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On A Novel Topological Space Based On Partially Ordered Ring Of Weak Fuzzy Complex Numbers and Its Relation With The Partially Ordered Neutrosophic Ring of Real Numbers

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Abstract: This work is dedicated to studying for the first time a novel topological space generated by weak fuzzy complex intervals based on the partial ordered ring structure of weak fuzzy complex numbers, where we build a topology structure over these intervals. Also, we present many novel results as mathematical theorems, mathematical proofs, and many related examples. On the other hand, we use the AH-isometry to determine the relationship between the weak fuzzy complex intervals, and neutrosophic intervals.

Keywords: weak fuzzy complex numbers, weak fuzzy interval, topological space, weak fuzzy complex topology.

Introduction

The algebraic extensions of real number field have attracted the attention of many different researchers from all over the world. In [9], Weak Fuzzy Complex numbers appeared for the first time, where they have been defined as a new generalization of classical real numbers through fuzzy operators. In [3, 5] authors studied weak fuzzy complex vector spaces and matrices. The weak fuzzy complex number theoretical concepts such as Diophantine equations and congruencies [2,7-8]. Also, in [1], authors have studied weak fuzzy complex linear Diophantine equations in two variables and presented an easy algorithm to solve them. In [10], the foundations of functions in weak fuzzy complex variables were studied. Actually, one of the important results of using this type of numbers, modeling the solutions of some vectorial equations defined by norms in 3-dimensional Euclidean space A-Curves [4]. However, Python introduced Weak Fuzzy Complex numbers in [11-20].

This has motivated us for the study of topological spaces based on the intervals defined over weak fuzzy complex numbers, where we study for the first time a novel topological space generated by weak fuzzy complex intervals based on the partial ordered ring structure of weak fuzzy complex numbers, where we build a topology structure over these intervals. Also, we present many novel results as mathematical theorems, mathematical proofs, and many related examples. On the other hand, we use the AH-isometry to determine the relationship between the weak fuzzy complex intervals, and neutrosophic intervals.

Main Discussion

Definition 1. ([9])

The set of Weak Fuzzy Complex numbers was defined as follows, where 'J' is the Weak Fuzzy Complex operator

($J \notin R$):

$$F_J = \{x_0 + x_1 J \,;\, x_0, x_1 \in R, J^2 = t \in]0,1[\,\}.$$

Properties of the Weak Fuzzy Complex numbers: ([9])

Let $X = x_0 + x_1 J$, $Y = y_0 + y_1 J \in F_J$, where $x_0, x_1, y_0, y_1 \in R$

- Addition: $X + Y = (x_0 + y_0) + (x_1 + y_1)J$.
- Multiplication $X \times Y = (x_0 y_0 + x_1 y_1 t) + (x_0 y_1 + x_1 y_0)J$.
- The conjugate of X is: $\bar{X} = x_0 + x_1 J$.

Definition 2. ([9])

Let φ be *the transformation function* from F_J to $R \times R$, which we define as follows:

$$\varphi: F_J \mapsto R \times R.$$

$$\varphi(x_0 + x_1 J) = (x_0 + x_1(-\sqrt{t}), x_0 + x_1(+\sqrt{t})) = (x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}).$$

Definition 3.

Let $\varphi: F_J \mapsto R \times R$ such that: $\varphi(X) = (a, b)$, the inverse function of φ is defined as follows:

$$\varphi^{-1}: R \times R \mapsto F_J$$

$$\varphi^{-1}(a, b) = \frac{1}{2}[a + b] + \frac{1}{2\sqrt{t}} J[b - a].$$

Definition 4.

The ring of weak fuzzy complex numbers is defined as follows:

$$\mathbb{C}_w = \{a + bJ \,;\, J^2 = t \in]0,1[\,; a, b \in \mathbb{R}\}.$$

Remark 1.

$\mathbb{C}_w$ is partially ordered set with $(\leq)$, with:

$$a + bJ \leq c + dJ \iff \begin{cases} a - b\sqrt{t} \leq c - d\sqrt{t} \\ a + b\sqrt{t} \leq c + d\sqrt{t} \end{cases}.$$

The relation $(<)$ can be defined as follows:

$$a + bJ < c + dJ \iff \begin{cases} a - b\sqrt{t} < c - d\sqrt{t} \\ a + b\sqrt{t} < c + d\sqrt{t} \end{cases}.$$

Definition 5.

