Neutrosophic Dynamic Network DEA: Efficient Allocation of Carryover Variables in Organizational Processes

Maryam Heydar[1] and Hadi Bagherzadeh Valami[1*]

[1]Department of Applied Mathematics, Yadegar-e-Imam Khomeini (RAH), Shahre Rey Branch, Islamic Azad University, Tehran, Iran, hadibagherzade0@gmail.com.

Abstract

Carryover activities in dynamic DEA refer to the persistence of resources, inputs, or outputs across periods in organizational processes, reflecting the impact of past decisions on current and future performance. In practical applications, some carryover variables can extend beyond the immediate next period, and their allocation is discretionary, controlled by the Decision-Maker (DM). This paper introduces a novel dynamic network DEA (DNDEA) model aimed at optimizing the allocation of these carryovers and identifying inefficiencies within a network system across multiple evaluation periods.

Recognizing the uncertainties present in real-world data, we incorporate neutrosophic sets to effectively process uncertain information, which adds complexity to our analysis. To address this, we transform the Neutrosophic Dynamic Network Slack-Based Measure (NDNSBM) model into a two-stage framework. By leveraging the concept of Pareto efficiency, our model establishes boundaries for overall and period scores across varying levels of truth, indeterminacy, and falsity. The key contribution of this work is the introduction of discretionary carryover variables in DNDEA models, facilitating strategic allocation across future periods. Additionally, the integration of neutrosophic data provides a more realistic approach to dynamic decision-making contexts. We validate our methodology through a numerical example evaluating the performance of Iranian bank branches, demonstrating that our proposed model is more discriminative and offers deeper insights into resource allocation strategies compared to the DNSBM model. This comprehensive approach enhances understanding of resource management in dynamic environments, offering valuable implications for decision-makers in various sectors.

Keywords: Neutrosophic Set; Dynamic Network DEA; Decision Making; Carryover Variables.

1 Introduction

Data Envelopment Analysis (DEA) proposed by Charnes et al. [1] is known as an effective tool for measuring the performance of homogeneous Decision-Making Units (DMUs) [2]–[5]. One key application of DEA is resource allocation [6]–[8]. Most studies about the resource allocation in DEA literature can be generally categorized into two groups [9]. The first group encompasses studies which concentrate on allocating fixed costs to DMUs. Fixed costs represent the total overhead cost utilized for common infrastructures across the subunits of a DMU. This issue is a typical challenge encountered in organizational budgeting and costing, involving the distribution of an overhead cost among various departments [10], [11]. The second group includes studies discuss the allocation of input resources (such as funds and personnel) to DMUs. In these studies, the Decision Maker (DM) allocates available resources to units or sub-units in order to reach a definite goal [12].

In today's dynamic and competitive economic environment, it's evident that the market undergoes continuous changes [13]. However, in both groups mentioned above, these dynamic changes are not adequately addressed [14]. There is a need to open a discussion on how these allocation methods can adapt to the evolving market conditions over time [15]. Dynamic DEA is recognized as an effective tool to capture temporal changes and fluctuations in market dynamics [16], [17]. Dynamic DEA models with network structure, DNDEA, analyze the internal processes of DMUs and monitor the fluctuation in both period efficiency and divisional efficiency of DMUs over time.

Integrating DNDEA principles into resource allocation frameworks could enhance the adaptability and responsiveness of allocation strategies to changing market conditionsn [18]. Carry-over activities are non-separable elements of dynamic DEA. They indicate how past decisions affect the current and future performance of organizations and provide a better understanding of performance evaluation over time periods by showing how resources and outcomes persist across different periods. In the literature of Dynamic DEA, carry-overs are typically viewed as variables or activities transferred from the current to the next period [19]. However, in real-world scenarios, some carryover variables in the production process can extend beyond the immediate next period [20]. In this paper, we refer to these variables as discretionary carryovers. For instance, in the banking industry, Loan Loss Reserves can be considered as such carryovers [21], [22]. Banks allocate reserves to cover potential losses from loans that may default in the future. If actual loan losses in a specific period are lower than expected, these reserves can be allocated to future periods to improve the bank's efficiency [23]. Another example is deferred tax assets or liabilities. Banks may have deferred tax assets or liabilities due to differences between accounting and tax laws. If the tax impact of certain transactions or events cannot be understood in the current period, these assets or liabilities can be carried forward to future periods when they are expected to be realized or addressed. Additionally, when a bank agrees with a borrower to postpone loan repayment to a future period with a specific interest rate, it's another instance of carryover allocation. Moreover, when a bank negotiates to extend the payment of taxes to a future period, it delays the tax obligation from the current period to a later one [24]. This adjustment affects financial planning and resource allocation over multiple periods, making it a form of carryover. Accrued revenues or expenses are also significant. Banks measure accumulated income from loans and investments and incurred expenses from deposits and borrowings over time. If the actual income or expenses differ from the accrued amounts in a specific period, the difference can be carried forward to future periods until reconciled [25]. These examples illustrate how carryover variables in the banking industry can extend to periods beyond the immediate next one and have implications for financial reporting, risk management, and regulatory compliance over multiple periods. There are also a lot of examples in other industries. For instance, in manufacturing, the carryover of raw materials inventory from one production cycle to the future can impact production efficiency and cost management [26], [27]. Therefore, strategic allocation of carryover activities and tactically distributing resources, considering both the present and the future, is crucial for managing resources effectively within organizations[2], [28].

On the other hand, in real-world problems, the uncertainty and vagueness present in available data often complicate performance evaluations [29], [30]. Neutrosophic sets proposed by Smarandache [31] offer innovative tools for modeling this uncertainty and incorporating it into calculations [32]–[34].

This paper introduces a novel approach within a neutrosophic environment to evaluate the overall, period, and divisional efficiencies of DMUs, incorporating excesses or shortfalls of carryover variables into the objective function. The key contribution is the introduction of discretionary carryover variables in DNDEA models, enabling strategic allocation across future periods. Additionally, the integration of neutrosophic data accounts for uncertainties, providing a more realistic analysis in dynamic decision-making environments. The methods in the paper were chosen based on their ability to address key challenges in DNDEA, particularly in handling carry-over activities that extend beyond consecutive periods. The SBM framework allows us to optimize efficiency while incorporating discretionary allocation of carryovers, a key feature of real-world decision-making. Additionally, the neutrosophic approach was introduced to handle uncertainty, as it provides a robust tool for dealing with indeterminate and vague data, which is often encountered in practical applications. This combination of methods ensures a comprehensive and flexible efficiency evaluation.

The rest of this paper is structured as follows. Section 2 presents an overview of previous studies and theoretical discussions on DNDEA. Section 3 provides an insightful exploration of the fundamentals of neutrosophic sets. Following this, Section 4 presents the proposed method. Subsequently, in Section 5, a case study is presented to illustrate the application of the proposed method, validate the proposed model and compare the results with those of an existing approach. Finally, Section 6 summarizes the key findings and concludes the paper.

2 Literature Review

Dynamic DEA modeling has become increasingly significant in the literature, offering valuable insights into the sources of inefficiencies over time periods. Network DEA, in particular, is notable for its investigation of inefficiency sources within DMUs. However, integrating dynamic considerations into network systems presents significant challenges. While network modeling provides a theoretical framework for examining the internal structure of DMUs, dynamic modeling clarifies the connections between periods via carry-over activities. DNDEA models handle the complexity of evaluating efficiency by incorporating multiple dynamic stages connected through network structures in each period. This involves comparing a set number of static models [35], which enables us to conduct a thorough analysis. We can observe changes in overall efficiency, dynamic adjustments in divisional efficiency, potential improvements, and efficiency estimates derived from a comprehensive assessment considering interactions between periods and divisions [36]. The innovative approach of DNDEA enables us to look into the black box of the traditional approach and model the interactions among DMUs across different time periods.

The seminal study of Chen [37] represents a pioneering proposition in the research area of DNDEA. This development integrates dynamic effects into the network structure to assess the efficiency of subunits and the overall system. Afterward, scholars and researchers developed many theoretical studies, resulting in a wide range of practical applications in various sectors such as healthcare [38], banking [39], transportation [40], [41], education [42], [43], research and development [44], [45], and energy [46]. One of the most famous models in the DNDEA literature is the one proposed by Tone and Tsutsui [47]. They developed their model based on the SBM approach, which builds upon the NSBM and the dynamic SBM model developed by the same authors in 2009 and 2010, respectively [19], [48]. They assume continuity of link flows between divisions and between periods in their proposed models. Lozano [49] relaxes the intermediate product constraints of the NSBM free link model in their study. In this approach more intermediate products can be produced internally than consumed.

Kuo et al. [50] introduced a DNDEA model to evaluate the sustainability and profitability efficiency of 53 multinational mining companies from 2016 to 2019. They found that companies prioritizing corporate social responsibility are often characterized by high levels of integrity, granting them access to greater financial resources and attracting more customers, thereby achieving satisfactory profitability. Chao et al. [35] introduced a two-stage methodology to evaluate the efficiency of container shipping companies, where the stages represented operational and marketing divisions sharing expenses and personnel. An et al. [51] conducted a study concentrating on the Chinese high-tech industry, employing a framework that shared inputs and outputs across stages.

Wang et al. [52] proposed a novel DNDEA model to measure the sustainable development efficiency of Jiangxi Province’s “Internet Plus Logistics” sector from 2002 to 2016. Their study utilized a comprehensive evaluation indicator system, with a particular emphasis on waterway, highway, and railway logistics outputs.

Liu et al. [53] proposed an evaluation indicator system from a stakeholder perspective and suggested a novel combined method applying DNDEA, cross-efficiency evaluation, and Shannon entropy method to evaluate the benefits of bus transit. They conducted an empirical study on bus transit systems in 33 major Chinese cities from 2016 to 2019. Samavati et al. [54] proposed a DNDEA model capable of measuring both optimistic and pessimistic efficiency and effectiveness. Their model accommodates undesirable outputs and enables the ranking of sustainable supply chains based on optimistic and pessimistic efficiency and effectiveness. Lobo et al. [55] introduced a DNDEA model to develop an evaluation framework for assessing the efficiency of university hospitals. Their model considers the coordination among teaching, patient care, and research activities, while also monitoring changes over time.

Table 1 presents a selection of studies in the DNDEA field, showcasing diverse methodologies and their applications across sectors such as banking, logistics, and public transportation. These studies examine various approaches to efficiency measurement, addressing key factors such as innovation, cost efficiency, and performance evaluation within dynamic environments, thereby improving decision-making processes in complex systems.

Table 1. Summary of studies on dynamic network data envelopment analysis.

Study	**Methodology**	**Application**	**Key Contributions**	**Flow of carry overs**
Tone & Tsutsui [19]	Dynamic DEA	Efficiency evaluation in various industries	Introduced carryover activities in dynamic DEA	Between Consecutive Periods
Fukuyama & Weber [56]	Dynamic Network DEA	Banking sector (Japanese banks)	Assessed production efficiency and financial stability, considering both period-specific and overall efficiency	Between Consecutive Periods
Herrera-Restrepo et al. [57]	Dynamic Network DEA with Slacks-based Model	Evacuation planning and transportation strategies	Integrated multi-perspective evacuation efficiency measurement.	Between Consecutive Periods
Soltanzadeh & Omrani [58]	Fuzzy Relational Dynamic Network DEA	Airline industry	Extended efficiency evaluation by considering interrelated processes over time in fuzzy environment	Between Consecutive Periods
Gharakhani et al. [59]	Dynamic Network DEA with Goal Programming	Performance assessment of insurance companies in Iran	Proposed common weights in dynamic network DEA.	Between Consecutive Periods
Samavati et al. [54]	Dynamic Network DEA	Performance evaluation in multiple sectors	Assessed both positive and negative performance aspects	Between Consecutive Periods
Wang et al. [52]	Enhanced DNDEA	Sustainable development in 'Internet Plus Logistics'	Proposed a custom algorithm in enhanced DNDEA to assess sustainable development in the logistics sector	Between Consecutive Periods
Tatlari et al. [44]	DNDEA with Game Theory	Stock selection in capital markets	Used DNDEA with game theory to rank stock performance and provided insights into efficient stock selection strategies	Between Consecutive Periods
Kaur & Puri [60]	Relational Dynamic Network DEA	Cost efficiency measurement in Indian banking sector	Developed cost-efficient targets in dynamic DEA.	Between Consecutive Periods
Anouze et al. [61]	DNDEA	National innovation system (NIS)	Developed a framework to evaluate NIS and proposed policy	Between Consecutive Periods

		in 23 countries	interventions for underperforming nations	
Liu et al. [53]	DNDEA, Cross-efficiency, Shannon entropy	Bus transit benefits in 33 Chinese cities	Showed inefficiencies in public transit services and suggested improvements focusing on service effectiveness	Between Consecutive Periods
Wu et al. [62]	DNDEA	Global airline performance during COVID-19	Evaluated airline performance across multiple dimensions with a focus on the COVID-19 pandemic impact	Between Consecutive Periods
Rashid et al. [63]	Dynamic Network DEA with Innovation Capital	Airline efficiency incorporating innovation capital	Assessed airline efficiency incorporating innovation capital.	Between Consecutive Periods
The current study	Neutrosophic Dynamic Network DEA	Banking industry	Developed a novel dynamic network SBM model allowing discretionary allocation of carry-overs and handling neutrosophic data	Discretionary

In the banking sector, numerous studies have investigated various aspects of performance and efficiency. In the following, we highlight some notable studies.

