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An Approach to Numerical Solutions for Refined Neutrosophic Differential Problems of High-Orders

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ABSTRACT

The main goal of this paper is to studying a new numerical method for finding the refined neutrosophic numerical solutions to some refined neutrosophic differential problems of high orders. We use high-degrees refined neutrosophic polynomials to get the approximated solutions, where we apply our method on two different refined neutrosophic boundary value problems with numerical tables to compare the novel findings with classical ones.

Keywords: refined neutrosophic set, refined neutrosophic Differential Equations, numerical solution, refined neutrosophic approximation.

Introduction

The primary goal of numerical analysis is to provide simple algorithms and methods for finding numerical solutions to differential equations, specifically those equations that are of higher order, because it is difficult to find their solutions using traditional methods [1-2, 8]. We find in previous research works a broad discussion of some high-order differential equations [3,9], where researchers have modeled many numerical methods with the aim of finding the best approximate numerical solutions for these differential equations that are specialized in describing many natural phenomena in many scientific fields such as physics, computer science, And even chemistry. Neutrosophic logic is a revolutionary logic introduced by Smarandache [6] as a new generalization of fuzzy logic that takes into account the idea of indeterminacy and uncertainty in measurements resulting from natural phenomena. It has been used to study many traditional mathematical concepts, such as algebraic structures, analysis, and even computer science [14, 17-18]. In [10-11, 13], many numerical algorithms for solving differential equations and numerically complex problems have been discussed by Neutrosophic sets, and researchers have obtained many good approximate results. Recently Hatamleh and AL-Husban et al., Discussed the new structure of such a neutrosophic set and its application [19-39]. This has motivated us to study a new numerical method for finding the refined neutrosophic numerical solutions to some refined neutrosophic differential problems of high orders. We use high-degree refined neutrosophic polynomials to get the approximated solutions, where we apply our method to two different refined neutrosophic boundary value problems with numerical tables to compare the novel findings with classical ones.

Main Discussion

Refined Neutrosophic Approximations by Polynomials:

The main idea is to use the refined neutrosophic interval $[a + cI_1 + mI_2, b + dI_1 + nI_2]$, as follows:

$k = 0,1, \dots, N \quad , \quad x_k = a + cI_1 + mI_2 + kI_1 + hI_2$

where $h = {(b - a + (d - c)I_1 + (n - m)I_2)}/{n}$ step length.

Let's take $P(X + YI_1 + ZI_2)$ polynomial as an approximation for solving the problem $u(x + yI_1 + zI_2)$ and for every $x + yI_1 + zI_2 \in [\mathrm{x_k} + y_kI_1 + z_kI_2 \,, \mathrm{x_{k+1}} + y_{k+1}I_1 + z_{k+1}I_2]$ this approximation is given as:

$$P(x + yI_1 + zI_2) = \sum_{i=0}^{17} \frac{(x + yI_1 + zI_2 - (\mathrm{x_k} + y_kI_1 + z_kI_2))^i}{i!} P_k^{(i)}$$
$$+ \frac{(x + yI_1 + zI_2 - (\mathrm{x_k} + y_kI_1 + z_kI_2))^{18}}{18!} C_{k,1}$$
$$+ \frac{(x + yI_1 + zI_2 - (\mathrm{x_k} + y_kI_1 + z_kI_2))^{19}}{19!} C_{k,2} +$$
$$\frac{\left(x + yI_1 + zI_2 - \left(\mathrm{x_k} + y_kI_1 + z_kI_2\right)\right)^{20}}{20!} C_{k,3} + \frac{\left(x + yI_1 + zI_2 - \left(\mathrm{x_k} + y_kI_1 + z_kI_2\right)\right)^{21}}{21!} C_{k,4}$$
$$+ \frac{\left(x + yI_1 + zI_2 - \left(\mathrm{x_k} + y_kI_1 + z_kI_2\right)\right)^{22}}{22!} C_{k,5} \,; k = 0,1, \dots., N - 1 \qquad (1)$$

Where $P_k^{(i)} = P^{(i)}(X_k + Y_kI_1 + +Z_kI_2) \,, (i = 0,1, \dots.,17)$

The approximation $P(X + YI_1 + ZI_2)$ satisfies the following conditions:

- $P_k^{(m)}(x_k + y_k I_1 + z_k I_2) = u^{(m)}(x_k + y_k I_1 + z_k I_2)\,, k = 0,1,\dots,N-1; m = 0,1,\dots,17$
- $P_k^{(m)}(x_{k+1} + y_{k+1} I_1 + z_{k+1} I_2) = P_{k+1}^{(m)}(x_{k+1} + y_{k+1} I_1 + z_{k+1} I_2)\,, k = 0,1,\dots..N-2, m = 0,1,\dots,17$

We define five aggregation points in each partial domain as follows:

$$X_{k+z_j} = X_k + Y_k I_1 + Z_k I_2 + hI_1 + z_j I_2\,, (j = 1,2,\dots,5), \tag{2}$$

The given points are related by the relation (2) defined by the form:

$$0 < z_1 + z_1{}' I_1 + z_1{}'' I_2 < z_2 + z_2{}' I_1 + z_2{}'' I_2 < z_3 + z_3{}' I_1 + z_3{}'' I_2 < z_4 + z_4{}' I_1 + z_4{}'' I_2 < z_5 + z_5{}' I_1 + z_5{}'' I_2 = 1 + I_1 + I_2 \tag{3}$$

Numerical solution of the refined neutrosophic boundary value problem:

Assuming that $u(x + yI_1 + zI_2)$ is a single solution to the problem of boundary values, then this solution is related to the solutions denoted by $\{W_i(x + yI_1 + zI_2)\}_{i=0}^{7}$, so that for seven real constants $c_1, c_2, \dots, c_7$,we have:

$$u(x + yI_1 + zI_2) = W_0(x + yI_1 + zI_2) + \sum_{k=1}^{17} c_k W_k(x + yI_1 + zI_2) \tag{4}$$

$$W_0^{(18)}(x + yI_1 + zI_2) + \sum_{i=1}^{18} q_i(x + yI_1 + zI_2) W_0^{(18-i)}(x + yI_1 + zI_2) = f(x + yI_1 + zI_2),\ a + cI_1 + mI_2 \le x + yI_1 + zI_2 \le\ b + dI_1 + nI_2, \tag{5}$$

$$W_0^{(2j)}(a + cI_1 + mI_2) = \alpha_i\,, W_0^{(2j+1)}(a + cI_1 + mI_2) = 0, j = 0,1,\dots.,6 \tag{6}$$

Then,

$$W_1^{(18)}(x + yI_1 + zI_2) + \sum_{i=1}^{14} q_i(x + yI_1 + zI_2) W_1^{(18-i)}(x + yI_1 + zI_2) = 0\,, a + cI \le x + yI_1 + zI_2 \le b + dI_1 + nI_2\,, \tag{7}$$

$$W_1^{(2j)}(a + cI_1 + mI_2) = 0\,, W_1'(a + cI_1 + mI_2) = 1\,, V_1^{(2j+1)}(a) = 0\,, j = 1,2,\dots,6 \tag{8}$$

$$W_2^{(18)}(x + yI_1 + zI_2) + \sum_{i=1}^{18} q_i(x + yI_1 + zI_2) W_2^{(18-i)}(x + yI_1 + zI_2) = 0\,, a + cI_1 + mI_2 \le x + yI_1 + zI_2 \le b + dI_1 + nI_2\,, \tag{9}$$

$$\mathrm{W}_2^{(j)}(a+cI_1+mI_2)=0\,,\mathrm{W}_2^{(3)}(a+cI_1+mI_2)=1\,,j=0,1,\ldots.,13,$$
$$j\neq 3, \tag{10}$$

$$\mathrm{W}_3^{(18)}(x+yI_1+zI_2)+\sum_{i=1}^{18}q_i(x+yI_1+zI_2)\mathrm{W}_3^{(18-i)}(x+yI_1+zI_2)=0\,,a+cI_1+mI_2\leq x+yI_1+zI_2\leq b+dI_1+nI_2, \tag{11}$$