We define the following types of weak fuzzy complex intervals:

$$I_1 = [a + bJ\,, c + dJ]$$

$$= \{x + yJ \in C_w\ ; a - b\sqrt{t} \leq x - y\sqrt{t} \leq c - d\sqrt{t}\,, a + b\sqrt{t} \leq x + y\sqrt{t} \leq c + d\sqrt{t}\}$$

$$I_2 =]a + bJ\,, c + dJ[$$

$$= \{x + yJ \in C_w\ ; a - b\sqrt{t} < x - y\sqrt{t} < c - d\sqrt{t}\,, a + b\sqrt{t} < x + y\sqrt{t} < c + d\sqrt{t}\}$$

$$I_3 = [a + bJ\,, c + dJ[$$

$$= \{x + yJ \in C_w\ ; a - b\sqrt{t} \leq x - y\sqrt{t} < c - d\sqrt{t}\,, a + b\sqrt{t} \leq x + y\sqrt{t} < c + d\sqrt{t}\}$$

$$I_4 =]a + bJ\,, c + dJ]$$

$$= \{x + yJ \in C_w\ ; a - b\sqrt{t} < x - y\sqrt{t} \leq c - d\sqrt{t}\,, a + b\sqrt{t} < x + y\sqrt{t} \leq c + d\sqrt{t}\}.$$

Remark 2.

1) The previous types of (WFC) intervals are defined under the condition $a + bJ \leq c + dJ$ in c_w.

2) (WFC) intervals with ∞ in any side can be recognized as:

$$]-\infty, a + bJ\,] = \{x + yJ \in C_w\ ; -\infty < x - y\sqrt{t} \leq a - b\sqrt{t}\,, -\infty < x + y\sqrt{t} \leq a + b\sqrt{t}\}.$$

$$[a + bJ\,, \infty[= \{x + yJ \in C_w\ ; a - b\sqrt{t} \leq x - y\sqrt{t} < \infty, a + b\sqrt{t} \leq x + y\sqrt{t} < \infty\}.$$

Example 1.

Take

$$I_1 =]1 + 2J, 3 + J[\ ; J^2 = t = \tfrac{1}{4}, \sqrt{t} = \tfrac{1}{2}.$$

$$\begin{cases} 1+2\sqrt{t}=2\leq 3+\sqrt{t}=\frac{7}{2} \\ 1-2\sqrt{t}=0\leq 3-\sqrt{t}=\frac{5}{2} \end{cases},$$

so that,

$$I_1=\left\{x+yJ\in C_w\ ;0<x-\tfrac{1}{2}y<\tfrac{5}{2}\ ,2<x+\tfrac{1}{2}y<\tfrac{7}{2}\right\}=\left\{(x,y)\in\mathbb{R}^2;\ 0<x-\tfrac{1}{2}y<\tfrac{5}{2}\ ,2<x+\tfrac{1}{2}y<\tfrac{7}{2}\right\}.$$

Take

$$I_5=\,]-\infty,-1+3J\,[\ ;\ J^2=\tfrac{1}{9}=t,\sqrt{t}=\tfrac{1}{3},$$

$$I_5=\left\{x+yJ\in C_w\ ;-\infty<x-\tfrac{1}{3}y<-2\ ,-\infty<x+\tfrac{1}{3}y<0\right\}=\left\{(x,y)\in\mathbb{R}^2;\ -\infty<x-\tfrac{1}{3}y<-2\ ,-\infty<x+\tfrac{1}{3}y<0\right\}.$$

Definition 6.

Let $a+bJ, c+dJ\in C_w$,

we define:

1) $\max(a+bJ,c+dJ)=f^{-1}\left[(\max\left(a-b\sqrt{t},c-d\sqrt{t}\right),\max\left(a+b\sqrt{t},c+d\sqrt{t}\right))\right]$.

2) $\min(a+bJ,c+dJ)=f^{-1}\left[(\min\left(a-b\sqrt{t},c-d\sqrt{t}\right),\min\left(a+b\sqrt{t},c+d\sqrt{t}\right))\right]$.

