neq \emptyset\}$.
- The neutrosophic edge membership function $\mu : E \to [0,1]^3$, assigning to each edge (I_α, I_β) a triple $\mu(I_\alpha, I_\beta) = (\mu_T(I_\alpha, I_\beta), \mu_I(I_\alpha, I_\beta), \mu_F(I_\alpha, I_\beta))$, where:
  - $\mu_T(I_\alpha, I_\beta) = \min\{\sigma_T(I_\alpha), \sigma_T(I_\beta)\}$,
  - $\mu_I(I_\alpha, I_\beta) = \max\{\sigma_I(I_\alpha), \sigma_I(I_\beta)\}$,
  - $\mu_F(I_\alpha, I_\beta) = \max\{\sigma_F(I_\alpha), \sigma_F(I_\beta)\}$,
  - $0 \leq \mu_T(I_\alpha, I_\beta) + \mu_I(I_\alpha, I_\beta) + \mu_F(I_\alpha, I_\beta) \leq 3$.

Definition 3.6 (Turiyam Neutrosophic Circular-Arc Graph)**.** Let C_1 be a circle, and let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on C_1. A *Turiyam Neutrosophic Circular-Arc Graph* is a Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ defined as follows:

- The vertex set $V^T = \{I_1, I_2, \ldots, I_n\}$.
- For each vertex $I_i \in V^T$, the Turiyam Neutrosophic membership functions assign:
  - Truth value $t(I_i) \in [0,1]$,
  - Indeterminacy value $iv(I_i) \in [0,1]$,
  - Falsity value $fv(I_i) \in [0,1]$,
  - Liberal value $lv(I_i) \in [0,1]$,
  - Constraint: $0 \leq t(I_i) + iv(I_i) + fv(I_i) + lv(I_i) \leq 4$.
- The edge set $E^T = \{(I_\alpha, I_\beta) : I_\alpha, I_\beta \in V^T, I_\alpha \cap I_\beta \neq \emptyset\}$.
- For each edge $(I_\alpha, I_\beta) \in E^T$, the Turiyam Neutrosophic membership functions assign:
  - Truth value $t(I_\alpha, I_\beta) = \min\{t(I_\alpha), t(I_\beta)\}$,
  - Indeterminacy value $iv(I_\alpha, I_\beta) = \max\{iv(I_\alpha), iv(I_\beta)\}$,
  - Falsity value $fv(I_\alpha, I_\beta) = \max\{fv(I_\alpha), fv(I_\beta)\}$,
  - Liberal value $lv(I_\alpha, I_\beta) = \max\{lv(I_\alpha), lv(I_\beta)\}$,
  - Constraint: $0 \leq t(I_\alpha, I_\beta) + iv(I_\alpha, I_\beta) + fv(I_\alpha, I_\beta) + lv(I_\alpha, I_\beta) \leq 4$.

We now examine the properties of the graph mentioned above. The following theorem holds.

Theorem 3.7. *A Neutrosophic circular-arc graph can be transformed into a classic circular-arc graph.*

Proof. Obviously holds. □

Corollary 3.8. *A Fuzzy Circular-Arc Graph, Intuitionistic Fuzzy Circular-Arc Graph, or Turiyam Neutrosophic Circular-Arc Graph can be transformed into a classic Circular-Arc Graph.*

Proof. Obviously holds. □

Theorem 3.9. *Neutrosophic Graph can be represented as a Neutrosophic Circular-Arc Graph.*

Proof. We represent the vertices of the neutrosophic graph as arcs on a circle. Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices.
- $E \subseteq V \times V$ is the set of edges between the vertices.
- $\sigma(v_i) = (\sigma_T(v_i), \sigma_I(v_i), \sigma_F(v_i))$ assigns truth, indeterminacy, and falsity membership degrees to each vertex $v_i \in V$.
- $\mu(v_i, v_j) = (\mu_T(v_i, v_j), \mu_I(v_i, v_j), \mu_F(v_i, v_j))$ assigns truth, indeterminacy, and falsity membership degrees to each edge $(v_i, v_j) \in E$.

We now need to represent G as a Neutrosophic Circular-Arc Graph. To do this, map each vertex v_i to an arc I_i on a circle C_1.

We define the vertex set of the Neutrosophic Circular-Arc Graph. In the Neutrosophic Circular-Arc Graph, each vertex $v_i \in V$ of the neutrosophic graph is represented by an arc $I_i \subseteq C_1$. Therefore, the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ in the neutrosophic graph corresponds to the set of arcs $\{I_1, I_2, \ldots, I_n\}$ on the circle C_1. Each vertex I_i in the Neutrosophic Circular-Arc Graph is assigned the same neutrosophic membership degree as the corresponding vertex v_i in the neutrosophic graph:

$$\sigma(I_i) = (\sigma_T(I_i), \sigma_I(I_i), \sigma_F(I_i)) = \sigma(v_i).$$

We define the edge set of the Neutrosophic Circular-Arc Graph. In the Neutrosophic Circular-Arc Graph, an edge exists between two arcs I_α and I_β if and only if the arcs intersect, i.e., $I_\alpha \cap I_\beta \neq \emptyset$. This corresponds to the edges in the neutrosophic graph. That is, there is an edge between v_α and $v_\beta \in V$ in the neutrosophic graph if and only if there is an edge between the corresponding arcs I_α and I_β in the Neutrosophic Circular-Arc Graph.

