A Study on Continuity functions in Neutro-Topological Spaces

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Abstract: The specific purpose of this study is to define continuity of mappings in neutro-topological spaces using neutro-open and neutro-closed sets and analyze the properties of continuous functions that are true in classical topological spaces in the neutro-topological space. Neutro-interior and neutro-closure in neutro-topological spaces have some properties that are somewhat different from those in classical topological spaces. However, with the definition of a new form of continuity, termed as weakly neutro-continuity, much of the properties of continuous functions could be established in neutro-topological spaces. Neutro-open map and neutro-closed maps are also defined on the basis of neutro-open and neutro-closed sets. The notion of weakly neutro-continuity has been used to define neutro-homeomorphism and many of the properties of homeomorphism are analyzed and found to be true in the case of neutro-homeomorphism. A comparison of some of the properties of continuity and homeomorphism in classical topological spaces have been done vis-à-vis the neutrosophic topological spaces and neutro-topological spaces.

Keywords: Neutro-topological space, neutro-continuity, weakly neutro-continuity, neutro-open map, neutro-closed map, neutro-homeomorphism

1. Introduction

Continuity of functions in topological spaces is a well-established notion. Kelley [1] characterized continuity of functions by eight equivalent definitions and characterizations. The validity of any one of the characterizations is equivalent for a function to be continuous. Halfar [2] in 1960 studied conditions that imply continuity of functions between spaces and studied continuity of functions in terms of connectedness and compactness of the spaces. Levine [3-4] introduced the concept of weakly continuity. In the later article he introduced semi-open sets and he defined semi-continuity of functions in terms of the semi-open sets. Other studies on semi-continuity can be seen in [5-7]. Studies on weakly, sub-weakly and semi-weakly continuous functions can be seen in [9-12]. Hussain [13] defined almost continuity of functions by the openness of image of inverse image of the neighborhood of a point. Almost continuity has been studied by many other scholars [14-18]. Gentry *et al*. [19] introduced somewhat open sets and defined somewhat continuous functions and studied the properties of such continuous functions. Noiri [20] introduced δ-closed sets and its complement, the δ-open sets and subsequently defined the δ-continuous functions and proved that δ-continuity implies weakly continuity. Mashour *et al*. [21] defined pre-continuous and weakly pre-continuous functions on the basis of pre-open sets. Further, Mashour *et al*. [22] defined α-open sets, and introduced α-continuous functions and α-open maps. More studies on the α-continuous functions have been done in [23-25]. Many other scholars defined many different types of open sets and defined continuity with regard to those open sets and the list is huge as new kinds of open sets and new kinds of closed sets have been defined over the years. Zadeh [26] defined the fuzzy set with the aim to overcome the shortfall of the Cantorian set in failing to provide a base to study ambiguity. Chang [27] defined fuzzy topological space based on the fuzzy set defined by Zadeh. Atanassov [28] refined the fuzzy set by introducing intuitionistic fuzzy set. Coker [29] defined intuitionistic fuzzy topological space. Smarandache [30] introduced the neutrosophic logic. Further, Smarandache [31] refined the fuzzy set and hence the intuitionistic fuzzy set by bringing about the neutrosophic set. The neutrosophic set has been defined to encompass three parts: "the Truth, the Indeterminacy and the Falsity" of an entity to belong to a set. The fuzzy set is based only on the grade of truth while the intuitionistic fuzzy set is based on both the grade of truth as well as the grade of falsity. Thus the third component indeterminacy refined the study of the fuzzy set. Further, Salama *et al.* [32] defined the neutrosophic topological space. Salama *et al.* [33] defined neutrosophic closed set and defined different types of neutrosophic continuity based on the neutrosophic closed sets and proposed many results on neutrosophic continuity. Al-Omeri *et al*. [34] defined different types of neutrosophic open sets in neutrosophic topological spaces and studied neutrosophic continuity on the basis of the new sets that they defined. Senyurt *et al*. [35] studied neutrosophic continuity by establishing the properties of continuous functions in classical topological spaces via neutrosophic topological spaces. Smarandache [36-39], introduced the concept of NeutroAlgebra and AntiAlgebra thereby laying the foundation for further studies in algebraic structures like groups, rings etc. He used the terms Neutrosophication and Antisophication to generalise the concepts in classical algebra to NeutroAlgebra and AntiAlgebra by defining terms like NeutroAxiom, AntiAxiom, NeutroTheorem, AntiTheorem, NeutroOperation, AntiOperation etc. Agboola *et al*. [40] realized the concept of neutro-algebra and anti-algebra [36-37] by studying them with the help of existing number systems. Further, Agboola [41-42] provided the definition of a neutro-group and neutro-rings. Smarandache *et al*. [43] studied BCK-algebra and extended the study to neutro-BCK-algebra by the application of neutrosophication of the underlying operations of the BCK-algebras. Agboola [44-45] dwelt on finite neutro-groups and finite and infinite neutro-rings. Agboola *et al*. [46-47] introduced anti-groups and anti-rings. Ibrahim *et al*. [48] introduced neutro-vector space and studied certain simple properties of only a particular type of neutro-vector space which they called type 4S. Ibrahim *et al*. [49] defined neutro-hypergroup and anti-hypergroup by neutrosophication and antisophication of the three axioms of a classical hypergroup. Mohammadzadeh *et al*. [50] introduced neutro-nilpotent groups and studied some of their properties and they found the quotient of the group in context and the intersection of two such groups are also of the same group. Al-Tahan *et al*. [51] studied the application of the new algebras to Semigroups by introducing partial order relation in the Neutro-algebras. Smarandache [52], diversified the study of the neutrosophic triplets Truth: <A>; Indeterminate: <Neut A>; Falsity: <Anti A> on various possible studies and analysis of space, event etc. and for any study on any structure, by neutrosophication one can always have the component neutro-structure and by antisophication, one can always have the anti-structure component. Analysis of any structure may be with anything like axiom, theorem, lemma, property, proposition etc. Smarandache [53] extended the study of the neutro and anti-algebras to neutro-geometry and anti-geometry. Rezaei *et al*. [54] extended the study of neutrosophication and antisophication to the study of Semi-hypergroups. Sahin *et al*. [55] defined a neutro-metric space and discussed the basic properties of the neutro-metric, studied various similarities between the neutro-metric and the classical metric and concluded that a neutro-metric is obtainable from every classical metric and also pointed out the variations. Further, Sahin *et al*. [56] introduced the conception of a neutro-topological space and an anti-topological space. The study compared the new topologies with the classical topology and concluded that neutro-topology has a more general structure than the classical topology. They also further concluded that a neutro-topology could be derived from any given classical topology and further that a neutro-topology could also be derived from any given anti-topology. Further, Basumatary *et al.* [57], extended the study on neutro-topological spaces by studying the traits of interior, closure and boundary in neutro-topological spaces and found many interesting results. Again, Basumatary *et al.* [58] defined neutro-bitopological spaces and studied the traits of interior, closure and boundary in the new spaces. Basumatary *et al*. [59] studied the traits of interior, closure and boundary in anti-topological spaces and found that most of the traits that are followed for the aspects in a classical topology are also true in an anti-topology. In a contemporary study made by Witczak [60], wherein study on interior, closure, door spaces was made and two types of continuity of functions was also defined in anti-topological spaces. Basumatary *et al*. [61-62] provided the definitions of neighborhood of a point, base and sub-base in a neutro-topological space and in an anti-topological space and have compared the properties of the aspects to that of the general topological spaces. Basumatary *et al.* [63] established formulae to evaluate the number of neutro-topological spaces based on the number of members in the universal set on which a neutro-topological space is defined.