Where *(f)* is the isomorphism defined in [6] as follows:

$$f: C_w\to R\times R\ : f(a+bJ)=(a-b\sqrt{t},a+b\sqrt{t}),$$

$$f^{-1}: R\times R\to C_w,$$

such that,

$$f^{-1}(a,b)=\frac{b+a}{2}+\frac{1}{\sqrt{t}}J\left(\frac{b-a}{2}\right)\ ;J^2=t\in\,]0,1[\ .$$

Theorem 1.

In C_w, we have:

1) $\min(a+bJ,c+dJ)\leq a+bJ,c+dJ$,

2) $\max(a+bJ,c+dJ)\geq a+bJ,c+dJ$,

3) If $a+bJ\leq c+dJ$, then $\min(a+bJ,c+dJ)=a+bJ$, and $\max(a+bJ,c+dJ)=c+dJ$,

4) $\min(a+bJ,c+dJ)\leq\max(a+bJ,c+dJ)$.

Theorem 2.

In C_w, we have:

1) If $x + yJ \leq a + bJ, x + yJ \leq c + dJ$, then

$$x + yJ \leq \min(a + bJ, c + dJ).$$

2) If $x + yJ \geq a + bJ, x + yJ \geq c + dJ$, then

$$x + yJ \geq \max(a + bJ, c + dJ).$$

Proof of theorem 1.

1) $\min(a + bJ, c + dJ) = \frac{1}{2}\left[\min\left(a - b\sqrt{t}, c - d\sqrt{t}\right) + \min\left(a + b\sqrt{t}, c + d\sqrt{t}\right)\right] + \frac{1}{2\sqrt{t}} J \left[\min\left(a + b\sqrt{t}, c + d\sqrt{t}\right) - \min\left(a - b\sqrt{t}, c - d\sqrt{t}\right)\right] = k_1 + k_2 J.$

$k_1 + k_2\sqrt{t} = \min\left(a + b\sqrt{t}, c + d\sqrt{t}\right) \leq a + b\sqrt{t}, c + d\sqrt{t}.$

$k_1 - k_2\sqrt{t} = \min\left(a - b\sqrt{t}, c - d\sqrt{t}\right) \leq a - b\sqrt{t}, c - d\sqrt{t},$

Hence,

$$\min(a + bJ, c + dJ) \leq a + bJ, c + dJ.$$

2) $\max(a + bJ, c + dJ) = \frac{1}{2}\left[\max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) + \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right)\right] + \frac{1}{2\sqrt{t}} J \left[\max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) - \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right)\right] = k_1 + k_2 J.$

$k_1 + k_2\sqrt{t} = \max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) \geq a + b\sqrt{t}, c + d\sqrt{t},$

$k_1 - k_2\sqrt{t} = \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right) \geq a - b\sqrt{t}, c - d\sqrt{t}.$

Hence,

$$\max(a + bJ, c + dJ) \geq a + bJ, c + dJ.$$

3) Assume that $a + bJ \leq c + dJ$,then

$$\begin{cases} a - b\sqrt{t} \leq c - d\sqrt{t} \\ a + b\sqrt{t} \leq c + d\sqrt{t} \end{cases}.$$

Hence,

$$\begin{cases} \min\left(a - b\sqrt{t}, c - d\sqrt{t}\right) = a - b\sqrt{t} \\ \min\left(a + b\sqrt{t}, c + d\sqrt{t}\right) = a + b\sqrt{t} \end{cases},$$

and

$$\begin{cases} \max\left(a - b\sqrt{t}, c - d\sqrt{t}\right) = c - d\sqrt{t} \\ \max\left(a + b\sqrt{t}, c + d\sqrt{t}\right) = c + d\sqrt{t} \end{cases}.$$

So that,

$$\min(a+bJ, c+dJ) = \frac{1}{2}\left(a-b\sqrt{t}+a+b\sqrt{t}\right)+\frac{1}{2\sqrt{t}}J\left(a+b\sqrt{t}-a+b\sqrt{t}\right) = a+bJ,$$

$$\max(a+bJ, c+dJ) = \frac{1}{2}\left(c-d\sqrt{t}+c+d\sqrt{t}\right)+\frac{1}{2\sqrt{t}}J\left(c+d\sqrt{t}-c+d\sqrt{t}\right) = c+dJ.$$

4) Since,

$$\min\left(a-b\sqrt{t}, c-d\sqrt{t}\right) \leq \max\left(a-b\sqrt{t}, c-d\sqrt{t}\right),$$

$$\min\left(a+b\sqrt{t}, c+d\sqrt{t}\right) \leq \max\left(a+b\sqrt{t}, c+d\sqrt{t}\right),$$

thus,

$$\min(a+bJ, c+dJ) \leq \max(a+bJ, c+dJ).$$

Proof of theorem 2.