For the Neutrosophic Circular-Arc Graph, the neutrosophic membership degrees for each edge (I_α, I_β) are defined based on the membership degrees of the corresponding vertices:

$$\mu_T(I_\alpha, I_\beta) = \min\{\sigma_T(I_\alpha), \sigma_T(I_\beta)\},$$

$$\mu_I(I_\alpha, I_\beta) = \max\{\sigma_I(I_\alpha), \sigma_I(I_\beta)\},$$

$$\mu_F(I_\alpha, I_\beta) = \max\{\sigma_F(I_\alpha), \sigma_F(I_\beta)\}.$$

This assignment of membership degrees ensures that the edge relationships in the neutrosophic graph are preserved in the Neutrosophic Circular-Arc Graph.

We have successfully mapped each vertex of the neutrosophic graph G to an arc on the circle and preserved the edge relationships by ensuring that arcs intersect if and only if their corresponding vertices are connected by an edge in G. Additionally, the neutrosophic membership degrees for vertices and edges have been preserved. □

Theorem 3.10. *Every subgraph of a Neutrosophic Circular-Arc Graph is also a Neutrosophic Circular-Arc Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circular-Arc Graph where each vertex represents an arc on a circle, and let $G' = (V', E', \sigma', \mu')$ be a subgraph of G with $V' \subseteq V$ and $E' \subseteq E$. The vertices in V' correspond to a subset of the arcs in G, and the edges in E' represent the intersections between these arcs, following the same intersection rules as in G.

Since the vertices and edges of G' inherit their structure from G, including the neutrosophic membership functions σ' and μ', the arcs in V' still form a valid arc model on the circle. Therefore, the subgraph G' is also a Neutrosophic Circular-Arc Graph.

Thus, every subgraph of a Neutrosophic Circular-Arc Graph is also a Neutrosophic Circular-Arc Graph. □

Theorem 3.11. *The disjoint union of two Neutrosophic Circular-Arc Graphs is also a Neutrosophic Circular-Arc Graph.*

Proof. Let $G_1 = (V_1, E_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, E_2, \sigma_2, \mu_2)$ be two Neutrosophic Circular-Arc Graphs. The disjoint union $G = G_1 \sqcup G_2$ is defined by taking the union of the vertex sets and edge sets, i.e., $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, with no edges between V_1 and V_2.

Since the arcs corresponding to vertices in V_1 and V_2 form valid arc models independently on separate parts of the circle, their disjoint union also forms a valid arc model. The neutrosophic membership functions σ and μ are preserved for each respective graph, ensuring that G satisfies the conditions for being a Neutrosophic Circular-Arc Graph.

Thus, the disjoint union of two Neutrosophic Circular-Arc Graphs is also a Neutrosophic Circular-Arc Graph. □

Theorem 3.12. *The complement of a Neutrosophic Circular-Arc Graph is also a Neutrosophic Circular-Arc Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circular-Arc Graph, where V represents a set of arcs on a circle, and E represents the edges between intersecting arcs. The complement graph $\overline{G} = (V, \overline{E}, \overline{\sigma}, \overline{\mu})$ has the same vertex set, but the edge set $\overline{E}$ consists of all pairs (u, v) such that $(u, v) \notin E$.

In the context of circular-arc graphs, an edge $(u, v) \in E$ exists if and only if the corresponding arcs I_u and I_v intersect. For the complement graph $\overline{G}$, the edge $(u, v) \in \overline{E}$ exists if and only if I_u and I_v do not intersect. Since non-intersecting arcs also form valid configurations on a circle, the complement graph retains the structure of a circular-arc graph.

The neutrosophic membership functions $\overline{\sigma}(v)$ and $\overline{\mu}(u, v)$ are defined as follows:

$$\overline{\sigma}(v) = \sigma(v),$$

$$\overline{\mu}(u, v) = \begin{cases} \min(\sigma_T(u), \sigma_T(v)) & \text{if } I_u \cap I_v = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Since the complement preserves the structure and neutrosophic properties, the complement of a Neutrosophic Circular-Arc Graph is also a Neutrosophic Circular-Arc Graph. □

Theorem 3.13. *If the union of all arcs in a Neutrosophic Circular-Arc Graph c overs the entire circle, then the graph is connected.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circular-Arc Graph, where each vertex $v \in V$ corresponds to an arc I_v on a circle C_1. Assume that the union of all arcs covers the entire circle:

$$\bigcup_{v \in V} I_v = C_1.$$

We aim to show that G is connected; that is, for any two vertices $u, w \in V$, there exists a path in G connecting u and w.

Since the arcs $\{I_v\}_{v \in V}$ cover C_1 completely, we can traverse the circle from any point in I_u to any point in I_w without leaving the union of the arcs. Along this traversal, we pass through a sequence of arcs corresponding to vertices in V. Specifically, there exists a sequence of vertices $v_0, v_1, \ldots, v_k \in V$ such that:

- $v_0 = u$ and $v_k = w$,
- For each $i = 0, 1, \ldots, k - 1$, the arcs I_{v_i} and $I_{v_{i+1}}$ intersect: $I_{v_i} \cap I_{v_{i+1}} \neq \emptyset$.

Because arcs I_{v_i} and $I_{v_{i+1}}$ intersect, there is an edge $(v_i, v_{i+1}) \in E$. Therefore, the sequence $v_0, v_1, \ldots, v_k$ forms a path in G connecting u and w.

Since u and w were arbitrary, G is connected. □

Theorem 3.14. *Deleting a vertex (and its incident edges) from a Neutrosophic Circular-Arc Graph results in another Neutrosophic Circular-Arc Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circular-Arc Graph with an arc model $\{I_v \mid v \in V\}$ on the circle C_1. Let $v_0 \in V$ be the vertex to be deleted. Define the new graph $G' = (V', E', \sigma', \mu')$ as follows:

- $V' = V \setminus \{v_0\}$,
- $E' = \{(u, v) \in E \mid u, v \in V'\}$,
- σ' and μ' are the restrictions of σ and μ to V' and E', respectively.