1.1 Motivation and method of study

The motivation behind this study comes from the fact that in literature, it has been observed that whenever any new set or topology is defined, the primary studies that are carried out in the new structure are studying the properties of interior, closure, boundary, exterior in the new space after which studies are done on continuity of functions in the space and other properties of the new space. After the definition of the neutro-topology has been provided in [56], the properties of interior, closure and boundary in neutro-topological spaces have been studied in [57]. The current study will utilize the notion of the neutro-open and neutro-closed sets defined and used in [57] to define continuity of functions in neutro-topological spaces. And taking advantage of the fact that a neutro-topology could be deduced from any classical topology [56], a new form of continuity called as weakly neutro continuity has also been defined and various underlying properties of continuity of functions are studied in neutro-topological spaces. Further, using the idea of weakly neutro-continuous functions, an attempt has been made to introduce neutro-homeomorphism. Also neutro-open map and neutro-closed map are introduced and they are used to further study the properties of neutro-homeomorphism. The study that has been undertaken has not yet been done by anyone as only a few studies have yet been done in neutro-topological and anti-topological spaces.

2. Preliminaries

Definition 2.1 [56] For a non-void universe X, with a class T of subsets of X, if one or more of {i, ii, iii} below are true, then T becomes a neutro-topology (nu-topology) over X and (X,T) will become a neutro-topological space (*NTTS*):

(i) [ϕ and $X \notin T$ simultaneously] or [$\phi , X \in_I T$]

(ii) For $p_\alpha \in T$, for a finite *n*, $\bigcap_{\alpha=1}^{n} p_\alpha \in T$ and for other $q_\alpha , r_\alpha \in T$, for a finite n, [$\bigcap_{\alpha=1}^{n} q_\alpha \notin T$ or $\bigcap_{\alpha=1}^{n} r_\alpha \in_I T$]

(iii) For $p_\alpha \in T$, $\bigcup_{\alpha \in I} p_\alpha \in T$, I being an arbitrary index set, and for other $q_\alpha , r_\alpha \in T$,[$\bigcup_{\alpha \in I} q_\alpha \notin T$ or, $\bigcup_{\alpha \in I} r_\alpha \in_I T$]

Remark 2.1 [56] The symbols “$=_I$” and “$\in_I$” are used to denote respectively circumstances when “equal to” and “belongs to” are not sure or not properly defined.

Theorem 2.1 [56] If T, a class of subsets of X becomes a topology over X, then $T \setminus \phi$ and $T \setminus X$ are n-topologies on X.

Theorem 2.2 [56] Finite union of n-topologies is again a n-topology.

Definition 2.2 [57] If (X,T) is a $NTTS$, then the components of T are termed neutro-open (nu-open) sets and complements of these nu-open sets are termed neutro-closed (nu-closed) sets.

Definition 2.3 [57] If (X,T) is a $NTTS$ over X and $O \subseteq X$, then the neutro-interior (nu-interior) of O is the join of every nu-open subsets of A and is identified by $NuInt(O)$. That is, $NuInt(O) = \bigcup\{Q_i : Q_i \subseteq O$ and each Q_i is nu-open$\}$.

Theorem 2.3 [57] If A is nu-open the $NuInt(O) = O.$ The result is however not always true the other way around in general.

Theorem 2.4 [57] If (X,T) is a $NTTS$ over X and $O, Q \subseteq X$ then:

(i) $NuInt(O) \subseteq O$

(ii) $NuInt(X) \subseteq X; NuInt(\phi) = \phi$

(iii) $NuInt(NuInt(O)) = NuInt(O)$

(iv) $O \subseteq Q \Rightarrow NuInt(O) \subseteq NuInt(Q)$

Definition 2.4 [57] If (X,T) is a $NTTS$ over X and $C \subseteq X$, then the neutro-closure (nu-closure) of C is the meet of the nu-closed sets that contains C and will be denoted by $NuCl(C)$. Thus, $NuCl(C) = \bigcap\{D_i : C \subseteq D_i$ and each D_i is nu-closed$\}$.

Theorem 2.5 [57] If C is nu-closed then $NuCl(C) = C$. The result is not true the other way around in general.