1) Assume that,

$$\begin{cases} x+yJ \leq a+bJ \\ x+yJ \leq c+dJ \end{cases},$$

then,

$$\begin{cases} x-y\sqrt{t} \leq a-b\sqrt{t} \\ x+y\sqrt{t} \leq a+b\sqrt{t} \end{cases} \text{and} \begin{cases} x-y\sqrt{t} \leq c-d\sqrt{t} \\ x+y\sqrt{t} \leq c+d\sqrt{t} \end{cases}.$$

Hence,

$$\begin{cases} x-y\sqrt{t} \leq \min\left(a-b\sqrt{t}, c-d\sqrt{t}\right) \\ x+y\sqrt{t} \leq \min\left(a+b\sqrt{t}, c+d\sqrt{t}\right) \end{cases},$$

and

$$x+yJ \leq \min(a+bJ, c+dJ).$$

2) Assume that:

$$\begin{cases} x+yJ \geq a+bJ \\ x+yJ \geq c+dJ \end{cases},$$

then,

$$\begin{cases} x-y\sqrt{t} \geq a-b\sqrt{t} \\ x+y\sqrt{t} \geq a+b\sqrt{t} \end{cases} \text{and} \begin{cases} x-y\sqrt{t} \geq c-d\sqrt{t} \\ x+y\sqrt{t} \geq c+d\sqrt{t} \end{cases}.$$

Hence,

$$\begin{cases} x-y\sqrt{t} \geq \max\left(a-b\sqrt{t}, c-d\sqrt{t}\right) \\ x+y\sqrt{t} \geq \max\left(a+b\sqrt{t}, c+d\sqrt{t}\right) \end{cases},$$

and

$$x+yJ \geq \max(a+bJ, c+dJ).$$

Theorem 3.

In C_w, we have:

If $a+bJ<c+dJ, x+yJ<m+nJ$, then

1) $]a+bJ\,, c+dJ[\,\cap\,]x+yJ, m+nJ[\,=\,]\max(a+bJ, x+yJ)\,, \min(c+dJ, m+nJ)[$,

2) $[a+bJ\,, c+dJ]\cap[x+yJ, m+nJ]=[\max(a+bJ, x+yJ)\,, \min(c+dJ, m+nJ)]$.

Theorem 4.

In C_w, we have:

1) $]a+bJ\,,\infty[\,\cap\,]x+yJ,\infty[\,=\,]\max(a+bJ, x+yJ)\,,\infty[$.

2) $[a+bJ\,,\infty[\,\cap\,[x+yJ,\infty[\,=\,[\max(a+bJ, x+yJ)\,,\infty[$.

3) $]a+bJ\,,\infty[\,\cup\,]x+yJ,\infty[\,=\,]\min(a+bJ, x+yJ)\,,\infty[$.

4) $[a+bJ\,,\infty[\,\cup\,[x+yJ,\infty[\,=\,[\min(a+bJ, x+yJ)\,,\infty[$.

Theorem 5.

In C_w, we have:

1) $]-\infty, a+bJ\,[\,\cap\,]-\infty, x+yJ[\,=\,]-\infty, \min(a+bJ, x+yJ)[$.

2) $]-\infty, a+bJ\,]\cap\,]-\infty, x+yJ]=\,]-\infty, \min(a+bJ, x+yJ)]$.

3) $]-\infty, a+bJ\,[\,\cup\,]-\infty, x+yJ[\,=\,]-\infty, \max(a+bJ, x+yJ)[$.

4) $]-\infty, a+bJ\,]\cup\,]-\infty, x+yJ]=\,]-\infty, \max(a+bJ, x+yJ)]$.

Proof of theorem 3.