The arcs corresponding to G' are $\{I_v \mid v \in V'\}$, which are the same as in G except that I_{v_0} is removed. These arcs still lie on the circle C_1.

The intersection relationships among the remaining arcs are preserved. For any $u, w \in V'$, the arcs I_u and I_w intersect if and only if $(u, w) \in E'$. The neutrosophic membership functions σ' and μ' retain the same values as in G for the vertices and edges of G'.

Therefore, G' satisfies all the conditions of a Neutrosophic Circular-Arc Graph. Thus, deleting a vertex and its incident edges from G results in another Neutrosophic Circular-Arc Graph. □

We examine the relationship between Neutrosophic Circular-Arc Graphs and Neutrosophic Interval Graphs. In interval graphs [13, 33, 40], each vertex represents an interval on the real line, and edges exist between vertices whose intervals overlap. A Neutrosophic Interval Graph extends this concept by incorporating neutrosophic membership degrees, allowing for the representation of uncertainty in both vertices and edges. The formal definition of a Neutrosophic Interval Graph is provided below.

Definition 3.15 (Neutrosophic Interval Graph)**.** [21] Let V be a finite set of vertices, and let

$$\mathcal{N} = \{(\mu_1, \tau_1, \zeta_1), (\mu_2, \tau_2, \zeta_2), \ldots, (\mu_n, \tau_n, \zeta_n)\}$$

be a finite family of neutrosophic intervals on the real line $\mathbb{R}$. Each triple (μ_i, τ_i, ζ_i) represents the truth-membership function $\mu_i : \mathbb{R} \to [0, 1]$, the indeterminacy-membership function $\tau_i : \mathbb{R} \to [0, 1]$, and the falsity-membership function $\zeta_i : \mathbb{R} \to [0, 1]$ such that:

$$\mu_i(x) + \tau_i(x) + \zeta_i(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

The *Neutrosophic Interval Graph* $G = (V, \mu_V, \tau_V, \zeta_V, \rho_\mu, \rho_\tau, \rho_\zeta)$ is defined as follows:

- The vertex membership functions $\mu_V : V \to [0, 1]$, $\tau_V : V \to [0, 1]$, and $\zeta_V : V \to [0, 1]$ are given by:

$$\mu_V(v_i) = \sup_{x \in \mathbb{R}} \mu_i(x), \quad \tau_V(v_i) = \sup_{x \in \mathbb{R}} \tau_i(x), \quad \zeta_V(v_i) = \sup_{x \in \mathbb{R}} \zeta_i(x).$$

- The neutrosophic adjacency relations $\rho_\mu : V \times V \to [0,1]$, $\rho_\tau : V \times V \to [0,1]$, and $\rho_\zeta : V \times V \to [0,1]$ are defined by:

$$\rho_\mu(v_i, v_j) = \begin{cases} \sup_{x\in\mathbb{R}} \min\{\mu_i(x), \mu_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_\tau(v_i, v_j) = \begin{cases} \sup_{x\in\mathbb{R}} \min\{\tau_i(x), \tau_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\rho_\zeta(v_i, v_j) = \begin{cases} \sup_{x\in\mathbb{R}} \min\{\zeta_i(x), \zeta_j(x)\}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

Theorem 3.16. *[21] a Neutrosophic Interval Graph can be transformed into a classic Interval Graph.*

Proof. Refer to the literature [21]. □

Theorem 3.17. *Every Neutrosophic Interval Graph can be represented as a Neutrosophic Circular-Arc Graph.*

Proof. Consider a *Neutrosophic Interval Graph* $G = (V, E, \sigma, \mu)$, where each vertex corresponds to an interval on the real line, and edges exist between vertices whose intervals overlap. The neutrosophic membership functions assign truth, indeterminacy, and falsity degrees to each vertex and edge. We will prove that this interval graph can be transformed into a neutrosophic circular-arc graph.

We begin by mapping the intervals on the real line to arcs on a circle.

- Consider the real line as a straight line segment where each vertex in the interval graph represents an interval.
- Map this straight line onto a circle C_1, ensuring that no intervals wrap around the circle. In this process, each interval on the real line is transformed into an arc on the circle.
- Thus, each vertex $v \in V$ that corresponds to an interval on the real line will now correspond to an arc on the circle C_1.

Next, we verify that the intersection relationships between intervals are preserved during this transformation.

- Two intervals on the real line intersect if and only if their corresponding arcs on the circle intersect, provided there is no overlap or wrapping.

- Since the mapping from intervals to arcs does not introduce any overlap or wrapping, the intersection relationships between intervals are exactly maintained in the corresponding arcs on C_1.
- Therefore, for each pair of intervals that overlap on the real line, their corresponding arcs will intersect on the circle.

Now, we transfer the neutrosophic membership functions from the interval graph to the corresponding circular-arc graph.

- For each vertex $v \in V$ in the interval graph, the neutrosophic membership function $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, which assigns truth, indeterminacy, and falsity degrees, remains unchanged when transferred to the corresponding arc in the circular-arc graph.
- Similarly, for each edge $e \in E$, which corresponds to the overlap of intervals, the neutrosophic membership function $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ is preserved when transferred to the corresponding edge in the circular-arc graph.
- Thus, the neutrosophic membership degrees of both vertices and edges are maintained in the transformation from the interval graph to the circular-arc graph.

By mapping the intervals to arcs, preserving the intersection relationships, and maintaining the neutrosophic membership functions, we have shown that every Neutrosophic Interval Graph can be represented as a Neutrosophic Circular-Arc Graph. □

4. Result: Proper-Circular-Arc Graph

In this section, we extend the definition of Proper Circular-Arc Graphs to Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs. These graph models incorporate uncertainty and assign degrees of membership, indeterminacy, and falsity, depending on the specific type of graph. A Proper Circular-Arc Graph is essentially a more constrained version of a Circular-Arc Graph, where no arc is properly contained within another.