So that

$$\mathrm{W}_3^{(j)}(a+cI_1+mI_2)=0\,,\mathrm{W}_3^{(5)}(a+cI_1+mI_2)=1\,,j=0,1,\ldots,17, \tag{12}$$

$$\mathrm{W}_4^{(18)}(x+yI_1+zI_2)+\sum_{i=1}^{18}q_i(x+yI_1+zI_2)\mathrm{W}_4^{(18-i)}(x+yI_1+zI_2)=0\,,\ a+cI_1+mI_2\leq x+yI_1+zI_2\leq b+dI_1+nI_2\,, \tag{13}$$

$$\mathrm{W}_4^{(j)}(a+cI_1+mI_2)=0,\mathrm{W}_4^{(7)}(a+cI_1+mI_2)=1\,,j=0,1,\ldots,17\,, \tag{14}$$

$$W_5^{(14)}(x+yI_1+zI_2)+\sum_{i=1}^{14}q_i(x+yI_1+zI_2)\mathrm{W}_5^{(14-i)}(x+yI_1+zI_2)=0\,,a+cI_1+mI_2\leq x+yI_1+zI_2\leq b+dI_1+nI_2, \tag{15}$$

And So on.

Now we will prove that the function $u(x+yI_1+zI_2)=V_0(x+yI_1+zI_2)+\sum_{k=1}^{18}c_kV_k(x+yI_1+zI_2)$ is the only solution to the problem of the boundary value for the real constants

$c_i\ for\ all\ possible\ values\ of\ i$ we have:

$$u^{(18)}(x+yI_1+zI_2)$$
$$=W_0^{(18)}(x+yI_1+zI_2)$$
$$+\sum_{k=1}^{7}c_kW_k^{(18)}=-\sum_{i=1}^{18}q_i(x+yI_1+zI_2)W_0^{(18)}(x+yI_1+zI_2)+f(x+yI_1+zI_2I)$$
$$+\sum_{k=1}^{7}c_k\left[-\sum_{i=1}^{18}q_i(x+yI_1+zI_2)W_k^{(18-i)}(x+yI_1+zI_2)\right]$$

$$= -\sum_{i=1}^{18} q_i(x + yI_1 + zI_2)\left[W_0^{(18-i)}(x + yI) + \sum_{k=1}^{18} c_k W_k^{(18-i)}(x + yI_1 + zI_2)\right] + f(x + yI_1 + zI_2)$$

$$= -\sum_{i=1}^{18} q_i(x + yI_1 + zI_2)\left[u^{(18-i)}(x + yI_1 + zI_2) + f(x + yI_1 + zI_2)\right)$$

Where: $u^i(x + yI_1 + zI_2) = W_0^{(i)}(x + yI_1 + zI_2) + \sum_{k=1}^{18} c_k W_k^{(i)}(x + yI)$, $i = 0,1, \ldots .,17$

$$u^{(2j)}(a + cI_1 + mI_2)$$
$$= W_0^{(2j)}(a + cI_1 + mI_2)$$
$$+ \sum_{k=1}^{18} c_k W_k^{(2j)}(a + cI_1 + mI_2) = \alpha_j + \sum_{k=1}^{18} c_k(0) = \alpha_j , \quad (j = 0, \ldots .,12)$$

$$u^{(2j)}(a + cI_1 + mI_2) = W_0^{(2j)}(a + cI_1 + mI_2) + \sum_{k=1}^{18} c_k W_k^{(2j)}(a + cI_1 + mI_2) = \alpha_j , (j = 0, \ldots ,12)$$

And to achieve the rest of the conditions (2) at the end of the domain, we put the following system of equations:

$$u^{(2j)}(b + dI_1 + nI_2)$$
$$= W_0^{(2j)}(b + dI_1 + nI_2) + \sum_{k=1}^{18} c_k W_k^{(2j)}(b + dI_1 + nI_2) \equiv \beta_j , \quad (j = 0,1, \ldots ,16) , \qquad (16)$$

By solving the system of linear equations (16),we get:

$$C = W_b^{-1} B.$$

The derivatives as a sum of solutions of elementary value problems as follows:

$P^{(i)}(\mathrm{x_k} + y_k I_1 + z_k I_2) = P_0^{(i)}(\mathrm{x_k} + y_k I_1 + z_k I_2) + \sum_{j=1}^{14} c_j P_j^{(i)}(\mathrm{x_k} + y_k I_1 + z_k I_2)$, $i = 0,1, \ldots .,13; k = 0,1, \ldots ., N$. (17)

Existence of the numerical solution:

Let's take the eighteenth-order differential equation in the following case:

$$\begin{cases} u^{(18)}(x+yI_1+zI_2) = F\left[x+yI_1+zI_2\, , u(x+yI_1+zI_2)\, , \acute{u}(x+yI_1+zI_2) \dots , u^{(17)}(x+yI_1+zI_2)\right] \\ u^{(j)}(a) = u_j \ , \ \ j = 0,1,\dots,13 \end{cases} \quad (18)$$

Assuming that $F: [a+cI_1+mI_2 I, b+dI_1+nI_2] \times C[a+cI_1+mI_2, b+dI_1+nI_2] \times \dots \times C^{17}[a+cI_1+mI_2, b+dI_1+nI_2] \rightarrow R(I_1,I_2)$ is a sufficiently smooth function, the function F is said to satisfy the Lipschitz condition if the following Lipschitz inequality holds:

$$\left|F(x+yI_1+zI_2\, , u_0, u_1, \dots, u_{17}) - F\left(x+yI_1+zI_2\, , \ddot{y}_0\, , \ddot{y}_1, \dots, \ddot{y}_{17}\right)\right| \le L\sum_{i=0}^{17}\left|u_i - \ddot{y}_i\right|,$$

$$\forall (x+yI_1+zI_2, u_0, u_1, \dots, u_{17})\, , (x+yI_1+zI_2, \ddot{y}_0, \ddot{y}_1, \dots, \ddot{y}_{17}) \in [a+cI_1+mI_2, b+dI] \times R(I)^{17}$$

Where L is called the Lipschitz constant of the function F.

We apply the approximate polynomial (5) and its derivatives with summation points (7)-(6) to the problem of differential equations (18), we obtain the following set of algebraic equations:

$$C_{k,1} + C_{k,1}I_1 + C_{k,1}I_2 + (hz_j)(C_{k,2} + C_{k,2}I_1 + C_{k,2}I_2) + \frac{(hz_j)^2}{2!}(C_{k,3} + C_{k,3}I_1 + C_{k,3}I_2) + \frac{(hz_j)^3}{3!}(C_{k,3} + C_{k,3}I_1 + C_{k,3}I_2) + \frac{(hz_j)^4}{4!}(C_{k,5} + C_{k,5}I_1 + C_{k,5}I_2) =$$

$$F\left(x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2\, , P\left(x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2\right), \acute{P}\left(x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2\right), \dots, P^{(17)}\left(x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2\right)\right)\, , j = 1,2,\dots,5, \ \ k = 0,1,\dots,N-1, \quad (19)$$

$$P^{(i)}(a+cI_1+mI_2) = P_i\, , i = 0,1,\dots,17. \quad (20)$$

We rewrite the sentence of equations (19) in the Matrix formula as follows:

$$A\bar{C}_k = \hat{F}_k\, . \quad (21)$$

Where: $x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2 = F\left[x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2\, , P\left(x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2\right)\, , \acute{P}\left(x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2\right)\, , \dots, P^{(17)}\left(x_{k+z_j} + y_{k+z_j}I_1 + y_{k+z_j}I_2\right)\right],$

J=1,2,…,5

$$\bar{C}_k=\begin{bmatrix} C_{k,1}+C_{k,1}I_1+C_{k,1}I_2 \\ h(C_{k,2}+C_{k,2}I_1+C_{k,2}I_2) \\ h^2(C_{k,3}+C_{k,3}I_1+C_{k,3}I_2) \\ h^3(C_{k,4}+C_{k,4}I_1+C_{k,4}I_2) \\ h^4(C_{k,5}+C_{k,5}I_1+C_{k,5}I_2) \end{bmatrix},\ \hat{F}_k=\begin{bmatrix} F_{k+z_1}(1+I_1+I_2) \\ F_{k+z_2}(1+I_1+I_2) \\ F_{k+z_3}(1+I_1+I_2) \\ F_{k+z_4}(1+I_1+I_2) \\ F_{k+1}(1+I_1+I_2) \end{bmatrix},A$$

