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On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices

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Abstract – In this paper, we present equivalent characterizations of k-kernel symmetric (k-KS) Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFMs). Additionally, we establish the necessary and sufficient conditions for the Schur complement (SC) within a k-KS QPNFM to be k-symmetric. The study also offers equivalent characterizations of both KS and k-KS QPNFMs. A few fundamental examples of KS QPNFMs are provided to clarify these concepts. It is shown that although k-symmetry implies k-KS, the converse does not necessarily hold. Several fundamental properties of k-KS QPNFMs are also derived. Finally, decision-making model utilizing QPNSMs has been successfully developed and validated through its application to real-world problems.

Keywords: QPNFM, Schur Complement, KS, k-KS.

1. Introduction

In the growing domain of fuzzy matrix theory, the Schur complement has emerged as a powerful mathematical tool, particularly for solving systems of linear equations, matrix decomposition, and optimization problems. The Schur complement has been widely applied in various branches of linear algebra and matrix theory, enabling the simplification of complex matrix structures by partitioning them into smaller submatrices and providing insights into their properties. When extended to fuzzy matrices, the Schur complement offers valuable methods for handling uncertainty and vagueness in system modeling, which is especially crucial in real-world applications where precise information is often unavailable. In recent years, the integration of neutrosophic logic into fuzzy matrices has further advanced the capacity to manage uncertainty. Neutrosophic fuzzy matrices introduce three distinct components—truth (T), falsity (F), and indeterminacy (I)—which allow for a more nuanced representation of uncertain or incomplete data. The k-kernel symmetric matrices, another essential class of matrices in fuzzy theory, provide structured approaches for modeling symmetry and balance within systems. These matrices maintain symmetry in relation to a kernel, offering valuable properties in matrix decomposition and transformation.

The concept of quadri partitioning within neutrosophic fuzzy matrices introduces a novel method for dividing the matrix into four distinct partitions, each of which can represent various aspects of the system being modeled. This partitioning method enhances the flexibility of matrix operations, enabling more efficient computation and improved structural analysis, especially in systems that exhibit complex or multidimensional relationships. The Schur complement in the context of k-kernel symmetric quadri partitioned neutrosophic fuzzy matrices represents a fusion of these powerful mathematical tools. It aims to extend the traditional Schur complement to this new matrix class, allowing for more sophisticated handling of uncertainty and system decomposition. The combination of neutrosophic logic, kernel symmetry, and quadri partitioning offers a unique framework for solving complex systems with multiple layers of uncertainty, making it particularly useful in fields such as decision-making, artificial intelligence, and network theory, where incomplete or conflicting information is common. In this study, we explore the application of the Schur complement in k-kernel symmetric quadri partitioned neutrosophic fuzzy matrices, aiming to extend its utility to more intricate and uncertain systems. This research contributes to the broader understanding of matrix theory by introducing novel theoretical frameworks and practical methodologies for solving systems with high degrees of uncertainty and complexity.

The growing complexity of real-world systems has posed significant challenges in modeling uncertainty and imprecision. Traditional set theory, which relies on binary logic, often falls short in capturing the ambiguity present in natural phenomena. To address these limitations, Lotfi A. Zadeh introduced the concept of fuzzy sets in 1965, which allowed for partial membership, offering a more flexible framework for uncertainty modeling [1]. This revolutionary idea laid the foundation for fuzzy logic and its widespread applications in control systems, decision-making, and artificial intelligence.Since Zadeh's pioneering work, several extensions and generalizations of fuzzy sets have emerged, each aimed at refining the ability to handle various forms of uncertainty. Atanassov's intuitionistic fuzzy sets, introduced in 1983, provided an additional degree of freedom by incorporating both membership and non-membership functions [2]. This model was soon followed by Smarandache's introduction of neutrosophic sets, which generalized intuitionistic fuzzy sets by adding an indeterminacy function to capture even more complex uncertainties [3]. These advancements have been crucial in various fields, such as medical diagnosis, pattern recognition, and engineering systems.

As fuzzy set theory evolved, so did the need for algebraic structures capable of representing and processing fuzzy data. Matrix theory emerged as a powerful tool for this purpose, enabling the structured manipulation of fuzzy information. One of the early contributions to this area was made by Kim and Roush, who generalized the concept of fuzzy matrices in 1980 [4]. This work opened up new possibilities for representing relationships between fuzzy quantities in matrix form, making it easier to perform operations such as addition, multiplication, and inversion on fuzzy data. Following this development, several researchers contributed to expanding the theory and applications of fuzzy matrices. Meenakshi's work in 2008 on fuzzy matrix theory significantly advanced the understanding of how matrices can be used in fuzzy logic and decision-making systems [5]. Matrix algebra provided a robust mathematical foundation for fuzzy models, and studies such as those by Hill and Waters [6] and Baskett and Katz [7] further explored the properties of κ-real and κ-Hermitian matrices, as well as EPr matrices. These investigations were essential in understanding the behavior of complex matrix operations, which are widely used in systems theory and optimization problems.

In parallel, research on special classes of matrices such as κ-EP matrices and their Schur complements gained traction. Meenakshi and Krishnamoorthy made substantial contributions in this domain, examining the properties and applications of κ-EP matrices and Schur complements [8-10]. Their findings have had significant implications for the study of matrix decompositions and applications in numerical analysis, signal processing, and optimization. Moreover, the exploration of secondary matrix structures, such as secondary κ-kernel symmetric matrices, introduced new directions for research in fuzzy mathematics. These matrices, first studied by Meenakshi and D. Jaya Shree, extend the concept of symmetry in matrices to fuzzy environments, allowing for more nuanced analysis of fuzzy systems [10-11]. Secondary symmetric matrices, including secondary skew-symmetric and orthogonal matrices, were further examined by An Lee [13], contributing to the understanding of how symmetry properties can influence matrix behavior in fuzzy systems.

More recent developments have focused on interval-valued fuzzy matrices, which account for uncertainty by allowing membership values to be represented as intervals rather than precise numbers. Shyamal and Pal's work in this area demonstrated the utility of interval-valued fuzzy matrices in dealing with imprecise data [12, 20]. Building on this, Meenakshi and Kalliraja introduced regular interval-valued fuzzy matrices, adding further refinement to the representation and manipulation of uncertain information [21]. The introduction of neutrosophic sets by Smarandache added another layer of complexity to fuzzy matrix theory. Neutrosophic fuzzy matrices, which incorporate indeterminacy in addition to membership and non-membership values, have gained considerable attention in recent years. Anandhkumar and colleagues made notable contributions to this field, exploring generalized symmetric neutrosophic fuzzy matrices [23] and secondary κ-column symmetric neutrosophic fuzzy matrices [24]. These studies have significantly expanded the applicability of fuzzy matrices in fields such as image processing, decision analysis, and communication systems.

Neutrosophic matrices have also been applied to new areas of research, including the study of pseudo-similarity in neutrosophic fuzzy matrices, as explored by Anandhkumar et al. in 2023 [27]. Their findings offer new perspectives on how similarity and dissimilarity can be quantified in fuzzy environments, providing valuable insights for applications in data mining, pattern recognition, and machine learning. Additionally, various inverse operations on neutrosophic fuzzy matrices, examined in [28], have opened up new possibilities for solving complex systems of equations in fuzzy environments, with potential applications in artificial intelligence, economics, and systems theory. As the field continues to grow, the intersection of matrix theory and fuzzy set theory remains a rich area of research. Ongoing studies on the properties of matrices, such as secondary κ-range symmetric fuzzy matrices [26] and interval-valued secondary κ-range symmetric neutrosophic fuzzy matrices [25], continue to push the boundaries of what is possible with fuzzy systems. Punithavalli et al [35] has discussed Reverse Sharp and Left-T Right-T Partial Ordering On IFM. These advancements highlight the importance of fuzzy matrix theory in addressing modern challenges in various scientific and engineering disciplines.

1.1 Literature Review

The concept of fuzzy sets, first introduced by Lotfi A. Zadeh in 1965, revolutionized the field of mathematical modeling by allowing for degrees of membership, thus accommodating uncertainty and imprecision in real-world systems [1]. Zadeh's work laid the foundation for fuzzy logic, which has since become essential in diverse applications, from control systems and decision-making to artificial intelligence and data analysis. Fuzzy set theory addressed the limitations of classical set theory, where membership was strictly binary. By allowing partial membership, fuzzy sets provided a more nuanced approach to modeling uncertainty.

Further developments in fuzzy set theory led to the introduction of intuitionistic fuzzy sets (IFS) by Atanassov in 1983 [22]. Intuitionistic fuzzy sets extended Zadeh's fuzzy sets by introducing an additional degree of uncertainty through a non-membership function, complementing the membership function. Unlike traditional fuzzy sets, which only consider the degree of membership, intuitionistic fuzzy sets account for both membership and non-membership, along with a degree of hesitation or indeterminacy.

The theory of intuitionistic fuzzy sets has found applications in areas such as decision-making, pattern recognition, and medical diagnosis [2]. For instance, intuitionistic fuzzy sets are particularly useful in situations where experts are unsure about the degree to which an element belongs to a set, allowing for a more comprehensive representation of uncertainty. Although the initial work focused on theoretical aspects, subsequent research has expanded into applications, such as multi-criteria decision-making and image processing [2, 12].