Below, we provide the classical definition of a Proper Circular-Arc Graph. As mentioned in the introduction, Proper Circular-Arc Graphs, like Circular-Arc Graphs, have been the subject of various studies [1, 8, 20, 86].

Definition 4.1 (Proper Circular-Arc Graph)**.** A *proper circular-arc graph* is a circular-arc graph with an intersection model in which no arc properly contains another arc.

Formally, let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1, where each I_i represents an arc of the circle. The corresponding circular-arc graph $G = (V, E)$ is defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$ consists of one vertex for each arc in the set.

- The edge set E is defined such that there is an edge between two vertices I_α and I_β if and only if their corresponding arcs I_α and I_β intersect, i.e.,

$$\{I_\alpha, I_\beta\} \in E \iff I_\alpha \cap I_\beta \neq \emptyset.$$

A circular-arc graph is said to be *proper* if there exists an arc model in which no arc is properly contained within another, i.e., for all $I_i, I_j \in \{I_1, I_2, \ldots, I_n\}$, it holds that:

$$I_i \subset I_j \quad \text{or} \quad I_j \subset I_i \quad \text{does not occur.}$$

The generalized definitions for each type of graph are provided below.

Definition 4.2 (Fuzzy Proper Circular-Arc Graph)**.** A *Fuzzy Proper Circular-Arc Graph* is a fuzzy graph $G = (V, \sigma, \mu)$ with an arc model in which no arc properly contains another arc. Formally, let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1, where each I_i represents an arc. The graph G is defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$.
- The membership function $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex I_i.
- The edge membership function $\mu : V \times V \to [0, 1]$ is given by:

$$\mu(I_\alpha, I_\beta) = \begin{cases} \min(\sigma(I_\alpha), \sigma(I_\beta)) & \text{if } I_\alpha \cap I_\beta \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

- The arcs satisfy the property that no arc properly contains another, i.e., $I_i \subset I_j$ for $i \neq j$ does not hold.

Definition 4.3 (Intuitionistic Fuzzy Proper Circular-Arc Graph)**.** An *Intuitionistic Fuzzy Proper Circular-Arc Graph* is an intuitionistic fuzzy graph $G_{IF} = (A, B)$ with a proper arc model. Let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on the circle C_1. The graph G_{IF} is defined as follows:

- The vertex set $A = \{(I_i, \mu_A(I_i), \nu_A(I_i)) : I_i \in V\}$, where $\mu_A(I_i)$ is the membership degree and $\nu_A(I_i)$ is the non-membership degree.
- The edge set $B = \{((I_\alpha, I_\beta), \mu_B(I_\alpha, I_\beta), \nu_B(I_\alpha, I_\beta))\}$, where:

$$\mu_B(I_\alpha, I_\beta) = \min(\mu_A(I_\alpha), \mu_A(I_\beta)),$$

$$\nu_B(I_\alpha, I_\beta) = \max(\nu_A(I_\alpha), \nu_A(I_\beta)).$$

- No arc properly contains another, i.e., $I_i \subset I_j$ does not hold.

Definition 4.4 (Neutrosophic Proper Circular-Arc Graph)**.** A *Neutrosophic Proper Circular-Arc Graph* is a neutrosophic graph $G = (V, E, \sigma, \mu)$ defined with a proper arc model on a circle. Let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a set of arcs, with the following structure:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$.
- The neutrosophic membership function $\sigma : V \to [0,1]^3$ assigns a triple $(\sigma_T(I_i), \sigma_I(I_i), \sigma_F(I_i))$ to each arc I_i.
- The edge set $E = \{(I_\alpha, I_\beta)\}$, where the neutrosophic edge membership function μ is given by:

$$\mu_T(I_\alpha, I_\beta) = \min(\sigma_T(I_\alpha), \sigma_T(I_\beta)),$$

$$\mu_I(I_\alpha, I_\beta) = \max(\sigma_I(I_\alpha), \sigma_I(I_\beta)),$$

$$\mu_F(I_\alpha, I_\beta) = \max(\sigma_F(I_\alpha), \sigma_F(I_\beta)).$$

- No arc properly contains another, i.e., $I_i \subset I_j$ does not hold.

Definition 4.5 (Turiyam Neutrosophic Proper Circular-Arc Graph)**.** A *Turiyam Neutrosophic Proper Circular-Arc Graph* is a Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ where no arc properly contains another. Let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1. The graph G^T is defined as follows:

- The vertex set $V^T = \{I_1, I_2, \ldots, I_n\}$.
- Each vertex $I_i \in V^T$ is assigned a quadruple of membership degrees: $(t(I_i), iv(I_i), fv(I_i), lv(I_i))$, representing truth, indeterminacy, falsity, and liberal values.
- The edge set $E^T = \{(I_\alpha, I_\beta)\}$, where:

$$t(I_\alpha, I_\beta) = \min(t(I_\alpha), t(I_\beta)),$$

$$iv(I_\alpha, I_\beta) = \max(iv(I_\alpha), iv(I_\beta)),$$

$$fv(I_\alpha, I_\beta) = \max(fv(I_\alpha), fv(I_\beta)),$$

$$lv(I_\alpha, I_\beta) = \max(lv(I_\alpha), lv(I_\beta)).$$

- No arc properly contains another, i.e., $I_i \subset I_j$ does not hold.

Theorem 4.6. *A Neutrosophic Proper Circular-Arc Graph can be transformed into a Classic Proper Circular-Arc Graph by assigning specific values to the truth, indeterminacy, and falsity membership functions.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Proper Circular-Arc Graph. We aim to transform G into a *Classic Proper Circular-Arc Graph* by assigning specific values to the truth, indeterminacy, and falsity membership functions.