Lahouel et al. [64] evaluated the efficiency score of 114 European banks from 2010 to 2019 employing a three-stage DNSBM model. Their study examines both the carryover characteristics of non-performing loans in the first stage of the production process and the carryover characteristics of net operating income in the final stage. Deposits are presented as intermediate products in the first stage, while loans and securities are considered intermediate products in the second stage. Wanke et al. [65] evaluated the performance score of the Indian banking sector applying a static dynamic DEA model. Similar to the study of Lahouel et al. [64], they considered both desirable (net profits) and undesirable (non-performing loans) carryover variables. Interest expenses and operating expenses were used as inputs in the single-stage DEA, while deposits, loans, and investments were considered outputs. In the study by Fukuyama & Tan [66] overall efficiency is broken down into five distinct components: innovation efficiency, primary business stability efficiency, strategic management stability efficiency, profitability efficiency, and corporate social responsibility efficiency. In another study, Fukuyama et al. [66] decomposed overall efficiency into input efficiency, output efficiency, and stability efficiency. Alongside their contributions to efficiency decomposition, their study analytically introduces market power in deposits and loans as intermediate products. Additionally, loan loss provisions are, for the first time, incorporated into the production process as a beneficial or good intermediate product Boussemart et al. [67], by considering carryover activities in the production process, proposed an approach where overall efficiency is decomposed into two sub-efficiencies, reflecting different aspects of banking operations. One aspect relates to economic performance, termed economic efficiency, and the other pertains to stability performance, termed credit risk efficiency. Zhou et al. [68] presented a comprehensive three-stage DEA model with network structure to evaluate bank efficiency in China. Their model includes the stages of capital organization, capital allocation, and profitability generation. They considered both carryover variables and shared inputs in their study to facilitate connections across different periods and production stages. Fukuyama et al. [66] analyzed efficiency in two significant ways: Firstly, they presented a novel model in DNDEA by proposing a sequential structure that incorporates the dual-role characteristics of production factors. Secondly, they improved their innovative sequential DNDEA proposal with a behavioral-causal analysis. To validate their model, they applied these two analyses to the banking industry, using a sample of

43 Chinese commercial banks between 2010 and 2018. Kweh et al. [69] proposed a DNDEA model to evaluate resource management and profitability efficiencies of 287 US commercial banks from 2010 to 2020. They assessed the efficiency score by incorporating five variables: capital adequacy, asset quality, management quality, earning ability, and liquidity, referred to as the CAMEL ratings. Their study contributes significantly to the discussion on measuring performance in the banking industry.

All studies mentioned above rely on accurate data. However, real-world data often lack clarity, and the gathered information is uncertain and indeterminate. Recognizing this challenge, many scholars in the literature of Decision-Making have begun to address uncertainty within their models by employing Zadeh's fuzzy sets [70] and developing various fuzzy approaches [71]–[76]. To handle imprecise data, Intuitionistic Fuzzy Sets (IFSs) proposed by Atanassov [77], an extension of Zadeh's fuzzy sets, have been frequently utilized to address imprecision and uncertainty, particularly in the DEA literature [78]–[81]. Al-Omeri [82] proposed a novel method for constructing a topological space by integrating two distinct fuzzy topologies. Ghareeb & Al-Omeri [83] introduced semiopenness, semicontinuity, preopenness, and related notions in (L, M)-fuzzy topological spaces using graded concepts. They explored their relationships with semi(pre)-compactness, semi(pre)-connectedness, and fuzzy separation axioms like Semi-T1 and Pre-T2. AL-Omeri et al. [84] investigated the complement of the maximum product of two neutrosophic graphs, analyzing vertex degrees. Additionally, they presented an application for locating online streaming services using normalized Hamming distance.

Smarandache [31] proposed the neutrosophic set to address the issues containing imprecise, incomplete, uncertain and indeterminate data. However, without a detailed definition, neutrosophic sets are challenging to apply in real-world scientific and technical applications. Wang et al. [85] defined Single-Valued Neutrosophic Sets (SVNSs), which are a special case of neutrosophic sets. In recent years, numerous research efforts have focused on scientific problems based on SVNSs and their extensions [84]. Al-Omeri et al. [86] presented common fixed-point theorems in neutrosophic cone metric spaces, introducing the concept of (Φ, Ψ)-weak contraction via altering distance functions. Several examples of cone metric spaces are reviewed to verify key properties. In the field of Multi-Criteria Decision-Making (MCDM), scholars have proposed various methodologies to tackle complex decision processes across different industries [87], [88]. Recent research has focused on handling uncertainty, especially through neutrosophic frameworks. Hesami [89] applied a hybrid ANP-TOPSIS method for supplier selection in Reverse Logistics, while Alharbi et al. [90] used the WASPAS method to evaluate leadership challenges in the energy sector. El-Douh et al. [91] employed a neutrosophic MCDM model to manage uncertainties in health sustainability during COVID-19, improving decision accuracy.

Edalatpanah [34] was the first to extend the DEA model using Single-Value Neutrosophic Numbers (SVNNs) in order to enhance the efficiency of private institutions. Abdelfattah [92] proposed a parametric approach to solve neutrosophic linear programming models. Kahraman et al. [93] developed the Neutrosophic Analytic Hierarchy Process (NAHP) and introduced a hybrid NAHP-NDEA algorithm for performance evaluation. Mao et al. [94] and Yang et al. [95] explored and enhanced the neutrosophic DEA model in various real-world applications; see also [96]. Rasinojehdehi & Valami [97] proposed a neutrosophic network SBM model to measure the relative efficiencies of Iranian Airlines. Their approach not only addresses the issue caused by the dual role of intermediate measures but also incorporates the inefficiency associated with intermediate measures into the objective function.

In the literature of DEA, researchers and scholars have increasingly extended DEA models with neutrosophic data. However, there are notable research gaps in this intriguing field. Neutrosophic sets, which encompass more uncertainty than other fuzzy numbers and are considered the most acceptable form of a fuzzy number, have rarely been employed in DNDEA with neutrosophic inputs, outputs, intermediate measures and carry over variables. All of the studies mentioned in this section and in Table 1 restrict carry-over variables to only two consecutive periods, limiting the flexibility in resource management across time. In the following, we propose a novel DNDEA model within the SBM framework, where carryover activities can be allocated to specific periods to enhance performance evaluation. The proposed model not only determines the optimal allocation of carryovers but also provides divisional, period, and overall efficiency within a unified framework. However, before presenting the proposed model, we review some fundamental background of Neutrosophic sets for a better understanding the idea of this paper in the next section.

3 Preliminaries

In this section, we will review some fundamental backgrounds of Neutrosophic logic required in this paper.

Definition 1 ([31]). Suppose X is a space of points with a generic element denoted by x. Then, a neutrosophic set A in X is indicated by $A = \{< x, T_A(x), I_A(x), F_A(x) | x \in X >\}$, where $T_A(x),\ I_A(x),\ F_A(x) \in]0^-, 1^+[$ signify the truth, indeterminacy and falsity-membership functions, respectively, such that $0^- \le sup\, T_A\,(x) + sup\, I_A\,(x) + sup\, F_A\,(x) \le 3^+$.

Wang et al. [85] recognizing the limitations in applying neutrosophic sets to practical problems, simplified the approach by converting nonstandard interval numbers into SVNSs of standard interval numbers.

Definition 2 ([85]). Let X be a space of points with a generic element in X, denoted by x. Then, An SVNS in A in X is denoted by $A = \{< x, T_A(x), I_A(x), F_A(x) | x \in X >\}$, where $T_A(x),\ I_A(x),\ F_A(x) \in [0,1]$, such that $0 \le sup\, T_A\,(x) + sup\, I_A\,(x) + sup\, F_A\,(x) \le 3$.

For an SVNS $\{< x, T_A(x), I_A(x), F_A(x) | x \in X >\}$, the ordered triple components $< T_A(x), I_A(x), F_A(x) >$ are described as an SVNN, and each SVNN can be characterized as $a =< T_a, I_a, F_a >$, where $T_a, I_a, F_a \in [0,1]$ and $0 \le T_a + I_a + F_a \le 1$.

Definition 3 ([98]). The Triangular Neutrosophic Number (TNNs) $\tilde{\Upsilon} =\prec (r_1, r_2, r_3), (\omega, \theta, \chi) \succ$ have the truth, indeterminacy, and falsehood membership functions of x as presented respectively in Equation (1) to Equation (3):

$$T_{\tilde{\Upsilon}}(x) = \begin{cases} \frac{(x - r_1)}{(r_2 - r_1)}\omega & r_1 \le x < r_2, \\ \omega & x = r_2, \\ \frac{(r_3 - x)}{(r_3 - r_2)}\omega & r_2 \le x < r_3, \\ 0 & otherwise. \end{cases} \tag{1}$$

$$I_{\tilde{\Upsilon}}(x) = \begin{cases} \frac{(r_2 - x)}{(r_2 - r_1)}\theta, & r_1 \le x < r_2, \\ \theta, & x = r_2, \\ \frac{(x - r_3)}{(r_3 - r_2)}\theta, & r_2 \le x < r_3, \\ 1, & otherwise. \end{cases} \tag{2}$$

$$F_{\tilde{\Upsilon}}(x) = \begin{cases} \frac{(r_2 - x)}{(r_2 - r_1)}\chi, & r_1 \le x < r_2, \\ \chi, & x = r_2, \\ \frac{(x - r_3)}{(r_3 - r_2)}\chi, & r_2 \le x < r_3, \\ 1, & otherwise. \end{cases} \tag{3}$$

Where, $0 \le T_{\tilde{\Upsilon}}(x) + I_{\tilde{\Upsilon}}(x) + F_{\tilde{\Upsilon}}(x)(x) \le 3, x \in \tilde{\Upsilon}$.

Definition 4 ([99]). Let $\tilde{\Upsilon} =\prec (r_1, r_2, r_3), (\omega, \theta, \chi) \succ$ is a TNN. Then the aggregate coefficient $\varphi_{(\alpha,\beta,\gamma)}$ is described as:

$$\varphi_{(\alpha,\beta,\gamma)} = [\tilde{\Upsilon}^L_{(\alpha,\beta,\gamma)}, \tilde{\Upsilon}^U_{(\alpha,\beta,\gamma)}] = [r_1 + (r_2 - r_3)\hbar, r_3 - (r_3 - r_2)\hbar] \tag{3}$$

Where, $\alpha \in [0, \eta], \beta \in [\lambda, 1], \gamma \in [\kappa, 1]$, and$\hbar = \frac{1}{4}(\frac{\alpha}{\omega} + 2\frac{(1-\beta)}{1-\theta} + \frac{(1-\gamma)}{1-\chi})$ is a variation degree of the TNNs.

In the upcoming section, we will introduce our innovative model and provide a detailed insight into the process of allocating carryovers.