Theorem 2.6 [57] If (X,T) is a $NTTS$ over X and $C, D \subseteq X$ then the following are true:

(i) $C \subseteq NuCl(C)$

(ii) $NuCl(NuCl(C)) = NuCl(C)$

(iii) $C \subseteq D \Rightarrow NuCl(C) \subseteq NuCl(D)$

3. Continuity in neutro-topological spaces

Definition 3.1. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be neutro-continuous (nu-continuous) if $O \in T_Y \Leftrightarrow \xi^{-1}(O) \in T_X$.

Definition 3.2. For the topological spaces (X,T_X) and (Y,T_Y) the spaces $(X,T_X \setminus \Delta)$ and $(Y,T_Y \setminus \Delta)$ where Δ stands for ϕ or X, are neutro-topological spaces on the sets X and Y by theorem 2.1. A function ξ which is continuous in these neutro-topologies will be termed as weakly neutro-continuous (w-nu-continuous).

Remark 3.1. Whenever w-nu-continuity is mentioned, the underlying nu-topology will be either $T \setminus \phi$ or $T \setminus X$, where T is a classical topology. We will denote the nu-topologies in the $NTTSs$ $(X,T_X \setminus \Delta)$ and $(Y,T_Y \setminus \Delta)$ simply by T_Δ in some of our future discussions.

Theorem 3.1. If ξ is nu-continuous then ξ is w-nu-continuous.

Proof: Straight from their definitions.

Remark 3.2. Theorem 3.1 is not always true the other way around because a classical topology cannot be obtained from a nu-topology by the inclusion of the whole set or the null set to the nu-topology.

Theorem 3.2. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be w-nu-continuous iff for each T_Y -nu-closed C, $\xi^{-1}(C)$ is a T_X -nu-closed.

Proof: If the map f is w-nu-continuous and C is any T_Y -nu-closed set then $C^c(=Y\setminus C)\in T_Y$ and ξ being w-nu-continuous $\xi^{-1}(Y\setminus C)\in T_X$. Now, $\xi^{-1}(Y\setminus C)=X\setminus\xi^{-1}(C)\in T_X$ -nu-open set and hence, $\xi^{-1}(C)$ is nu-closed in T_X.

Conversely, if $f^{-1}(C)$ is nu-closed in T_Y for every nu-closed set C in T_X, let $D\in T_Y$. Then $Y\setminus D$ is nu-closed in T_Y and as such $\xi^{-1}(Y\setminus D)$ is nu-closed in T_X. Now, $\xi^{-1}(Y\setminus D)=X-\xi^{-1}(D)$ will be a nu-closed set in T_X thereby showing that $\xi^{-1}(D)\in T_X$. Hence as per definition the map ξ is w-nu-continuous.

Theorem 3.3. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be w-nu-continuous if and only if $NuCl(\xi^{-1}(B)\subseteq\xi^{-1}(NuCl(B))$ for any subset B of Y.

Proof: Assume that ξ is w-nu-continuous and assume that B is any nu-closed subset of Y, then $NuCl(B)$ is nu-closed in T_Y and so by theorem 3.2, $\xi^{-1}(NuCl(B))$ is nu-closed in T_X and hence $NuCl[\xi^{-1}(NuCl(B))]=\xi^{-1}(NuCl(B))$ … (1)

Now, $B\subseteq NuCl(B)\Rightarrow\xi^{-1}(B)\subseteq\xi^{-1}(NuCl(B))$

$\Rightarrow NuCl(\xi^{-1}(B))\subseteq NuCl(\xi^{-1}(NuCl(B)))$, by theorem 2.6 (iii)

$\Rightarrow NuCl(\xi^{-1}(B))\subseteq\xi^{-1}(NuCl(B))$, by (1)

Conversely, the condition is assumed to be true and let C be any nu-closed set in T_Y. Then $NuCl(C)=C$ and so by the given condition $NuCl[\xi^{-1}(C)]\subseteq\xi^{-1}(NuCl(C))=\xi^{-1}(C)$

That is, $NuCl[\xi^{-1}(C)]\subseteq\xi^{-1}(C)$.

But, we have $\xi^{-1}(C)\subseteq NuCl[\xi^{-1}(C)]$ in general by theorem 2.6 (i).

Thus, $NuCl[\xi^{-1}(C)]=\xi^{-1}(C)$, thereby showing that $\xi^{-1}(C)$ is nu-closed in T_X and so by theorem 3.3, the function ξ is w-nu-continuous.

Remark 3.4. In theorem 3.3, the function ξ will not be nu-continuous because in a $NTTS$, the nu-closure of a set is not necessarily a nu-closed set (see theorem 2.5).

Theorem 3.4. For two $NTTSs$ (X,T_X) and (Y,T_Y), a mapping ξ defined between T_X and T_Y will be w-nu-continuous iff $\xi(NuCl(C))\subseteq NuCl(\xi(C))$ for any subset C of X.

Proof: Consider ξ to be w-nu-continuous and $C\subseteq X$ and assume $\xi(C)=B\subseteq Y$. Then by theorem 3.4, we have $NuCl(\xi^{-1}(B))\subseteq\xi^{-1}(NuCl(B)))$

$\Rightarrow NuCl(\xi^{-1}(\xi(C)))\subseteq\xi^{-1}(NuCl(\xi(C)))$

$\Rightarrow NuCl(C)\subseteq\xi^{-1}(NuCl(\xi(C)))$, since ξ is w-nu-continuous $\xi^{-1}(\xi(C))=C$

$\Rightarrow\xi[NuCl(C)]\subseteq\xi[\xi^{-1}(NuCl(\xi(C)))]$

$\Rightarrow\xi(NuCl(C))\subseteq NuCl(\xi(C))$

Conversely, if the condition is true then let B be any arbitrary nu-closed set in T_Y, then $\xi^{-1}(B)\subseteq X$ and hence by the condition, we have:

$\xi(NuCl(\xi^{-1}(B)))\subseteq NuCl(\xi(\xi^{-1}(B)))\subseteq\xi(\xi^{-1}(NuCl(B)))=\xi(\xi^{-1}(B))$, since B is nu-closed.