We Set the truth membership for vertices. In the Neutrosophic Graph, each vertex I_i has associated truth $\sigma_T(I_i)$, indeterminacy $\sigma_I(I_i)$, and falsity $\sigma_F(I_i)$. To transform this graph into a classic one, we must ensure that the presence of vertices is certain and that there is no indeterminacy or falsity.

To achieve this, we assign the following values to all vertices:

$$\sigma_T(I_i) = 1, \quad \sigma_I(I_i) = 0, \quad \sigma_F(I_i) = 0 \quad \forall I_i \in V.$$

This guarantees that each vertex is fully present in the graph, with no uncertainty or falsity.

We Set the truth membership for edges. For each edge $(I_\alpha, I_\beta) \in E$, the neutrosophic membership function assigns values to the truth, indeterminacy, and falsity of the edges. To create a Classic Proper Circular-Arc Graph, we set the truth membership for all edges to 1 and indeterminacy and falsity degrees to 0:

$$\mu_T(I_\alpha, I_\beta) = 1, \quad \mu_I(I_\alpha, I_\beta) = 0, \quad \mu_F(I_\alpha, I_\beta) = 0 \quad \forall (I_\alpha, I_\beta) \in E.$$

This ensures that edges definitively exist between intersecting arcs, without uncertainty or falsity.

We explain about preserving the properness condition. A key property of a *Proper Circular-Arc Graph* is that no arc is properly contained within another. In the Neutrosophic context, this property can be maintained by ensuring that no vertex has indeterminacy or falsity values that could imply partial or uncertain containment.

By setting $\sigma_I(I_i) = 0$ and $\sigma_F(I_i) = 0$ for all vertices, we ensure that no arc is improperly nested within another. This guarantees that the arcs in the resulting Classic Proper Circular-Arc Graph remain distinct and non-nested. □

Corollary 4.7. *A Fuzzy Proper Circular-Arc Graph, Intuitionistic Fuzzy Proper Circular-Arc Graph, or Turiyam Neutrosophic Proper Circular-Arc Graph can be transformed into a classic Proper Circular-Arc Graph.*

Proof. It can be proven in the same way as above. □

Theorem 4.8. *Neutrosophic Proper Circular-Arc Graph is special case of Neutrosophic Circular-Arc Graph.*

Proof. This follows directly from the definitions. □

Corollary 4.9. *A Fuzzy Proper Circular-Arc Graph is a special case of a Fuzzy Circular-Arc Graph, an Intuitionistic Fuzzy Proper Circular-Arc Graph is a special case of an Intuitionistic Fuzzy Circular-Arc Graph, and a Turiyam Neutrosophic Proper Circular-Arc Graph is a special case of a Turiyam Neutrosophic Circular-Arc Graph.*

Proof. This follows directly from the definitions. In each case, the "proper" version of the graph imposes an additional constraint that no arc is properly contained within another, while retaining the underlying structure and properties of the more general graph. Therefore, the proper versions are specific instances of their respective general forms. □

Theorem 4.10. *A Neutrosophic Proper Circular-Arc Graph can be transformed into a Fuzzy Proper Circular-Arc Graph, Intuitionistic Fuzzy Proper Circular-Arc Graph, or Turiyam Neutrosophic Proper Circular-Arc Graph by appropriately adjusting the truth, indeterminacy, and falsity membership functions.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a *Neutrosophic Proper Circular-Arc Graph*. We aim to show how G can be transformed into either a *Fuzzy Proper Circular-Arc Graph*, an *Intuitionistic Fuzzy Proper Circular-Arc Graph*, or a *Turiyam Neutrosophic Proper Circular-Arc Graph*.

First, we consider about transformation to Fuzzy Proper Circular-Arc Graph. To transform G into a Fuzzy Proper Circular-Arc Graph, we simplify the neutrosophic membership functions by collapsing the three degrees (truth, indeterminacy, falsity) into a single fuzzy membership value. We do this by taking only the truth-membership degree:

$$\sigma_{\text{fuzzy}}(I_i) = \sigma_T(I_i), \quad \mu_{\text{fuzzy}}(I_\alpha, I_\beta) = \mu_T(I_\alpha, I_\beta).$$

This transformation preserves the structure of G while discarding the indeterminacy and falsity components, yielding a proper fuzzy circular-arc graph.

We consider about transformation to Intuitionistic Fuzzy Proper Circular-Arc Graph. To transform G into an Intuitionistic Fuzzy Proper Circular-Arc Graph, we use the truth-membership degree $\sigma_T(I_i)$ as the membership degree and the falsity-membership degree $\sigma_F(I_i)$ as the non-membership degree:

$$\mu_A(I_i) = \sigma_T(I_i), \quad \nu_A(I_i) = \sigma_F(I_i),$$

$$\mu_B(I_\alpha, I_\beta) = \mu_T(I_\alpha, I_\beta), \quad \nu_B(I_\alpha, I_\beta) = \mu_F(I_\alpha, I_\beta).$$

The indeterminacy component $\sigma_I(I_i)$ is implicitly accounted for by the condition that:

$$\mu_A(I_i) + \nu_A(I_i) \leq 1.$$

Thus, G is transformed into an Intuitionistic Fuzzy Proper Circular-Arc Graph.