4 The Proposed Neutrosophic Dynamic Network SBM Model

In this section, we introduce our novel neutrosophic DNDEA model within the SBM framework. As discussed previously, our proposed model allocates carryover activities to one of the subsequent periods in a manner that maximizes overall inefficiencies while the data are available in neutrosophic numbers. Equations

(4-17) illustrate the proposed model in which $\tilde{x}_{ij}^{tk}$,$\tilde{y}_{rj}^{tk}$,$\tilde{z}_{cj}^{tcarry_\alpha}$and $z_{dp}^{t(k,h)\beta}$ are neutrosophic numbers. This paper utilizes the notations for the data and variables as presented in Table 2 and Table 3, respectively:

Table 2. Data notation.

data	symbol	definition
input	x_{ip}^{tk}	Input resource i to Division k (Divk) of DMUj at time period t
output	y_{rp}^{tk}	Output product r from Divk of DMUj at time period t
link	$z_{dp}^{t(k,h)\beta}$	link d from category β between interconnected Divk and h of DMUp in period t. β stands for free, fixed, as-input, and as-output
Carry-over	$z_{ckj}^{tcarry\alpha}$	Carry-over c from category α of DMUj at Divk from period t

Table 3. Variable notation.

Variable	Symbol	Definition
Input slack	s_{ip}^{tk-}	Slack of input i of DMU_p for Divk at period t.
Output slack	s_{rp}^{tk+}	Slack of output r of DMUp for Divk at period t.
Link slack	$s_{dp}^{t(k,h)\beta}$	Slack of link d between Divk and Divh of DMUp at period t. β stands for "fixed", "free", "as-input", and" as-output".
Carry-over slack	$s_{cp}^{(t,f)k\alpha}$	Slack of carry-over c for Divk of DMUp from period t to period f. α stands for free, fixed, good and bad.
intensity	λ_j^{tk}	Intensity of DMUj corresponding to Divk at period t.
Carry-over allocator	$t_{kc\alpha}^{(t,f)}$	a binary variable that equals 1 if the carry-over c in category α from Divk in period t is assigned to the corresponding division in period f; otherwise, it equals zero.
Carry-over value control	$zz_{cj}^{tcarry\_bad}$	A free variable which controls the value of the corresponding carry over.
symbol	$h_k^{(a,t)}$	The number of all carry overs from Divk in period a to Divk in period t ($\forall a < t$).
symbol	$h_k^{(t,a)}$	The number of all carry overs from Divk in period t to Divk in period a ($\forall a$, $t < a \leq T$).

The objective function presented in the Equation (4) measures non-oriented overall efficiency, where W^t and W^k denote the weights for period t and *Divk*, respectively, determined exogenously by the DM.

Equations (5) and (6) represent input and output constraints, respectively, for *Divk* of DMUj in period t.

For linking activities, we present four scenarios, each offering distinct characteristics: the as-input value case (Equation (7)), the as-output value case (Equation (8)), the fixed-link value case (Equation (9)), and the free-link value case (Equation (10)). For a comprehensive understanding, please refer to the DNSBM model in [47].

In this study, discretionary carryover activities are defined as those that can be allocated not only to the next period but also extended beyond it. Figure 1 illustrates the carry-over activities in a multi-stage dynamic network system across several periods. Outputs from earlier periods (such as Period t) are carried over as inputs to subsequent periods (like Period t+1 and beyond), ensuring the interconnection between stages and the influence of past decisions on future performance. We categorize these activities into four distinct types, following the framework outlined by Tone and Tsutsui (2014). However, unlike the carryovers in their study, which were limited to the next period, the carryovers in our study can be allocated at the discretion of the DM, to the future periods beyond the immediate next one. The categories are presented as follows;

a. Good or Desirable Carry-over:

This category encompasses favorable carry-over such as profit transferred to the one of the subsequent periods. In our model, desirable carry-overs are considered outputs from the corresponding period, and their magnitudes are controlled to not fall below the observed values. Any shortfall in comparative carry-over within this category is considered as an inefficiency.

b. Bad or Undesirable Carry-over:

This category comprises unfavorable instances of carry-over such as losses carried forward, bad debts, and dead stock. In our model, undesirable carry-overs are treated as inputs to one of the next periods determined by the model, with their magnitudes constrained to not exceed the observed values. Any surplus in comparative carry-over within this category is considered as an inefficiency.

c. Free Carry-over:

The carry-overs in this category are managed freely by the DMU. Their value can be lower or higher than the observed value, and this variation does not directly influence the efficiency. They indirectly impact the efficiency score through the continuity assumption between periods.

It's important to highlight that in the study of Tone and Tsutsui (2014), free carryovers are also referred to as discretionary. To simplify matters in our study, we reserve the term "discretionary" for carryovers whose allocation is determined freely by the DM. Meanwhile, we simply refer to the other carryovers as free carryovers, indicating that their values can be managed without restrictions.

d. Fixed Carry-over:

The carry-overs in this category are beyond the control of the DMU and their values are fixed at the observed level. Like free carry-overs, fixed carry-overs influence the efficiency score through the continuity assumption between periods.

The carry-over constraints are formulated as presented in Equation (11) to Equation (17). Equation (11) maintains the continuity assumption between the periods t and f if there exists a connection between them. In the case $t_{kc\alpha}^{(t,f)} = 1$ where, indicating the allocation of carry-over c from category α and *Divk* in period t to period f, the replacement of $t_{kc\alpha}^{(t,f)} = 1$ results in:

$$\sum_{j=1}^{n} \lambda_j^{kf} \tilde{z}_{kcj}^{tcarry\_\alpha} = \sum_{j=1}^{n} \lambda_j^{kt} \tilde{z}_{kcj}^{tcarry\_\alpha} \tag{3}$$

In the scenario where $t_{kc\alpha}^{(t,f)} = 0$, Equation (11) becomes redundant.

Equation (12) ensures that the carry over $\tilde{z}_{kcj}^{tcarry\_\alpha}$ from period t is allocated to only one of the subsequent periods.

Equation (13) ensures that the carryovers in "fixed" category do not deviate from their observed values.

Equation (14) illustrates that the carry over $\tilde{z}_{kcj}^{tcarry\_good}$ is considered as an output from period t and its shortfall from the observed values is measured by $s_{cp}^{(t,f)kgood}$.

Equations (15) and (16) demonstrate that if the undesirable carry over $\tilde{z}_{kcj}^{tcarry\_bad}$ is allocated to period f ($t_{kc\alpha}^{(t,f)} = 1$), it is regarded as an input to that period and any excess from the observed values is quantified by $s_{cp}^{(t,f)kbad}$.

$$\tilde{\psi}_p^* = Min \frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk-}}{\tilde{x}_{ip}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{t(f,k)as-input}}{\tilde{z}_{dp}^{t(f,k)as-input}} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{s_{cp}^{(t}}{\tilde{z}_{cj}^{tc}} \right.\right.}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk+}}{\tilde{y}_{rp}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{t(k,h)as-output}}{\tilde{z}_{dp}^{t(k,h)as-output}} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{s_{c}^{(}}{\tilde{z}_{c}^{t}} \right.\right.} \tag{4}$$

$$\text{s.t. } \sum_{j=1}^{n} \lambda_j^{tk} \tilde{x}_{ij}^{tk} + s_{ip}^{tk-} = \tilde{x}_{ip}^{tk}, \ (k = 1, \dots, K), (i = 1, \dots, m_k), \tag{5}$$

$$\sum_{j=1}^{n} \lambda_j^{tk} \tilde{y}_{rj}^{tk} - s_{rp}^{tk+} = \tilde{y}_{rp}^{tk}, \ (k = 1, \dots, K), (r = 1, \dots, r_k), \tag{6}$$

$$\sum_{j=1}^{n} \lambda_j^{th} \tilde{z}_{dj}^{t(k,h)in} + s_{dp}^{t(k,h)as-input} = \tilde{z}_{dp}^{t(k,h)as-input}, \left(d \in l_{(k,h)in}\right), \forall (k,h), \quad (7)$$

$$\sum_{j=1}^{n} \lambda_j^{tk} \tilde{z}_{dj}^{t(k,h)out} - s_{dp}^{t(k,h)as-output} = \tilde{z}_{dp}^{t(k,h)as-output}, \quad \left(d \in l_{(k,h)out}\right), \forall (k,h), \quad (8)$$

$$\sum_{j=1}^{n} \lambda_j^{tk} \tilde{z}_{dj}^{t(k,h)fixed} = \tilde{z}_{dp}^{t(k,h)fixed}, \left(d \in l_{(k,h)fix}\right), \forall (k,h), \quad (9)$$

$$\sum_{j=1}^{n} \lambda_j^{tk} \tilde{z}_{dj}^{t(k,h)} = \sum_{j=1}^{n} \lambda_j^{th} \tilde{z}_{dj}^{t(k,h)}, \left(d = 1, \ldots, l_{(k,h)}\right), \forall (k,h), \quad (10)$$

$$\sum_{j=1}^{n} \lambda_j^{kf} \tilde{z}_{kcj}^{tcarry\_\alpha} - M\left(1 - t_{kc\alpha}^{(t,f)}\right) \le \sum_{j=1}^{n} \lambda_j^{kt} \tilde{z}_{cj}^{tcarry_\alpha} \le \sum_{j=1}^{n} \lambda_j^{kf} \tilde{z}_{cj}^{tcarry_\alpha} + M\left(1 - t_{kc\alpha}^{(t,f)}\right), f > t, \alpha \in \{bad, good, free, fixed\}, \quad (11)$$

$$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1, \forall f > t, (k = 1, \ldots, K), \alpha \in \{bad, good, free, fixed\}, \quad (12)$$

$$\sum_{j=1}^{n} \lambda_j^{kt} \tilde{z}_{cj}^{tcarry\_fixed} = \tilde{z}_{cp}^{tfixed}, \forall t, \forall c, \forall k, \quad (13)$$

$$\sum_{j=1}^{n} \lambda_j^{kt} \tilde{z}_{cj}^{tcarry\_good} - s_{cp}^{(t,f)kgood} = \tilde{z}_{cj}^{tcarry_{good}}, \forall t, \forall c, \forall k, \quad (14)$$

$$\sum_{j=1}^{n} \lambda_j^{kf} \tilde{z}_{cj}^{tcarry\_bad} + s_{cp}^{(t,f)kbad} = \widetilde{zz}_{cj}^{tcarry\_bad}, \forall t, \forall c, \forall k, \quad (15)$$

$$\tilde{z}_{cj}^{tcarry\_bad} - M\left(1 - t_{kcbad}^{(t,f)}\right) \le \widetilde{zz}_{cj}^{tcarry_{bad}} \le \tilde{z}_{cj}^{tcarry_{bad}} + M\left(1 - t_{kcbad}^{(t,f)}\right), \forall t, \forall c, \forall k, \forall f > t, \quad (16)$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \ge 0, s_{rp}^{k+} \ge 0, s_{ip}^{k-} \ge 0, s_{dp}^{(k,h)-} \ge 0, s_{dp}^{(k,h)+} \ge 0, s_{cp}^{(t,f)kgood} \ge 0, s_{cp}^{(t,f)kbad} \ge 0. \quad (17)$$

In the following, we propose a theorem to compare the results of the proposed model with those of the DNSBM. The aim is to demonstrate that the scores obtained by the proposed model do not exceed those of the DNSBM.

Theorem: The scores produced by the proposed model are not greater than those obtained by the DNSBM.

Proof: By substituting $f = t + 1$ in the proposed model, it simplifies to the DNSBM model. This implies that the proposed model can, under certain conditions, replicate the behavior of the DNSBM. Moreover, the feasible region of the proposed model fully contains the feasible region of the DNSBM, meaning that the proposed model has at least as many feasible solutions as the DNSBM. The objective function of the proposed model, when $f = t + 1$, can achieve values identical to those of the DNSBM objective function. Consequently, the scores obtained by the proposed model cannot exceed those produced by the DNSBM.

In the proposed model, given that all inputs, outputs, intermediate products, and carry-overs are of the neutrosophic data type, the efficiency obtained from the model is also expressed as a neutrosophic number. In the next section, we aim to transform the uncertain model into a deterministic model.