In 2005, Smarandache introduced neutrosophic sets, a generalization of intuitionistic fuzzy sets that incorporated an additional indeterminacy function [3]. Neutrosophic sets allow for the representation of truth, falsity, and indeterminacy independently, making them more flexible for handling complex uncertainty. This development addressed limitations in intuitionistic fuzzy sets, where the sum of membership and non-membership functions had to equal 1. Neutrosophic sets relaxed this restriction, allowing for more comprehensive modeling of uncertain, inconsistent, and incomplete information.

Neutrosophic sets have since been applied to a wide range of fields, including decision-making, optimization, and image processing. The flexibility provided by the independent consideration of truth, falsity, and indeterminacy has made neutrosophic sets particularly valuable in environments with high uncertainty and ambiguity [23]. More recent work has focused on integrating neutrosophic sets with other mathematical frameworks, such as matrices, to improve their applicability in complex systems.

Matrix theory has long been a fundamental tool in various branches of mathematics, and its extension to fuzzy environments has expanded its utility. Fuzzy matrices represent relationships between fuzzy quantities in a structured way, enabling matrix operations such as addition, multiplication, and inversion to be applied to uncertain data [4]. One of the early works in this area was Kim and Roush's 1980 exploration of generalized fuzzy matrices [4]. Their work established the foundational concepts needed for the manipulation of fuzzy matrices, setting the stage for further research.

Subsequent developments in fuzzy matrix theory include A. R. Meenakshi's work in 2008, which contributed significantly to the application of fuzzy matrices in decision-making and optimization [5]. Meenakshi's work extended the theory of fuzzy matrices to include new classes of matrices, such as κ-EP matrices, which have important applications in system theory, numerical analysis, and signal processing [8-10].

Research on the Schur complement of κ-EP matrices by Meenakshi and Krishnamoorthy has been particularly influential, as it addresses the properties and behavior of matrices under certain transformations [9]. The Schur complement plays a vital role in the decomposition of matrices and has applications in solving systems of linear equations, making it an essential tool in fields such as optimization, control theory, and econometrics.

One significant extension of fuzzy matrix theory is the development of interval-valued fuzzy matrices, which allow for the representation of uncertainty in the form of intervals rather than precise membership values. This approach was introduced by Shyamal and Pal in 2006 [12], who demonstrated the advantages of interval-valued fuzzy matrices in handling data with a high degree of imprecision. By allowing membership values to vary within a range, interval-valued fuzzy matrices offer a more flexible and realistic representation of uncertainty in real-world problems, where exact probabilities or membership degrees may be difficult to obtain.

Meenakshi and Kalliraja further developed this concept by introducing regular interval-valued fuzzy matrices, which added an additional layer of structure and consistency to the representation of uncertainty [21]. These matrices have found applications in areas such as decision analysis and pattern recognition, where handling imprecise data is critical.The introduction of neutrosophic sets into matrix theory led to the development of neutrosophic fuzzy matrices, which incorporate the three components of neutrosophic logic: truth, falsity, and indeterminacy [23]. This extension allows matrices to model even more complex forms of uncertainty, where the degree of indeterminacy plays a significant role.

Anandhkumar and colleagues have been at the forefront of research on neutrosophic fuzzy matrices, exploring several new classes of matrices, such as generalized symmetric neutrosophic fuzzy matrices and secondary κ-column symmetric neutrosophic fuzzy matrices [23-25]. These studies have highlighted the potential of neutrosophic fuzzy matrices in fields such as image processing, decision support systems, and cryptography, where uncertainty and indeterminacy are inherent.In addition to traditional and neutrosophic fuzzy matrices, researchers have investigated the properties of secondary matrices, which introduce a new level of symmetry to matrix theory. Secondary symmetric, secondary skew-symmetric, and secondary orthogonal matrices, first explored by An Lee in 1976 [13], have unique properties that make them useful in specialized mathematical and engineering applications. These matrices have been further studied in fuzzy environments, particularly in the context of κ-EP matrices and κ-kernel symmetric matrices [10, 11].

Recent advances in the field have focused on the application of fuzzy and neutrosophic matrices in modern technologies such as artificial intelligence, machine learning, and blockchain. The work of Anandhkumar et al. on pseudo-similarity in neutrosophic fuzzy matrices [27] and various inverse operations on these matrices [28] has opened up new possibilities for their use in data mining, pattern recognition, and cryptographic systems. Additionally, the exploration of time-varying fuzzy matrices and their applications in dynamic systems presents a promising area of future research. Despite these advancements, gaps remain in the practical application of fuzzy and neutrosophic matrices, particularly in terms of computational efficiency and scalability for large-scale systems. Moreover, the unification of various fuzzy matrix frameworks, such as intuitionistic, neutrosophic, and interval-valued matrices, into a single, cohesive structure remains an open research question.

1.2 Novelties

In recent years, significant advancements have emerged in the field of fuzzy matrix theory, particularly with the integration of neutrosophic logic, which extends the traditional fuzzy set framework to handle more complex forms of uncertainty. One of the most notable innovations is the development of neutrosophic fuzzy matrices, which incorporate the three elements of truth, falsity, and indeterminacy, offering a more comprehensive approach to modeling uncertainty in real-world systems [23]. This innovation has paved the way for the creation of new matrix classes, such as secondary κ-column symmetric neutrosophic fuzzy matrices and generalized symmetric neutrosophic fuzzy matrices [24, 23]. These novel matrix structures enable the representation of intricate data relationships, particularly in fields like communication systems and cryptography, where uncertainty is pervasive.

Additionally, the introduction of interval-valued neutrosophic matrices has further enhanced the ability to model uncertain data through ranges of values, making them particularly useful in decision-making and machine learning applications [25]. Recent research also explores the pseudo-similarity of neutrosophic matrices, providing new methods to analyze relationships between matrices that share approximate similarities, a significant leap for pattern recognition and clustering techniques [27]. Moreover, the extension of inverse operations in neutrosophic matrices [28] and the application of these matrices in cryptography and blockchain systems represent cutting-edge developments, highlighting the vast potential of neutrosophic logic in securing and managing decentralized systems. These advancements not only expand the theoretical foundations of fuzzy matrices but also demonstrate their growing relevance in modern technologies, offering robust tools for handling complex uncertainty across various scientific and engineering domains.

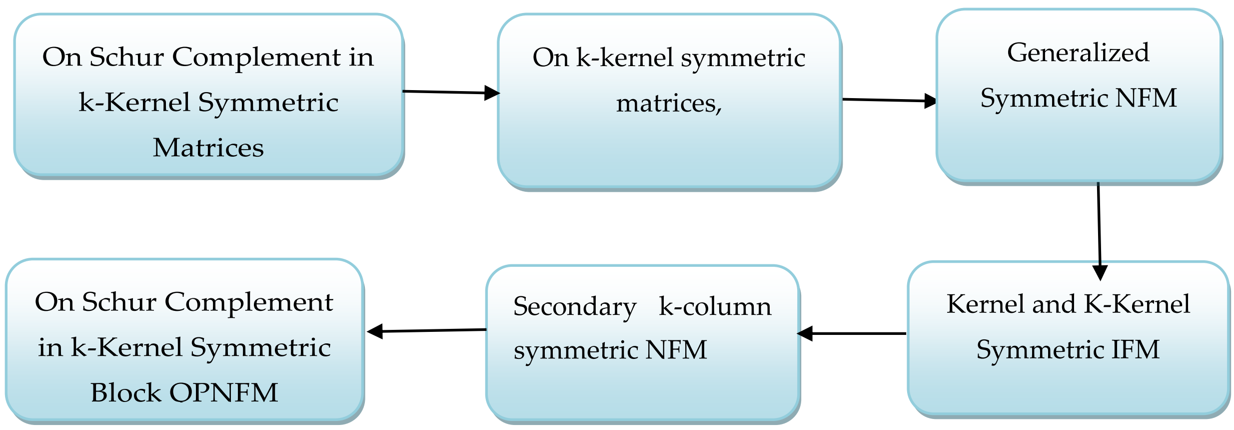
1.3 Research Gap:

Meenakshi, Jayasree [10,17] and Anandhkumar et al [23,24,30] introduced the concepts of k-kernel symmetric matrices and the Schur complement in the context of k-KS matrices.

Building on their work, we have applied these principles to Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFMs). In this study, we examine several results and extend these concepts to QPNFMs. First, we present equivalent characterizations for the Schur complement within k-KSQPNFMs and provide equivalent conditions that QPNFMs must satisfy to exhibit kernel symmetry (KS). Additionally, we explore the relationship between KS and k-KS in this framework.

Table:1 Review of the Extension of QPNFM.