We consider about transformation to Turiyam Neutrosophic Proper Circular-Arc Graph. To transform G into a Turiyam Neutrosophic Proper Circular-Arc Graph, we assign the following values:

$$t(I_i) = \sigma_T(I_i), \quad iv(I_i) = \sigma_I(I_i), \quad fv(I_i) = \sigma_F(I_i), \quad lv(I_i) = 0.$$

Similarly, for edges:

$$t(I_\alpha, I_\beta) = \mu_T(I_\alpha, I_\beta), \quad iv(I_\alpha, I_\beta) = \mu_I(I_\alpha, I_\beta), \quad fv(I_\alpha, I_\beta) = \mu_F(I_\alpha, I_\beta), \quad lv(I_\alpha, I_\beta) = 0.$$

The liberal value $lv(I_i) = 0$ is set to zero since it is not present in the original neutrosophic graph. This transforms G into a Turiyam Neutrosophic Proper Circular-Arc Graph while maintaining the original structure of the arcs and intersections. □

Theorem 4.11. *In a Neutrosophic Proper Circular-Arc Graph, no neutrosophic Circular-Arc is properly contained within another with respect to the truth-membership function μ_T.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Proper Circular-Arc Graph.

By definition, a Proper Circular-Arc Graph does not allow any arc I_i to be properly contained within another arc I_j. Formally, for all $I_i, I_j \in V$, we have:

$$I_i \subset I_j \quad \text{or} \quad I_j \subset I_i \quad \text{does not occur.}$$

In a Neutrosophic Proper Circular-Arc Graph, the truth-membership function μ_T respects this containment restriction. If $I_i \subset I_j$ were to hold, we would expect $\mu_T(I_i) \leq \mu_T(I_j)$, but since proper containment does not occur, this situation is impossible.

Therefore, no neutrosophic arc is properly contained within another with respect to μ_T, and the theorem follows. □

Next, we define p-Proper Circular-Arc Graphs and q-Improper Circular-Arc Graphs, and extend these concepts to neutrosophic graphs by introducing Neutrosophic p-Proper Circular-Arc Graphs and Neutrosophic q-Improper Circular-Arc Graphs. The generalized concepts of p-Proper Circular-Arc Graphs and q-Improper Circular-Arc Graphs are natural extensions of the classical circular-arc graph. Below, we provide the formal definitions.

Definition 4.12 (p-Proper Circular-Arc Graph)**.** A graph $G = (V, E)$ is called a *p-proper circular-arc graph* if there exists a circular-arc representation of G such that no arc in the representation is properly contained within more than p other arcs.

Formally, let $\mathcal{A} = \{A_v \mid v \in V\}$ be a set of arcs on a circle C_1 corresponding to the vertices of G. The graph G is p-proper if for every arc $A_u \in \mathcal{A}$, the number of other arcs $A_v \in \mathcal{A}$ such that $A_v \subset A_u$ is at most p:

$$\forall u \in V, \quad |\{v \in V \mid A_v \subset A_u\}| \leq p.$$

Definition 4.13 (q-Improper Circular-Arc Graph)**.** A graph $G = (V, E)$ is called a *q-improper circular-arc graph* if there exists a circular-arc representation of G such that no arc in the representation properly contains more than q other arcs.

Formally, let $\mathcal{A} = \{A_v \mid v \in V\}$ be a set of arcs on a circle C_1 corresponding to the vertices of G. The graph G is q-improper if for every arc $A_u \in \mathcal{A}$, the number of other arcs $A_v \in \mathcal{A}$ such that $A_u \subset A_v$ is at most q:

$$\forall u \in V, \quad |\{v \in V \mid A_u \subset A_v\}| \leq q.$$

Definition 4.14 (Neutrosophic p-Proper Circular-Arc Graph)**.** A graph $G = (V, E, \sigma, \mu)$ is called a *Neutrosophic p-Proper Circular-Arc Graph* if there exists a circular-arc representation of G such that no arc in the representation is properly contained within more than p other arcs with respect to the truth-membership function μ_T.

Formally, let $\mathcal{A} = \{A_v \mid v \in V\}$ be a set of arcs on a circle C_1 corresponding to the vertices of G, where each arc A_v is assigned a truth-membership degree $\mu_T(v)$. The graph G is p-proper if for every arc $A_u \in \mathcal{A}$, the number of other arcs $A_v \in \mathcal{A}$ such that $A_v \subset A_u$ with respect to $\mu_T(v)$ is at most p:

$$\forall u \in V, \quad |\{v \in V \mid A_v \subset A_u \text{ and } \mu_T(A_v) \leq \mu_T(A_u)\}| \leq p.$$

Definition 4.15 (Neutrosophic q-Improper Circular-Arc Graph)**.** A graph $G = (V, E, \sigma, \mu)$ is called a *Neutrosophic q-Improper Circular-Arc Graph* if there exists a circular-arc representation of G such that no arc in the representation properly contains more than q other arcs with respect to the truth-membership function μ_T.

Formally, let $\mathcal{A} = \{A_v \mid v \in V\}$ be a set of arcs on a circle C_1 corresponding to the vertices of G, where each arc A_v is assigned a truth-membership degree $\mu_T(v)$. The graph G is q-improper if for every arc $A_u \in \mathcal{A}$, the number of other arcs $A_v \in \mathcal{A}$ such that $A_u \subset A_v$ with respect to $\mu_T(u)$ is at most q:

$$\forall u \in V, \quad |\{v \in V \mid A_u \subset A_v \text{ and } \mu_T(A_u) \leq \mu_T(A_v)\}| \leq q.$$

Theorem 4.16. *Neutrosophic p-Proper circular-arc graph is a Neutrosophic graph. Also Neutrosophic q-Improper circular-arc graph is a Neutrosophic graph.*

Proof. Obviously holds. □

Theorem 4.17. *A Neutrosophic p-Proper circular-arc graph can be transformed into a classic p-Proper circular-arc graph.*

Proof. Obviously holds. □

Theorem 4.18. *A Neutrosophic q-Improper circular-arc graph can be transformed into a classic q-Improper circular-arc graph.*

Proof. Obviously holds. □

Theorem 4.19. *A Neutrosophic* 0*-Proper Circular-Arc Graph is exactly a Neutrosophic Proper Circular-Arc Graph.*

Proof. By definition, a Neutrosophic p-Proper Circular-Arc Graph is a graph in which no arc is properly contained within more than p other arcs with respect to the truth-membership function μ_T.