4.1 Transformation of the proposed model into deterministic model

As we mentioned earlier the efficiency obtained from the proposed model is a neutrosophic number. The neutrosophic values of the model are denoted by the convex neutrosophic numbers $\tilde{z}_{ckj}^{tcarry\alpha}$, $\tilde{x}_{ij}^{tk}$, $\tilde{y}_{rj}^{tk}$ and $\tilde{z}_{dj}^{t(k,h)\beta}$ with the following truth, indeterminacy, and falsity membership functions presented in Equation (18) to Equation (21):

$$\tilde{X}_{ij}^{tk} = \left\{ \tilde{x}_{ij}^{tk}, T_{\tilde{x}_{ij}^{tk}}, I_{\tilde{x}_{ij}^{tk}}, F_{\tilde{x}_{ij}^{tk}} | \tilde{x}_{ij}^{k} \in S(\tilde{x}_{ij}^{tk}) \right\}, \quad (18)$$

$$\tilde{Z}_{ckj}^{tcarry\alpha} = \left\{ \tilde{z}_{ckj}^{tcarry\alpha}, T_{\tilde{z}_{ckj}^{tcarry\alpha}}, I_{\tilde{z}_{ckj}^{tcarry\alpha}}, F_{\tilde{z}_{ckj}^{tcarry\alpha}} | \tilde{z}_{ckj}^{tcarry\alpha} \in S(\tilde{z}_{ckj}^{tcarry\alpha}) \right\}, \quad (19)$$

$$\tilde{Y}_{rj}^{tk} = \left\{ \tilde{y}_{rj}^{tk}, T_{\tilde{y}_{rj}^{tk}}, I_{\tilde{y}_{rj}^{tk}}, F_{\tilde{y}_{rj}^{tk}} | \tilde{y}_{rj}^{tk} \in S(\tilde{y}_{rj}^{tk}) \right\}, \quad (20)$$

$$\tilde{Z}_{dj}^{t(k,h)\beta} = \left\{\tilde{z}_{dj}^{t(k,h)\beta}, T_{\tilde{z}_{dj}^{t(k,h)\beta}}, I_{\tilde{z}_{dj}^{t(k,h)\beta}}, F_{\tilde{z}_{dj}^{t(k,h)\beta}} | \tilde{z}_{dj}^{t(k,h)\beta} \in S(\tilde{Z}_{dj}^{t(k,h)\beta})\right\}. \quad (21)$$

Where $S(.)$ denotes the support of the set. Therefore, it is adequate to determine the left shape and the right shape function of the aggregate coefficient of Definition 4. Hence, for all $i, j, r, d, (k, h)$, the upper and the lower limits for each neutrosophic variable in a φ variation degree are obtained as Equations (22) to (25):

$$(X_{ij}^{tk})_{\varphi}^{L} \le x_{ij}^{tk} \le (X_{ij}^{tk})_{\varphi}^{U}, \quad (22)$$

$$(Y_{rj}^{tk})_{\varphi}^{L} \le y_{rj}^{tk} \le (Y_{rj}^{tk})_{\varphi}^{U}, \quad (23)$$

$$(Z_{ckj}^{tcarry\alpha})_{\varphi}^{L} \le z_{ckj}^{tcarry\alpha} \le (Z_{ckj}^{tcarry\alpha})_{\varphi}^{U}, \quad (24)$$

$$(Z_{dj}^{t(k,h)\beta})_{\varphi}^{L} \le z_{dj}^{t(k,h)\beta} \le (Z_{dj}^{t(k,h)\beta})_{\varphi}^{U}. \quad (25)$$

Using φ variation degree of neutrosophic numbers, the proposed neutrosophic DNSBM model easily can be converted into an interval problem. To do this, we created two-stage mathematical programming that measures the upper efficiency of the *DMUp* as presented in Equations (26) to (39).

$$(\psi_p^*)_{\varphi}^{U} = \max_{\substack{(X_{ij}^{tk})_{\varphi}^{L} \le x_{ij}^{tk} \le (X_{ij}^{tk})_{\varphi}^{U} \\ (Y_{rj}^{tk})_{\varphi}^{L} \le y_{rj}^{tk} \le (Y_{rj}^{tk})_{\varphi}^{U} \\ (Z_{ckj}^{tcarry\alpha})_{\varphi}^{L} \le z_{ckj}^{tcarry\alpha} \le (Z_{ckj}^{tcarry\alpha})_{\varphi}^{U} \\ (Z_{dj}^{t(k,h)\beta})_{\varphi}^{L} \le z_{dj}^{t(k,h)\beta} \le (Z_{dj}^{t(k,h)\beta})_{\varphi}^{U}}} = \quad (26)$$

$$\left\{ Min \frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk-}}{x_{ip}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{t(f,k)as-input}}{z_{dp}^{t(f,k)as-input}} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{s_{cp}^{(t,f)kbad}}{z_{cj}^{tcarry_{bad}}}\right)\right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk+}}{y_{rp}^{tk}} + \sum_{d=1}^{l_{(f,k)}} \frac{s_{dp}^{t(k,h)as-output}}{z_{dp}^{t(k,h)as-output}} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{s_{cp}^{(t,a)kgood}}{z_{cj}^{tcarry\_good}}\right)\right]} \right. ,$$

s.t. $\sum_{j=1}^{n} \lambda_j^{tk} x_{ij}^{tk} + s_{ip}^{tk-} = x_{ip}^{tk}, (k = 1, \ldots, K), (i = 1, \ldots, m_k),$ (27)

$$\sum_{j=1}^{n} \lambda_j^{tk} y_{rj}^{tk} - s_{rp}^{tk+} = y_{rp}^{tk}, (k = 1, \ldots, K), (r = 1, \ldots, r_k), \quad (28)$$

$$\sum_{j=1}^{n} \lambda_j^{th} z_{dj}^{t(k,h)in} + s_{dp}^{t(k,h)as-input} = z_{dp}^{t(k,h)as-input}, \left(d \in l_{(k,h)in}\right), \forall (k,h), \quad (29)$$

$$\sum_{j=1}^{n} \lambda_j^{tk} z_{dj}^{t(k,h)out} - s_{dp}^{t(k,h)as-output} = z_{dp}^{t(k,h)as-output}, \left(d \in l_{(k,h)out}\right), \forall (k,h) \quad (30)$$

$$\sum_{j=1}^{n} \lambda_j^{tk} z_{dj}^{t(k,h)fixed} = z_{dp}^{t(k,h)fixed}, \left(d \in l_{(k,h)fix}\right), \forall (k,h), \quad (31)$$

$$\sum_{j=1}^{n} \lambda_j^{tk} z_{dj}^{t(k,h)} = \sum_{j=1}^{n} \lambda_j^{th} z_{dj}^{t(k,h)}, \left(d = 1, \ldots, l_{(k,h)}\right), \forall (k,h), \quad (32)$$

$$\sum_{j=1}^{n} \lambda_j^{kf} z_{kcj}^{tcarry\_\alpha} - M\left(1 - t_{kc\alpha}^{(t,f)}\right) \le \sum_{j=1}^{n} \lambda_j^{kt} z_{cj}^{tcarry_{\alpha}} \le \sum_{j=1}^{n} \lambda_j^{kf} z_{cj}^{tcarry_{\alpha}} + M\left(1 - t_{kc\alpha}^{(t,f)}\right), \quad (33)$$
$$\forall f > t, \alpha \in \{bad, good, free, fixed\},$$

$$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1, \forall f > t, (k = 1, \ldots, K), \alpha \in \{bad, good, free, fixed\}, \quad (34)$$

$$\sum_{j=1}^{n} \lambda_j^{kt} z_{cj}^{tcarry\_fixed} = z_{cp}^{tfixed}, \forall t, \forall c, \forall k, \quad (35)$$

$$\sum_{j=1}^{n} \lambda_j^{kt} z_{cj}^{tcarry\_good} - s_{cp}^{(t,f)kgood} = z_{cj}^{tcarry_{good}}, \forall t, \forall c, \forall k, \forall f > t, \quad (36)$$

$$\sum_{j=1}^{n} \lambda_j^{kf} z_{cj}^{tcarry\_bad} + s_{cp}^{(t,f)kbad} = zz_{cj}^{tcarry\_bad}, \forall t, \forall c, \forall k, \forall f > t, \quad (37)$$

$$z_{cj}^{tcarry\_bad} - M\left(1 - t_{kcbad}^{(t,f)}\right) \le zz_{cj}^{tcarry_{bad}} \le z_{cj}^{tcarry_{bad}} + M\left(1 - t_{kcbad}^{(t,f)}\right), \forall t, \forall c, \forall k, \forall f > t, \quad (38)$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^{k} \ge 0, s_{rp}^{k+} \ge 0, s_{ip}^{k-} \ge 0, s_{dp}^{(k,h)-} \ge 0, s_{dp}^{(k,h)+} \ge 0, s_{cp}^{(t,f)kgood} \ge 0, s_{cp}^{(t,f)kbad} \ge 0. \quad (39)$$

Similarly, we developed a two-step mathematical programming model to capture the lower efficiencies of the DMUp, as presented in Equation (40).

$$(\psi_p^*)_\varphi^L = min$$
$$(X_{ij}^{tk})_\varphi^L \le x_{ij}^{tk} \le (X_{ij}^{tk})_\varphi^U$$
$$(Y_{rj}^{tk})_\varphi^L \le y_{rj}^{tk} \le (Y_{rj}^{tk})_\varphi^U \quad =$$
$$(Z_{ckj}^{tcarry\alpha})_\varphi^L \le z_{ckj}^{tcarry\alpha} \le (Z_{ckj}^{tcarry\alpha})_\varphi^U$$
$$(Z_{dj}^{t(k,h)\beta})_\varphi^L \le z_{dj}^{t(k,h)\beta} \le (Z_{dj}^{t(k,h)\beta})_\varphi^U$$

$$\begin{cases} Min \dfrac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k}\left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk-}}{x_{ip}^{tk}} + \sum_{d=1}^{l(f,k)} \frac{s_{dp}^{t(f,k)as-input}}{z_{dp}^{t(f,k)as-input}} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{s_{cp}^{(t,f)kbad}}{z_{cj}^{tcarry_{bad}}}\right)\right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k}\left(\sum_{i=1}^{m_k} \frac{s_{ip}^{tk+}}{y_{rp}^{tk}} + \sum_{d=1}^{l(f,k)} \frac{s_{dp}^{t(k,h)as-output}}{z_{dp}^{t(k,h)as-output}} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{s_{cp}^{(t,f)kgood}}{z_{cj}^{tcarry_{good}}}\right)\right]}, \\ s.t. \\ constarints \quad (24-36) \\ s_{cp}^{(t,f)kgood} \ge 0, s_{cp}^{(t,f)kbad} \ge 0. \end{cases} \tag{40}$$

By comparing the two-stage models, it becomes evident that $(\psi_p^*)_\varphi^L \le (\psi_p^*)_\varphi^U$ .Consequently, based on this finding, the range $\left[(\psi_p^*)_\varphi^L, (\psi_p^*)_\varphi^U\right]$ is deemed as the variation degree for φ in assessing DMUp performance. We will simplify by merging the two-stage models into one stage. Following the Pareto efficiency idea, for DMUp with a specific aggregate coefficient $\varphi_{(\alpha,\beta,\gamma)} = \varphi$, maximum efficiency occurs when its inputs are minimized and outputs are maximized. Conversely, for other units, inputs should be maximized while outputs should be minimized. Likewise, for DMUp, the lowest performance will happen when inputs are at their maximum and outputs are at their minimum. Conversely, for other units, inputs should be minimized while outputs should be maximized. The model presented in Equations (41) to (54) computes the upper level for efficiency of DMUp at aggregate coefficient of φ.