1) Let $k+hJ\in\,]a+bJ\,, c+dJ[\,\cap\,]x+yJ, m+nJ[$. Then

$$\begin{cases}k+hJ>a+bJ\\k+hJ>x+yJ\end{cases},\begin{cases}k+hJ<c+dJ\\k+hJ<m+nJ\end{cases}.$$

Hence,

$$\begin{cases}k+hJ>\max(a+bJ, x+yJ)\\k+hJ<\min(c+dJ, m+nJ)\end{cases},$$

so that

$$k+hJ\in\,]\max(a+bJ, x+yJ)\,, \min(c+dJ, m+nJ)[.$$

Now, let

$$k+hJ\in\,]\max(a+bJ, x+yJ)\,, \min(c+dJ, m+nJ)[.$$

Then,

$$\begin{cases} k+hJ > \max(a+bJ, x+yJ) \\ k+hJ < \min(c+dJ, m+nJ) \end{cases} \Rightarrow \begin{cases} k+hJ > a+bJ, x+yJ \\ k+hJ < c+dJ, m+nJ \end{cases} \Rightarrow k+hJ \in \,]a+bJ, c+dJ[\, \cap$$

$$]x+yJ, m+nJ[.$$

2) The proof is exactly similar to 1 by replacing $(<)$ with $(\leq)$ and $(>)$ with $(\geq)$.

Proof of theorem 4.

1) $]a+bJ\,,\infty[\,\cap\,]x+yJ,\infty[\, = \,]\max(a+bJ, x+yJ)\,, \min(\infty,\infty)[\, = \,]\max(a+bJ, x+yJ)\,,\infty[$,

2) $[a+bJ\,,\infty[\,\cap\,[x+yJ,\infty[\, = [\max(a+bJ, x+yJ)\,, \min(\infty,\infty)[\, = [\max(a+bJ, x+yJ)\,,\infty[$,

3) Let $+hJ \in \,]a+bJ,\infty[\,\cup\,]x+yJ,\infty[$. Then

$$\begin{cases} k+hJ > a+bJ \\ \quad or \\ k+hJ > x+yJ \end{cases} \Rightarrow k+hJ > \min(a+bJ, x+yJ) \Rightarrow k+hJ \in \,]\min(a+bJ, x+yJ), \infty[.$$

Conversely, suppose that $k+hJ \in \,]\min(a+bJ, x+yJ), \infty[$, then

$$\begin{cases} k+hJ > a+bJ \\ \quad or \\ k+hJ > x+yJ \end{cases} \Rightarrow k+hJ \in \,]a+bJ\,,\infty[\,\cup\,]x+yJ,\infty[.$$

4) It is similar to 3.

Proof of theorem 5.

1) $]-\infty, a+bJ\,[\,\cap\,]-\infty, x+yJ[\, = \,]\max(-\infty,-\infty)\,, \min(a+bJ, x+yJ)[\, = \,]-\infty, \min(a+bJ, x+yJ)[$,

2) $]-\infty, a+bJ\,]\,\cap\,]-\infty, x+yJ] = \,]\max(-\infty,-\infty)\,, \min(a+bJ, x+yJ)] = \,]-\infty, \min(a+bJ, x+yJ)]$,

3) Let $k+hJ \in \,]-\infty, a+bJ[\,\cup\,]-\infty, x+yJ[$ .Then

$$\begin{cases} k+hJ < a+bJ \\ \quad or \\ k+hJ < x+yJ \end{cases} \Rightarrow k+hJ < \max(a+bJ, x+yJ) \Rightarrow k+hJ \in \,]-\infty, \max(a+bJ, x+yJ)[.$$

Conversely, assume that $k+hJ \in \,]-\infty, \max(a+bJ, x+yJ)[$, then

$$\begin{cases} k+hJ < a+bJ \\ \quad or \\ k+hJ < x+yJ \end{cases} \Rightarrow k+hJ \in \,]-\infty, a+bJ\,[\,\cap\,]-\infty, x+yJ[.$$

4) It is similar to 3.

Definition7.

We define $\tau = \{\emptyset, C_w, S_i\}$; $S_i = \cup_{i\in I}]a_i + b_iJ, c_i + d_iJ[$.

Theorem 6.

(C_w, τ) is a topological space.

Proof.

We have:

$\emptyset, C_w \in \tau$ by a definition.