For the case $p = 0$, this condition simplifies to no arc being properly contained within any other arc. Formally, for each arc $A_u \in \mathcal{A}$, the number of arcs $A_v \in \mathcal{A}$ such that $A_v \subset A_u$ is zero:

$$\forall u \in V, \quad |\{v \in V \mid A_v \subset A_u \text{ and } \mu_T(A_v) \leq \mu_T(A_u)\}| = 0.$$

This directly matches the definition of a Neutrosophic Proper Circular-Arc Graph, which states that no arc is properly contained within another. Therefore, a Neutrosophic 0-Proper Circular-Arc Graph is precisely a Neutrosophic Proper Circular-Arc Graph. □

Theorem 4.20. *A Neutrosophic* 0*-Improper Circular-Arc Graph is a Neutrosophic Proper Circular-Arc Graph.*

Proof. By definition, a Neutrosophic q-Improper Circular-Arc Graph is a graph in which no arc properly contains more than q other arcs with respect to the truth-membership function μ_T.

For the case $q = 0$, this condition simplifies to no arc properly containing any other arc. Formally, for each arc $A_u \in \mathcal{A}$, the number of arcs $A_v \in \mathcal{A}$ such that $A_u \subset A_v$ is zero:

$$\forall u \in V, \quad |\{v \in V \mid A_u \subset A_v \text{ and } \mu_T(A_u) \leq \mu_T(A_v)\}| = 0.$$

This condition ensures that no arc properly contains any other arc, which is exactly the definition of a Neutrosophic Proper Circular-Arc Graph. Thus, a Neutrosophic 0-Improper Circular-Arc Graph is a Neutrosophic Proper Circular-Arc Graph. □

Theorem 4.21. *A Neutrosophic* 1*-Proper Circular-Arc Graph allows each arc to be properly contained within at most one other arc with respect to the truth-membership function μ_T.*

Proof. Obviously holds. □

5. Future tasks: Graph Power

A Power Graph is a concept that examines the powers of a graph. The general definition is provided below [6, 58].

Definition 5.1 (Graph Power)**.** Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. The k-th power of G, denoted G^k, is defined as a graph with the same vertex set V, but with an edge between two vertices $u, v \in V$ if and only if the distance between u and v in G is at most k. Formally, the edge set E^k of G^k is given by:

$$E^k = \{(u, v) \in V \times V : d_G(u, v) \leq k\},$$

where $d_G(u, v)$ denotes the shortest path distance between u and v in the original graph G.

For example, G^2 is called the square of G, and G^3 is called the cube of G.

At the conceptual stage, we outline the definitions for Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs below. Moving forward, we plan to investigate the relationships between these concepts and various types of graphs, such as Circular-Arc Graphs.

Definition 5.2 (Fuzzy Graph Power)**.** Let $G = (V, E, \sigma, \mu)$ be a Fuzzy Graph, where:

- $\sigma : V \to [0, 1]$ assigns membership degrees to vertices.
- $\mu : E \to [0, 1]$ assigns membership degrees to edges.

The k-th power of G, denoted $G^k = (V, E^k, \sigma^k, \mu^k)$, is defined as:

(1) $\sigma^k(v) = \sigma(v), \quad \forall v \in V.$

(2) For each $(u, v) \in E^k$, the edge membership degree $\mu^k(u, v)$ is defined as:

$$\mu^k(u, v) = \max_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\min_{e \in P_{uv}} \mu(e) \right),$$

where:

- P_{uv} is a path from u to v in G,
- $\ell(P_{uv})$ is the length of the path P_{uv} (number of edges),
- The maximum is taken over all paths of length at most k connecting u and v,
- The minimum is taken over the membership degrees of edges along each path.

Definition 5.3 (Intuitionistic Fuzzy Graph Power)**.** Let $G = (V, E, \mu_A, v_A, \mu_B, v_B)$ be an Intuitionistic Fuzzy Graph, where:

- $\mu_A : V \to [0, 1]$ and $v_A : V \to [0, 1]$ assign degrees of membership and non-membership to vertices, respectively.
- $\mu_B : E \to [0, 1]$ and $v_B : E \to [0, 1]$ assign degrees of membership and non-membership to edges, respectively.

The k-th power of G, denoted $G^k = (V, E^k, \mu_A^k, v_A^k, \mu_B^k, v_B^k)$, is defined as:

(1) $\mu_A^k(v) = \mu_A(v), \quad v_A^k(v) = v_A(v), \quad \forall v \in V.$

(2) For each $(u, v) \in E^k$, the edge membership degrees are defined as:

$$\mu_B^k(u,v) = \max_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\min_{e \in P_{uv}} \mu_B(e) \right),$$

$$v_B^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\max_{e \in P_{uv}} v_B(e) \right).$$

Definition 5.4 (Neutrosophic Graph Power)**.** Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a Neutrosophic Graph, where:

- $\sigma_T, \sigma_I, \sigma_F : V \to [0, 1]$ assign truth, indeterminacy, and falsity membership degrees to vertices.
- $\mu_T, \mu_I, \mu_F : E \to [0, 1]$ assign truth, indeterminacy, and falsity membership degrees to edges.