$$(\psi_p^*)_\varphi^U$$
$$=$$
$$Min \frac{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 - \frac{1}{m_k + l_{(k,h)in} + nbad_k}\left(\sum_{i=1}^{m_k} \frac{(s_{ip}^{tk-})_\varphi^L}{(x_{ip}^{tk})_\varphi^L} + \sum_{d=1}^{l(f,k)} \frac{(s_{dp}^{t(f,k)as-input})_\varphi^L}{(z_{dp}^{t(k,h)as-input})_\varphi^L} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{(s_{cp}^{(t,f)kbad})_\varphi^L}{(z_{cp}^{tcarry_{bad}})_\varphi^L}\right)\right]}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k}\left(\sum_{i=1}^{m_k} \frac{(s_{rp}^{tk+})_\varphi^U}{(y_{rp}^{tk})_\varphi^U} + \sum_{d=1}^{l(f,k)} \frac{(s_{dp}^{t(k,h)as-output})_\varphi^U}{(z_{dp}^{t(k,h)as-output})_\varphi^U} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{(s_{cp}^{(t,f)kgood})_\varphi^U}{(z_{cp}^{tcarry\_good})_\varphi^U}\right)\right]} \tag{41}$$

$$\text{s.t.} \sum_{\substack{j=1 \\ j \ne p}}^{n} \lambda_j^{tk} (x_{ij}^{tk})_\varphi^U + \lambda_p^{tk} (x_{ip}^{tk})_\varphi^L + (s_{ip}^{tk-})_\varphi^L = (x_{ip}^{tk})_\varphi^L \tag{42}$$
$$(k = 1, \dots, K), (i = 1, \dots, m_k)$$

$$\sum_{\substack{j=1 \\ j \ne p}}^{n} \lambda_j^{tk} (y_{ij}^{tk})_\varphi^L + \lambda_p^{tk} (y_{ip}^{tk})_\varphi^U + (s_{rp}^{tk+})_\varphi^U = (y_{rp}^{tk})_\varphi^U \tag{43}$$
$$(k = 1, \dots, K), (r = 1, \dots, r_k)$$

$$\sum_{\substack{j=1 \\ j \ne p}}^{n} \lambda_j^{th} (z_{dj}^{t(k,h)in})_\varphi^U + \lambda_p^{th} (z_{dj}^{t(k,h)in})_\varphi^L + (s_{dp}^{t(k,h)as-input})_\varphi^L = (z_{dp}^{t(k,h)as-input})_\varphi^L \tag{44}$$
$$(d \in l_{(k,h)in}), \forall (k, h)$$

$$\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^{tk}(z_{dp}^{t(k,h)as-output})_{\varphi}^{L}+\lambda_p^{tk}(z_{dp}^{t(k,h)as-output})_{\varphi}^{U}-(s_{dp}^{t(k,h)as-output})_{\varphi}^{U}=(z_{dp}^{t(k,h)as-output})_{\varphi}^{U} \quad (45)$$

$$(d\in l_{(k,h)out}),\forall(k,h)$$

$$\sum_{j=1}^{n}\lambda_j^{tk}(z_{dj}^{t(k,h)fixed})_{\varphi}^{U}=(z_{dp}^{t(k,h)fixed})_{\varphi}^{U} \quad (46)$$

$$(d\in l_{(k,h)fix}),\forall(k,h)$$

$$\sum_{j=1}^{n}\lambda_j^{tk}(z_{dj}^{t(k,h)})_{\varphi}^{L}=\sum_{j=1}^{n}\lambda_j^{th}(z_{dj}^{t(k,h)})_{\varphi}^{U} \quad (47)$$

$$(d=1,\dots,l_{(k,h)}),\forall(k,h)$$

$$\sum_{j=1}^{n}\lambda_j^{kf}\,(z_{kcj}^{tcarry\_\alpha})_{\varphi}^{L}-M(1-t_{kc\alpha}^{(t,f)})\leq\sum_{j=1}^{n}\lambda_j^{kt}(z_{cj}^{tcarry\_\alpha})_{\varphi}^{U} \quad (48)$$

$$\leq\sum_{j=1}^{n}(\lambda_j^{kf}z_{cj}^{tcarry\_\alpha})_{\varphi}^{L}+M(1-t_{kc\alpha}^{(t,f)})$$

$$\forall f>t\ ,\ \alpha\in\{bad,good,free,fixed\}$$

$$\sum_{f=t+1}^{T}t_{kc\alpha}^{(t,f)}=1 \quad (49)$$

$$\forall f>t,\ (k=1,\dots,K),$$

$$\alpha\in\{bad,good,free,fixed\}$$

$$\sum_{j=1}^{n}\lambda_j^{kt}\,(z_{cj}^{tcarry\_fixed})_{\varphi}^{U}=(z_{cp}^{tfixed})_{\varphi}^{U} \quad (50)$$

$$\forall t,\forall c,\forall k$$

$$\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^{kt}\,(z_{cj}^{tcarry\_good})_{\varphi}^{L}+\lambda_p^{kt}(z_{pj}^{tcarry\_good})_{\varphi}^{U}-(s_{cp}^{(t,f)kgood})_{\varphi}^{U}=(z_{cp}^{tcarry\_good})_{\varphi}^{U} \quad (51)$$

$$\forall t,\forall c,\forall k,\forall f>t$$

$$\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^{kf}\,(z_{cj}^{tcarry\_bad})_{\varphi}^{U}+\lambda_p^{kf}(z_{cj}^{tcarry\_bad})_{\varphi}^{L}+(s_{cp}^{(t,f)kbad})_{\varphi}^{L}=zz_{cj}^{tcarry\_bad} \quad (52)$$

$$\forall t,\forall c,\forall k,\forall f>t$$

$$(z_{cj}^{tcarry\_bad})_{\varphi}^{L}-M\left(1-t_{kcbad}^{(t,f)}\right)\leq zz_{cj}^{tcarry_{bad}}\leq(z_{cj}^{tcarry_{bad}})_{\varphi}^{L}+M\left(1-t_{kcbad}^{(t,f)}\right) \quad (53)$$

$$\forall t,\forall c,\forall k,\forall f>t$$

$$t_{k\alpha}^{(t,f)}=\{0,1\};\widetilde{zz}_{cj}^{tcarry_{bad}}:free,\lambda_j^{k}\geq0,s_{rp}^{k+}\geq0,s_{ip}^{k-}\geq0,s_{dp}^{(k,h)-}\geq0,s_{dp}^{(k,h)+}\geq0 \quad (54)$$

The lower score can be derived from the model outlined in Equations (55) to (68).

$$(\psi_p^{*})_{\varphi}^{L}=Min\frac{\sum_{t=1}^{T}W^t\sum_{k=1}^{K}W^k\left[1-\frac{1}{m_k+l_{(k,h)in}+nbad_k}(\sum_{i=1}^{m_k}\frac{(s_{ip}^{tk-})_{\varphi}^{U}}{(x_{ip}^{tk})_{\varphi}^{U}}+\sum_{d=1}^{l_{(f,k)}}\frac{(s_{dp}^{t(f,k)as-input})_{\varphi}^{U}}{(z_{dp}^{t(k,h)as-input})_{\varphi}^{U}}+\sum_{a=1}^{t}\sum_{c}^{h}\right.}{\sum_{t=1}^{T}W^t\sum_{k=1}^{K}W^k\left[1+\frac{1}{m_k+l_{(k,h)out}+ngood_k}(\sum_{i=1}^{m_k}\frac{(s_{rp}^{tk+})_{\varphi}^{L}}{(y_{rp}^{tk})_{\varphi}^{L}}+\sum_{d=1}^{l_{(f,k)}}\frac{(s_{dp}^{t(k,h)as-output})_{\varphi}^{L}}{(z_{dp}^{t(k,h)as-output})_{\varphi}^{L}}+\sum_{a=t}^{T}\sum\right.} \quad (55)$$

$$\text{s.t.} \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{tk} (x_{ij}^{tk})_{\varphi}^{L} + \lambda_p^{tk} (x_{ip}^{tk})_{\varphi}^{U} + (s_{ip}^{tk-})_{\varphi}^{U} = (x_{ip}^{tk})_{\varphi}^{U} \quad (k=1,\dots,K), (i=1,\dots,m_k) \tag{56}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{tk} (y_{ij}^{tk})_{\varphi}^{U} + \lambda_p^{tk} (y_{ip}^{tk})_{\varphi}^{L} + (s_{rp}^{tk+})_{\varphi}^{L} = (y_{rp}^{tk})_{\varphi}^{L} \quad (k=1,\dots,K), (r=1,\dots,r_k) \tag{57}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{th} (z_{dj}^{t(k,h)in})_{\varphi}^{L} + \lambda_p^{th} (z_{dj}^{t(k,h)in})_{\varphi}^{U} + (s_{dp}^{t(k,h)as-input})_{\varphi}^{U} = (z_{dp}^{t(k,h)as-input})_{\varphi}^{U} \quad (d \in l_{(k,h)in}), \forall (k,h) \tag{58}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{tk} (z_{dp}^{t(k,h)as-output})_{\varphi}^{U} + \lambda_p^{tk} (z_{dp}^{t(k,h)as-output})_{\varphi}^{L} - (s_{dp}^{t(k,h)as-output})_{\varphi}^{L} = (z_{dp}^{t(k,h)as-output})_{\varphi}^{L} \quad (d \in l_{(k,h)out}), \forall (k,h) \tag{59}$$

$$\sum_{j=1}^{n} \lambda_j^{tk} (z_{dj}^{t(k,h)fixed})_{\varphi}^{L} = (z_{dp}^{t(k,h)fixed})_{\varphi}^{L} \quad (d \in l_{(k,h)fix}), \forall (k,h) \tag{60}$$

$$\sum_{j=1}^{n} \lambda_j^{tk} (z_{dj}^{t(k,h)})_{\varphi}^{U} = \sum_{j=1}^{n} \lambda_j^{th} (z_{dj}^{t(k,h)})_{\varphi}^{L} \quad (d=1,\dots,l_{(k,h)}), \forall (k,h) \tag{61}$$

$$\sum_{j=1}^{n} \lambda_j^{kf} (z_{kcj}^{tcarry\_\alpha})_{\varphi}^{U} - M(1 - t_{kc\alpha}^{(t,f)}) \le \sum_{j=1}^{n} \lambda_j^{kt} (z_{cj}^{tcarry\_\alpha})_{\varphi}^{L} \le \sum_{j=1}^{n} (\lambda_j^{kf} z_{cj}^{tcarry\_\alpha})_{\varphi}^{U} + M(1 - t_{kc\alpha}^{(t,f)}) \quad \forall f > t,\ \alpha \in \{bad, good, free, fixed\} \tag{62}$$

$$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1 \quad \forall f > t, (k=1,\dots,K), \alpha \in \{bad, good, free, fixed\} \tag{63}$$

$$\sum_{j=1}^{n} \lambda_j^{kt} (z_{cj}^{tcarry\_fixed})_{\varphi}^{L} = (z_{cp}^{tfixed})_{\varphi}^{L} \quad \forall t, \forall c, \forall k \tag{64}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{kt} (z_{cj}^{tcarry\_good})_{\varphi}^{U} + \lambda_p^{kt} (z_{pj}^{tcarry\_good})_{\varphi}^{L} - (s_{cp}^{(t,f)kgood})_{\varphi}^{L} = (z_{cp}^{tcarry\_good})_{\varphi}^{L} \quad \forall t, \forall c, \forall k, \forall f > t \tag{65}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^{kf} (z_{cj}^{tcarry\_bad})_{\varphi}^{L} + \lambda_p^{kf} (z_{cj}^{tcarry\_bad})_{\varphi}^{U} + (s_{cp}^{(t,f)kbad})_{\varphi}^{U} = zz_{cj}^{tcarry\_bad} \tag{66}$$

$$\forall t, \forall c, \forall k, \forall f > t$$
$$(z_{cj}^{tcarry\_bad})_{\varphi}^{U} - M\left(1 - t_{kcbad}^{(t,f)}\right) \leq zz_{cj}^{tcarry_{bad}} \leq (z_{cj}^{tcarry_{bad}})_{\varphi}^{U} + M\left(1 - t_{kcbad}^{(t,f)}\right) \tag{67}$$
$$\forall t, \forall c, \forall k, \forall f > t$$
$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \geq 0, s_{rp}^{k+} \geq 0, s_{ip}^{k-} \geq 0, s_{dp}^{(k,h)-} \geq 0, s_{dp}^{(k,h)+} \geq 0,\ s_{cp}^{(t,f)kgood} \geq 0, s_{cp}^{(t,f)kbad} \geq 0. \tag{68}$$

Due to the nonlinear nature of objective function presented in Equation (4) and the binary variables $t_{kc\alpha}^{(t,f)}$, the proposed model is a Mixed-integer nonlinear programming (MINLP) problems, which can be computationally complex. To address this, in the next subsection, we apply the Charnes-Cooper transformation, which linearizes the model, converting it into a linear programming problem. This transformation significantly reduces the computational burden, allowing for more efficient solving of the model while maintaining the accuracy of the results.

