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# Correlation and Regression Coefficients of Neutrosophic Graph Products using Minimum Spanning Tree for Enhancing Healthcare Strategies

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Abstract. This paper investigates the correlation and regression coefficients ($\mathbb{CRC}s$) of single-valued neutrosophic graph products ($SVNGPs$) with a focus on the minimum spanning tree ($MST$) and their applications in enhancing healthcare strategies. Single-valued neutrosophic Graph ($SVNG$) is an extension of traditional graph theory that uses neutrosophic logic to solve uncertainties and indeterminacy. We have introduced the modular and strong products of $SVNGs$ and calculated the $\mathbb{CRC}s$ with and without using $MST$ to evaluate the relationships within healthcare networks. A comparative analysis has been done for $\mathbb{CRC}s$ with and without $MST$ which helps to increase the accuracy and efficiency of $\mathbb{CRC}s$. The gained results offer valuable insights into optimizing resource allocation and improving patient care by identifying critical factors influencing healthcare outcomes. This study demonstrates the applicability of $SVNGPs$ in decision-making processes, with $MST$ playing a pivotal role in enhancing overall healthcare efficiency and effectiveness.

**Keywords:** Single-valued neutrosophic graph products; Correlation coefficient; Regression coefficient; Minimum spanning tree

## 1. Introduction

A sophisticated mathematical framework known as neutrosophic fuzzy graphs manages uncertainty, imprecision, and indeterminacy in complex systems. Neutrosophic fuzzy graphs, which combine fuzzy graphs and neutrosophic logic, offer a powerful tool for modeling and analyzing situations involving inconsistent, inaccurate, or insufficient information. Neutrosophic sets suggested by Smarandache (1) and (2) as an extension of fuzzy sets and intuitionistic fuzzy sets, are effective tools for dealing with incomplete, ambiguous, and inconsistent information in the real world. Neutrosophic sets are differentiated by the truth membership function ($\mathbb{T}$), indeterminacy membership function ($\mathbb{I}$), and falsity membership function ($\mathbb{F}$). The concept of single-valued neutrosophic sets was established by Wang et al. (3). Ye (4) introduced a correlation coefficient for single-valued neutrosophic sets applied to multiple-attribute decision-making. Broumi and Smarandache (5) and (6) proposed an improved similarity measure for neutrosophic sets using the cosine function. The many similarity metrics of neutrosophic sets were also covered, along with a complete discussion of existing methodologies and applications. Ye (7) have studied the single-valued neutrosophic graph using a minimum spanning tree and its clustering method. Sahoo and Pal (8) studied several types of products on intuitionistic fuzzy graphs and investigated various graph product methods and impacts. Banerjee et al. (9) conducted grey relational analysis (GRA) for multi-attribute decisions in a neutrosophic cubic set context. Mohanta et al.,(10) paper presents a comprehensive investigation of various forms of neutrosophic graph products, an area of growing interest due to its applicability in managing uncertainty and complexity in graph-based models. The correlation coefficient of hesitancy fuzzy graphs in decision-making processes was examined by Reddy and Basha (11). Akula and Basha (12) introduced a regression coefficient measure for intuitionistic fuzzy graphs, which they applied to soil selection for optimal paddy crop development, highlighting its agricultural decision-making relevance. Bajaj and Kumar (13) introduced a novel intuitionistic fuzzy correlation coefficient technique, demonstrating its effectiveness in complex multi-criteria decision-making. Single valued neutrosophic graphs ($SVNGs$) and their applications were thoroughly examined by Dey et al. (14). Meenakshi et al. (15) and (16) examined neutrosophic social networks with max product networks and investigated neutrosophic optimal networks using various operations. Meenakshi and Mishra (17) explored the modular product of two cubic fuzzy graph structures. Meenakshi et al. (18) conducted a comparative study of fuzzy domination and fuzzy coloring in an optimal approach. Meenakshi et al. (19) examined neutrosophic optimum networks and applied efficient domination techniques to intuitionistic fuzzy networks. Akula and Shaik (20) developed a correlation coefficient for intuitionistic fuzzy graphs to analyze financial investment strategies, demonstrating the practical use of fuzzy graph theory in financial decision-making. Ye (21) provided an improved correlation coefficient for intuitionistic fuzzy sets, representing a substantial development in fuzzy set theory. Christianto and Smarandache (22) have explored quantum communication advancements, focusing on China's Tiantong satellite and its potential to revolutionize communication through quantum entanglement. Christianto and Smarandache (23) also discussed the intersection of Ikigai and Design Thinking using a conceptual framework.

Single Valued Neutrosophic Graph ($SVNG$) expands on classical graph theory by including degrees of truth, indeterminacy, and falsity for each vertex and edge. This research addresses the necessity for a robust framework by enhancing graph theory through $SVNGs$, enabling a more thorough analysis of $\mathbb{CRC}s$ in uncertain environments. Graph products of $SVNGs$, like the modular and strong products, are an important field of study as they enable the construction of new graphs with better structural properties by combining two $SVNGs$. By utilizing $SVNGs$ and exploring the relationship between $\mathbb{CRC}s$ for $SVNGPs$, this study aims to improve decision-making processes. This research carried out a sensitivity analysis to test the effectiveness of $\mathbb{CRC}s$ applied on two different types of operations of $SVNGPs$ and their $MST$. The analysis results indicate that the $\mathbb{CRC}s$ have moderate to high levels of truth and falsity memberships and less indeterminacy. The modular product is more efficient in minimizing indeterminacy, making it a more reliable option in scenarios where clarity is crucial. The motivation for this research is to provide advanced mathematical models that can deal with uncertainty, especially in areas like healthcare where patient outcomes and resource management are critical. The current research lacks sufficient study on graph products in $SVNGs$, particularly in terms of analyzing correlation and regression in uncertain conditions. The present analysis draws attention to the relevance of the parameters in $SVNGPs$ and their usage in decision-making processes. Specifically, the research focuses on identifying key factors that affect outcomes, such as patient care, and optimizing resource allocation, ultimately contributing to enhanced treatment protocols and patient management.

This research work is organized into nine sections. Section 1 provides a brief introduction to the research. In section 2, we have provided the preliminary definitions of the research. In section 3, we have introduced the concept of $\mathbb{CRC}s$ of the modular product of $SVNGs$. In section 4, we have introduced the $\mathbb{CRC}s$ for the modular product of $SVNGs$ using $MST$. In section 5, we have analyzed the strong product of $SVNGs$ and its $\mathbb{CRC}s$. The $\mathbb{CRC}s$ of the strong product of $SVNGs$ using $MST$ has been explored in section 6. In section 7, we present a flowchart that outlines the detailed approach used in the research. In section 8, we have discussed the applications of $\mathbb{CRC}s$ of $SVNGPs$ to identify significant factors affecting patient outcomes and develop focused strategies for improving treatment protocols and patient management, ultimately improving patient care and resource allocation. Section 9 offers a comprehensive summary of the key conclusions derived from the research findings.

## 2. **Preliminaries**

This section covers the key definitions and ideas needed for the discussions in this manuscript.

**Definition 2.1.**(1) Let $\xi$ represent the universe. A neutrosophic set $\mathcal{A}$ in $\xi$ is characterized by a truth membership function $\mathbb{T}_{\mathcal{A}}$, an indeterminacy membership function $\mathbb{I}_{\mathcal{A}}$, and a falsity membership function $\mathbb{F}_{\mathcal{A}}$, where $\mathbb{T}_{\mathcal{A}}$, $\mathbb{I}_{\mathcal{A}}$, and $\mathbb{F}_{\mathcal{A}}$ are real standard elements of $[0,1]$. It can be written as:

$$\mathcal{A} = \{\langle \mathbb{N}\mathcal{G}, x, \mathbb{T}_{\mathcal{A}}(x), \mathbb{I}_{\mathcal{A}}(x), \mathbb{F}_{\mathcal{A}}(x)\rangle : x \in \xi\}.$$

There is no restriction on the sum of $\mathbb{T}_{\mathcal{A}}(x)$, $\mathbb{I}_{\mathcal{A}}(x)$, and $\mathbb{F}_{\mathcal{A}}(x)$, so $0 \leq \mathbb{T}_{\mathcal{A}}(x) + \mathbb{I}_{\mathcal{A}}(x) + \mathbb{F}_{\mathcal{A}}(x) \leq 3$.