Ref	**Journal Name**	**Authors Name**	**Extension of NFM.**	**Year**
[10]	Int. Journal of Math. Analysis	A. R. Meenakshi and D. Jaya Shree	On Schur Complement in k-Kernel Symmetric Matrices,	1989
[17]	International Journal of Mathematics and Mathematical Sciences	AR.Meenakshi and D.Jaya Shree,	On k-kernel symmetric matrices,	2009
[23]	Neutrosophic Sets and Systems	M.Anandhkumar et al	Generalized Symmetric NFM	2023
[30]	TWMS J. App. and Eng. Math	G. Punithavalli, and M. Anandhkumar,	Kernel and K-Kernel Symmetric IFM	2024
[24]	Neutrosophic Sets and Systems	M. Anandhkumar et al	Secondary k-column symmetric NFM	2024
Proposed	Neutrosophic Sets and Systems	K. Radhika et al	On Schur Complement in k-Kernel Symmetric Block QPNFM	2024



1.3.1 Validation and Comparison of QPNFM with Existing Models

The Quadripolar Neutrosophic Fuzzy Matrix (QPNFM) demonstrates significant advancements when compared to existing models, such as traditional fuzzy matrices and intuitionistic fuzzy matrices (IFM), especially in handling uncertainty and indeterminacy. Fuzzy matrices rely on a single membership value to represent truth, making them suitable for well-defined problems but often inadequate in contexts involving ambiguity or conflicting information. Similarly, while IFMs extend this framework by introducing truth and falsity components, they lack a robust mechanism to explicitly account for indeterminacy.

QPNFM addresses these limitations by incorporating quadripolar neutrosophic elements, enabling the simultaneous representation of truth, falsity, and indeterminacy as independent parameters. This added dimension allows QPNFM to model complex relational data with greater precision, particularly in scenarios characterized by incomplete or vague information. For example, in social network analysis, where relationships between entities may not be clearly defined, QPNFM provides a more nuanced representation compared to traditional fuzzy matrices.

Moreover, QPNFM outperforms existing models in practical case studies by effectively capturing and processing high degrees of indeterminacy. While this added complexity increases computational requirements, the trade-off is justified by the enhanced capability to derive meaningful insights and support decision-making. Through empirical validation, QPNFM has demonstrated superior accuracy and reliability in modeling and analyzing uncertain systems, establishing itself as a comprehensive and effective alternative to traditional and intuitionistic fuzzy matrices.

1.4 Notation

let $(Q^T,Q^C,Q^I,Q^F)^T$ transpose of (Q^T,Q^C,Q^I,Q^F),

$R((Q^T,Q^C,Q^I,Q^F))$ Row space of (Q^T,Q^C,Q^I,Q^F)

$N((Q^T,Q^C,Q^I,Q^F))$ Null space of (Q^T,Q^C,Q^I,Q^F)

$(Q^T,Q^C,Q^I,Q^F)^+$ Moore-Penrose inverse of (Q^T,Q^C,Q^I,Q^F)

$C((Q^T,Q^C,Q^I,Q^F))$ column space of (Q^T,Q^C,Q^I,Q^F)

QPNFM = Quadri Partitioned Neutrosophic fuzzy matrices,
KSNFM=Kernel symmetric Neutrosophic fuzzy matrices,
KS = Kernel symmetric,
RS = Range symmetric,

PM =Permutation matrices.

1.5 The structure of the article is as follows.

In section 1, We present intraduction, Literature Review, Novelties, Research Gap, Notation.

In section 2, We present some elementary definitions and findings.

In section 3, we provided k-kernel symmetric QPNFM.

In section 4, we introduced schur complement in k-kernel symmetric QPNFM.

In section 5, Application of Fuzzy Quadripartitioned Neutrosophic Soft Matrix in Medical Decision Making Problem

2. Definitions and Theorems

Definition : 2.1 Let Q =< (Q^T, Q^C, Q^I, Q^F) > ∈ (QPNFM)n then Q is called as kernel or null if

N (Q^T, Q^C, Q^I, Q^F) = {x ∈ $(QPNFM)_{1n}$ / x (Q^T, Q^C, Q^I, Q^F) = (0,0,1,1) } = (0,0,1,1).

Example: 2.1 Consider a QPNFM

$$Q = (Q^T, Q^C, Q^I, Q^F) = \begin{bmatrix} <0.5,0.6,0.7,0.4> & <0.3,0.3,0.7,0.9> & <0.9,0.4,0.3,0.1> \\ <0.2,0.6,0.4,0.8> & <0.6,0.1,0.7,0.6> & <0.1,0.6,0.7,0.2> \\ <0.9,0.6,0.7,0.4> & <0.4,0.6,0.7,0.8> & <0.2,0.4,0.7,0.1> \end{bmatrix}$$

Choose, $x = [<a,b,c,d>, <e,f,g,h>, <i,j,k,l>] \in (QPNFM)_{1n}$

Such that $x(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

Therefore, $N(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

Definition 2.2 A QPNFM $(Q^T, Q^C, Q^I, Q^F) \in (QPNFM)_n$ is said to be k-kernel symmetric if

$\boldsymbol{N}((Q^T, Q^C, Q^I, Q^F))$ = $N(K(Q^T, Q^C, Q^I, Q^F)^T K)$.

Definition 2.3 For QPNFM $(Q^T, Q^C, Q^I, Q^F) \in (QPNFM)_n$ is kernel symmetric if

$N(Q^T, Q^C, Q^I, Q^F) = N((Q^T, Q^C, Q^I, Q^F)^T)$.

Example: 2.2 Consider a QPNFM

$$Q = (Q^T, Q^C, Q^I, Q^F) = \begin{bmatrix} <0.7,0.6,0.2,0.4> & <0.6,0.3,0.5,0.9> & <0.1,0.4,0.4,0.1> \\ <0.3,0.6,0.4,1> & <0.6,0.1,0.2,0.6> & <0.4,0.6,0.7,0.2> \\ <0.2,0.6,0.5,0.4> & <0.4,0.6,0.5,0.8> & <0.2,0.4,0.7,0.1> \end{bmatrix}$$

Choose, $x = [<a,b,c,d>, <e,f,g,h>, <i,j,k,l>] \in (QPNFM)_{1n}$

Such that $x(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1)$

Therefore, $N(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1)$

$$Q^T=(Q^T,Q^C,Q^I,Q^F)^T=\begin{bmatrix} <0.7,0.6,0.2,0.4> & <0.3,0.6,0.4,1> & <0.2,0.6,0.5,0.4> \\ <0.6,0.3,0.5,0.9> & <0.6,0.1,0.2,0.6> & <0.4,0.6,0.5,0.8> \\ <0.1,0.4,0.4,0.1> & <0.4,0.6,0.5,0.8> & <0.2,0.4,0.7,0.1> \end{bmatrix}$$

Choose, $x=\left[<a,b,c,d>,<e,f,g,h>,<i,j,k,l>\right]\in\left(QPNFM\right)_{1n}$

Such that $x(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1)$

Therefore, $N(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1)$

Therefore, $N(Q^T,Q^C,Q^I,Q^F)=N((Q^T,Q^C,Q^I,Q^F)^T)$.

Definition 2.4 For (Q^T,Q^C,Q^I,Q^F), $(R^T,R^C,R^I,R^F)\in$ QPNFMn, (Q^T,Q^C,Q^I,Q^F) is k-similar to (R^T,R^C,R^I,R^F) if there exists a permutation matrix (PM) P such that $(Q^T,Q^C,Q^I,Q^F)=(KP^TK)\,(R^T,R^C,R^I,R^F)P$.

Theorem 2.1 For A ∈ Fn, the following subsequence are equivalent:

(i) A is KS,

(ii) PAP^T is KS for some PM P,

(iii) There exists a PM such that $PAP=\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ with det D > 0

3. k-kernel symmetric Quadri Partitioned Neutrosophic fuzzy matrices

Consider (Q^T,Q^C,Q^I,Q^F) to be a QPNFM. If (Q^T,Q^C,Q^I,Q^F) is a part of (QPNFM)n is known as k-kernel symmetric QPNFM if N((Q^T,Q^C,Q^I,Q^F)) is more significant than N(K $(Q^T,Q^C,Q^I,Q^F)^T$K). Matrices are essential in a variety of areas of research in engineering and science. The conventional matrix theory must address issues with a wide range of uncertainty. Let $QPNFM_{mn}$ indicates the set of every m×n QPNFM over the QPNF algebra (QPNF)n. We denote a solution Z of the equation $(Q^T,Q^C,Q^I,Q^F)\,Z\,(Q^T,Q^C,Q^I,Q^F)=(Q^T,Q^C,Q^I,Q^F)$ by $(Q^T,Q^C,Q^I,Q^F)^-$.

For a complex matrix Q subdivided in the form $Q = \begin{bmatrix} (Q^T, Q^C, Q^I, Q^F) & (R^T, R^C, R^I, R^F) \\ (S^T, S^C, S^I, S^F) & (T^T, T^C, T^I, T^F) \end{bmatrix}$.