Let $S_i, S_j \in \tau$. Then

$$\begin{cases} S_i = \cup_{i \in I}]a_i + b_i J, c_i + d_i J[\\ S_j = \cup_{j \in L}]a_j + b_j J, c_j + d_j J[\end{cases},$$

$$S_i \cap S_j = \cup_{l,k \in M}]\max(a_k + b_k J, a_l + b_l J), \min(c_k + d_k J, c_l + d_l J)[\ \in \tau.$$

Also, $\cup_{i=1}^{\infty} S_i$ is a union of open intervals in C_w, hence $\cup_{i=1}^{\infty} S_i \in \tau$, and τ is a topology on C_w, thus $(C_w, \tau)$is a topological space.

Remark 3.

1) Open sets in (C_w, τ) are the union of open intervals in C_w.

2) Closed sets in (C_w, τ) are the complements of open sets, i.e.

$$D_i = \{C_w / S_i \ ; i \in I, S_i \in \tau\}.$$

3) open intervals in C_w are open sets.

Example 2.

Take $S_1 =]1 + 2J, 3 + 5J[, S_2 =]2 + J, 3 + J[\ ; J^2 = t = \frac{1}{4}$.

$S_1 = S_1 \cup \emptyset, S_2 = S_2 \cup \emptyset$, thus S_1, S_2 are open sets.

$S_1 \cap S_2 =]\max(1 + 2J, 2 + J), \min(3 + 5J, 3 + J)[;$

where,

$\max(1 + 2J, 2 + J) = \frac{1}{2}\left[\max\left(1 + 2\left(\frac{1}{2}\right), 2 + (1)\left(\frac{1}{2}\right)\right) + \max\left(1 - 2\left(\frac{1}{2}\right), 2 - (1)\left(\frac{1}{2}\right)\right)\right] + \frac{1}{2\left(\frac{1}{2}\right)} J \left[\max\left(1 + 2\left(\frac{1}{2}\right), 2 + (1)\left(\frac{1}{2}\right)\right) - \max\left(1 - 2\left(\frac{1}{2}\right), 2 - (1)\left(\frac{1}{2}\right)\right)\right] = \frac{1}{2}\left[\max\left(2, \frac{5}{2}\right) + \max\left(0, \frac{3}{2}\right)\right] + J\left[\max\left(2, \frac{5}{2}\right) - \max\left(0, \frac{3}{2}\right)\right] = \frac{1}{2}\left(\frac{5}{2} + \frac{3}{2}\right) + J\left[\frac{5}{2} - \frac{3}{2}\right] = 2 + J.$

$\min(3 + 5J, 3 + J) = \frac{1}{2}\left[\min\left(3 + \frac{5}{2}, 3 + \frac{1}{2}\right) + \min\left(3 - \frac{5}{2}, 3 - \frac{1}{2}\right)\right] + \frac{1}{2\left(\frac{1}{2}\right)} J \left[\min\left(3 + \frac{5}{2}, 3 + \frac{1}{2}\right) - \min\left(3 - \frac{5}{2}, 3 - \frac{1}{2}\right)\right] = \frac{1}{2}\left[\min\left(\frac{11}{2}, \frac{7}{2}\right) + \min\left(\frac{1}{2}, \frac{5}{2}\right)\right] + J\left[\min\left(\frac{11}{2}, \frac{7}{2}\right) - \min\left(\frac{1}{2}, \frac{5}{2}\right)\right] = \frac{1}{2}\left(\frac{7}{2} + \frac{1}{2}\right) + J\left[\frac{7}{2} - \frac{1}{2}\right] = 2 + 3J.$

$S_1 \cap S_2 =]2+J, 2+3J[\in \tau$.

$S_1 \cup S_2 =]1+2J, 3+5J[\cup]2+J, 3+J[\in \tau$.

$\overline{S_1} = C_w /]1+2J, 3+5J[$, is a closed set.

$\overline{S_2} = C_w /]2+J, 3+J[$, is a closed set.

$\overline{S_1} \cap \overline{S_2} = (\overline{S_1 \cup S_2}), \overline{S_1} \cup \overline{S_2} = \overline{S_1 \cap S_2} \in \tau$.

Weak Fuzzy complex intervals representation as neutrosophic intervals:

Definition 8. [17]

Let $R(I) = \{a + bI \;; a, b \in \mathbb{R}, I^2 = I\}$ be the neutrosophic ring of real numbers, then it is partially ordered with:

$$a + bI \leq c + dI \Leftrightarrow \begin{cases} a + b \leq c + d \\ a \leq c \end{cases}.$$

Remark. [17]

$h: R(I) \to R \times R: h(a + bI) = (a, a + b)$ is a ring isomorphism, and it is called the AH- isometry.