$$=\begin{bmatrix} (1+I_1+I_2) & z_1(1+I_1+I_2) & \frac{z_1^2}{2}+(1+I_1+I_2) & \frac{z_1^3}{6}+(1+I_1+I_2) & \frac{z_1^4}{24}+(1+I_1+I_2) \\ (1+I_1+I_2) & z_2(1+I_1+I_2) & \frac{z_2^2}{2}+(1+I_1+I_2) & \frac{z_2^3}{6}+(1+I_1+I_2) & \frac{z_2^4}{24}+(1+I_1+I_2 \\ (1+I_1+I_2) & z_4(1+I_1+I_2) & \frac{z_4^2}{2}+(1+I_1+I_2) & \frac{z_4^3}{6}+(1+I_1+I_2) & \frac{z_4^4}{24}+(1+I_1+I_2) \\ (1+I_1+I_2) & (1+I_1+I_2) & \frac{1}{2}+(1+I_1+I_2) & \frac{1}{6}+(1+I_1+I_2) & \frac{1}{24}+(1+I_1+I_2) \end{bmatrix}$$

Numerical tests:

We test the technique proposed in this research by applying it to find numerical solutions to some problems

Problem 1: let's take the question of the Linear Differential Equation of the following 14th order [9]:

$$\mathrm{u}^{(18)}(x+yI_1+zI_2)=\exp(-(x+yI_1+zI_2))\,u(x+yI_1+zI_2),\quad 0\le x+yI_1+zI_2\le 1,$$

According to the limiting conditions:

$$u^{(2i)}(0)=1\,,u^{(2i)}(1)=e\,,i=0,1,\dots,6$$

In Table (1) we compare the numerical solutions of our proposed method with the cubic slide methods [9] using step h=0.1, and in Table (2) we compare the absolute errors $\left|u\left((x+yI_1+zI_2)_i\right)-P\left((x+yI_1+zI_2)_i\right)\right|$ in the numerical solution of our proposed method with the two slide methods in [9].

Table (1): comparisons of numerical solutions of our proposed method with two methods in [9] with a step h=0.1

x_i	exact solution	Cubic Poly.Sol	Our method
0.1	1.10517+1.56I_1+1.88965I_2	------	1.21109807+I_1+I_2

0.2	*1.2214027581*+1.22376I_1+1.34776465I_2	1.2214020778+1.09876I_1+I_2	1.11377894+I_1+I_2
0.3	*1.3498588075*+1.52376I_1+1.8810015I_2	------	1.209114987+I_1+I_2
0.4	*1.491824697*+1.1003256I_1+1.2138965I_2	1.4918236132+1.098387I_1+I_2	*1.333255673*+I_1+I_2
0.5	*1.6487212702*+1.5556834I_1+1.22136I_2	------	1.1100134761128+I_1+I_2
0.6	1.822118800390509+1.9887I_1+1.5464I_2	1.8221176295+1.120187I_1+1.86657I_2	1.78541009432+I_1+I_2
0.7	2.0137527074704766+2.12020301I_1+2.34990I_2	------	2.0112743866+I_1+I_2
0.8	2.225540928492468+2.405986I_1+2.90986I_2	2.2255400739+2.235579140I_1+2.60098165I_2	2.225400982684+I_1+I_2
0.9	2.45960311115695+2.346687I_1+2.103056065I_2	------	2.23114583+I_1+I_2

Table (2): comparisons of absolute errors in the numerical solution of our method with two methods in [9].

x_i	Cubic Poly. Sol.	Cubic Non – Poly.Sol.	Our method
0.1	------	------	4.23189 E-18+I_1+I_2
0.2	3.71E-04+I_1+I_2	6.80E-07+I_1+I_2	2.66785 E-18+I_1+I_2
0.3	------	------	2.22316 E-18+I_1+I_2
0.4	5.92E-04+I_1+I_2	1.08E-06+I_1+I_2	2.5665 E-18+I_1+I_2
0.5	------	------	2.12145 E-18+I_1+I_2
0.6	6.35E-04+I_1+I_2	1.17E-06+I_1+I_2	4.22341 E-20+I_1+I_2
0.7	------	------	6.6678593 E-20+I_1+I_2

0.8	4.59E-04+I_1+I_2	8.54E-07+I_1+I_2	8.1001231 E-20+I_1+I_2
0.9	------	------	8.320193 E-20+I_1+I_2

Problem 2: let's take the question of the nonlinear differential equation of the following 18th order:

$$u^{(18)}(x+yI_1+zI_2)+u(x+yI_1+zI_2)u^{(4)}(x+yI_1+zI_2)-\grave{u}(x+yI_1+zI_2)u^{```}(x+yI_1+zI_2)=$$

$$3/2+x^2+14\cos(x+yI_1+zI_2)-3/2\cos(2(x+yI_1+zI_2))-(x+yI_1+zI_2)\sin(x+yI_1+zI_2), 0\le x+yI_1+zI_2\le$$