Schur complement of (Q^T, Q^C, Q^I, Q^F) in Q denoted by $(T^T, T^C, T^I, T^F) - (S^T, S^C, S^I, S^F)\ (Q^T, Q^C, Q^I, Q^F)^-\ (R^T, R^C, R^I, R^F)$. This is known as the QPNFM generalized Schur complement. If (Q^T, Q^C, Q^I, Q^F) Z (Q^T, Q^C, Q^I, Q^F) = (Q^T, Q^C, Q^I, Q^F) has a solution, (Q^T, Q^C, Q^I, Q^F) is considered regular. A solution Z to the equation (Q^T, Q^C, Q^I, Q^F) Z (Q^T, Q^C, Q^I, Q^F) = (Q^T, Q^C, Q^I, Q^F) is defined by $(Q^T, Q^C, Q^I, Q^F)^-$ and is referred to as a generalized inverse, or generalized inverse of (Q^T, Q^C, Q^I, Q^F).

A QPNFM (Q^T, Q^C, Q^I, Q^F) is range symmetric if R (Q^T, Q^C, Q^I, Q^F) = R (Q^T, Q^C, Q^I, Q^F) T and kernel symmetric if N (Q^T, Q^C, Q^I, Q^F) = N (Q^T, Q^C, Q^I, Q^F) T. For QPNFM (Q^T, Q^C, Q^I, Q^F) ∈ QPNFMn, (Q^T, Q^C, Q^I, Q^F) is range symmetric, that is, R (Q^T, Q^C, Q^I, Q^F) = R (Q^T, Q^C, Q^I, Q^F) T implies N (Q^T, Q^C, Q^I, Q^F) = N (Q^T, Q^C, Q^I, Q^F) T but converse needs not be true.

Definition: 3.1 Let $R = <[R_T, R_C, R_I, R_F]>$ be a QPNFM, if R [$[R_T, R_C, R_I, R_F]$] = R [$[R_T, R_C, R_I, R_F]$ T] then $R = <[R_T, R_C, R_I, R_F]>$ is called as RS.

Example: 3.1 Consider an QPNFM

$$[R_T, R_C, R_I, R_F] = \begin{bmatrix} <0.4, 0, 1, 0.6> & <1, 0, 1, 1> & <0.2, 0.3, 0.4, 0.7> \\ <1, 0, 1, 1> & <1, 0, 1, 1> & <1, 0, 1, 1> \\ <0.2, 0.3, 0.4, 0.7> & <1, 0, 1, 1> & <1, 0, 1, 1> \end{bmatrix}$$

Here, R $[R_T, R_C, R_I, R_F]$ = R [$[R_T, R_C, R_I, R_F]$ T] Range symmetric

Therefore N $[R_T, R_C, R_I, R_F]$ = N[$[R_T, R_C, R_I, R_F]$ T] kernel symmetric

Therefore the above example both Range symmetric and kernel symmetric QPNFM.

The following matrices does not satisfy the range symmetric condition

$$[R_T, R_C, R_I, R_F] = \begin{bmatrix} <0.4,0,1,0.7> & <1,0,1,1> & <0.2,0.3,0.4,0.6> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.4,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$$[R_T, R_C, R_I, R_F]^T = \begin{bmatrix} <0.4,0,1,0.7> & <1,0,1,1> & <0.2,0.4,0.4,0.6> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$[(0.4,0,1,0.7)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)] \in R([R_T, R_C, R_I, R_F])$,

$[(0.4,0,1,0.7)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)] \notin R([R_T, R_C, R_I, R_F]^T)$,

$[(1,0,1,1)\ (1,0,1,1)\ (1,0,1,1)] \in R[R_T, R_C, R_I, R_F]$,

$[(1,0,1,1)\ (1,0,1,1)\ (1,0,1,1)] \in R([R_T, R_C, R_I, R_F]^T)$,

$[(0.2,0.4,0.4,0.6)\ (1,0,1,1)\ (1,0,1,1)] \in R([R_T, R_C, R_I, R_F])$,

$[(0.2,0.4,0.4,0.6)\ (1,0,1,1)\ (1,0,1,1)] \notin R([R_T, R_C, R_I, R_F]^T)$,

Here, R $[R_T, R_C, R_I, R_F]$ $\neq$ R $[[R_T, R_C, R_I, R_F]$T] not Range symmetric

Therefore N $[R_T, R_C, R_I, R_F]$ = N$[[R_T, R_C, R_I, R_F]$T] kernel symmetric

Remark: 3.1 The above example shows that a kernel-symmetric QPNFM is not necessarily range-symmetric. Consequently, every range-symmetric QPNFM is also kernel-symmetric.

Remark 3.2. In particular, when κ(i)= i for each i = 1 to n, the associated PM K reduces to the identity matrix reduces to N((Q^T, Q^C, Q^I, Q^F)) = N((Q^T, Q^C, Q^I, Q^F))^T , that is, (Q^T, Q^C, Q^I, Q^F) is kernel symmetric. If (Q^T, Q^C, Q^I, Q^F) is symmetric, then (Q^T, Q^C, Q^I, Q^F) is k-KS for all transpositions k in S_n.

Further, (Q^T, Q^C, Q^I, Q^F) is k-Symmetric implies it is k-KS, for (Q^T, Q^C, Q^I, Q^F) = K$(Q^T, Q^C, Q^I, Q^F)^T$K automatically implies N((Q^T, Q^C, Q^I, Q^F)) = N(K$(Q^T, Q^C, Q^I, Q^F)^T$K).

The opposite need not be true, though.

Therefore, $(Q^T, Q^C, Q^I, Q^F) \neq K(Q^T, Q^C, Q^I, Q^F)^T K$

But, N ((Q^T, Q^C, Q^I, Q^F)) = N(K $(Q^T, Q^C, Q^I, Q^F)^T$ K) = $(0,0,1,1)$

Therefore, (Q^T, Q^C, Q^I, Q^F) is not k-symmetric. For this (Q^T, Q^C, Q^I, Q^F) , since (Q^T, Q^C, Q^I, Q^F) has no zero rows and no zero columns. N(K $(Q^T, Q^C, Q^I, Q^F)^T$K)= $(0,0,1,1)$.

Hence (Q^T, Q^C, Q^I, Q^F) is k-KS, but (Q^T, Q^C, Q^I, Q^F) is not k-symmetric.

Theorem 3.1 Let $(R^T, R^C, R^I, R^F) = \begin{bmatrix} (T^T, T^C, T^I, T^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}$ where (T^T, T^C, T^I, T^F) is r × r QPNFM with no zero rows and no zero columns, then the following equivalent conditions hold:

(i) (R^T, R^C, R^I, R^F) is k-KS

(ii) $N(R^T, R^C, R^I, R^F)^T = N\left((R^T, R^C, R^I, R^F)K\right)^T$

(iii) $K = \begin{bmatrix} K_1 & (0,0,1,1) \\ (0,0,1,1) & K_2 \end{bmatrix}$ where K_1 and K_2 are QPNFPM of order r and n-r, respectively,

Proof: Since (T^T, T^C, T^I, T^F) has no zero rows and no zero columns $N(T^T, T^C, T^I, T^F) = N(T^T, T^C, T^I, T^F)^T = (0,0,1,1).$

Therefore, $N(R^T, R^C, R^I, R^F) = N(R^T, R^C, R^I, R^F)^T \neq (0,0,1,1)$ and (R^T, R^C, R^I, R^F) is KS.

Now we will show that (i),(ii) and (iii) . (R^T, R^C, R^I, R^F) is k-KS⇔ $N(R^T, R^C, R^I, R^F) = N\left((R^T, R^C, R^I, R^F)K\right)^T$.

Choose z =[0 y] with all element of $y \neq (0,0,1,1)$ and subdivided in conformity with that of $(R^T, R^C, R^I, R^F) = \begin{bmatrix} (T^T, T^C, T^I, T^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}$

Clearly,

$$z \in N(R^T, R^C, R^I, R^F) = N(R^T, R^C, R^I, R^F)^T = N\left((R^T, R^C, R^I, R^F)K\right)^T.$$

Let us subdivided K as $K = \begin{bmatrix} K_1 & K_3 \\ K^T{}_3 & K_2 \end{bmatrix}$

Then

$$K(R^T, R^C, R^I, R^F) = \begin{bmatrix} K_1 & K_3 \\ K^T{}_3 & K_2 \end{bmatrix} \begin{bmatrix} (T^T, T^C, T^I, T^F)^T & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}$$

$$= \begin{bmatrix} K_1(T^T, T^C, T^I, T^F)^T & (0,0,1,1) \\ K^T{}_3(T^T, T^C, T^I, T^F)^T & (0,0,1,1) \end{bmatrix}$$

$$z = [0 \ y] \in N(R^T, R^C, R^I, R^F) = N\left(K(R^T, R^C, R^I, R^F)^T\right)$$

$$\Rightarrow [0 \ y] \begin{bmatrix} K_1(T^T, T^C, T^I, T^F)^T & (0,0,1,1) \\ K^T{}_3(T^T, T^C, T^I, T^F)^T & (0,0,1,1) \end{bmatrix} = (0,0,1,1)$$

$$\Rightarrow yK^T{}_3(T^T, T^C, T^I, T^F)^T = (0,0,1,1)$$

Since $N(T^T, T^C, T^I, T^F)^T = (0,0,1,1)$, it follows that $yK^T{}_3 = (0,0,1,1)$.

Since all element of y $\neq (0,0,1,1)$ under max-min arrangement $yK^T{}_3 = (0,0,1,1)$ this implies $K^T{}_3 = (0,0,1,1) \Rightarrow K_3 = (0,0,1,1)$.