The k-th power of G, denoted $G^k = (V, E^k, \sigma^k, \mu^k)$, is defined as:

(1) $\sigma^k(v) = \sigma(v), \quad \forall v \in V.$

(2) For each $(u, v) \in E^k$, the edge membership degrees are defined as:

$$\mu_T^k(u,v) = \max_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\min_{e \in P_{uv}} \mu_T(e) \right),$$

$$\mu_I^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\max_{e \in P_{uv}} \mu_I(e) \right),$$

$$\mu_F^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\max_{e \in P_{uv}} \mu_F(e) \right).$$

Definition 5.5 (Turiyam Neutrosophic Graph Power)**.** Let $G = (V, E, \sigma = (t, iv, fv, lv), \mu = (t_e, iv_e, fv_e, lv_e))$ be a Turiyam Neutrosophic Graph, where:

- $t, iv, fv, lv : V \to [0, 1]$ assign truth, indeterminacy, falsity, and liberal state values to vertices.
- $t_e, iv_e, fv_e, lv_e : E \to [0, 1]$ assign truth, indeterminacy, falsity, and liberal state values to edges.

The k-th power of G, denoted $G^k = (V, E^k, \sigma^k, \mu^k)$, is defined as:

(1) $\sigma^k(v) = \sigma(v), \quad \forall v \in V.$

(2) For each $(u, v) \in E^k$, the edge membership degrees are defined as:

$$t^k(u,v) = \max_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\min_{e \in P_{uv}} t_e(e) \right),$$

$$iv^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\max_{e \in P_{uv}} iv_e(e) \right),$$

$$fv^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\max_{e \in P_{uv}} fv_e(e) \right),$$

$$lv^k(u,v) = \min_{\substack{P_{uv} \\ \ell(P_{uv}) \leq k}} \left(\max_{e \in P_{uv}} lv_e(e) \right).$$

Theorem 5.6. *The Turiyam Neutrosophic Graph Power reduces to the Neutrosophic Graph Power when the liberal state functions are set to zero.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Turiyam Neutrosophic Graph. Define a Neutrosophic Graph $G' = (V, E, \sigma', \mu')$ by setting:

$$\sigma'(v) = (t(v), iv(v), fv(v)), \quad \forall v \in V,$$
$$\mu'(e) = (t_e(e), iv_e(e), fv_e(e)), \quad \forall e \in E.$$

Similarly, the k-th power G^k transforms to $(G')^k$ with edge membership degrees:

$$t'^k(u,v) = t^k(u,v),$$
$$iv'^k(u,v) = iv^k(u,v),$$
$$fv'^k(u,v) = fv^k(u,v),$$

for all $(u,v) \in E^k$.

Since the liberal state functions $lv(v)$ and $lv_e(e)$ are set to zero and omitted, the Turiyam Neutrosophic Graph Power reduces to the Neutrosophic Graph Power. □

Theorem 5.7. *The Neutrosophic Graph Power reduces to the Intuitionistic Fuzzy Graph Power when the falsity-membership functions are set to one minus the truth-membership functions, and indeterminacy-membership functions are set to zero.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Graph. Define an Intuitionistic Fuzzy Graph $G'' = (V, E, \mu_A, v_A, \mu_B, v_B)$ by setting:

$$\mu_A(v) = t(v), \quad v_A(v) = fv(v), \quad \forall v \in V,$$
$$\mu_B(e) = t_e(e), \quad v_B(e) = fv_e(e), \quad \forall e \in E.$$

Assuming $iv(v) = 0$ and $iv_e(e) = 0$ for all $v \in V$ and $e \in E$, the indeterminacy is removed.

The k-th power $(G')^k$ transforms to $(G'')^k$ with edge membership degrees:

$$\mu_B^k(u,v) = t'^k(u,v),$$
$$v_B^k(u,v) = fv'^k(u,v),$$

for all $(u,v) \in E^k$.

Thus, the Neutrosophic Graph Power reduces to the Intuitionistic Fuzzy Graph Power under these conditions. □

Theorem 5.8. *The Intuitionistic Fuzzy Graph Power reduces to the Fuzzy Graph Power when the non-membership functions are set to one minus the membership functions.*

Proof. Let $G'' = (V, E, \mu_A, v_A, \mu_B, v_B)$ be an Intuitionistic Fuzzy Graph. Define a Fuzzy Graph $G''' = (V, E, \sigma, \mu)$ by setting:

$$\sigma(v) = \mu_A(v), \quad \forall v \in V,$$
$$\mu(e) = \mu_B(e), \quad \forall e \in E.$$

Assuming $v_A(v) = 1 - \mu_A(v)$ and $v_B(e) = 1 - \mu_B(e)$, the non-membership functions are determined by the membership functions.

The k-th power $(G'')^k$ transforms to $(G''')^k$ with edge membership degrees:

$$\mu^k(u, v) = \mu_B^k(u, v),$$

for all $(u, v) \in E^k$.

Thus, the Intuitionistic Fuzzy Graph Power reduces to the Fuzzy Graph Power under these conditions. □

Theorem 5.9. *The Fuzzy Graph Power reduces to the classical Graph Power when all membership degrees are binary (0 or 1).*

Proof. Let $G''' = (V, E, \sigma, \mu)$ be a Fuzzy Graph where $\sigma(v) \in \{0, 1\}$ and $\mu(e) \in \{0, 1\}$ for all $v \in V$ and $e \in E$.

The k-th power $(G''')^k$ becomes a classical graph $G^k = (V, E^k)$ where:

$$(u, v) \in E^k \iff \text{there exists a path from } u \text{ to } v \text{ of length } \leq k \text{ in } G.$$

This aligns with the definition of the classical Graph Power. Hence, the Fuzzy Graph Power reduces to the classical Graph Power when membership degrees are binary. □

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Data Availability

This paper does not involve any data analysis.

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