**Definition 2.2.** (1)
Let $X$ be a space of points (objects) with generic elements in $\xi$ represented by $x$. The truth-membership function $\mathbb{T}_{\mathcal{A}}(x)$, the indeterminacy-membership function $\mathbb{I}_{\mathcal{A}}(x)$, and the falsity-membership function $\mathbb{F}_{\mathcal{A}}(x)$ define a single valued neutrosophic set $\mathcal{A}$ ($SVNS$). For each point $x$ in $\xi$, $\mathbb{T}_{\mathcal{A}}(x), \mathbb{I}_{\mathcal{A}}(x), \mathbb{F}_{\mathcal{A}}(x) \in [0,1]$. The $SVNS$ $\mathcal{A}$ can be expressed as:

$$\mathcal{A} = \{\langle x, \mathbb{T}_{\mathcal{A}}(x), \mathbb{I}_{\mathcal{A}}(x), \mathbb{F}_{\mathcal{A}}(x)\rangle : x \in \xi\}$$

**Definition 2.3.**(11) For any two single valued neutrosophic sets ($SVNSs$) $\mathcal{A}$ and $\mathcal{B}$ in the universe of discourse $\mathcal{X} = \{t_1, t_2, \ldots, t_n\}$, the correlation coefficient $\mathcal{N}(\mathcal{A},\mathcal{B})$ is defined by:

$$\mathcal{N}(\mathcal{A},\mathcal{B}) = \frac{\sum_{i=1}^{n}\left[\mathbb{T}_{\mathcal{A}}(t_i)\mathbb{T}_{\mathcal{B}}(t_i) + \mathbb{I}_{\mathcal{A}}(t_i)\mathbb{I}_{\mathcal{B}}(t_i) + \mathbb{F}_{\mathcal{A}}(t_i)\mathbb{F}_{\mathcal{B}}(t_i)\right]}{\sqrt{\sum_{i=1}^{n}\left(\mathbb{T}_{\mathcal{A}}(t_i)^2 + \mathbb{I}_{\mathcal{A}}(t_i)^2 + \mathbb{F}_{\mathcal{A}}(t_i)^2\right)} \cdot \sqrt{\sum_{i=1}^{n}\left(\mathbb{T}_{\mathcal{B}}(t_i)^2 + \mathbb{I}_{\mathcal{B}}(t_i)^2 + \mathbb{F}_{\mathcal{B}}(t_i)^2\right)}} \quad (1)$$

where $\mathbb{T}_{\mathcal{A}}(t_i)$, $\mathbb{I}_{\mathcal{A}}(t_i)$, $\mathbb{F}_{\mathcal{A}}(t_i)$ and $\mathbb{T}_{\mathcal{B}}(t_i)$, $\mathbb{I}_{\mathcal{B}}(t_i)$, $\mathbb{F}_{\mathcal{B}}(t_i)$ are the truth-membership, indeterminacy-membership, and falsity-membership functions of $\mathcal{A}$ and $\mathcal{B}$ respectively at element $t_i$.

**Definition 2.4.**(11)
Assume that $\mathbb{N}\mathcal{G} = (V, E, \mathbb{T}, \mathbb{I}, \mathbb{F})$ is a single valued Neutrosophic Fuzzy Graph ($SVNFG$); and the energy of two $NFG$ $\mathbb{N}\mathcal{G}_1$ and $\mathbb{N}\mathcal{G}_2$ is defined as follows:

$$\mathcal{E}(\mathbb{N}\mathcal{G}_1) = \sum_{i=1}^{n}\left[\mathbb{T}^2_{\mathbb{N}\mathcal{G}_1}(t_i) + \mathbb{I}^2_{\mathbb{N}\mathcal{G}_1}(t_i) + \mathbb{F}^2_{\mathbb{N}\mathcal{G}_1}(t_i)\right]$$

$$\mathcal{E}(\mathbb{N}\mathcal{G}_2) = \sum_{i=1}^{n}\left[\mathbb{T}^2_{\mathbb{N}\mathcal{G}_2}(t_i) + \mathbb{I}^2_{\mathbb{N}\mathcal{G}_2}(t_i) + \mathbb{F}^2_{\mathbb{N}\mathcal{G}_2}(t_i)\right]$$

The covariance of $\mathbb{N}\mathcal{G}_1$ and $\mathbb{N}\mathcal{G}_2$ is defined as follows:

$$\mathrm{Cov}(\mathbb{N}\mathcal{G}_1, \mathbb{N}\mathcal{G}_2) = \sum_{i=1}^{n}\left[\mathbb{T}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{T}_{\mathbb{N}\mathcal{G}_2}(t_i) + \mathbb{I}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{I}_{\mathbb{N}\mathcal{G}_2}(t_i) + \mathbb{F}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{F}_{\mathbb{N}\mathcal{G}_2}(t_i)\right]$$

The correlation coefficients of $SVNGs$ $\mathbb{N}\mathcal{G}_1$ and $\mathbb{N}\mathcal{G}_2$ are defined as follows:

$$\mathrm{CC}(\mathbb{N}\mathcal{G}_1, \mathbb{N}\mathcal{G}_2) = \frac{\mathrm{Cov}(\mathbb{N}\mathcal{G}_1, \mathbb{N}\mathcal{G}_2)}{\sqrt{\mathcal{E}(\mathbb{N}\mathcal{G}_1)\mathcal{E}(\mathbb{N}\mathcal{G}_2)}}$$

$$= \frac{\sum_{i=1}^{n}\left[\mathbb{T}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{T}_{\mathbb{N}\mathcal{G}_2}(t_i)+\mathbb{I}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{I}_{\mathbb{N}\mathcal{G}_2}(t_i)+\mathbb{F}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{F}_{\mathbb{N}\mathcal{G}_2}(t_i)\right]}{\sqrt{\sum_{i=1}^{n}\left[\mathbb{T}^2_{\mathbb{N}\mathcal{G}_1}(t_i)+\mathbb{I}^2_{\mathbb{N}\mathcal{G}_1}(t_i)+\mathbb{F}^2_{\mathbb{N}\mathcal{G}_1}(t_i)\right]}\sqrt{\sum_{i=1}^{n}\left[\mathbb{T}^2_{\mathbb{N}\mathcal{G}_2}(t_i)+\mathbb{I}^2_{\mathbb{N}\mathcal{G}_2}(t_i)+\mathbb{F}^2_{\mathbb{N}\mathcal{G}_2}(t_i)\right]}}$$

**Definition 2.5.** (21) Xu et al. provided an alternative formula for the correlation coefficient of *NFGs* $\mathbb{N}\mathcal{G}_1$ and $\mathbb{N}\mathcal{G}_2$, defined as follows:

$$\mathrm{CC}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)=\frac{\mathrm{Cov}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)}{\max\left(\sqrt{\mathbb{E}(\mathbb{N}\mathcal{G}_1)},\sqrt{\mathbb{E}(\mathbb{N}\mathcal{G}_2)}\right)}$$

$$= \frac{\sum_{i=1}^{n}\left[\mathbb{T}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{T}_{\mathbb{N}\mathcal{G}_2}(t_i)+\mathbb{I}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{I}_{\mathbb{N}\mathcal{G}_2}(t_i)+\mathbb{F}_{\mathbb{N}\mathcal{G}_1}(t_i)\mathbb{F}_{\mathbb{N}\mathcal{G}_2}(t_i)\right]}{\max\left(\sqrt{\sum_{i=1}^{n}\left[\mathbb{T}^2_{\mathbb{N}\mathcal{G}_1}(t_i)+\mathbb{I}^2_{\mathbb{N}\mathcal{G}_1}(t_i)+\mathbb{F}^2_{\mathbb{N}\mathcal{G}_1}(t_i)\right]},\sqrt{\sum_{i=1}^{n}\left[\mathbb{T}^2_{\mathbb{N}\mathcal{G}_2}(t_i)+\mathbb{I}^2_{\mathbb{N}\mathcal{G}_2}(t_i)+\mathbb{F}^2_{\mathbb{N}\mathcal{G}_2}(t_i)\right]}\right)}$$