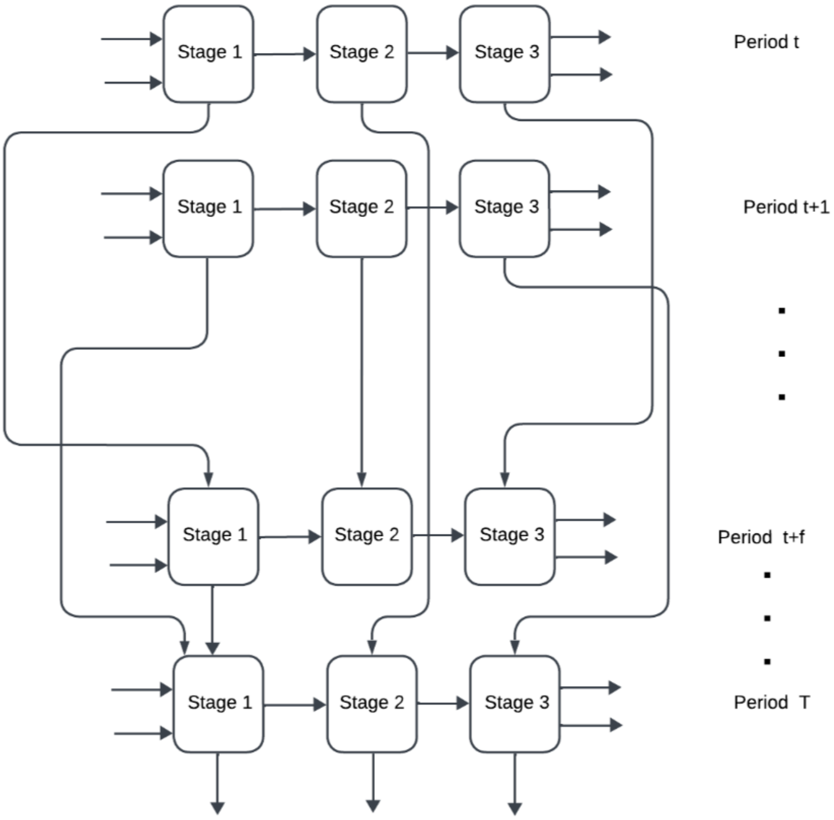


Figure 1. Allocation of carry overs.

4.2 Linearization of the proposed model

The models described by equations (41-54) and (55-68) are nonlinear programming because of its nonlinear objective function. To simplify and linearize the models, we utilize the Charnes-Cooper transformation method (Charnes et al., 1978). To enhance clarity, we'll concentrate on the linearization of the model delineated in equations (41) - (54), computing the upper level of *DMUp* efficiency at the aggregate coefficient of φ. Given its similarity, we'll omit the linearization of the lower efficiency model.

Equation (69) introduces u as a factor modifying the original expression to achieve linearity.

$$\text{u} = \frac{1}{\sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[1 + \frac{1}{m_k + l_{(k,h)out} + ngood_k} \left(\sum_{i=1}^{m_k} \frac{(s_{rp}^{tk+})_{\varphi}^{U}}{(y_{rp}^{tk})_{\varphi}^{U}} + \sum_{d=1}^{l_{(f,k)}} \frac{(s_{dp}^{t(k,h)as-output})_{\varphi}^{U}}{(z_{dp}^{t(k,h)as-output})_{\varphi}^{U}} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{(s_{cp}^{(t,f)kgood})_{\varphi}^{U}}{(z_{cp}^{tcarry\_good})_{\varphi}^{U}}\right)\right]}, \tag{69}$$
$$s_{cp}^{(t,f)kgood} \geq 0, s_{cp}^{(t,f)kbad} \geq 0.$$

Applying $u > 0$ to both the numerator and denominator of the objective function (38) keeps the variable $(\psi_p^*)_\varphi^U$ unchanged, while adjusting u effectively standardizes the denominator to 1. This leads to the reformulation of the model, resulting in Equations (70) to (84).

$$(\psi_p^*)_\varphi^U = Min \sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[u - \frac{1}{m_k+l_{(k,h)in}+nbad_k}\left(\sum_{i=1}^{m_k} \frac{u(s_{ip}^{tk-})_\varphi^L}{(x_{ip}^{tk})_\varphi^L} + \sum_{d=1}^{l_{(f,k)}} \frac{u(s_{dp}^{t(f,k)as-input})_\varphi^L}{(z_{dp}^{t(k,h)as-input})_\varphi^L} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{(s_{cp}^{(t,f)kbad})_\varphi^L}{(z_{cp}^{tcarry_{bad}})_\varphi^L}\right)\right]. \quad (70)$$

$$\text{s.t.} \sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \left[u + \frac{1}{m_k + l_{(k,h)out} + ngood_k}\left(\sum_{i=1}^{m_k} \frac{u(s_{rp}^{tk+})_\varphi^U}{(y_{rp}^{tk})_\varphi^U} + \sum_{d=1}^{l_{(f,k)}} \frac{u(s_{dp}^{t(k,h)as-output})_\varphi^U}{(z_{dp}^{t(k,h)as-output})_\varphi^U} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{u(s_{cp}^{(t,f)kgood})_\varphi^U}{(z_{cp}^{tcarry\_good})_\varphi^U}\right)\right] = 1 \quad (71)$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} u\lambda_j^{tk}(x_{ij}^{tk})_\varphi^U + u\lambda_p^{tk}(x_{ip}^{tk})_\varphi^L + u(s_{ip}^{tk-})_\varphi^L = u(x_{ip}^{tk})_\varphi^L \quad (72)$$
$$(k = 1, \dots, K), (i = 1, \dots, m_k)$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} u\lambda_j^{tk}(y_{ij}^{tk})_\varphi^L + u\lambda_p^{tk}(y_{ip}^{tk})_\varphi^U + u(s_{rp}^{tk+})_\varphi^U = u(y_{rp}^{tk})_\varphi^U \quad (73)$$
$$(k = 1, \dots, K), (r = 1, \dots, r_k)$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} u\lambda_j^{th}(z_{dj}^{t(k,h)in})_\varphi^U + u\lambda_p^{th}(z_{dj}^{t(k,h)in})_\varphi^L + u(s_{dp}^{t(k,h)as-input})_\varphi^L = u(z_{dp}^{t(k,h)as-input})_\varphi^L \quad (74)$$
$$(d \in l_{(k,h)in}), \forall(k,h)$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} u\lambda_j^{tk}(z_{dp}^{t(k,h)as-output})_\varphi^L + u\lambda_p^{tk}(z_{dp}^{t(k,h)as-output})_\varphi^U - u(s_{dp}^{t(k,h)as-output})_\varphi^U = u(z_{dp}^{t(k,h)as-output})_\varphi^U \quad (75)$$
$$(d \in l_{(k,h)out}), \forall(k,h)$$

$$\sum_{j=1}^{n} u\lambda_j^{tk}(z_{dj}^{t(k,h)fixed})_\varphi^U = u(z_{dp}^{t(k,h)fixed})_\varphi^U \quad (76)$$
$$(d \in l_{(k,h)fix}), \forall(k,h)$$

$$\sum_{j=1}^{n} u\lambda_j^{tk}(z_{dj}^{t(k,h)})_\varphi^L = \sum_{j=1}^{n} u\lambda_j^{th}(z_{dj}^{t(k,h)})_\varphi^U \quad (77)$$
$$(d = 1, \dots, l_{(k,h)}), \forall(k,h)$$

$$\sum_{j=1}^{n} u\lambda_j^{kf}\,(z_{kcj}^{tcarry\_\alpha})_{\varphi}^{L} - M(u - ut_{kc\alpha}^{(t,f)}) \le \sum_{j=1}^{n} \lambda_j^{kt} u(z_{cj}^{tcarry\_\alpha})_{\varphi}^{U}$$
$$\le \sum_{j=1}^{n} u\lambda_j^{kf}(z_{cj}^{tcarry\_\alpha})_{\varphi}^{L} + M(u - ut_{kc\alpha}^{(t,f)})$$
$$\forall f > t\ ,\ \alpha \in \{bad, good, free, fixed\} \tag{78}$$

$$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1$$
$$\forall f > t,\ (k = 1, \dots, K),$$
$$\alpha \in \{bad, good, free, fixed\} \tag{79}$$

$$\sum_{j=1}^{n} u\lambda_j^{kt}\,(z_{cj}^{tcarry\_fixed})_{\varphi}^{U} = u(z_{cp}^{tfixed})_{\varphi}^{U}$$
$$\forall t, \forall c, \forall k \tag{80}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} u\lambda_j^{kt}\,(z_{cj}^{tcarry\_good})_{\varphi}^{L} + u\lambda_p^{kt}(z_{pj}^{tcarry\_good})_{\varphi}^{U} - u(s_{cp}^{(t,f)kgood})_{\varphi}^{U} = u(z_{cp}^{tcarry\_good})_{\varphi}^{U}$$
$$\forall t, \forall c, \forall k, \forall f > t \tag{81}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} u\lambda_j^{kf}\,(z_{cj}^{tcarry\_bad})_{\varphi}^{U} + u\lambda_p^{kf}\,(z_{cj}^{tcarry\_bad})_{\varphi}^{L} + u(s_{cp}^{(t,f)kbad})_{\varphi}^{L} = u.\,zz_{cj}^{tcarry\_bad}$$
$$\forall t, \forall c, \forall k, \forall f > t \tag{82}$$

$$u(z_{cj}^{tcarry\_bad})_{\varphi}^{L} - M\left(u - ut_{kcbad}^{(t,f)}\right) \le uzz_{cj}^{tcarry_{bad}} \le u(z_{cj}^{tcarry_{bad}})_{\varphi}^{L} + M\left(u - ut_{kcbad}^{(t,f)}\right)$$
$$\forall t, \forall c, \forall k, \forall f > t \tag{83}$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^{k} \ge 0, s_{rp}^{k+} \ge 0, s_{ip}^{k-} \ge$$
$$0, s_{dp}^{(k,h)-} \ge 0, s_{dp}^{(k,h)+} \ge 0,\ s_{cp}^{(t,f)kgood} \ge 0, s_{cp}^{(t,f)kbad} \ge 0. \tag{84}$$

Given that $zz_{cj}^{tcarry\_bad}$ is an unrestricted variable and u is positive ($u > 0$), the variable $ZZ_{cj}^{tcarry\_bad}$, defined as $ZZ_{cj}^{tcarry\_bad} = u(zz_{cj}^{tcarry\_bad})$, maintains the same sign as $zz_{cj}^{tcarry\_bad}$. To transform Equation (83) into a linear form, we utilize the relationships outlined in Equations (85) to (88).

$$u.\,t_{kc\alpha}^{(t,f)} = h_{kc\alpha}^{(t,f)} \tag{85}$$
$$h_{kc\alpha}^{(t,f)} \le u \tag{86}$$
$$h_{kc\alpha}^{(t,f)} \le M.\,t_{kc\alpha}^{(t,f)} \tag{87}$$
$$h_{kc\alpha}^{(t,f)} \ge u - M(1 - t_{kC\alpha}^{(t,f)}) \tag{88}$$

Now, in the last stage of linearization of the model, we define $u(s_{ip}^{tk-})_{\varphi}^{L} = (S_{ip}^{tk-})_{\varphi}^{L}$, $u(s_{dp}^{t(f,k)as-input})_{\varphi}^{L} = (S_{dp}^{t(f,k)as-input})_{\varphi}^{L}, u(s_{rp}^{tk+})_{\varphi}^{U} = (S_{rp}^{tk+})_{\varphi}^{U}, u(s_{dp}^{t(k,h)as-output})_{\varphi}^{U} = (S_{dp}^{t(k,h)as-output})_{\varphi}^{U}$ ‹$u(s_{cp}^{(t,f)kgood})_{\varphi}^{U} = (S_{cp}^{(t,f)kgood})_{\varphi}^{U}, u(s_{cp}^{(t,f)kbad})_{\varphi}^{L} = (S_{cp}^{(t,f)kbad})_{\varphi}^{L}\ , u\lambda_j^{kf} = \Lambda_j^{kf}$.