Therefore $K = \begin{bmatrix} K_1 & (0,0,1,1) \\ (0,0,1,1) & K_2 \end{bmatrix}$

Thus, (iii) holds, conversely, if (iii) holds, then

$$K(R^T,R^C,R^I,R^F)^T=\begin{bmatrix} K_1(T^T,T^C,T^I,T^F)^T & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix},$$

$$N\left(K(R^T,R^C,R^I,R^F)^T\right)=N\left((R^T,R^C,R^I,R^F)\right)$$

Thus (i)⇔(ii)⇔(iii) holds.

Theorem 3.2. For (Q^T,Q^C,Q^I,Q^F) ∈ Fn and k = k_1k_2 . Then the subsequence are equivalent:

(i) (Q^T,Q^C,Q^I,Q^F) is k-KS of rank r,

(ii) (Q^T,Q^C,Q^I,Q^F) is k-similar to a diagonal block matrix

$\begin{bmatrix} (T^T,T^C,T^I,T^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}$ with det $(T^T,T^C,T^I,T^F) >$ $(0,0,1,1)$,

(iii) (Q^T,Q^C,Q^I,Q^F) = $KGLG^T$ and L ∈ F_r with det L > $(0,0,1,1)$ and $G^TG = I_r$.

Proof. (i) ⇔(ii) (Q^T,Q^C,Q^I,Q^F) is k-KS⇔ K(Q^T,Q^C,Q^I,Q^F) is KS

$$\Leftrightarrow PK(Q^T,Q^C,Q^I,Q^F)P^T=\begin{bmatrix} (E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix} \text{ with det}$$

$(E^T,E^C,E^I,E^F) >$ $(0,0,1,1)$, for some Permutation matrix P

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=KP^T\begin{bmatrix} (E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)K\begin{bmatrix} (E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)\begin{bmatrix} K_1 & (0,0,1,1) \\ (0,0,1,1) & K_2 \end{bmatrix}\begin{bmatrix} (E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)\begin{bmatrix} K_1(E^T,E^C,E^I,E^F) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) \end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)\begin{bmatrix}(T^T,T^C,T^I,T^F) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1)\end{bmatrix}P$$

Thus (Q^T,Q^C,Q^I,Q^F) is k-similar to a diagonal block matrix $\begin{bmatrix}(T^T,T^C,T^I,T^F) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1)\end{bmatrix}$, where $(T^T,T^C,T^I,T^F)=K_1(E^T,E^C,E^I,E^F)$ and det $(T^T,T^C,T^I,T^F)>(0,0,1,1)$.

However, (ii) $\Leftrightarrow$ (iii)

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=\left(KP^TK\right)\begin{bmatrix}K_1(E^T,E^C,E^I,E^F) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1)\end{bmatrix}P$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=K\begin{bmatrix}P_1^T & P_3^T\\ P_2^T & P_4^T\end{bmatrix}\begin{bmatrix}K_1 & (0,0,1,1)\\ (0,0,1,1) & K_2\end{bmatrix}\begin{bmatrix}(T^T,T^C,T^I,T^F) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1)\end{bmatrix}\begin{bmatrix}P_1 & P_2\\ P_3 & P_4\end{bmatrix}$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=K\begin{bmatrix}P_1^T\\ P_2^T\end{bmatrix}K_1(T^T,T^C,T^I,T^F)\begin{bmatrix}P_1 & P_2\end{bmatrix}$$

$$\Leftrightarrow (Q^T,Q^C,Q^I,Q^F)=KGLG^T\ ,G=\begin{bmatrix}P_1^T\\ P_2^T\end{bmatrix},G^T=\begin{bmatrix}P_1 & P_2\end{bmatrix},L=K_1(T^T,T^C,T^I,T^F)\in(QPNFM)_r$$

$$G^TG=\begin{bmatrix}P_1 & P_2\end{bmatrix}\begin{bmatrix}P_1^T\\ P_2^T\end{bmatrix}=P_1P_1^T+P_2P_2^T=I_r,L\in(QPNFM)_r$$

4. Schur Complement in k-Kernel QPNFM

Consider block QPNFM of the following form throughout $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$ with respect to this subdividing a SC of (Q^T,Q^C,Q^I,Q^F) in M is a QPNFM of the form $M/(Q^T,Q^C,Q^I,Q^F)$ = (T^T,T^C,T^I,T^F) – $(S^T,S^C,S^I,S^F)(Q^T,Q^C,Q^I,Q^F)^-$ (R^T,R^C,R^I,R^F) .where (Q^T,Q^C,Q^I,Q^F) and (T^T,T^C,T^I,T^F) are square QPNFM. Here M/ (Q^T,Q^C,Q^I,Q^F) is a QPNFM iff $(T^T,T^C,T^I,T^F)\geq(S^T,S^C,S^I,S^F)(Q^T,Q^C,Q^I,Q^F)^-(R^T,R^C,R^I,R^F)$ that is

$(T^T,T^C,T^I,T^F)=(T^T,T^C,T^I,T^F)+(S^T,S^C,S^I,S^F)(T^T,T^C,T^I,T^F)^{-}(R^T,R^C,R^I,R^F)$. A subdivided QPNFM M of the form $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$ is k-KS then it is not true in general that a SC of (Q^T,Q^C,Q^I,Q^F) in M , $M/(Q^T,Q^C,Q^I,Q^F)$ is k-KSQPNFM. Here the both if and only if conditions for $M/(Q^T,Q^C,Q^I,Q^F)$ to be k-KS are obtained.

Theorem 4.1 Let M be a matrix of the form $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$ with $N((Q^T,Q^C,Q^I,Q^F)) \subseteq N((R^T,R^C,R^I,R^F))$ and $N(M/(Q^T,Q^C,Q^I,Q^F)) \subseteq N((S^T,S^C,S^I,S^F))$, then the subsequent statement are equivalent.

(i) M is k – KSQPNFM with $k = k_1k_2$.

(ii) (Q^T,Q^C,Q^I,Q^F) is k -KS, $M/(Q^T,Q^C,Q^I,Q^F)$ is k -KS,

$N((Q^T,Q^C,Q^I,Q^F)^T) \subseteq N((S^T,S^C,S^I,S^F)^T)$ and $N((M/(Q^T,Q^C,Q^I,Q^F))^T) \subseteq N((R^T,R^C,R^I,R^F)^T)$

(iii) Both the matrices $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (0,0,1,1)\\ (S^T,S^C,S^I,S^F) & M/(Q^T,Q^C,Q^I,Q^F)\end{bmatrix}$ and $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (0,0,1,1) & M/(Q^T,Q^C,Q^I,Q^F)\end{bmatrix}$ are k- kernel symmetric.

Proof:(i)⇒ (ii):

To prove (Q^T,Q^C,Q^I,Q^F) is k is k-KS, $M/(Q^T,Q^C,Q^I,Q^F)$ is k-KS

Let $y_1 \in N((Q^T,Q^C,Q^I,Q^F))$ and $y_2 \in N(M/(Q^T,Q^C,Q^I,Q^F))$.

Hence $y_1(Q^T,Q^C,Q^I,Q^F) = (0,0,1,1)$ and $y_2(M/(Q^T,Q^C,Q^I,Q^F)) = (0,0,1,1)$

Define $y=[y_1 \quad y_2]$

we claim that $yM = [y_1\, y_2]\begin{bmatrix} (Q^T, Q^C, Q^I, Q^F) & (R^T, R^C, R^I, R^F) \\ (S^T, S^C, S^I, S^F) & (T^T, T^C, T^I, T^F) \end{bmatrix} = (0,0,1,1)$

Since $N(M/(Q^T, Q^C, Q^I, Q^F)) \subseteq \mathrm{N}(S^T, S^C, S^I, S^F)$,

$y_2(M/(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1)$

Implies, $y_2\,(S^T, S^C, S^I, S^F) = (0,0,1,1)$

$N((Q^T, Q^C, Q^I, Q^F)) \subseteq N((R^T, R^C, R^I, R^F))$,

$y_1\,(Q^T, Q^C, Q^I, Q^F) = (0,0,1,1) \Rightarrow y_1\,(R^T, R^C, R^I, R^F) = (0,0,1,1)$

Hence, $y_1\,(Q^T, Q^C, Q^I, Q^F) + y_2\,(S^T, S^C, S^I, S^F) = (0,0,1)$ and

$y_1(R^T, R^C, R^I, R^F) + y_2(T^T, T^C, T^I, T^F) = (0,0,1,1)$.