$\Rightarrow NuCl(\xi^{-1}(B))\subseteq\xi^{-1}(B)$

But $\xi^{-1}(B) \subseteq NuCl(\xi^{-1}(B))$, by theorem 2.6 (i).
Hence, $NuCl(\xi^{-1}(B)) = \xi^{-1}(B)$ thereby showing that $\xi^{-1}(B)$ is nu-closed in T_X and so by theorem 3.2, the map ξ is w-nu-continuous.

Remark 3.5. In theorem 3.4, the function ξ will not be nu-continuous because in a $NTTS$, the nu-closure of a set being equal to the set need not necessarily imply that the set is nu-closed (see theorem 2.5).

Theorem 3.5. For two $NTTSs$ (X, T_X) and (Y, T_Y), a mapping ξ defined between T_X and T_Y will be w-nu-continuous iff $\xi^{-1}(NuInt(A)) \subseteq NuInt(\xi^{-1}(A))$ for any subset A of Y.

Proof: Let ξ be w-nu-continuous with $A \subseteq Y$. Then $NuInt(A) \in T_Y$ and so $\xi^{-1}(NuInt(A)) \in T_X$.
Hence $NuInt(\xi^{-1}(NuInt(A))) = \xi^{-1}(NuInt(A))$.
Now $NuInt(A) \subseteq A \Rightarrow \xi^{-1}(NuInt(A)) \subseteq \xi^{-1}(A)$
$\Rightarrow NuInt(\xi^{-1}(NuInt(A))) \subseteq NuInt(\xi^{-1}(A))$
$\Rightarrow \xi^{-1}(NuInt(A)) \subseteq NuInt(\xi^{-1}(A))$, since $\xi^{-1}(NuInt(A)) \in T_X$.
Conversely, assume that the condition is true and let $B \in T_2$ so that $NuInt(B) = B$, then by the given condition we have $\xi^{-1}(NuInt(B)) \subseteq NuInt(\xi^{-1}(B))$ which gives $\xi^{-1}(B) \subseteq NuInt(\xi^{-1}(B))$. But $NuInt(\xi^{-1}(B)) \subseteq \xi^{-1}(B)$ by theorem 2.4 (i) and as such we must have $NuInt(\xi^{-1}(B)) = \xi^{-1}(B)$ thereby showing that $\xi^{-1}(B) \in T_X$ and hence ξ is w-nu-continuous.

Remark 3.6. In theorem 3.5 the function ξ will not be nu-continuous because in a $NTTS$, the nu-interior of a set is not necessarily nu-open (see theorem 2.3).

Theorem 3.6. For three $NTTSs$ $(X, T_X), (Y, T_Y)$, and (Z, T_Z), if the maps ξ from T_X to T_Y and η from T_Y to T_Z are nu-continuous, then the map from T_X to T_Z given by $\eta \circ \xi : (X, T_X) \to (Z, T_Z)$ is also nu-continuous.

Proof: Assume that C be a nu-open set in T_Z, then $\eta^{-1}(C) \in T_Y$ and $\xi^{-1}(\eta^{-1}(C)) \in T_X$. But $\xi^{-1}(\eta^{-1}(C)) = (\xi^{-1} \circ \eta^{-1})(C) = (\eta \circ \xi)^{-1}(C)$. Thus, $(\eta \circ \xi)^{-1}(C) \in T_X$ whenever $C \in T_Z$ and so the map $(\eta \circ \xi)$ is nu-continuous.

Theorem 3.7. If (X, T_1) and (Y, T_2) are two $NTTSs$ and if $\{x\}$ is a singleton subset of X, then the map $\xi : (X, T_X) \to (Y, T_Y)$ is nu-continuous at $x \in X$.

Proof: Let $\xi(x) \in B \subseteq Y \Rightarrow x \in \xi^{-1}(B) \Rightarrow \{x\} \in \xi^{-1}(B) \Rightarrow \xi$ is nu-continuous at $x \in X$.

Theorem 3.8. For a $NTTS$ (X, T), the map $\xi : X \to X$ defined by $\xi(x) = x$ for every $x \in X$ is nu-continuous.

Proof: Let $B \in T \Rightarrow B \subseteq X$. Now, $\xi(x) = x \quad \forall x \in X$ and $B \subseteq X$
$\Rightarrow \xi^{-1}(B) = \{x \in X : \xi(x) \in B\} \Rightarrow \xi^{-1}(B) = \{x \in X : x \in B\} \Rightarrow \xi^{-1}(B) = \{x\}$
$\Rightarrow \xi$ is nu-continuous.

Definition 3.3. A map ξ between two $NTTSs$ (X, T_X) and (Y, T_Y) is called a neutro-open (nu-open) map if ξ images of nu-open sets in T_X are nu-open sets in T_Y. The map

$\xi:(X,T_X)\to(Y,T_Y)$ will be called a neutro-closed (nu-closed) map if the ξ images of T_X nu-closed sets are T_Y nu-closed sets.

Definition 3.4. If (X,T_X) and (Y,T_Y) are two $NTTSs$, then a map $\xi:(X,T_X)\to(Y,T_Y)$ will be called a neutro-homeomorphism (nu-homeomorphism) iff:

(i) ξ is one-one and onto;

(ii) $\xi: X\to Y$ is w-nu-continuous; and

(iii) $\xi^{-1}: Y\to X$ is w-nu-continuous.

If such a map ξ exists between the $NTTSs$ (X,T_X) and (Y,T_Y), then the two $NTTSs$ will be neutro-homomorphic to each other.

Theorem 3.9. For two $NTTSs$ (X,T_X) and (Y,T_Y), if ξ is one-one and onto mapping of X to Y, then ξ is a nu-homeomorphism iff ξ is w-nu-continuous and a nu-open map.