As a result, we obtain the equivalent linear representation of the model, as described in Equation (89).

$$(\psi_p^*)_{\varphi}^{U}$$

$$= Min \sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \Bigg[u - \frac{1}{m_k + l_{(k,h)in} + nbad_k} (\sum_{i=1}^{m_k} \frac{(S_{ip}^{tk-})_\varphi^L}{(x_{ip}^{tk})_\varphi^L} + \sum_{d=1}^{l_{(f,k)}} \frac{(S_{dp}^{t(f,k)as-input})_\varphi^L}{(z_{dp}^{t(k,h)as-input})_\varphi^L} + \sum_{a=1}^{t} \sum_{c=1}^{h_k^{(a,t)}} \frac{(S_{cp}^{(t,f)kbad})_\varphi^L}{(z_{cp}^{tcarry_{bad}})_\varphi^L}) \Bigg],$$

$$\text{s.t.} \sum_{t=1}^{T} W^t \sum_{k=1}^{K} W^k \Bigg[u + \frac{1}{m_k + l_{(k,h)out} + ngood_k} (\sum_{i=1}^{m_k} \frac{(S_{rp}^{tk+})_\varphi^U}{(y_{rp}^{tk})_\varphi^U} + \sum_{d=1}^{l_{(f,k)}} \frac{(S_{dp}^{t(k,h)as-output})_\varphi^U}{(z_{dp}^{t(k,h)as-output})_\varphi^U} + \sum_{a=t}^{T} \sum_{c=1}^{h_k^{(t,a)}} \frac{(S_{cp}^{(t,f)kgood})_\varphi^U}{(z_{cp}^{tcarry\_good})_\varphi^U}) \Bigg] = 1,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^{kt} (x_{ij}^{tk})_\varphi^U + \Lambda_p^{kt} (x_{ip}^{tk})_\varphi^L + (S_{ip}^{tk-})_\varphi^L = u(x_{ip}^{tk})_\varphi^L, (k = 1, \dots, K), (i = 1, \dots, m_k),$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^{th} (y_{ij}^{tk})_\varphi^L + \Lambda_p^{kt} (y_{ip}^{tk})_\varphi^U + (S_{rp}^{tk+})_\varphi^U = u(y_{rp}^{tk})_\varphi^U, (k = 1, \dots, K), (r = 1, \dots, r_k),$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^{th} (z_{dj}^{t(k,h)in})_\varphi^U + \Lambda_p^{th} (z_{dj}^{t(k,h)in})_\varphi^L + (S_{dp}^{t(k,h)as-input})_\varphi^L = u(z_{dp}^{t(k,h)as-input})_\varphi^L, \left(d \in l_{(k,h)in}\right), \forall (k,h), \tag{89}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \Lambda_j^{tk} (z_{dp}^{t(k,h)as-output})_\varphi^L + \Lambda_p^{tk} (z_{dp}^{t(k,h)as-output})_\varphi^U - \left(S_{dp}^{t(k,h)as-output}\right)_\varphi^U = u\left(z_{dp}^{t(k,h)as-output}\right)_\varphi^U, \left(d \in l_{(k,h)out}\right), \forall (k,h),$$

$$\sum_{j=1}^{n} \Lambda_j^{tk} (z_{dj}^{t(k,h)fixed})_\varphi^U = u(z_{dp}^{t(k,h)fixed})_\varphi^U, \left(d \in l_{(k,h)fix}\right), \forall (k,h),$$

$$\sum_{j=1}^{n} \Lambda_j^{tk} (z_{dj}^{t(k,h)})_\varphi^L = \sum_{j=1}^{n} \Lambda_j^{tk} \left(z_{dj}^{t(k,h)}\right)_\varphi^U, \left(d = 1, \dots, l_{(k,h)}\right), \forall (k,h),$$

$$\sum_{j=1}^{n} \Lambda_j^{fk} (z_{kcj}^{tcarry\_\alpha})_\varphi^L - M\left(u - ut_{kc\alpha}^{(t,f)}\right) \le \sum_{j=1}^{n} \Lambda_j^{tk} \left(z_{cj}^{tcarry_\alpha}\right)_\varphi^U \le \sum_{j=1}^{n} \Lambda_j^{fk} \left(z_{cj}^{tcarry_\alpha}\right)_\varphi^L + M\left(u - ut_{kc\alpha}^{(t,f)}\right), \forall f > t\ , \alpha \in \{bad, good, free, fixed\},$$

$$\sum_{f=t+1}^{T} t_{kc\alpha}^{(t,f)} = 1\ , \forall f > t, (k = 1, \dots, K), \alpha \in \{bad, good, free, fixed\}$$

$$\sum_{j=1}^{n} \Lambda_j^{tk} \left(z_{cj}^{tcarry\_fixed}\right)_{\varphi}^{U} = u\left(z_{cp}^{tfixed}\right)_{\varphi}^{U}, \forall t, \forall c, \forall k,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^{tk} \left(z_{cj}^{tcarry\_good}\right)_{\varphi}^{L} + \Lambda_p^{tk}\left(z_{pj}^{tcarry\_good}\right)_{\varphi}^{U} - \left(S_{cp}^{(t,f)kgood}\right)_{\varphi}^{U} = u\left(z_{cp}^{tcarry\_good}\right)_{\varphi}^{U},$$

$$\forall t, \forall c, \forall k, \forall f > t,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^{fk} \left(z_{cj}^{tcarry\_bad}\right)_{\varphi}^{U} + \Lambda_p^{fk} \left(z_{cj}^{tcarry\_bad}\right)_{\varphi}^{L} + \left(S_{cp}^{(t,f)kbad}\right)_{\varphi}^{L} = u.zz_{cj}^{tcarry\_bad},$$

$$\forall t, \forall c, \forall k, \forall f > t,$$

$$u\left(z_{cj}^{tcarry\_bad}\right)_{\varphi}^{L} - M\left(u - h_{kCbad}^{(t,f)}\right) \le ZZ_{cj}^{tcarry_{bad}} \le u\left(z_{cj}^{tcarry_{bad}}\right)_{\varphi}^{L} + M\left(u - h_{kCbad}^{(t,f)}\right),$$

$$\forall t, \forall c, \forall k, \forall f > t,$$

$$h_{kc\alpha}^{(t,f)} \le u, \forall t, \forall c, \forall k, \forall f > t,$$

$$h_{kc\alpha}^{(t,f)} \le M.t_{kc\alpha}^{(t,f)}, \forall t, \forall c, \forall k, \forall f > t,$$

$$h_{kc\alpha}^{(t,f)} \ge \mathrm{u} - M\left(1 - t_{kC\alpha}^{(t,f)}\right), \forall t, \forall c, \forall k, \forall f > t,$$

$$t_{k\alpha}^{(t,f)} = \{0,1\}, \alpha \in \{bad, good, free, fixed\}; \widetilde{zz}_{cj}^{tcarry_{bad}}: free, \lambda_j^k \ge 0, s_{rp}^{k+} \ge 0, s_{ip}^{k-} \ge$$

$$0, s_{dp}^{(k,h)-} \ge 0, s_{dp}^{(k,h)+} \ge 0, s_{cp}^{(t,f)kgood} \ge 0, s_{cp}^{(t,f)kbad} \ge 0.$$

In this section, we derived linear models to measure the upper and lower efficiency scores at specific levels of truth, indeterminacy, and falsity. The mathematical contribution of the presented model lies in proposing a novel DNDEA model that incorporates discretionary carryover variables extending beyond the immediate next period, with binary variables determining their allocation. The model utilizes neutrosophic numbers to handle uncertainties in efficiency scores, transforming these into deterministic forms through a two-stage mathematical programming approach that captures both upper and lower efficiency bounds for decision-making units. The use of Pareto efficiency principles simplifies the two-stage model into a single-stage framework for performance evaluation. Additionally, the model addresses the computational complexity of Mixed-Integer Nonlinear Programming (MINLP) problems by applying the Charnes-Cooper transformation, converting the problem into a linear programming model, thus significantly improving computational efficiency while maintaining accuracy. These contributions advance resource allocation and efficiency analysis in dynamic environments.

In the next section, we will present an empirical study to validate the proposed model.

5 Empirical Study

The proposed DNDEA model has broad applications across various sectors where carryover activities extend beyond immediate periods. In the banking sector, the model can optimize loan portfolios, risk management, and capital allocation by considering how loans or financial risks carry over from one fiscal period to the next. Similarly, in supply chain management, it can address inventory levels and production capacity, ensuring efficient resource allocation over time to minimize costs. In healthcare, the model can help optimize resource distribution (staff, equipment, and patient care) by factoring in the carryover effects of ongoing treatments and future healthcare demands. The energy sector can benefit from the model in optimizing the distribution and storage of energy resources across periods, taking into account fluctuations in supply and demand. Additionally, manufacturing industries can use the model to manage work-in-progress inventories and production schedules, ensuring better efficiency and reduced waste. Finally, in public infrastructure planning, the model can strategically allocate resources across multi-year construction projects, accounting for delays, budget fluctuations, and other uncertainties. These applications highlight the versatility of the DNDEA model in optimizing resource allocation across various dynamic and uncertain environments.

In this section, we apply the proposed DNDEA model to compute the overall, divisional, and period performance of ten major Iranian private banks. The DMUs are represented by the following banks: Bank Pasargad (DMU1), Parsian Bank (DMU2), Saman Bank (DMU3), Karafarin Bank (DMU4), Tejarat Bank (DMU5), Sina Bank (DMU6), Bank Mellat (DMU7), Eghtesad Novin Bank (DMU8), Ansar Bank (DMU9), and Shahr Bank (DMU10), over the years 1396, 1397, 1398, 1399, 1400, 1401, and 1402. Firstly, based on what recorded in the literature and consulting experts, we consider the banks as a network structure with three divisions. Each division has inputs, outputs, links and carry-overs presented in Table 4. Loan loss reserves and non-performing loans are considered as carry over variables. As we discussed earlier, this carry-over can be allocated to future periods to improve the bank's overall efficiency. This carry-over is considered as a good carry over and its value is under control of the current period. The model allocates this carry-over to one of the future periods in a way that enhances the objective function. In contrast, the carry-over Non-Performing Loans (NPL) is considered as a bad link carried to the next period and its allocation is not discretionary.

The related data are collected from various sources including the bank's records unit, the statistical center of Iran[1], online resources, insights from experts, and the Codel website[2]. During data collection, inconsistencies, indeterminacy, and incompleteness were observed. Our investigation revealed that several banking reforms and other factors have led to considerable uncertainty and indeterminacy in the data. As a result, these uncertainties were identified and modeled using Neutrosophic Numbers.

Table 4. Factors of production utilized in this study.

Input	Input 1 consumed at the first stage in period t	Personnel expenses
Input	Input 2 consumed at the first stage in period t	Fixed assets
Intermediate variable	Intermediate variable which is output from the first stage in period t and input to the second stage in the same period	Total deposits
Intermediate variable	Intermediate variable which is output from the second stage in period t and input to the third stage in the same period	Gross loans
Intermediate variable	Intermediate variable which is output from the second stage in period t and input to the third stage in the same period	Total securities investment
Output	Output from the third stage in period t	Income
carry over	The generated carry over form the 3rd stage in period t and is considered as good carry over to one of the subsequent periods.	Loan loss reserves
carry over	The generated carry over form the 3rd stage in period t and is considered as bad carry over to one of the subsequent periods.	Non-Performing Loans (*NPLs*)-bad loans

5.1 Performance evaluation of bank branches

In this section, we utilize our proposed model to assess the relative efficiency of bank branches. Employing the Generalized Algebraic Modeling System (GAMS), we solve the proposed model to measure efficiency bounds for branches and allocate carryovers. The subsequent results obtained are presented in Tables 5 and 6, illustrating the lower and upper performance limits of 10 branches at φ $(0,1.0,1.0)$. Table 5 also represents the optimum allocation of carryovers during 1396-1402.

Table 5. Lower bound of efficiency at $\varphi_{(0,1.0,1.0)}$.