Therefore yM $= (0,0,1,1)$ i.e) $y \in N(M)$.
Since M is k - KS, $N(M) = N(KM^T K)$
Then, $yKM^T K = (0,0,1,1)$

$$[y_1\, y_2]\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} (Q^T, Q^C, Q^I, Q^F)^T & (R^T, R^C, R^I, R^F)^T \\ (S^T, S^C, S^I, S^F)^T & (T^T, T^C, T^I, T^F)^T \end{bmatrix}\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} = (0,0,1,1)$$

$\Rightarrow y_1 K(Q^T, Q^C, Q^I, Q^F)^T K + y_2 K(R^T, R^C, R^I, R^F)^T K = (0,0,1,1)$

$\Rightarrow y_1 K(Q^T, Q^C, Q^I, Q^F)^T K = (0,0,1,1)$ and $y_2 K(R^T, R^C, R^I, R^F)^T K = (0,0,1,1)$ and

$\Rightarrow y_1 K(S^T, S^C, S^I, S^F)^T K + y_2 K(T^T, T^C, T^I, T^F)^T K = (0,0,1,1)$

$\Rightarrow y_1 K(S^T, S^C, S^I, S^F)^T K = (0,0,1)$ and $y_2 K(T^T, T^C, T^I, T^F)^T K = (0,0,1,1)$

Hence

$y_1 \in N\left[K(Q^T, Q^C, Q^I, Q^F)K\right], y_2 \in N\left[K(R^T, R^C, R^I, R^F)^T K\right]$ and

$y_2 \in N\left[K(T^T, T^C, T^I, T^F)K\right]$

Since $y_1 \in N\left[(Q^T, Q^C, Q^I, Q^F)\right]$ and $y_2 \in N\left[M / (Q^T, Q^C, Q^I, Q^F)\right]$ it follows that

$N\left[(Q^T, Q^C, Q^I, Q^F)\right] \subseteq N\left[K(Q^T, Q^C, Q^I, Q^F)K\right]$,

$N\left[M/(Q^T,Q^C,Q^I,Q^F)\right]\subseteq N\left[K(R^T,R^C,R^I,R^F)^T K\right]$

$N\left[M/(Q^T,Q^C,Q^I,Q^F)\right]\subseteq N\left[K(T^T,T^C,T^I,T^F)^T K\right]$ implies

$N\left[M/(Q^T,Q^C,Q^I,Q^F)\right]\subseteq N\left[K(M/(Q^T,Q^C,Q^I,Q^F))K\right]$

Likewise, it may be demonstrated that

$N\left[K(Q^T,Q^C,Q^I,Q^F)K\right]\subseteq N\left[(Q^T,Q^C,Q^I,Q^F)\right]$

Thus (Q^T,Q^C,Q^I,Q^F) is k-KS

Since, $y_1\in N\left[K(S^T,S^C,S^I,S^F)K\right]$ and (Q^T,Q^C,Q^I,Q^F) is k-KS

$N\left[(Q^T,Q^C,Q^I,Q^F)\right]=N\left[K(Q^T,Q^C,Q^I,Q^F)^T K\right]\subseteq N\left[K(S^T,S^C,S^I,S^F)^T K\right]$

$N\left[(Q^T,Q^C,Q^I,Q^F)^T\right]\subseteq N\left[(S^T,S^C,S^I,S^F)^T\right]$

$M/(Q^T,Q^C,Q^I,Q^F) = (T^T,T^C,T^I,T^F) - (S^T,S^C,S^I,S^F)(Q^T,Q^C,Q^I,Q^F)^-$

(R^T,R^C,R^I,R^F).

Thus (i) implies (ii) holds

(ii) implies (iii)

$$M_1=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (0,0,1,1)\\ (S^T,S^C,S^I,S^F) & M/(Q^T,Q^C,Q^I,Q^F)\end{bmatrix} \text{and}$$

$$M_1=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (0,0,1,1) & M/(Q^T,Q^C,Q^I,Q^F)\end{bmatrix}$$

are k-KS.

Let $y\in N\left(M_1\right)$. Partition y conformity with that M_1 as y = [y_1 y_2] then,

$$[y_1\, y_2]\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (0,0,1,1)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}=(0,0,1,1)$$

$y_1(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1), y_2(S^T,S^C,S^I,S^F)=(0,0,1,1),$

$\Rightarrow y_2\left[M/(Q^T,Q^C,Q^I,Q^F)\right]=(0,0,1,1)$

Since (Q^T, Q^C, Q^I, Q^F) and $M/(Q^T, Q^C, Q^I, Q^F)$ are k-KS

$$y_1 \in N\left[(Q^T, Q^C, Q^I, Q^F)\right] = N\left[K(Q^T, Q^C, Q^I, Q^F)^T K\right]$$

$$\Rightarrow y_1 K(Q^T, Q^C, Q^I, Q^F)^T K = (0,0,1,1)$$

$$y_2 \in N\left[M/(Q^T, Q^C, Q^I, Q^F)\right]^T = N\left[K\left(M/(Q^T, Q^C, Q^I, Q^F)\right)^T K\right]$$

$$\Rightarrow y_2\left[K\left(M/(Q^T, Q^C, Q^I, Q^F)\right)^T K\right] = (0,0,1,1)$$

Since, $N\left[(Q^T, Q^C, Q^I, Q^F)^T\right] \subseteq N\left[(S^T, S^C, S^I, S^F)^T\right]$

$$N\left[K(Q^T, Q^C, Q^I, Q^F)^T K\right] \subseteq N\left[K(S^T, S^C, S^I, S^F)^T K\right]$$

$$\Rightarrow y_1 K(S^T, S^C, S^I, S^F)^T K = (0,0,1,1)$$

Now by using $y_1 K(Q^T, Q^C, Q^I, Q^F)^T K = (0,0,1,1), y_1 K(S^T, S^C, S^I, S^F)^T K = (0,0,1,1)$

and $y_2 K\left[M/(Q^T, Q^C, Q^I, Q^F)\right]^T K = (0,0,1,1)$ it can be verified that

$$[y_1\, y_2]\begin{bmatrix} K(Q^T, Q^C, Q^I, Q^F)^T K & K(S^T, S^C, S^I, S^F)K \\ (0,0,1,1) & K(M/(Q^T, Q^C, Q^I, Q^F)^T K \end{bmatrix} = (0,0,1,1)$$

Thus $N\left[M_1\right] \subseteq N\left[K M_1{}^T K\right]$.

$$N\left[K M_1{}^T K\right] \subseteq N\left[M_1\right]$$

Hence Therefore, $N\left[K M_1{}^T K\right] = N\left[M_1\right]$

Hence M_1 is k-KS.

Similarly, it may be demonstrated that M_2 is k-KS.

Thus (ii)⇒ (iii) holds.

(iii)⇒ (i) M_1 is k - KS ⇒ $N(M_1) = N(KM^T K)$

M_2 is k- KS⇒ $N(M_2) = N(KM^T K)$

To prove, M is k-KS that is $N(M) = N(KM^T K)$.

Let $y \in N(M) \Rightarrow yM = 0$.

M as y = $[y_1\ y_2]$ then,

$$[y_1\, y_2]\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}=(0,0,1,1)$$

$$y_1(Q^T,Q^C,Q^I,Q^F)+y_2(S^T,S^C,S^I,S^F)=(0,0,1,1)$$

$$\Rightarrow y_1(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1), y_2(S^T,S^C,S^I,S^F)=(0,0,1,1)$$

$$y_1(R^T,R^C,R^I,R^F)+y_2(T^T,T^C,T^I,T^F)=(0,0,1,1)$$

$$\Rightarrow y_1(R^T,R^C,R^I,R^F)=(0,0,1,1), y_2(T^T,T^C,T^I,T^F)=(0,0,1,1)$$

From the definition of M/ (Q^T,Q^C,Q^I,Q^F) = (T^T,T^C,T^I,T^F) $-(S^T,S^C,S^I,S^F)$ $(Q^T,Q^C,Q^I,Q^F)^-$ (R^T,R^C,R^I,R^F)

We have, $y_2(T^T,T^C,T^I,T^F)=(0,0,1,1), y_2(S^T,S^C,S^I,S^F)=(0,0,1,1)$

$$\Rightarrow y_2\left(M/(Q^T,Q^C,Q^I,Q^F)\right)=(0,0,1,1)$$

$y_1(Q^T,Q^C,Q^I,Q^F)+y_2(S^T,S^C,S^I,S^F)=(0,0,1,1)$ and $y_2\left(M/(Q^T,Q^C,Q^I,Q^F)\right)=(0,0,1,1)$

And $y_1(Q^T,Q^C,Q^I,Q^F)=(0,0,1,1)$,

$$y_1(R^T,R^C,R^I,R^F)+y_2\left[M/(Q^T,Q^C,Q^I,Q^F)\right]=(0,0,1,1)$$

$$y\in N(M_1)\Rightarrow y\in N(KM_1^{\,T}K)$$

$$y\in N(M_2)\Rightarrow y\in N(KM_2^{\,T}K)$$

Hence, $y\in N(KM^TK)$.

$$N(M)\subseteq N(KM^TK)$$

Similarly, $N(M)\supseteq N(KM^TK)$

$N(M) = N(KM^T K)$

Therefore M is k-KSNFM.

Theorem 4.2 Let M be a QPNFM of the form $M = \begin{bmatrix} (Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F) \\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F) \end{bmatrix}$ with $N((Q^T,Q^C,Q^I,Q^F)^T) \subseteq$ $N((S^T,S^C,S^I,S^F)^T)$ and $N\left(\left(M/(Q^T,Q^C,Q^I,Q^F)\right)^T\right) \subseteq N((R^T,R^C,R^I,R^F)^T)$, then the subsequence are equivalent.