The correlation coefficient function $\mathrm{CC}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)$ satisfies the following conditions:

$$0 \le \mathrm{CC}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2) \le 1$$

$$\mathrm{CC}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)=\mathrm{CC}(\mathbb{N}\mathcal{G}_2,\mathbb{N}\mathcal{G}_1)$$

$$\mathrm{CC}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)=1 \text{ if } \mathbb{N}\mathcal{G}_1=\mathbb{N}\mathcal{G}_2$$

**Definition 2.6.** (13)

The regression coefficient of $y$ on $x$ is defined as:

$$\mathbb{R}_{yx}=\mathcal{K}\frac{\sigma_y}{\sigma_x}=\frac{\mathrm{Cov}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)}{\sigma_x\sigma_y}\frac{\sigma_y}{\sigma_x}=\frac{\mathrm{Cov}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)}{\sigma_x^2}$$

The regression coefficient of $x$ on $y$ is defined as:

$$\mathbb{R}_{xy}=\mathcal{K}\frac{\sigma_x}{\sigma_y}=\frac{\mathrm{Cov}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)}{\sigma_x\sigma_y}\frac{\sigma_x}{\sigma_y}=\frac{\mathrm{Cov}(\mathbb{N}\mathcal{G}_1,\mathbb{N}\mathcal{G}_2)}{\sigma_y^2}$$

Also, the relationship between the correlation and regression coefficients can be written as:

$$\mathcal{K}=\sqrt{\mathbb{R}_{yx}\times\mathbb{R}_{xy}}$$

## 3. $\mathbb{CRC}s$ of Modular Product of *SVNGs*

This section examines the modular product of two *SVNGs* and the $\mathbb{CRC}s$.

3.1. *Definition (Modular Product of SVNGs)*

Let $\mathbb{P}_1 = (\sigma_1, \mu_1)$ and $\mathbb{P}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of graph $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Then, the modular product $\mathbb{P}_1 \star \mathbb{P}_2 = (\sigma_1 \star \sigma_2, \mu_1 \star \mu_2)$ is defined as follows:

(1) For all $(a, b) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$$\mathbb{T}_{\sigma_1 \star \sigma_2}(a, b) = \mathbb{T}_{\sigma_1}(a) \wedge \mathbb{T}_{\sigma_2}(b),$$
$$\mathbb{I}_{\sigma_1 \star \sigma_2}(a, b) = \mathbb{I}_{\sigma_1}(a) \wedge \mathbb{I}_{\sigma_2}(b),$$
$$\mathbb{F}_{\sigma_1 \star \sigma_2}(a, b) = \mathbb{F}_{\sigma_1}(a) \vee \mathbb{F}_{\sigma_2}(b).$$

(2) For all $(a_1, b_1) \in \mathbb{E}_1$ and $(a_2, b_2) \in \mathbb{E}_2$:

$$\mathbb{T}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{T}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2),$$
$$\mathbb{I}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{I}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2),$$
$$\mathbb{F}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{F}_{\mu_1}(a_1, b_1) \vee \mathbb{F}_{\mu_2}(a_2, b_2),$$

(3) For all $(a_1, b_1) \notin \mathbb{E}_1$ and $(a_2, b_2) \notin \mathbb{E}_2$:

$$\mathbb{T}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{T}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2),$$
$$\mathbb{I}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{I}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2),$$
$$\mathbb{F}_{\mu_1 \star \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{F}_{\mu_1}(a_1, b_1) \vee \mathbb{F}_{\mu_2}(a_2, b_2),$$

3.2. *Example*

Consider the *SVNGs* $\mathbb{P}_1$ and $\mathbb{P}_2$ as shown in figure 1 and figure 2 and their modular product $\mathbb{P}_1 \star \mathbb{P}_2$ as shown in figure 3.

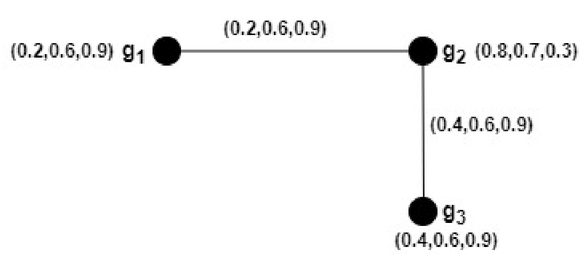



FIGURE 1. *SVNG* $\mathbb{P}_1$

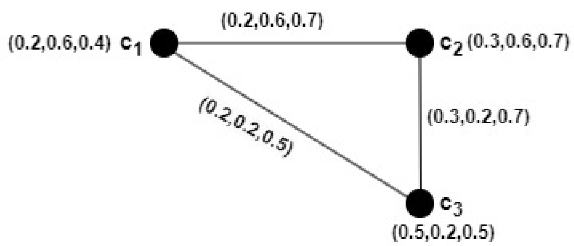



FIGURE 2. *SVNG* $\mathbb{P}_2$

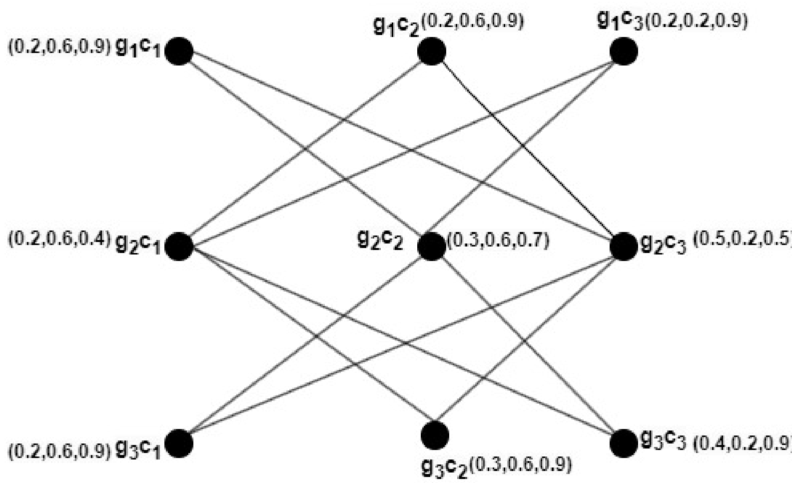



FIGURE 3. $\mathbb{P}_1 \star \mathbb{P}_2$

3.3. *Working Procedure*

Below is a working procedure to find the $\mathbb{CRC}s$ of Modular product $SVNGs$.

Step 1: Let $\mathbb{P}_1 = (\sigma_1, \mu_1)$ and $\mathbb{P}_2 = (\sigma_2, \mu_2)$ be two $SVNGs$ of $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Construct a modular product of $\mathbb{P}_1$ and $\mathbb{P}_2$, $\mathbb{P}_1 \star \mathbb{P}_2$. Then, find the adjacency matrix of $\mathbb{P}_1 \star \mathbb{P}_2$ .