Proof: Assume that ξ is a nu-homeomorphism and let $\xi^{-1}=\eta$ and $\eta^{-1}=\xi$. Since ξ is one-one and onto, so η will also be one-one and onto. Now, if $O\in T_X$, then $\eta^{-1}(O)\in T_Y$. But since $\eta^{-1}=\xi$ so $\eta^{-1}(O)=\xi(O)\in T_Y$. Thus $O\in T_1\Rightarrow\xi(O)\in T_2$ and hence it follows that ξ is a nu-open map and since ξ is a nu-homeomorphism, so it is w-nu-continuous.

Conversely, let ξ be w-nu-continuous and a nu-open map. Also by condition ξ is one-one and onto, so it suffices to prove that $\xi^{-1}=\eta$ is w-nu-continuous. Let $O\in T_X$, then $\xi(O)\in T_Y$ since ξ is nu-open. That is $\eta^{-1}(O)\in T_Y$ thereby showing that $\eta=\xi^{-1}$ is w-nu-continuous. Hence the map ξ is a nu-homeomorphism.

Theorem 3.10. For two $NTTSs$ (X,T_X) and (Y,T_Y), if ξ is one-one and onto mapping of X to Y, then ξ is a nu-homeomorphism iff ξ is w-nu-continuous and a nu-closed map.

Proof: Let ξ be a nu-homeomorphism and let C be any nu-closed set in T_X then $X\setminus C\in T_X$. Since $\eta=\xi^{-1}$ is w-nu-continuous, it follows that $\eta^{-1}(X\ \setminus\ C)\in\ T_Y$. Now, $\eta^{-1}(X\ \setminus\ C)=Y\setminus\eta^{-1}(C)$ and hence $\eta^{-1}(C)$ is nu-open in T_Y and as such $\eta^{-1}(C)$ is nu-closed in T_Y. That is, $\eta^{-1}(C)=\xi(C)$ is nu-closed in T_Y. Hence ξ is w-nu-continuous and a nu-closed map.
Conversely, let the conditions hold and let O be any nu-open set in T_X, then $X\setminus O$ is nu-closed set and ξ is a nu-closed map, so $\xi(X\setminus O)=\xi(X\setminus O)=Y\setminus\eta^{-1}(O)$ is a nu-closed set in T_Y which implies that $\eta^{-1}(O)\in T_Y$. Thus the image of every nu-open set in T_X under the function η is nu-open in T_Y. Thus, $\eta=\xi^{-1}$ is w-nu-continuous and hence ξ is a nu-homeomorphism.

Theorem 3.11. For two $NTTSs$ (X,T_X) and (Y,T_Y), if a mapping ξ from T_X to T_Y is one-one onto and w-nu-continuous, then ξ is a nu-homeomorphism if ξ is nu-open or nu-closed.

Proof: We assume that ξ is one-one onto and w-nu-continuous and also that ξ is either nu-open or nu-closed. We have to show that ξ^{-1} is w-nu-continuous. By theorem 3.4, it suffices to verify that $\xi^{-1}(NuCl(B))\subseteq NuCl(\xi^{-1}(B))$ for any subset B of Y.

Now, $B\subseteq Y\Rightarrow NuCl(\xi^{-1}(B))\subseteq X$ and is nu-closed. Also since ξ is given to be nu-closed, we have: $\xi(NuCl(\xi^{-1}(B)))=NuCl(\xi(NuCl(\xi^{-1}(B))))$ … (1)
Also, $\xi^{-1}(B)\subseteq NuCl(\xi^{-1}(B))\Rightarrow\xi(\xi^{-1}(B))\subseteq\xi(NuCl(\xi^{-1}(B)))$

$\Rightarrow NuCl(\xi(\xi^{-1}(B))) \subseteq NuCl(\xi(NuCl(\xi^{-1}(B)))) = \xi(NuCl(\xi^{-1}(B)))$ by (1)
$\Rightarrow NuCl(B) \subseteq \xi(NuCl(\xi^{-1}(B))) \Rightarrow \xi^{-1}(NuCl(B)) \subseteq NuCl(\xi^{-1}(B))$
Hence by theorem 3.3 the map ξ^{-1} is w-nu-continuous and hence ξ is a nu-homeomorphism.

Theorem 3.12. For two $NTTSs\ (X, T_{X,\Delta})$ and $(Y, T_{Y,\Delta})$, a function ξ between the two spaces is nu-open iff $\xi(NuInt(A)) \subseteq NuInt(\xi(A))$ for any $A \subseteq X$.

Proof: Assume ξ to be nu-open and $A \subseteq X$ then $\xi(NuInt(A))$ is nu-open since $NuInt(A) \in T_{X,\Delta}$. Now $NuInt(A) \subseteq A \Rightarrow \xi(NuInt(A)) \subseteq \xi(A)$.
Again $\xi(NuInt(A)) \in T_{Y,\Delta}$, so $NuInt(\xi(NuInt(A))) = \xi(NuInt(A))$ … (1)
Also, $\xi(NuInt(A)) \subseteq \xi(A) \Rightarrow NuInt(\xi(NuInt(A))) \subseteq NuInt(\xi(A))$
$\Rightarrow \xi(nInt(A)) \subseteq nInt(\xi(A))$ by (1)
Conversely, let the condition be true and let $O \in T_{X,\Delta}$ so that $NuInt(O) = O$. Then $\xi(O) = \xi(NuInt(O)) \subseteq NuInt(\xi(O))$, by the given condition. But we have by theorem 2.4 (i) $NuInt(\xi(O)) \subseteq \xi(O)$. Thus, we have $NuInt(\xi(O)) = \xi(O)$ thereby showing that $\xi(O) \in T_{Y,\Delta}$. Thus ξ is nu-open.

Remark 3.7. In the above theorem T_X and T_Y are two different nu-topologies both having the property of T_Δ (see remark 3.1) and so have been denoted by $(X, T_{X,\Delta})$ and $(Y, T_{Y,\Delta})$.