(i) M is k – KSQPNFM with $k = k_1k_2$.

(ii) (Q^T,Q^C,Q^I,Q^F) is k -KS, $M/(Q^T,Q^C,Q^I,Q^F)$ is k -KS, $N(Q^T,Q^C,Q^I,Q^F) \subseteq$ $N(R^T,R^C,R^I,R^F)$ and $N((M/(Q^T,Q^C,Q^I,Q^F)) \subseteq N(S^T,S^C,S^I,S^F)$

(iii) Both the matrices $M = \begin{bmatrix} (Q^T,Q^C,Q^I,Q^F) & (0,0,1,1) \\ (S^T,S^C,S^I,S^F) & M/(Q^T,Q^C,Q^I,Q^F) \end{bmatrix}$ and

$$M = \begin{bmatrix} (Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F) \\ (0,0,1,1) & M/(Q^T,Q^C,Q^I,Q^F) \end{bmatrix}$$ are k- kernel symmetric.

Proof: This theorem is directly supported by Theorem (4.1) and the observation that M is k-KS⇔ M^T is k - KS.

Theorem 4.3 Let M be a QPNFM of the form $M = \begin{bmatrix} (Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F) \\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F) \end{bmatrix}$ with $N((Q^T,Q^C,Q^I,Q^F)) \subseteq N((R^T,R^C,R^I,R^F)^T)$ and $N\left(M/(Q^T,Q^C,Q^I,Q^F)\right) \subseteq$ $N((R^T,R^C,R^I,R^F))$, then the following are equivalent.

(i) M is k – KSQPNFM with $k = k_1k_2$.

(ii) (Q^T,Q^C,Q^I,Q^F) is k -KS, $M/(Q^T,Q^C,Q^I,Q^F)$ is k -KS,

(iii) The matrices $M = \begin{bmatrix} (Q^T,Q^C,Q^I,Q^F) & (0,0,1,1) \\ (S^T,S^C,S^I,S^F) & M/(Q^T,Q^C,Q^I,Q^F) \end{bmatrix}$ is k- KS.

Remark 4.1 It is crucial to consider the condition that is placed on M in Theorems 4.1 and 4.2. The example that follows serves to illustrate this.

Example:4.1 Let us consider a QPNFM

$$M=\begin{bmatrix}(1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (0,0,1,1) & (1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\end{bmatrix} \text{ and}$$

$$M=\begin{bmatrix} & v_1 & v_2 & v_3 & v_4\\ v_1 & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\\ v_2 & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\\ v_3 & (1,1,0,0) & (0,0,1,1) & (1,1,0,0) & (1,1,0,0)\\ v_4 & (1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\end{bmatrix}$$

$$K=\begin{bmatrix}(0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\end{bmatrix}$$

$$M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\ (S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$$

$$(Q^T,Q^C,Q^I,Q^F)=\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix},(R^T,R^C,R^I,R^F)=\begin{bmatrix}(1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix},$$

$$(S^T,S^C,S^I,S^F)=\begin{bmatrix}(1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix},\ (T^T,T^C,T^I,T^F)=\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix}$$

For this M, since M has ho zero rows and no zero columns N(M) = (0,0,1,1).

$N(KM^T K)$ = (0,0,1,1).Thus N(M) = $N(KM^T K) \Rightarrow$ M is k- KS.

$(Q^T,Q^C,Q^I,Q^F)^- =\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix}$ is a g-inverse, with respect to $(Q^T,Q^C,Q^I,Q^F)^-$

$M/(Q^T,Q^C,Q^I,Q^F) = (T^T,T^C,T^I,T^F) - (S^T,S^C,S^I,S^F)\ (Q^T,Q^C,Q^I,Q^F)^-$

(R^T,R^C,R^I,R^F).

$$M/(Q^T,Q^C,Q^I,Q^F) = \begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix} - \begin{bmatrix}(1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix}$$

$$\begin{bmatrix}(1,1,0,0) & (1,1,0,0)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix}\begin{bmatrix}(1,1,0,0) & (0,0,1,1)\\ (1,1,0,0) & (1,1,0,0)\end{bmatrix},$$

$M/(Q^T,Q^C,Q^I,Q^F) = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) \\ (1,1,0,0) & (1,1,0,0) \end{bmatrix}$,

M/(Q^T,Q^C,Q^I,Q^F) is k - KS,

Since $N(M/(Q^T,Q^C,Q^I,Q^F)) = N(K(M/(Q^T,Q^C,Q^I,Q^F))^T K)$ = (0,0,1,1).

(Q^T,Q^C,Q^I,Q^F) is k - KS, since $N(M) = N(KM^T K)$ = (0,0,1,1) for all K.

$N((Q^T,Q^C,Q^I,Q^F)) \subseteq N((R^T,R^C,R^I,R^F))$ and $N((Q^T,Q^C,Q^I,Q^F)^T) \subseteq N((S^T,S^C,S^I,S^F)^T)$.

Here, N(M/(Q^T,Q^C,Q^I,Q^F)) =(0,0,1,1) $= N((M/(Q^T,Q^C,Q^I,Q^F))^T)$,

N((S^T,S^C,S^I,S^F)) = (0,0,1,1) , $N((R^T,R^C,R^I,R^F)^T)$ = (0,0,1,1)

N(M/(Q^T,Q^C,Q^I,Q^F)) contained in N($(S^T,S^C,S^I,S^F)^T$) and $N((M/(Q^T,Q^C,Q^I,Q^F))^T$ contained in $N((R^T,R^C,R^I,R^F)^T)$.

Further

$$M_1 = \begin{bmatrix} (1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \end{bmatrix} \text{ and}$$

$$K = \begin{bmatrix} (0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \end{bmatrix}$$

$N(M_1) = (0,0,1,1)$.

$$KM_1^T K = \begin{bmatrix} (0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \end{bmatrix}$$

$$\begin{bmatrix}(1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\\(1,1,0,0) & (1,1,0,0) & (0,0,1,1) & (1,1,0,0)\\(0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\(0,0,1,1) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\end{bmatrix}\begin{bmatrix}(0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\(1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\(0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\\(0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\end{bmatrix}$$

$$KM_1^{\,T}K=\begin{bmatrix}(1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\\(1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\\(0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\(0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\end{bmatrix}$$

$$N(KM_1^{\,T}K)=\left\{(0,0,1,1),(0,0,1,1),(0,0,1,1),(Q^T,Q^C,Q^I,Q^F):(Q^T,Q^C,Q^I,Q^F)\in(QPNFM)\right\}$$

$\Rightarrow M_1$ is not k- Kernel Symmetric.

$$M_2=\begin{bmatrix}(1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\\(1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (1,1,0,0)\\(0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\\(0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\end{bmatrix}$$

M_2 is not k- Kernel Symmetric.

Remark:4.2 For a KSQPNFM M of the form $M=\begin{bmatrix}(Q^T,Q^C,Q^I,Q^F) & (R^T,R^C,R^I,R^F)\\(S^T,S^C,S^I,S^F) & (T^T,T^C,T^I,T^F)\end{bmatrix}$

with $k = k_1k_2$ following are equivalent.

$N((Q^T,Q^C,Q^I,Q^F)) \subseteq N((R^T,R^C,R^I,R^F))$ and $N(M/(Q^T,Q^C,Q^I,Q^F)) \subseteq N((S^T,S^C,S^I,S^F))$,

$N((Q^T,Q^C,Q^I,Q^F)^T) \subseteq N((S^T,S^C,S^I,S^F)^T)$, and $N((M/(Q^T,Q^C,Q^I,Q^F))^T) \subseteq N((R^T,R^C,R^I,R^F)^T)$

Example:4.2 Let us consider a QPNFM

$$M=\begin{bmatrix}(1,1,0,0) & (1,1,0,0) & (1,1,0,0) & (0,0,1,1)\\(1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\\(0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\(1,1,0,0) & (0,0,1,1) & (1,1,0,0) & (1,1,0,0)\end{bmatrix}\text{ and}$$

$$K=\begin{bmatrix}(0,0,1,1) & (1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\(1,1,0,0) & (0,0,1,1) & (0,0,1,1) & (0,0,1,1)\\(0,0,1,1) & (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\\(0,0,1,1) & (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\end{bmatrix}$$

$$N(M)=\{(0,0,1,1),(0,0,1,1),(Q^T,Q^C,Q^I,Q^F),(0,0,1,1):(Q^T,Q^C,Q^I,Q^F)\in(QPNFM)\}$$

$$KM^TK=\begin{bmatrix}(0,0,1,1)&(1,1,0,0)&(0,0,1,1)&(0,0,1,1)\\(1,1,0,0)&(0,0,1,1)&(0,0,1,1)&(0,0,1,1)\\(0,0,1,1)&(0,0,1,1)&(0,0,1,1)&(1,1,0,0)\\(0,0,1,1)&(0,0,1,1)&(1,1,0,0)&(0,0,1,1)\end{bmatrix}$$