Step 2: Compute the energy $\mathcal{E}(\mathbb{N}_i^{(r)})$ of an adjacency matrix $\mathbb{N}_i^{(r)}$ where $\mathcal{E}(\mathbb{N}_i^{(r)}) = \sum_{i=1}^{n} |\lambda_i|$

Step 3: Compute the weight scores $\mathcal{W}(\mathbb{S}_i^{(r)})$ determined by $\mathcal{E}(\mathbb{N}_i^{(r)})$

$$\mathcal{W}(\mathbb{S}_i^{(r)}) = \left( \frac{\mathcal{E}(\mathbb{N}_1^{(r)})}{\sum_{r=1}^{3} \mathcal{E}(\mathbb{N}_i^{(r)})}, \frac{\mathcal{E}(\mathbb{N}_2^{(r)})}{\sum_{r=1}^{3} \mathcal{E}(\mathbb{N}_i^{(r)})}, \frac{\mathcal{E}(\mathbb{N}_3^{(r)})}{\sum_{l=1}^{3} \mathcal{E}(\mathbb{N}_i^{(r)})} \right)$$

Step 4: Compute Correlation Coefficient $CC(\mathbb{N}_i^{(r)}, \mathbb{S}_i^{(r)})$ between $\mathbb{N}_i^{(r)}$ and $\mathbb{S}_i^{(r)}$.

$$CC(\mathbb{N}_i^{(r)}, \mathbb{S}_i^{(r)})) = \frac{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{N}_i^{(r)}}(t_i)\mathbb{T}_{\mathbb{S}_i^{(r)}}(t_i) + \mathbb{I}_{\mathbb{N}_i^{(r)}}(t_i)\mathbb{I}_{\mathbb{S}_i^{(r)}}(t_i) + \mathbb{F}_{\mathbb{N}_i^{(r)}}(t_i)\mathbb{F}_{\mathbb{S}_i^{(r)}}(t_i))}{\sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{N}_i^{(r)}}(t_i)^2 + \mathbb{I}_{\mathbb{N}_i^{(r)}}(t_i)^2 + \mathbb{F}_{\mathbb{N}_i^{(r)}}(t_i)^2)}\sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{S}_i^{(r)}}(t_i)^2 + \mathbb{I}_{\mathbb{S}_i^{(r)}}(t_i)^2 + \mathbb{F}_{\mathbb{S}_i^{(r)}}(t_i)^2)}}$$

step 5: Find the regression coefficient $\mathbb{R}_{(\mathbb{N}_i^{(r)},\mathbb{S}_i^{(r)})}$ of $\mathbb{P}_1 \star \mathbb{P}_2$. The regression coefficient of $\mathbb{N}_i^{(r)}$ on $\mathbb{S}_i^{(r)}$ is defined as:

$$\mathbb{R}_{(\mathbb{N}_i^{(r)},\mathbb{S}_i^{(r)})} = \frac{\text{Cov}(\mathbb{N}_i^{(r)}, \mathbb{S}_i^{(r)})}{(\mathbb{N}_i^{(r)})^2}$$

The regression coefficient of $\mathbb{S}_i^{(r)}$ on $\mathbb{N}_i^{(r)}$ is defined as:

$$\mathbb{R}_{(\mathbb{S}_i^{(r)},\mathbb{N}_i^{(r)})} = \frac{\text{Cov}(\mathbb{N}_i^{(r)}, \mathbb{S}_i^{(r)})}{(\mathbb{S}_i^{(r)})^2}$$

step 6: Calculate the relationship between the $\mathbb{CRC}s$

$$\mathcal{K}(\mathbb{S}_i^{(r)}, \mathbb{N}_i^{(r)}) = \sqrt{\mathbb{R}_{(\mathbb{N}_i^{(r)},\mathbb{S}_i^{(r)})} \times \mathbb{R}_{(\mathbb{S}_i^{(r)},\mathbb{N}_i^{(r)})}}$$

### 3.4. *Illustration of* $\mathbb{P}_1 \star \mathbb{P}_2$

#### 3.4.1. *Adjacency matrix of* $\mathbb{P}_1 \star \mathbb{P}_2$

$$\mathbb{T}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0.2 \\ 0.2 & 0 & 0.2 & 0 & 0 & 0 & 0.2 & 0 & 0.3 \\ 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{I}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0.2 & 0 & 0 & 0 & 0 & 0.6 & 0.2 \\ 0.6 & 0 & 0.2 & 0 & 0 & 0 & 0.6 & 0 & 0.2 \\ 0.6 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{F}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0.9 & 0 & 0 & 0 & 0 & 0.9 & 0.9 \\ 0.9 & 0 & 0.9 & 0 & 0 & 0 & 0.9 & 0 & 0.9 \\ 0.9 & 0.9 & 0 & 0 & 0 & 0 & 0.9 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.4.2. *Energy of* $\mathbb{P}_1 \star \mathbb{P}_2$

The energy $\mathcal{E}(\mathbb{N}_1^{(r)})$ of an adjacency matrix $\mathbb{N}_1^{(r)}$:

$$\mathcal{E}(\mathbb{N}_1^{(r)}) = \begin{bmatrix} 2.032 & 4.1854 & 4.1854 \end{bmatrix}$$

3.4.3. *Score function of* $\mathbb{P}_1 \star \mathbb{P}_2$

The weight score $\mathbb{S}_1^{(r)}$ determined by $\mathcal{E}(\mathbb{N}_1^{(r)})$:

$$\mathcal{W}(\mathbb{S}_1^{(r)}) = \begin{bmatrix} 0.1953 & 0.4023 & 0.4023 \end{bmatrix}$$

3.4.4. *Correlation Coefficient of* $\mathbb{P}_1 \star \mathbb{P}_2$

To Compute Correlation Coefficient $CC(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)})$ between $\mathbb{N}_1^{(r)}$ and $\mathbb{S}_1^{(r)}$:

$$\mathrm{CC}(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)}) = \frac{3.764}{3.763} = 1.00026$$

3.4.5. *Regression coefficients of* $\mathbb{P}_1 \star \mathbb{P}_2$

The regression coefficient measure of $\mathbb{N}_1^{(r)}$ on $\mathbb{S}_1^{(r)}$ calculated as:

$$\mathbb{R}_{(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)})} = \frac{3.764}{0.3617} = 10.406$$

The regression coefficient measure of $\mathbb{S}_1^{(r)})$ on $\mathbb{N}_1^{(r)}$ computed as:

$$\mathbb{R}_{(\mathbb{S}_1^{(r)}), \mathbb{N}_1^{(r)})} = \frac{3.764}{39.164} = 0.096$$

3.4.6. *Relationship between* $\mathbb{CRC}s$ *of* $\mathbb{P}_1 \star \mathbb{P}_2$

We defined the relationship between $\mathbb{CRC}s$ of $\mathbb{P}_1 \star \mathbb{P}_2$.

$$\mathcal{K}(\mathbb{N}_1^{(r)}, \mathbb{S}_1^{(r)}) = \sqrt{10.406 \times 0.096} = 0.999$$

## 4. $\mathbb{CRC}s$ of $\mathbb{P}_1 \star \mathbb{P}_2$ using $MST$

The $MST$ of the modular product of $SVNGs$ are explored in this section. The analysis is expanded to address the relationship between the $\mathbb{CRC}s$ of the modular product $SVNGs$.

### 4.1. *Working Procedure*

A working procedure to find the $\mathbb{CRC}s$ of $\mathbb{P}_1 \star \mathbb{P}_2$ using $MST$ is given below.
Step 1: Let $\mathbb{P}_1 = (\sigma_1, \mu_1)$ and $\mathbb{P}_2 = (\sigma_2, \mu_2)$ be two $SVNGs$ of $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Construct a modular product of $\mathbb{P}_1$ and $\mathbb{P}_2$, $\mathbb{P}_1 \star \mathbb{P}_2$. Then, find the $MST$ using the Kruskal algorithm .
Step 2: Compute the energy $\mathcal{E}(\mathbb{N}^{(r)}_{MST})$ of $MST$ using its adjacency matrix $\mathbb{N}^{(r)}_{MST}$:

$$\mathcal{E}(\mathbb{N}^{(r)}_{MST}) = \sum_{i=1}^{n} |\lambda_i|$$

Step 3: Compute the weight score $\mathcal{W}(\mathbb{S}^{(r)}_{MST})$ determined by $\mathcal{E}(\mathbb{N}^{(r)}_{MST})$ .