The inverse isometry is defined as:

$h^{-1}: R \times R \to R(I): h^{-1}(a, b) = a + (b - a)I$.

Definition 9.

The neutrosophic interval $[a + bJ, c + dJ]$ is defined under the condition $a + bI \leq c + dI$.

$$x + yI \in [a + bJ, c + dJ] \Leftrightarrow \begin{cases} a \leq x \leq c \\ a + b \leq x + y \leq c + d \end{cases}.$$

Remark 4.

The ring of weak fuzzy complex numbers is isomorphic to $R \times R$ with $f: C_w \to R \times R$ such that:

$f(a + bJ) = \left(a - b\sqrt{t}, a + b\sqrt{t}\right) \;; J^2 = t$.

Theorem 7.

Every weak fuzzy complex number can be represented with one neutrosophic real number.

Proof.

Let $h: R(I) \to R \times R$ be the neutrosophic AH- isometry and $f: W_c \to R \times R$ be the (WFC) isomorphism, then we can define:

$$\varphi: C_w \to R(I)\;; \quad \varphi = h^{-1} \circ f.$$

It is clear that (φ) is a ring isomorphism, with:

$$\varphi(a+bJ)=h^{-1}\circ f(a+bJ)=h^{-1}\left(a-b\sqrt{t},a+b\sqrt{t}\right)=a-b\sqrt{t}+\left[\left(a+b\sqrt{t}\right)-\left(a-b\sqrt{t}\right)\right]I=\left(a-b\sqrt{t}\right)+2b\sqrt{t}I$$

Hence, the proof is complete.

Result 1.

Every open interval in C_w $]a+bJ,c+dJ[$ can be represented with exactly one open interval in $R(I)$ as follows:

$$]a+bJ,c+dJ[\ \leftrightarrow\ ]\varphi(a+bJ),\varphi(c+dJ)[\ =\ \left]\left(a-b\sqrt{t}\right)+2b\sqrt{t}I,c-d\sqrt{t}+2d\sqrt{t}I\right[.$$

Remark 5.

We call $\varphi: C_w \to R(I)$ the weak fuzzy neutrosophic AH- isometry.

Theorem 8.

Let $\varphi: C_w \to R(I)$ be the weak fuzzy neutrosophic AH- isometry, then:

If $a+bJ<c+dJ$, $\varphi(a+bJ)<\varphi(c+dJ)$.

Proof: Assume that $a+bJ<c+dJ$, then:

$$\begin{cases}a-b\sqrt{t}<c-d\sqrt{t}\\ a+b\sqrt{t}<c+d\sqrt{t}\end{cases}\Rightarrow\begin{cases}a-b\sqrt{t}<c-d\sqrt{t}\\ \left(a-b\sqrt{t}\right)+2b\sqrt{t}<\left(c-d\sqrt{t}\right)+2d\sqrt{t}\end{cases}$$
$$\Rightarrow a-b\sqrt{t}+2b\sqrt{t}I<c-d\sqrt{t}+2d\sqrt{t}I\Rightarrow\varphi(a+bJ)<\varphi(c+dJ).$$

Result 2.

The weak fuzzy complex interval topological space is isomorphic to the neutrosophic interval topology.

Proof.

Since every (WFC) interval can be linked with only one neutrosophic interval, then it is isomorphic to the neutrosophic topological space of intervals.

Result 3.

Every open set in $C_w, A=\bigcup_{i\in k}]a_i+b_iJ,c_i+d_iJ[$ can be linked with exactly one open set in $R(I)$ as follows:

$$\varphi(A) = \bigcup_{i \in k}]\varphi(a_i + b_i J), \varphi(c_i + d_i J)[.$$

Conclusion

In this work we studied for the first time a novel topological space generated by weak fuzzy complex intervals based on the partial ordered ring structure of weak fuzzy complex numbers, where we built a topology structure over these intervals. Also, we presented many novel results as mathematical theorems, mathematical proofs, and many related examples. On the other hand, we use the AH-isometry to determine the relationship between the weak fuzzy complex intervals, and neutrosophic intervals. For future studies, these mathematical tools can be used effectively with other tools and techniques that can be observed through [21-45].

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