Theorem 3.13. For two $NTTSs\ (X, T_{X,\Delta})$ and $(Y, T_{Y,\Delta})$, a function ξ between the two spaces is nu-closed iff $NuCl(\xi(C)) \subseteq \xi(NuCl(C))$ for any $C \subseteq X$.

Proof: Let ξ be nu-closed and $C \subseteq X$, then $NuCl(C)$ is nu-closed in $T_{X,\Delta}$ and since ξ is nu-closed $\xi(NuCl(C))$ is nu-closed in $T_{Y,\Delta}$ and thus $NuCl(\xi(NuCl(C))) = \xi(NuCl(C))$ … (1)
Again,
$C \subseteq NuCl(C) \Rightarrow \xi(C) \subseteq \xi(NuCl(C)) \Rightarrow NuCl(\xi(C)) \subseteq NuCl(\xi(NuCl(C))) = \xi(NuCl(C))$
by (1). That is $NuCl(\xi(C)) \subseteq \xi(NuCl(C))$.
Conversely, let the condition hold and suppose that D is some nu-closed set in $T_{X,\Delta}$ so that $NuCl(D) = D$. Then $\xi(NuCl(D)) = \xi(D)$ … (2)
Now, by the given condition $NuCl(\xi(D)) \subseteq \xi(NuCl(D)) = \xi(D)$ by (2)
That is, $NuCl(\xi(D)) \subseteq \xi(D)$. But by theorem 2.6 (i) we have $\xi(D) \subseteq NuCl(\xi(D))$ as a result of which we must have $NuCl(\xi(D)) = \xi(D)$ thereby showing that $\xi(D)$ is nu-closed in $T_{Y,\Delta}$. Hence ξ is nu-closed.

Theorem 3.14. For two $NTTSs\ (X, T_X)$ and (Y, T_Y), if the function ξ between the two spaces is one-one onto, then ξ is a nu-homeomorphism iff $NuCl(\xi(C)) = \xi(NuCl(C))$ for every $C \subseteq X$.

Proof: Assume that ξ is a nu-homeomorphism. Then ξ is one-one, onto, and also by theorem 3.11 ξ is nu-closed and w-nu-continuous. Now, if $C \subseteq X$, then by theorem 3.4 $\xi(NuCl(C)) \subseteq NuCl(\xi(C))$ … (1)
Also, $C \subseteq NuCl(C) \Rightarrow \xi(C) \subseteq \xi(NuCl(C)) \Rightarrow NuCl(\xi(C)) \subseteq NuCl(\xi(nCl(C))$ … (2)
Now, ξ is nu-closed and $NuCl(C)$ is nu-closed in T_X and so $\xi(NuCl(C))$ is nu-closed in T_Y and hence $NuCl(\xi(NuCl(C))) = \xi(NuCl(C))$ … (3)

(2) and (3) gives $NuCl(\xi(C)) \subseteq \xi(NuCl(C))$ … (4)
(1) and (4) gives $\xi(NuCl(C)) = NuCl(\xi(C))$
Conversely, let $\xi(NuCl(C)) = NuCl(\xi(C))$ for every $C \subseteq X$.
Then obviously $\xi(NuCl(C)) \subseteq NuCl(\xi(C))$ which from theorem 3.4 means that the function ξ is w-nu-continuous. Again, if D is any nu-closed set in T_X so that $NuCl(D) = D$ which implies $\xi(NuCl(D)) = \xi(D) \Rightarrow \xi(D) = NuCl(\xi(D))$ by the given condition, thereby showing that $\xi(D)$ is nu-closed in T_Y whenever D is closed in T_X thereby showing that ξ is nu-closed. Thus ξ is nu-closed as well as one-one onto and w-nu-continuous, so ξ is a nu-homeomorphism.

4. Comparative view of neutro-continuity, neutrosophic continuity vis-à-vis classical continuity

In a neutro-topological space the interior of a set is not open in general and the closure of a set is also not generally closed. However, if we choose the neutro-topological space (X, T_Δ), where T_Δ is the neutro-topology obtained by either the exclusion of the whole set or the null set from the topology T , then we can have the interior of any subset open and the closure of any subset closed by choosing the exclusion of the null set or the whole set in the neutro-topology appropriately. Below we list some basic properties that are true in the three topological spaces:

Classical Continuity	Neutrosophic Continuity	Nu-continuity and w-nu-continuity
If $(X,T_X),(Y,T_Y)$, and (Z,T_Z) are three topological spaces and $\xi : X \to Y$ and $\eta : Y \to Z$ are continuous functions then $\eta \circ \xi : X \to Z$ is a continuous function.	True. Theorem 3.1 [35].	True. Theorem 3.6.
A mapping ξ from a space X into another space Y is continuous if and only if $\xi(\overline{A}) \subset \overline{\xi(A)}$ for every $A \subset X$.	True. Theorem 3.3 [35].	True only for w-nu-continuity. Theorem 3.4 and remark 3.5.
A mapping ξ from a space X into another space Y is continuous if and only if $\overline{\xi^{-1}(B)} \subset \xi^{-1}(\overline{B})$ for every $B \subset Y$.	True. Theorem 3.4 [35].	True only for w-nu-continuity. Theorem 3.3 and remark 3.4.
A mapping from a space X into another space Y is continuous if and only if $\xi^{-1}[B^\circ] \subset [\xi^{-1}(B)]^\circ$ for every $B \subset Y$.	True. Theorem 3.6 [35].	True only for w-nu-continuity. Theorem 3.5 and remark 3.6.
If a mapping $\xi : X \to Y$ between the spaces X and Y is one-one onto then ξ is a homeomorphism if and only if ξ is continuous and closed.	True. Theorem 3.12 [35]	True for w-nu-homeomorphism. Theorem 3.10.