$$\mathcal{W}(\mathbb{S}^{(r)}_{MST}) = \left( \frac{\mathcal{E}(\mathbb{N}^{1}_{MST})}{\sum_{r=1}^{3} \mathcal{E}(\mathbb{N}^{r}_{MST})}, \frac{\mathcal{E}(\mathbb{N}^{2}_{MST})}{\sum_{r=1}^{3} \mathcal{E}(\mathbb{N}^{r}_{MST})}, \frac{\mathcal{E}(\mathbb{N}^{3}_{MST})}{\sum_{l=1}^{3} \mathcal{E}(\mathbb{N}^{r}_{MST})} \right)$$

Step 4: Compute Correlation Coefficient of $MST$ of $\mathbb{P}_1 \star \mathbb{P}_2$ as shown in figure 4 $CC(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})$ between $\mathbb{N}^{(r)}_{MST}$ and $\mathbb{S}^{(r)}_{MST}$.

$$CC(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})) = \frac{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{N}^{(r)}_{MST}}(t_i)\mathbb{T}_{\mathbb{S}^{(r)}_{MST}}(t_i) + \mathbb{I}_{\mathbb{N}^{(r)}_{MST}}(t_i)\mathbb{I}_{\mathbb{S}^{(r)}_{MST}}(t_i) + \mathbb{F}_{\mathbb{N}^{(r)}_{MST}}(t_i)\mathbb{F}_{\mathbb{S}^{(r)}_{MST}}(t_i))}{\sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{N}^{(r)}_{MST}}(t_i)^2 + \mathbb{I}_{\mathbb{N}^{(r)}_{MST}}(t_i)^2 + \mathbb{F}_{\mathbb{N}^{(r)}_{MST}}(t_i)^2)}\sqrt{\sum_{i=1}^{n} (\mathbb{T}_{\mathbb{S}^{(r)}_{MST}}(t_i)^2 + \mathbb{I}_{\mathbb{S}^{(r)}_{MST}}(t_i)^2 + \mathbb{F}_{\mathbb{S}^{(r)}_{MST}}(t_i)^2}$$

step 5: Find the regression coefficients $\mathbb{R}_{(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})}$ of $MST$ of modular product of $SVNGs$.
The regression coefficient of $\mathbb{N}^{(r)}_{MST}$ on $\mathbb{S}^{(r)}_{MST}$ is defined as:

$$\mathbb{R}_{(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})} = \frac{\mathrm{Cov}(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})}{(\mathbb{N}^{(r)}_{MST})^2}$$

The regression coefficient of $\mathbb{S}^{(r)}_{MST}$ on $\mathbb{N}^{(r)}_{MST}$ is defined as:

$$\mathbb{R}_{(\mathbb{S}^{(r)}_{MST}, \mathbb{N}^{(r)}_{MST})} = \frac{\mathrm{Cov}(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})}{(\mathbb{S}^{(r)}_{MST})^2}$$

step 6: Calculate the relationship between the $\mathbb{CRC}s$ using $MST$ of $\mathbb{P}_1 \star \mathbb{P}_2$.

$$\mathcal{K}(\mathbb{S}^{(r)}_{MST}, \mathbb{N}^{(r)}_{MST}) = \sqrt{\mathbb{R}_{(\mathbb{N}^{(r)}_{MST}, \mathbb{S}^{(r)}_{MST})} \times \mathbb{R}_{(\mathbb{S}^{(r)}_{MST}, \mathbb{N}^{(r)}_{MST})}}$$

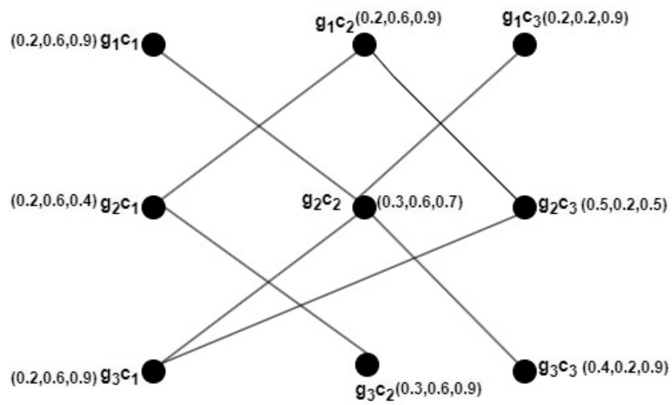



FIGURE 4. *MST* of $\mathbb{P}_1 \star \mathbb{P}_2$

4.2. *Illustration of* $\mathbb{CRC}s$ *using MST of* $\mathbb{P}_1 \star \mathbb{P}_2$

Step 1: Let $\mathbb{P}_1 = (\sigma_1, \mu_1)$and$\mathbb{P}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Construct a modular product of $\mathbb{P}_1$ (figure 1) and $\mathbb{P}_2$ (figure 2), $\mathbb{P}_1 \star \mathbb{P}_2$ as shown in figure 3. Then, the find *MST* using kurskal algorithm for $\mathbb{P}_1 \star \mathbb{P}_2$.
Step 2: Adjacency matrix of *MST* of $\mathbb{P}_1 \star \mathbb{P}_2$

$$\mathbb{T}^{(1)}_{MT} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0.2 & 0 & 0.2 & 0 & 0 & 0 & 0.2 & 0 & 0.3 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{I}^{(1)}_{MT} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0.6 & 0 & 0.2 & 0 & 0 & 0 & 0.6 & 0 & 0.2 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{F}_{MT}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0 \\ 0.9 & 0 & 0.9 & 0 & 0 & 0 & 0.9 & 0 & 0.9 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3: Energy of the $MST$ of $\mathbb{P}_1 \star \mathbb{P}_2$.

The energy $\mathcal{E}(\mathbb{N}_{MT}^{(1)})$ of an adjacency matrix $\mathbb{N}_{MT}^{(1)}$:

$$\mathcal{E}(\mathbb{N}_{MT}^{(1)}) = \begin{bmatrix} 1.9182 & 2.8686 & 2.8686 \end{bmatrix}$$

Step 4: Weight score of the $MST$ of $\mathbb{P}_1 \star \mathbb{P}_2$.

The score $\mathbb{N}_{MT}^{(1)}$ determined by $\mathcal{E}(\mathbb{N}_{MT}^{(1)})$ of the energy:

$$\mathcal{W}(\mathbb{S}_{MT}^{(1)}) = \begin{bmatrix} 0.2505 & 0.3747 & 0.3747 \end{bmatrix}$$

Step 5: Compute the covariance of the $MST$ $\mathbb{P}_1 \star \mathbb{P}_2$.

$$Cov(\mathcal{E}(\mathbb{N}_{MT}^{(1)}), \mathcal{W}(\mathbb{S}_{MT}^{(1)})) = \begin{bmatrix} 0.4805 & 1.0748 & 1.0748 \end{bmatrix} = 2.6301$$

Step 6: Calculate the correlation coefficient of the $MST$ of $\mathbb{P}_1 \star \mathbb{P}_2$.

$$\text{CC}(\mathcal{E}(\mathbb{N}_{MT}^{(1)}), \mathcal{W}(\mathbb{S}_{MT}^{(1)})) = \frac{2.6301}{2.6295} = 1.0002$$

Step 7: Compute the regression coefficients of the $MST$ of $\mathbb{P}_1 \star \mathbb{P}_2$.

The regression coefficient measure of $(\mathcal{E}(\mathbb{N}_{MT}^{(1)})$ on $\mathcal{W}(\mathbb{S}_{MT}^{(1)}))$ calculated as:

$$\mathbb{R}_{(\mathbb{N}_{MT}^{(1)}, \mathbb{S}_{MT}^{(1)})} = \frac{2.63}{0.343} = 7.66$$

The regression coefficient measure of $\mathcal{W}(\mathbb{S}_{MT}^{(1)}))$ on $(\mathcal{E}(\mathbb{N}_{MT}^{(1)})$ computed as:

$$\mathbb{R}_{(\mathbb{S}_{MT}^{(1)}, \mathbb{N}_{MT}^{(1)})} = \frac{2.63}{20.136} = 0.1306$$

Step 8: Compute the relationship between $\mathbb{CRC}s$ of the $MST$ of $\mathbb{P}_1 \star \mathbb{P}_2$.

$$\mathcal{K}(\mathbb{N}_{MT}^{(1)}, \mathbb{S}_{MT}^{(1)}) = \sqrt{7.66 \times 0.1306} = 1.00$$

## 5. $\mathbb{CRC}s$ of Strong Product of *SVNGs*

This section examines the strong product of two *SVNGs* and investigates the relationship between the $\mathbb{CRC}s$.