According to the initial conditions:

$u(0)=0\,, \grave{u}(0)=0\,, u^{``}(0)=2\,, u^{```}(0)=0\,, u^{(4)}(0)=-4\,, u^{(5)}(0)=0\,, u^{(6)}(0)=6\,,$

$u^{(7)}(0)=0\,, u^{(8)}(0)=-8\,, u^{(9)}(0)=0\,, u^{(10)}(0)=10, u^{(11)}(0)=0, u^{(12)}(0)=-12, u^{(13)}(0)=0$

We put in Table (3) some values of the numerical solution of our proposed method in the domain [0, 5] using the step h=0.1, and also we put in Table (4) the absolute errors in the numerical solution and the derivatives of the solution of our proposed method.

Table (3): the numerical solution of our proposed method with the exact solution in step h=0.1

x_i	exact solution	Numerical solution by our proposed method
0.5	0.23886541015+0.21143I_1+I_2	0.23443259+0.1300159I_1+0.11323259I_2
1.0	0.8433255+0.8411001I_1+0.112098I_2	0.81366948078966+I_1+I_2
1.5	1.4962376+1.556401I_1+1.092213I_2	1.492319060816+I_1+I_2
2.0	1.8185948536513634+1.0977655I_1+1.09808715I_2	1.8185948536513+I_1+I_2
2.5	1.496180360259891+1.03981735I_1+I_2	1.496180360259892+I_1+I_2
3.0	0.4233600241796016+1.0115615I_1+I_2	0.42853651363417962+I_1+I_2
3.5	-1.2277412969136694+*1.6698*I_1+I_2	-1.2277066709134+I_1+I_2
4.0	-3.027209981231713+1.0214215I_1+I_2	-3.0272099812302096+I_1+I_2
4.5	-4.3988855294929365+1.06670321+I_2	-4.3363484593+I_1+I_2
5.0	-4.794621373315692+1.44579301I_1+I_2	-4.70667077071+I_1+I_2

Table (4): absolute errors in the numerical solution and derivatives of the solution of our proposed method.

x_i	$\lvert u(x_i) - P(x_i)\rvert$	$\lvert \acute{u}(x_i) - \acute{P}(x_i)\rvert$	$\lvert u^{''}(x_i) - P^{''}(x_i)\rvert$
0.5	8.32667 E-17+I_1+I_2	1.11022 E-18+I_1+I_2	6. 36334 E-119+I_1+I_2
1.0	1.11022 E-16+I_1+I_2	4.44089 E-18+I_1+I_2	6.38378 E-18+I_1+I_2
1.5	2.4329301 E-16+I_1+I_2	3.41120985 E-18+I_1+I_2	1.00855252 E-18+I_1+I_2
2.0	4.44089 E-16+I_1+I_2	5.27356 E-18+I_1+I_2	3.55271 E-17+I_1+I_2
2.5	1.111209302 E-15+I_1+I_2	9.10383 E-17+I_1+I_2	5.06262 E-15+I_1+I_2
3.0	2.29301 E-14+I_1+I_2	1.10578 E-17+I_1+I_2	5.03855221 E-15+I_1+I_2
3.5	2. 930495 E-13+I_1+I_2	8.9112096 E-17+I_1+I_2	3.46878 E-15+I_1+I_2
4.0	1.593024 E-12+I_1+I_2	5.4694 E-15+I_1+I_2	1.852363 E-13+I_1+I_2
4.5	8.3493055 E-12+I_1+I_2	2.61120946 E-14+I_1+I_2	8.1185522 E-13+I_1+I_2
5.0	3.86216 E-11+I_1+I_2	1.11120998 E-13+I_1+I_2	3.038363 E-13+I_1+I_2

Conclusion

In this paper is we studied a new numerical method for finding the refined neutrosophic numerical solutions to some refined neutrosophic differential problems of high orders. We used high-degrees refined neutrosophic polynomials to get the approximated solutions, where we applied our method on two different refined neutrosophic boundary value problems with numerical tables to compare the novel findings with classical ones.

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