4. Conclusions

In this article, we have introduced the definition of neutro-continuous functions in neutro-topological spaces with the use of the open sets called neutro-open sets. Taking advantage of the fact that a neutro-topology could be deduced from any classical topology with the exclusion of the empty set or the whole set, we have defined a new form of continuity of functions in such neutro-topological spaces and named it weakly neutro-continuity. In a general neutro-topology the union and intersection of the neutro-open sets may or may not be neutro-open. However, in the neutro-topology which is obtained by the simple exclusion of the empty set from a topology, the unions of the neutro-open sets are neutro-open and in the neutro-topology obtained from a topology by the exclusion of the whole set, the intersection of the neutro-open sets are neutro-open. It has been assumed that whenever properties of interior of neutro-open sets are involved, the exclusion of the whole set is considered in the neutro-topology and whenever closure properties are involved the exclusion of the empty set should be considered. Thus, we have found the results involving the interior and closure of subsets valid via the weakly neutro-continuity of the functions. It has also been observed that whenever a function is weakly neutro-continuous, it is not always neutro-continuous because in a neutro-topology the union and intersection of the open and closed sets involved in the very definition of the neutro-topology may not be included in the collection that forms the neutro-topology.

4.1 Limitations

In our analysis we have found that most of the properties of continuous functions are true only in the case of weakly neutro-continuity, which has been defined in a special type of a neutro-topology. Neutro-homeomorphism has also been defined on the basis of weakly neutro-continuity and will not be generally meaningful if defined with the help of neutro-continuity as most of the properties that have been established might not be true in a general neutro-topological space.

4.2 Scope for further studies in the field

Further, one could investigate the continuity of functions with regard to the neutro-neighborhood, neutro-base, neutro-sub-base, neutro-relative topology and via other aspects of study of continuity of functions. The current study can be extended to study uniform continuity that will help study connectedness and compactness in neutro-topological spaces.

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References

1. Kelley, J. L. *General Topology*, 4th Edition. D. Van Nostrand Company, Inc., New York, USA, **1955**.
2. Halfar, E. Conditions implying continuity of functions. *Proc. Amer. Math. Soc.* **1960**, *11(5),* 688-691.
3. Levine, N. A decomposition of continuity in topological space. *Amer. Math. Monthly* **1961**, *68*, 44-46.
4. Levine, N. Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly* **1963**, *70*, 36-41.
5. Noiri, T. On semi-continuous mappings. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti* **1973**, *54(2)*, 210-214.
6. Noiri, T. A note on semi-continuous mappings. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti* **1973**, *55(5),* 400-403.
7. Rose, D. A. On Levine' s decomposition of continuity. *Canad. Math. Bull.* **1978**, *21(4)*, 477-481.
8. Espelie, M. S., & Joseph, J. E. Remarks on two weak forms of continuity. *Canad. Math. Bull.* **1982**, *25(1),* 59-63.
9. Baker, C. W. Properties of subweakly continuous functions. *Yokohama Math. J.* **1984**, *32*, 39-43.
10. Dorsett, C. Semi-continuous and semi-weakly continuous functions. *Kyungpook Math. J.* **1990**, *30(1),* 95-100.
11. Noiri, T. On weakly continuous mappings. *Proc. Amer. Math. Soc.* **1974**, *46*, 120-124.

12. Noiri, T., & Popa, V. Weak forms of faint continuity. *Bull. Math. Soc. Sc. Math. Roumanie* **1990**, *34(82)*, 263-270.
13. Hussain, T. Almost continuous mappings. *Prac. Mat*. **1966**. *10*, 1-7.
14. Lin, S-Y, T. Almost continuity of mappings. *Canad. Math. Bull.* **1968**, *11(3)*, 453-455.
15. Singal, M. K., & Singal, A. R. Almost-continuous mappings. *Yokohama Math. J.* **1968**, *16*, 63-73.
16. Long, P. E., & McGehee, E. E. Properties of almost continuous functions. *Proc. Amer. Math. Soc.* **1970**, *24(1)*, 175-180.
17. Long, P. E., & Carnahan, D. A. Comparing almost continuous functions. *Proc. Amer. Math. Soc.* **1973**, *38(2)*, 413-418.
18. Rose, D. A. Weak continuity and almost continuity. *Int. J. Math. Math. Sc.* **1984**, *7*, 311-318.
19. Gentry, K. R., Hughes, H. B., III. Somewhat continuous functions. *Czechoslovak Math. J.* **1971**, *21(1)*, 5-12.
20. Noiri, T. On δ-continuous functions. *J. of Korean Math. Soc.* **1980**, *16*, 161-166.
21. Mashhour, A. S., Abd El-Monsef, M. E., & El-Deeb, S. N. On pre-continuous and weak pre-continuous mappings, *Proc. Math. Phys. Soc. Egypt* **1982**, *53*, 47-53.
22. Mashhour, A. S., Hasanein, I. A., & El-Deeb, S. N. (1983). α-Continuous and α-open mappings. *Acta Math. Hungar.* **1983**, *41*, 213-218.
23. Noiri, T. On α-continuous functions. *Časopis pro pěstovánímatematiky* **1984**, *109(2)*, 118-126.
24. Tong, J. A decomposition of continuity. *Acta Math. Hungar.* **1986**, *48* (1-2), 11-15.
25. Tong, J. On decomposition of continuity in topological spaces. *Acta Math. Hungar.* **1989**, *54(1-2)*, 51-55.
26. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338-353.
27. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, *24(1)*, 182-190.
28. Atanassov, K. T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20(1)*, 87-96.
29. Coker, D. An Introduction to intuitionistic fuzzy topological spaces. *Fuzzy Sets Syst.* **1997**, *88(1)*, 81-89.
30. Smarandache, F. Neutrosophy, neutrosophic probability, set, and logic. *Proquest Information & Learning*, Ann Arbor, Michigan, USA, **1998**; *105*, 118-123.
31. Smarandache, F. Neutrosophic set-a generalisation of the intuitionistic fuzzy set. *Int. J. Pure Appl. Math.* **2005**, *24(3)*, 287.
32. Salama, A. A., & Alblowi, S. A. Neutrosophic set and neutrosophic topological spaces. *IOSR J. Math.* **2012**, *3(4)*, 31-35.
33. Salama, A. A., Smarandache, F., & Kromov, V. Neutrosophic closed sets and neutrosophic continuous functions. *Neutrosophic Sets Systs.* **2014**, *4*, 4-8.
34. Al-Omeri, W. & Smarandache, F. New neitrosophic sets via neutrosophic topological spaces. *Neutrosophic Operational Research*, *Volume 1*, Pons, Brussels, **2017**, 189-209.
35. Senyurt, S., & Kaya, G. On neutrosophic continuity. *Ordu University Journal of Science and Technology,* **2017**, *7(2)*, 330-339.
36. Smarandache, F. *Advances of Standard and Nonstandard Neutrosophic Theories*, Brussels, Pons Edition, **2019.**
37. Smarandache, F. Introduction to NeutroAlgebraic structures and AntiAlgebraic structures (revisited). *Neutrosophic Sets Syst.* **2020**, *31*, 1-16.
38. Smarandache, F. Generalizations and alternatives of classical algebraic structures to NeutroAlgebraic structures and AntiAlgebraic structures. *J. Fuzzy Ext. Appl.* **2020**, *1(2)*, 85.
39. Smarandache, F. NeutroAlgebra is a generalization of partial algebra. *Int. J. Neutrosophic Sc.* **2020**, **2***(1)*, 8-17.