*Definition: Strong Product of SVNGs*

Let $\mathbb{S}_1 = (\sigma_1, \mu_1)$ and $\mathbb{S}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of graph $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. The strong product $\mathbb{S}_1 \boxtimes \mathbb{S}_2 = (\sigma_1 \boxtimes \sigma_2, \mu_1 \boxtimes \mu_2)$ is defined as follows:

(1) For all $(a, b) \in \mathbb{V}_1 \times \mathbb{V}_2$:

$$\mathbb{T}_{\sigma_1 \boxtimes \sigma_2}(a, b) = \mathbb{T}_{\sigma_1}(a) \wedge \mathbb{T}_{\sigma_2}(b),$$

$$\mathbb{I}_{\sigma_1 \boxtimes \sigma_2}(a, b) = \mathbb{I}_{\sigma_1}(a) \wedge \mathbb{I}_{\sigma_2}(b),$$

$$\mathbb{F}_{\sigma_1 \boxtimes \sigma_2}(a, b) = \mathbb{F}_{\sigma_1}(a) \vee \mathbb{F}_{\sigma_2}(b),$$

(2) For all $a_1 \in \mathbb{V}_1$ and $(b_1, b_2) \in \mathbb{E}_2$:

$$\mathbb{T}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_1, b_2)) = \mathbb{T}_{\mu_1}(a_1, b_1) \wedge \mathbb{T}_{\mu_2}(b_1, b_2),$$

$$\mathbb{I}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_1, b_2)) = \mathbb{I}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(b_1, b_2),$$

$$\mathbb{F}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_1, b_2)) = \mathbb{F}_{\mu_1}(a_1, b_1) \vee \mathbb{F}_{\mu_2}(b_1, b_2),$$

(3) For all $b_1 \in \mathbb{V}_2$ and $(a_1, a_2) \in \mathbb{E}_1$:

$$\mathbb{T}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_1)) = \mathbb{T}_{\mu_1}(a_1, a_2) \wedge \mathbb{T}_{\mu_2}(b_1, b_1),$$

$$\mathbb{I}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_1)) = \mathbb{I}_{\mu_1}(a_1, a_2) \wedge \mathbb{I}_{\mu_2}(b_1, b_1),$$

$$\mathbb{F}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_1)) = \mathbb{F}_{\mu_1}(a_1, a_2) \vee \mathbb{F}_{\mu_2}(b_1, b_1),$$

(4) For all $(a_1, b_1) \in \mathbb{E}_1$ and $(a_2, b_2) \in \mathbb{E}_2$:

$$\mathbb{T}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{T}_{\mu_1}(a_1, b_1) \wedge \mathbb{T}_{\mu_2}(a_2, b_2),$$

$$\mathbb{I}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{I}_{\mu_1}(a_1, b_1) \wedge \mathbb{I}_{\mu_2}(a_2, b_2),$$

$$\mathbb{F}_{\mu_1 \boxtimes \mu_2}((a_1, b_1), (a_2, b_2)) = \mathbb{F}_{\mu_1}(a_1, b_1) \vee \mathbb{F}_{\mu_2}(a_2, b_2),$$

5.1. *Example*

Consider the *SVNGs* $\mathbb{S}_1$ and $\mathbb{S}_2$ as shown in figure 5 and figure 6 and their strong product $\mathbb{S}_1 \boxtimes \mathbb{S}_2$ as shown in figure 7.

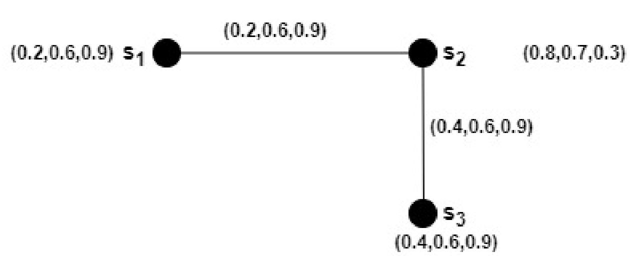



FIGURE 5. *SVNGs* $\mathbb{S}_1$

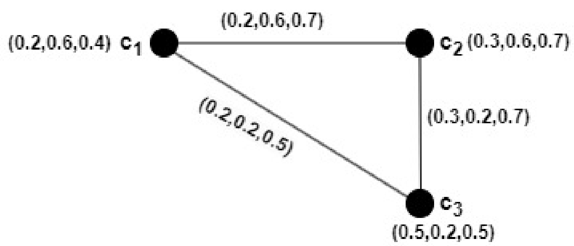



FIGURE 6. *SVNGs* $\mathbb{S}_2$

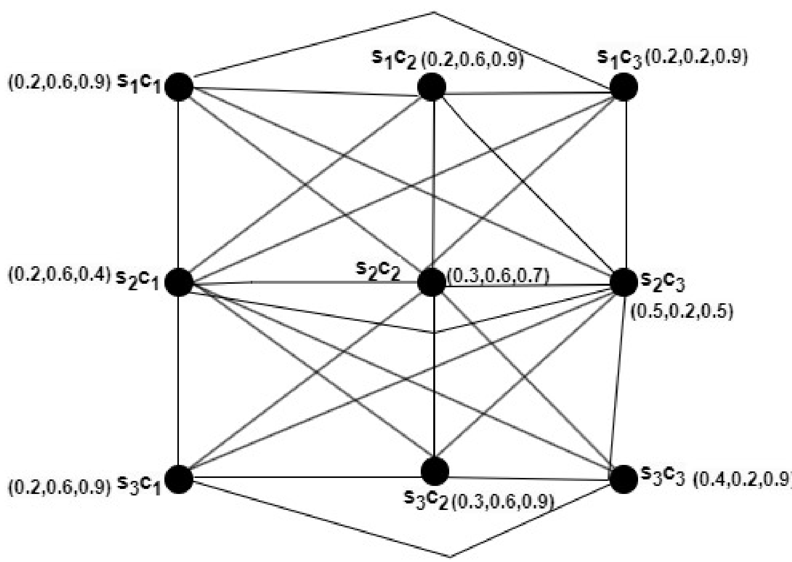



FIGURE 7. $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

5.2. *Adjacency matrix of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

$$\mathbb{T}^{(2)} = \begin{pmatrix} 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0.3 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.3 & 0 & 0.2 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.3 & 0.2 & 0 & 0.3 \\ 0 & 0 & 0 & 0.2 & 0.3 & 0.4 & 0.2 & 0.3 & 0 \end{pmatrix}$$

$$\mathbb{I}^{(2)} = \begin{pmatrix} 0 & 0.6 & 0.2 & 0.6 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0.6 & 0 & 0.2 & 0.6 & 0.6 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0 & 0 & 0 \\ 0.6 & 0.6 & 0.2 & 0 & 0.6 & 0.2 & 0.6 & 0.6 & 0.2 \\ 0.6 & 0.6 & 0.2 & 0.6 & 0 & 0.2 & 0.6 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.6 & 0.2 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.6 & 0.2 & 0.6 & 0 & 0.2 \\ 0 & 0 & 0 & 0.6 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \end{pmatrix}$$

$$\mathbb{F}^{(2)} = \begin{pmatrix} 0 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0.9 & 0 & 0.9 & 0.9 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0.9 & 0.9 & 0 & 0.9 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0.9 & 0.9 & 0.9 & 0 & 0.7 & 0.5 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.9 & 0.7 & 0 & 0.7 & 0.9 & 0.9 & 0.9 \\ 0.9 & 0.9 & 0.9 & 0.5 & 0.7 & 0 & 0.9 & 0.9 & 0.9 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0.9 & 0 & 0.9 & 0.9 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0.9 & 0.9 & 0 & 0.9 \\ 0 & 0 & 0 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0 \end{pmatrix}$$