$$\begin{bmatrix}(1,1,0,0)&(1,1,0,0)&(0,0,1,1)&(1,1,0,0)\\(1,1,0,0)&(0,0,1,1)&(0,0,1,1)&(0,0,1,1)\\(1,1,0,0)&(0,0,1)&(0,0,1)&(1,1,0,0)\\(0,0,1,1)&(1,1,0,0)&(0,0,1,1)&(1,1,0,0)\end{bmatrix}\begin{bmatrix}(0,0,1,1)&(1,1,0,0)&(0,0,1,1)&(0,0,1,1)\\(1,1,0,0)&(0,0,1,1)&(0,0,1,1)&(0,0,1,1)\\(0,0,1,1)&(0,0,1,1)&(0,0,1,1)&(1,1,0,0)\\(0,0,1,1)&(0,0,1,1)&(1,1,0,0)&(0,0,1,1)\end{bmatrix}$$

$$KP^TK=\begin{bmatrix}(1,1,0,0)&(1,1,0,0)&(0,0,1,1)&(0,0,1,1)\\(1,1,0,0)&(1,1,0,0)&(1,1,0,0)&(0,0,1,1)\\(1,1,0,0)&(0,0,1,1)&(1,1,0,0)&(0,0,1,1)\\(0,0,1,1)&(1,1,0,0)&(1,1,0,0)&(0,0,1,1)\end{bmatrix}$$

$$N(KM^TK)=(0,0,1,1)$$

$$N(M)\neq N(KM^TK)$$

Therefore M is not k - KS.

$$\text{P}/(Q^T,Q^C,Q^I,Q^F)=\begin{bmatrix}(0,0,1,1)&(0,0,1,1)\\(1,1,0,0)&(0,0,1,1)\end{bmatrix}$$

Here

$$N(Q^T,Q^C,Q^I,Q^F)\subseteq N(R^T,R^C,R^I,R^F),N(M/(Q^T,Q^C,Q^I,Q^F))\subseteq N(S^T,S^C,S^I,S^F)$$

But $N(Q^T,Q^C,Q^I,Q^F)^T$ is not contained $N(S^T,S^C,S^I,S^F)^T$.

$N(M/(Q^T,Q^C,Q^I,Q^F))^T$ is not contained $N(R^T,R^C,R^I,R^F)^T$.

5. Application of Fuzzy QuadripartitionedNeutrosophic Soft Matrix in Medical Decision Making Problem.

The process of medical decision-making is among the most complex, as it requires comprehensive knowledge of a patient's medical history, the symptoms they are experiencing, and the treatments they have undergone under the guidance of medical experts. Often, the information provided by patients can be incomplete, unclear, or ambiguous, making it challenging for healthcare professionals to accurately diagnose the condition. This uncertainty may lead to misdiagnosis and improper treatment. To address this, experts should compile a thorough list of the patient's symptoms and continuously monitor their health status from the onset. In cases involving complex or atypical symptoms, a panel of specialists may be required for an accurate diagnosis and effective treatment.

While biopsies are widely regarded as a definitive diagnostic tool, they are invasive and come with certain risks. It is essential to develop methodologies that can minimize the need for such procedures, especially for patients who are not at significant risk. Given the inherent uncertainty in medical treatment, the application of Fuzzy Quadripartition Neutrosophic Soft Sets (FQNSSs) is proposed. This approach offers a high-precision decision-making framework, which is crucial for accurate medical investigations. Below, we outline an algorithm to apply FQNSSs in the context of medical diagnosis.

Algorithm:

Step 1- Input the FQNSSs (M,S) and (N,D) called the patient- symptom and symptom-disease sets, respectively, and write their corresponding matrices P,Q.

Step 2- Compute $P*Q$ and $P\nabla *Q$.

$$P*Q =< \max\left(\min(T^P,T^Q)\right), \max\left(\min(C^P,C^Q)\right), \min\left(\max(U^P,U^Q)\right), \min\left(\max(F^P,F^Q)\right) >$$

$$P\nabla *Q =< \max\left\{\frac{T^A+T^B}{2}\right\}, \max\left\{\frac{C^A+C^B}{2}\right\}, \min\left\{\frac{U^A+U^B}{2}\right\}, \min\left\{\frac{F^A+F^B}{2}\right\} >$$

Step 3- Compute $S\left(P*Q\right)$ and $S\left(P\nabla *Q\right)$

$$S\left(P*Q\right) =< \frac{T^{(P*Q)}+C^{(P*Q)}-U^{(P*Q)}-F^{(P*Q)}}{2} >.$$

$$S\left(P\nabla *Q\right) =< \frac{T^{(P*Q)}+C^{(P*Q)}-U^{(P*Q)}-F^{(P*Q)}}{2} >$$

Step 4- Find the total score T_S = $S\left(P*Q\right)+S\left(P\nabla *Q\right)$

Step 5- Identify the maximum total score T_S for each patient P_i and conclude that the patient P_i is surely suffering from the disease D_j.

5.1 Example

Suppose there are three patients denoted by the set P ={ p_1, p_2, p_3 } with symptoms denoted by the set S = e_1: Fewer; e_2: Muscle aches, e_3: Fatigue; Let the possible diseases denoted by d_1 = Dengue fewer ; d_2 = Malaria, d_3 = Typhoid. Let the FQNSS (M,S), S over P and Q is given by

$$P = \begin{pmatrix} <0.4,0.3,0.5,0.6> & <0.2,0.4,0.3,0.4> & <0.3,0.6,0.4,0.5> \\ <0.2,0.4,0.7,0.3> & <0.4,0.6,0.5,0.4> & <0.7,0.2,0.1,0.2> \\ <0.5,0.4,0.1,0.5> & <0.6,0.7,0.3,0.8> & <0.5,0.3,0.6,0.7> \end{pmatrix}$$

$$Q=\begin{pmatrix} <0.5,0.4,0.7,0.2> & <0.2,0.5,0.6,0.5> & <0.4,0.5,0.6,0.7> \\ <0.6,0.2,0.4,0.4> & <0.6,0.4,0.7,0.2> & <0.6,0.3,0.4,0.5> \\ <0.2,0.6,0.4,0.6> & <0.5,0.6,0.4,0.5> & <0.4,0.2,0.4,0.6> \end{pmatrix}$$

$$P*Q=\begin{pmatrix} <0.4,0.6,0.4,0.4> & <0.3,0.6,0.4,0.4> & <0.4,0.3,0.4,0.5> \\ <0.2,0.4,0.4,0.3> & <0.5,0.4,0.4,0.4> & <0.4,0.4,0.4,0.5> \\ <0.6,0.4,0.4,0.5> & <0.6,0.5,0.6,0.5> & <0.6,0.4,0.4,0.7> \end{pmatrix}$$

$$P\nabla *Q=\begin{pmatrix} <0.45,0.6,0.35,0.4> & <0.4,0.6,0.4,0.3> & <0.4,0.4,0.35,0.45> \\ <0.5,0.4,0.25,0.25> & <0.6,0.5,0.25,0.3> & <0.55,0.45,0.25,0.4> \\ <0.6,0.45,0.35,0.35> & <0.6,0.55,0.35,0.5> & <0.6,0.5,0.35,0.6> \end{pmatrix}$$

$$S\left(P*Q\right)=\begin{pmatrix} 0.1 & 0.05 & -0.1 \\ -0.05 & 0.05 & -0.05 \\ 0.05 & 0 & -0.05 \end{pmatrix}$$

$$S\left(P\nabla *Q\right)=\begin{pmatrix} 0.15 & 0.15 & 0 \\ 0.2 & 0.27 & 0.17 \\ 0.17 & 0.15 & 0.07 \end{pmatrix}$$

$$T_S=S\left(P*Q\right)+S\left(P\nabla *Q\right)=\begin{pmatrix} & d_1 & d_2 & d_3 \\ p_1 & 0.25 & 0.2 & -0.1 \\ p_2 & 0.15 & 0.32 & 0.12 \\ p_3 & 0.22 & 0.15 & 0.02 \end{pmatrix}$$

From the given matrix T_S, it is evident that patients p_1, p_2, and p_3 are diagnosed with diseases d_1, d_2, and d_2, respectively. Furthermore, no patient has been identified with disease d_3.

Conclusion: 6

This work presents theorems that describe the properties of k-kernel symmetry (k-KS) and the Schur complement in the context of k-KSQPNFMs. We introduce the concepts of kernel symmetry (KS) and k-KS QPNFMs, exploring various properties and providing examples to illustrate these findings. Several equivalent characterizations of kernel symmetric and k-KSNFMs are discussed, along with fundamental examples that enhance the understanding of KSQPNFMs. It is shown that while k-symmetry implies k-KS, the reverse implication does not hold. Future work will aim to prove additional properties related to the generalized inverses of the Schur complement and k-kernel symmetric QPNFMs.

An algorithm has been developed to facilitate real-world decision-making within the FQNSM framework. To demonstrate its practical utility, the algorithm has been successfully applied to a medical diagnosis problem. Future research could explore extending this work by incorporating interval-based FQNSM theory, where the truth, contradiction, ignorance, and false membership degrees are represented as intervals rather than crisp values. Additionally, the proposed approach has potential applications in areas such as game theory, similarity measures, risk management, and group decision-making problems.

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