40. Agboola, A. A. A., Ibrahim, M. A., & Adeleke, E. O. Elementary examination of NeutroAlgebras and AntiAlgebras viz-a-viz the classical number system. *Int. J. Neutrosophic Sc.* **2020**, *4(1)*, 16-19.
41. Agboola, A. A. A. Introduction to NeutroGroups. *Int. J. Neutrosophic Sc.* **2020**, *6(1)*, 41-47.
42. Agboola, A. A. A. Introduction to NeutroRings. *Int. J. Neutrosophic Sc.* **2020**, *7(2)*, 62-73.
43. Smarandache, F., & Hamidi, M. Neutro-BCK-algebra. *Int. J. Neutrosophic Sc.* **2020**, *8(2)*, 110-117.
44. Agboola, A. A. A. On finite NeutroGroups of type NG [1, 2, 4]. *Int. J. Neutrosophic Sc.* **2020**, *10(2)*, 84-95.
45. Agboola, A. A. A. On finite and infinite NeutroRings of type NR. *Int. J. Neutrosophic Sc.* 2020, *11(2)*, 87-99.
46. Agboola, A. A. A. Introduction to AntiGroups. *Int. J. Neutrosophic Sc.* **2020**, *12(2)*, 71-80.
47. Agboola, A. A. A., & Ibrahim, M. A. Introduction to AntiRings. *Neutrosophic Sets Syst.* **2020**, *36(1)*, 292-307.
48. Ibrahim, M. A., & Agboola, A. A. A. NeutroVector spaces I. *Neutrosophic Sets Syst.* **2020**, *36(1)*, 328-350.
49. Ibrahim, M. A., & Agboola, A. A. A. Introduction to NeutroHyperGroups. *Neutrosophic Sets Syst.* **2020**, *38*, 15-32.
50. Mohammadzadeh, E., & Rezaei, A. On NeutroNilpotentGroups. *Neutrosophic Sets Syst.* **2020**, *38*, 33-40.
51. Al-Tahan, M., Davvaz, B., Smarandache, F., & Anis, O. On Some NeutroHyperStructures. *Symmetry* **2021**, *13(4)*, 535.
52. Smarandache, F. Structure, NeutroStructure, and AntiStructure in science. *Int. J. Neutrosophic Sc.* **2021**, *13(1)*, 28-33.
53. Smarandache, F. NeutroGeometry & AntiGeometry are alternatives and generalizations of the non-Euclidean Geometries (revisited). *Neutrosophic Sets Syst.* **2021**, *46*, 456-477.
54. Rezaei, A., Smarandache, F., & Mirvakili, S. Applications of (Neutro/Anti) Sophications to Semihypergroups. *J. Math.* **2021**, 1-7.
55. Sahin, M., Kargin, A., & Altun, A. Neutro-Metric spaces. *NeutroAlgebra Theory*, 2021; *1*, 71-84.
56. Sahin, M., Kargin, A., & Yucel, M. Neutro-Topological space and Anti-Topological space. *NeutroAlgebra Theory,* **2021***;* *1*, 16-30.
57. Basumatary, B., Khaklary, J.K., Wary, N., & Smarandache, F. On Neutro-Topological Spaces and Their Properties. *Theory and Applications of NeutroAlgebras as Generalisations of Classical Algebras,* IGI Global, **2022***;* 180-201.
58. Basumatary, B., Khaklary, J.K., & Said, B. On NeutroBitopological Spaces. *Int. J. Neutrosophic Sc.* **2022**, *18(2)*, 254-261.
59. Basumatary, B., & Khaklary, J.K. A study on the properties of anti-topological spaces. *Neutrosophic Algebraic Structures and their Applications,* IGI Global, **2022***;* 16-27.
60. Witczak, T. Some remarks on anti-topological spaces. *Asia Mathematika* **2022**, *6(1)*, 23-34.
61. Basumatary, B., & Talukdar, A. A Study on neutro-topological neighbourhood and neutro-topological base. In *NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World*, IGI Global, **2023**; 187-201.
62. Talukdar, A., Basumatary, B., & Hossain, S. A study on anti-topological neighbourhood and anti-topological base. In *Neutrosophic SuperHyperAlgebra and New Types of Topologies,* Global Knowledge Publishing House, Florida, USA, **2023**; 76-96.
63. Basumatary, B., Basumatary, J., & Wary, N. Formulae for finding the number of NeutroTopological apaces. In *NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World*, IGI Global, **2023**; 202-220.

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