### 5.3. *Energy of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

The energy $\mathcal{E}(\mathbb{N}_2^{(r)})$ of an adjacency matrix $\mathbb{N}_2^{(r)}$:

$$\mathcal{E}(\mathbb{N}_2^{(r)}) = \begin{bmatrix} 3.6266 & 6.2948 & 14.3234 \end{bmatrix}$$

### 5.4. *Weight Score function of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

The scores $\mathbb{S}_2^{(r)}$ determined by $\mathcal{E}(\mathbb{N}_2^{(r)})$ of the energy:

$$\mathcal{W}(\mathbb{S}_2^{(r)}) = \begin{bmatrix} 0.1495 & 0.2596 & 0.5907 \end{bmatrix}$$

### 5.5. *Correlation Coefficient of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

To Compute Correlation Coefficient $CC(\mathbb{N}_2^{(r)}, \mathbb{S}_2^{(r)})$ between $\mathbb{N}_2^{(r)}$ and $\mathbb{S}_2^{(r)}$:

$$\mathrm{CC}(\mathbb{N}_2^{(r)}, \mathbb{S}_2^{(r)}) = \frac{10.637}{10.633} = 1.0003$$

5.6. *Regression Coefficients of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

The regression coefficient measure of $\mathbb{N}_2^{(r)}$ on $\mathbb{S}_2^{(r)}$ calculated as:

$$\mathbb{R}_{(\mathbb{N}_2^{(r)},\mathbb{S}_2^{(r)})} = \frac{10.63}{257.936} = 0.0412$$

The regression coefficient measure of $\mathbb{S}_2^{(r)})$ on $\mathbb{N}_2^{(r)}$ computed as:

$$\mathbb{R}_{(\mathbb{S}_2^{(r)}),\mathbb{N}_2^{(r)})} = \frac{10.63}{0.4385} = 24.2417$$

5.7. *Relationship between* $\mathbb{CRC}s$ *of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

$$\mathcal{K}(\mathbb{N}_2^{(r)}, \mathbb{S}_2^{(r)}) = \sqrt{0.0412 \times 24.2417} = 0.999$$

## 6. $\mathbb{CRC}s$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$ using *MST*

In this section, we have discussed the *MST* of the strong product of *SVNGs* in figure 7. The $\mathbb{CRC}s$ of *MST* of the strong product of *SVNGs* is introduced.

6.1. *The* $\mathbb{CRC}s$ *of MST of* $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

Step 1: Let $\mathbb{S}_1 = (\sigma_1, \mu_1)$ and $\mathbb{S}_2 = (\sigma_2, \mu_2)$ be two *SVNGs* of $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2)$, respectively. Construct a strong product of $\mathbb{S}_1$ (figure 5) and $\mathbb{S}_2$ (figure 6) which is denoted as $\mathbb{S}_1 \boxtimes \mathbb{S}_2$ as shown in figure 7. Then, find the *MST* using the Kruskal algorithm for $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

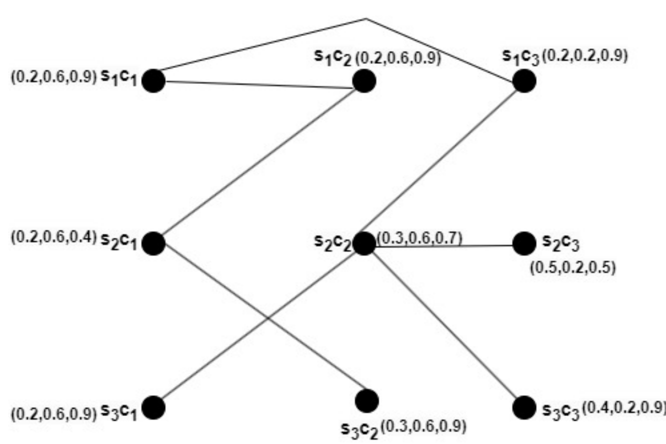



Figure 8. *MST* of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

Step 2: Adjacency matrix of $MST$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$

$$\mathbb{T}^{(2)}_{ST} = \begin{pmatrix} 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.3 & 0.2 & 0 & 0.3 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{I}^{(2)}_{ST} = \begin{pmatrix} 0 & 0.6 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.2 & 0.2 & 0 & 0.2 \\ 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{F}^{(2)}_{ST} = \begin{pmatrix} 0 & 0.9 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.9 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0.7 & 0.9 & 0 & 0.9 \\ 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3: Energy of the $MST$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

The energy $\mathcal{E}(\mathbb{N}^{(2)}_{ST})$ of an adjacency matrix $\mathbb{N}^{(2)}_{ST}$:

$$\mathcal{E}(\mathbb{N}^{(2)}_{ST}) = \begin{bmatrix} 1.8256 & 3.4306 & 3.4306 \end{bmatrix}$$

Step 4: Weight score of the $MST$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

The score $\mathbb{N}^{(2)}_{ST}$ determined by $\mathcal{E}(\mathbb{N}^{(r)}_{ST})$ of the energy:

$$\mathcal{W}(\mathbb{S}^{(2)}_{ST}) = \begin{bmatrix} 0.2101 & 0.3949 & 0.3949 \end{bmatrix}$$

Step 5: Compute the covariance of the $MST$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

$$Cov(\mathcal{E}(\mathbb{N}_{ST}^{(2)}), \mathcal{W}(\mathbb{S}_{ST}^{(2)})) = 3.092$$

Step 6: Calculate the correlation coefficient of the $MST$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

$$\mathrm{CC}(\mathcal{E}(\mathbb{N}_{ST}^{(2)}), \mathcal{W}(\mathbb{S}_{ST}^{(2)})) = \frac{3.092}{3.031} = 1.02012$$

Step 7: Compute the regression coefficients of the $MST$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.
The regression coefficient measure of $(\mathcal{E}(\mathbb{N}_{ST}^{(2)})$ on $\mathcal{W}(\mathbb{S}_{ST}^{(2)}))$ calculated as:

$$\mathbb{R}_{(\mathbb{N}_{ST}^{(2)}, \mathbb{S}_{ST}^{(2)})} = \frac{3.092}{0.355} = 8.709$$

The regression coefficient measure of $\mathcal{W}(\mathbb{S}_{ST}^{(2)}))$ on $(\mathcal{E}(\mathbb{N}_{ST}^{(2)})$ computed as:

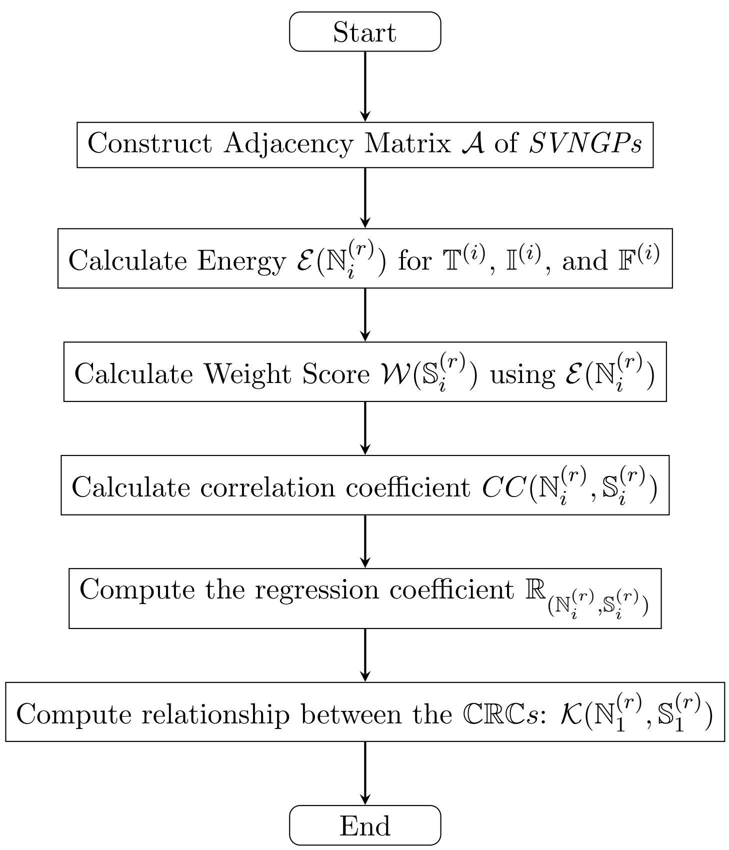
$$\mathbb{R}_{(\mathbb{S}_{ST}^{(2)}, \mathbb{N}_{ST}^{(2)})} = \frac{3.092}{25.830} = 0.119$$

Step 8: Calculate the relationship between $\mathbb{CRC}s$ of the $MST$ of $\mathbb{S}_1 \boxtimes \mathbb{S}_2$.

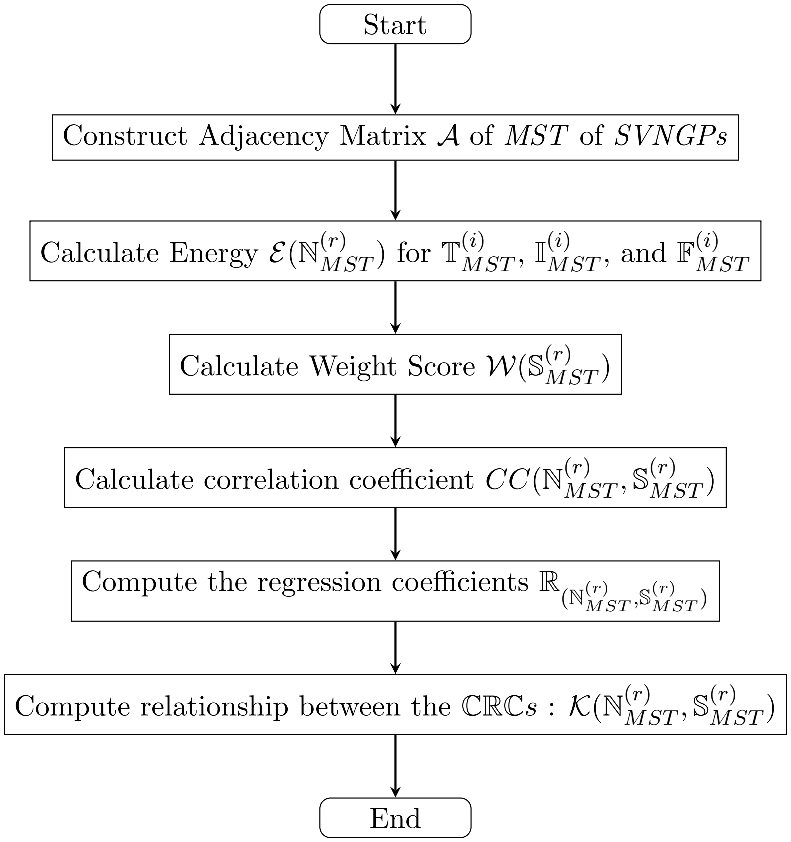
$$\mathcal{K}(\mathbb{N}_{ST}^{(2)}, \mathbb{S}_{ST}^{(2)}) = \sqrt{8.709 \times 0.119} = 1.01$$

## 7. **Flowchart**

### 7.1. *Flowchart for CRCs of SVNGPs without MST*



7.2. *Flowchart for CRCs of SVNGPs using MST*




## 8. **Applications**

In the healthcare system, it is essential to recognize the complex dynamics between patients, medical professionals, and healthcare institutions to deliver high-quality care and improve outcomes. To examine these interactions with inherent uncertainties and indeterminacies, we used the modular product of $SVNGs$ and strong product $SVNGs$, as well as their minimal spanning tree ($MST$). Using $\mathbb{CRC}s$, we can identify significant interaction patterns, optimize resource allocation, and analyze patient outcomes. The modular product $SVNGs$ enables us to combine many healthcare-related aspects into a complete graph that captures the complicated interdependence of diverse components. By calculating the $\mathbb{CRC}$ for this modular product of $SVNGs$, We can explore the relationships among different healthcare interactions. For instance, we can discover how interactions between patients and medical personnel affect each other behaviors, providing information about the critical variations of interaction that enhance patient care.

In addition, the critical connections in the healthcare network are precisely represented by the $MST$ of this modular product of $SVNGs$. The dynamics of the healthcare network can be completely understood by combining the correlation and regression analyses of $SVNGPs$ and their *MST*. This reduces redundancies and boosts overall efficiency by ensuring that medical supplies, staff, and equipment are dispersed appropriately. The strong product *SVNGs* yielded $\mathbb{CRC}s$ which are useful in understanding how interactions between healthcare personnel impact patient recovery rates and overall health outcomes. This study assists healthcare organizations in developing effective plans by identifying times and locations with high demand. Identifying critical characteristics that affect patient outcomes and developing focused techniques to enhance therapies and patient care are made easier with this all-inclusive strategy. Combining these *SVNGPs* with their *MST* offers valuable insights that enhance healthcare systems, ultimately leading to better resource management and patient care.

## 9. Comparison and Discussion

In this section, we compare and discuss the $\mathbb{CRC}s$ values for both the modular and strong products of *SVNGs*, with and without the use of *MST*.

### 9.1. *Comparison of $\mathbb{CRC}$ Values for Modular and Strong Products with and without MST*

Table 1. Comparison of $\mathbb{CRC}$ Values for Modular and Strong Product of *SVNGs*

| **Operations on SVNGs** | $\mathbb{CRC}$ **Value Without *MST*** | $\mathbb{CRC}$ **Value With *MST*** |
|---|---|---|
| **Modular Product of *SVNG*** | 0.999 | 1.00 |
| **Strong Product of *SVNG*** | 0.999 | 1.01 |

Comparison of the modular product of *SVNGPs* with and without *MST* of $\mathbb{CRC}$ shows that the value rises from 0.999 to 1.00, indicating that *MST* enhances graph efficiency. Similarly, for the strong product of *SVNGPs*, the $\mathbb{CRC}$ value increases from 0.999 to 1.01, highlighting that *MST* has a greater accuracy in capturing key relationships.

### 9.2. *Discussion*

The modular product of *SVNGPs* with and without *MST* provides a slight improvement in accuracy as $\mathbb{CRC}$ value 0.999 increases to 1.00. This subtle difference suggests that *MST* removes redundant edges and creates more efficient graphs where identifying key relationships is crucial for decision-making. For a strong product of *SVNGPs* with and without *MST* the $\mathbb{CRC}$ value 0.999 increases to 1.01. This provides a more accurate and interpretable correlation analysis, particularly in highly interconnected systems. As a result, the strong product with *MST* outperforms the version without *MST*. A negative correlation does not align with our research objectives. Since the $\mathbb{CRC}$ of the *SVNGPs* lies within the range [0,1], this presents a limitation for our research. Our study aims to identify positive relationships between elements, such as patient care factors and resource management, where an increase in one factor positively affects another.

## 10. Conclusion

This study provides $\mathbb{CRC}s$ for the modular and strong product of *SVNGs* and their *MST*. This analysis offers insights into the comparative study of *SVNGPs* with and without *MST*, determining $\mathbb{CRC}s$. The $\mathbb{CRC}s$ of *SVNGPs* helps to identify the key factors affecting patient outcomes and develop better treatment strategies and patient management. This study ultimately enhances patient care and optimizes resource allocation in healthcare. Future research will investigate the use of *SVNGPs* in various sectors to show the robustness of the proposed methodologies, including different types of neutrosophic graphs like bipolar or interval-valued graphs.

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