DMU	Overall score	Rank	Overall Efficiency in 1396	Overall Efficiency in 1397	Overall Efficiency in 1398	Overall Efficiency in 1399	Overall Efficiency in 1400	Overall Efficiency in 1401	Overall Efficiency in 1402	$\mathbf{h}_{kc\alpha}^{*(96,f)} = 1$	$\mathbf{h}_{kc\alpha}^{*(97,f)} = 1$	$\mathbf{h}_{kc\alpha}^{*(98,f)} = 1$	$\mathbf{h}_{kc\alpha}^{*(99,f)} = 1$	$\mathbf{h}_{kc\alpha}^{*(1400,f)} = 1$	$\mathbf{h}_{kc\alpha}^{*(1401,f)} = 1$
DMU1	0.868	5	0.88	1	0.96	0.78	0.75	0.69	0.82	$h_{kc}^{*(96,99)}$	$h_{kc}^{*(97,1400)}$	$h_{kc}^{*(98,1400)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU2	0.874	4	0.85	0.89	0.88	0.92	0.77	1	0.81	$h_{kc}^{*(96,97)}$	$h_{kc}^{*(97,1401)}$	$h_{kc}^{*(98,99)}$	$h_{kc}^{*(99,1402)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU3	0.738	7	0.77	0.87	0.74	0.61	0.92	0.57	0.45	$h_{kc}^{*(96,99)}$	$h_{kc}^{*(97,1400)}$	$h_{kc}^{*(98,1400)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1401)}$	$h_{kc}^{*(1401,1402)}$
DMU4	0.960	1	0.985	1	0.95	0.86	0.95	1	0.95	$h_{kc}^{*(96,1401)}$	$h_{kc}^{*(97,99)}$	$h_{kc}^{*(98,1401)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU5	0.954	2	0.85	0.87	0.95	1	1	1	1	$h_{kc}^{*(96,1400)}$	$h_{kc}^{*(97,98)}$	$h_{kc}^{*(98,1402)}$	$h_{kc}^{*(99,1400)}$	$h_{kc}^{*(1400,1401)}$	$h_{kc}^{*(1401,1402)} = 1$

[1] https://amar.org.ir
[2] www.codal.ir

DMU6	0.578	9	0.66	0.695	0.78	0.45	0.58	0.36	0.58	$h_{kc}^{*(96,1402)}$	$h_{kc}^{*(97,98)}$	$h_{kc}^{*(98,1402)}$	$h_{kc}^{*(99,1400)}$	$h_{kc}^{*(1400,1401)}$	$h_{kc}^{*(1401,1402)}$
DMU7	0.611	8	0.81	0.635	0.61	0.56	0.61	0.5	0.61	$h_{kc}^{*(96,1400)}$	$h_{kc}^{*(97,1400)}$	$h_{kc}^{*(98,1401)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1401)}$	$h_{kc}^{*(1401,1402)}$
DMU8	0.900	3	0.96	0.925	0.95	0.89	0.85	0.97	0.85	$h_{kc}^{*(96,1400)}$	$h_{kc}^{*(97,1400)}$	$h_{kc}^{*(98,99)}$	$h_{kc}^{*(99,1402)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU9	0.771	6	0.96	0.829	0.79	0.77	0.74	0.62	0.74	$h_{kc}^{*(96,99)}$	$h_{kc}^{*(97.98)}$	$h_{kc}^{*(98,1400)}$	$h_{kc}^{*(99,1401)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
DMU10	0.568	10	0.69	0.49	0.68	0.64	0.45	0.64	0.44	$h_{kc}^{*(96,97)}$	$h_{kc}^{*(97,99)}$	$h_{kc}^{*(98,1400)}$	$h_{kc}^{*(99,1402)}$	$h_{kc}^{*(1400,1402)}$	$h_{kc}^{*(1401,1402)}$
Average	0.785	_	0.84	0.82	0.82	0.77	0.76	0.75	0.72	_	_	_	_	_	_

As seen in Table 5, DMU10 has the lowest overall score, whereas DMU4 obtained the highest overall score at φ (0,1.0,1.0). Specifically, for DMU4, the allocated carryover loan loss reserve is assigned from the year 1396 to 1401, from 1397 to 1399, from 1398 to 1401, from 1399 to 1401, from 1400 to 1402, and from 1401 to 1402. Based on the results derived from the proposed model, the allocation of carryover activities, particularly evident in the case of DMU4, sheds light on significant insights. These findings indicate not only the temporal distribution of the carryover loan loss reserve but also highlight potential patterns and trends within the financial operations of the organization. Understanding these patterns can empower managers to make informed decisions regarding resource allocation, enabling them to optimize efficiency and effectiveness in managing financial risks and obligations over time. Figure 2 compares the lower and upper bounds of efficiencies at φ (0,1.0,1.0). The results show significant variations between lower and upper efficiency bounds for certain DMUs, such as DMU6 and DMU10, indicating a higher level of uncertainty or inefficiency in these units. Conversely, DMUs like DMU4 and DMU5 demonstrate minimal difference between lower and upper efficiencies, reflecting more stable and optimal performance.

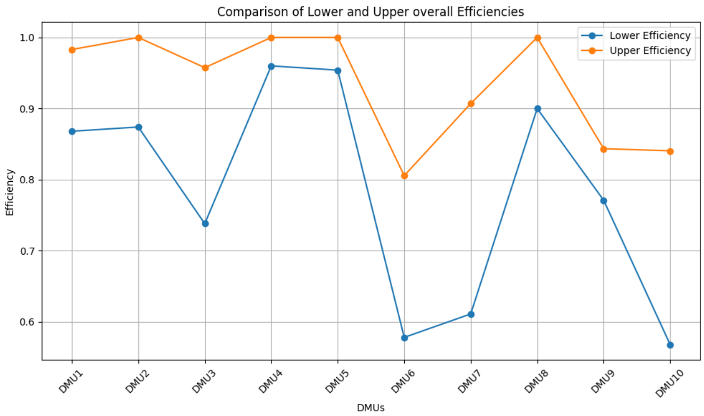


Figure 2. Comparison of the lower and upper bounds of scores at φ (0,1.0,1.0).

Table 6. Upper bound of efficiency at $\varphi_{(0,1.0,1.0)}$.

DMU	Overall Efficiency	Overall Efficiency in 1396	Overall Efficiency in 1397	Overall Efficiency in 1398	Overall Efficiency in 1399	Overall Efficiency in 1400	Overall Efficiency in 1401	Overall Efficiency in 1402
DMU1	1	1.00	1.00	1.00	0.94	0.96	1.00	1.00
DMU2	0.97	1.00	1.00	1.00	1.00	0.89	1.00	1.00
DMU3	0.89	0.96	1.00	0.95	0.85	1.00	0.76	0.65
DMU4	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU5	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU6	0.86	0.89	0.91	0.90	0.69	0.87	0.52	0.87
DMU7	0.81	1.00	0.79	0.89	0.77	0.81	0.65	0.85
DMU8	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU9	0.95	1.00	1.00	0.96	0.95	0.94	0.82	0.94
DMU10	0.84	0.94	0.68	0.68	0.81	0.65	0.84	0.65
average	0.932	0.979	0.938	0.938	0.901	0.913	0.859	0.896

Table 7 represents the overall scores of DMUs at different levels of truth, indeterminacy and falsity. According to the results DMUs with higher overall scores tend to have a narrower range between their lower and upper scores, indicating higher levels of efficiency and consistency in performance across different levels of uncertainty. For instance, DMU5 maintains an almost consistent perfect score across all levels, indicating exceptionally high efficiency.

Table 7. Scores of the DMUs at different levels of truth, indeterminacy and falsity.

DMU	Overall lower Score at $\boldsymbol{\varphi}_{(0.2,0.7,0.9)}$	Overall upper Score at $\boldsymbol{\varphi}_{(0.2,0.7,0.9)}$	Overall lower Score at $\boldsymbol{\varphi}_{(0.5,0.8,0.9)}$	Overall upper Score at $\boldsymbol{\varphi}_{(0.5,0.8,0.9)}$
DMU1	0.88	0.97	0.93	1.00
DMU2	0.89	0.97	0.93	1.00
DMU3	0.75	0.89	0.78	1.00
DMU4	1.00	1.00	1.00	1.00
DMU5	1.00	1.00	1.00	1.00
DMU6	0.68	0.82	0.78	0.8
DMU7	0.64	0.78	0.71	0.78
DMU8	0.94	1.00	0.97	1.00
DMU9	0.85	0.95	0.87	0.95
DMU10	0.64	0.79	0.64	0.74
Average	0.827	0.917	0.861	0.927

5.2 Comparison of the Results between Proposed Model and DNSBM

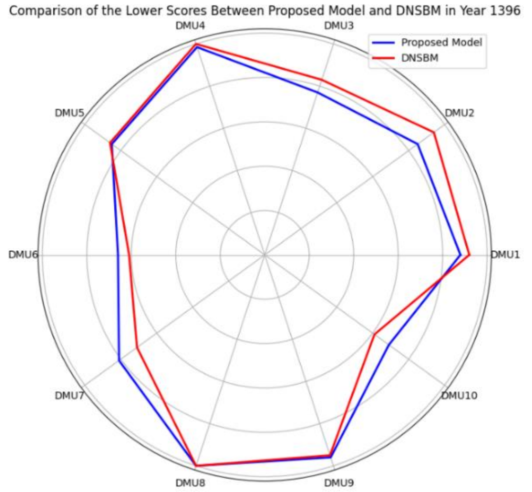
In this subsection, to validate our proposed model, we conduct a comparative analysis between the outcomes generated by our proposed model and those derived from the DNSBM model. Table 8 represents the overall and period efficiencies for the DMUs under consideration with the data used for determination of the lower score at $\varphi_{(0,1.0,1.0)}$ for the proposed model.

Table 8. The period and overall lower scores by DNSBM at $\varphi_{(0,1.0,1.0)}$.

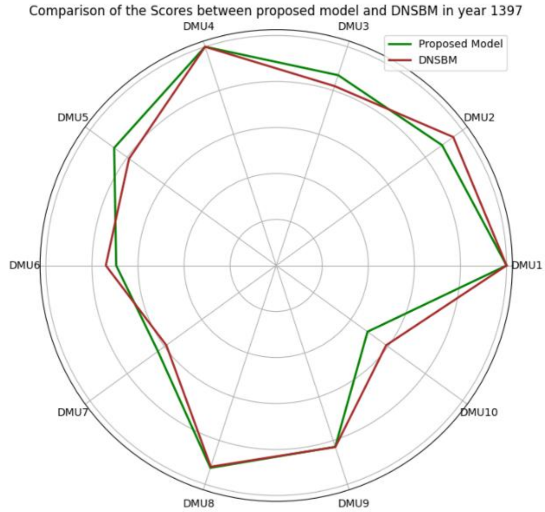
DMU	Overall score	Efficiency in 1396	Efficiency in 1397	Efficiency in 1398	Efficiency in 1399	Efficiency in 1400	Efficiency in 1401	Efficiency in 1402
DMU1	0.89	0.92	1.00	0.95	0.81	0.79	0.9	0.88
DMU2	0.94	0.94	0.95	0.92	0.95	0.88	1.00	0.91
DMU3	0.76	0.83	0.82	0.86	0.78	0.95	0.5	0.49
DMU4	0.98	1.00	1.00	1.00	0.95	0.97	1.00	0.98
DMU5	0.94	0.86	0.79	1.00	1.00	1.00	1.00	1.00
DMU6	0.59	0.61	0.74	0.69	0.56	0.46	0.38	0.59
DMU7	0.63	0.71	0.56	0.56	0.68	0.72	0.51	0.62
DMU8	0.92	1.00	0.92	0.87	1.00	0.89	0.89	0.86
DMU9	0.78	0.95	0.83	0.86	0.8	0.78	0.54	0.75
DMU10	0.57	0.61	0.59	0.51	0.72	0.46	0.65	0.44
average	0.801	0.843	0.823	0.822	0.825	0.804	0.737	0.752

According to the results, the overall scores obtained by the proposed model are lower than those of the DNSBM. The findings suggest that the proposed model exhibits greater discrimination than DNSBM. Across the years 1396-1402, DNSBM achieved 13 perfect scores, while the proposed model attained 8 perfect scores. DMU5 achieves a perfect score in both models from 1398 to 1402. In the year 1397, DMU4, and in 1399, DMU5 obtained perfect scores in both models. In the year 1396, both models recorded the lowest score for DMU6 and the highest score for DMU8. However, in the year 1397, the lowest score was achieved by DMU10 in the proposed model, while DMU7 obtained the lowest score in DNSBM. Notably, there is a sharp difference between the scores obtained by the proposed model and DNSBM during 1398 and 1399, contributing significantly to the difference in overall scores. This difference must be caused by the different assumption on allocation of carry over variables. Figure 3 compares the and period scores obtained by the proposed model and DNSBM at $\varphi_{(0,1.0,1.0)}$. The overall score of the proposed model is lower than that of the DNSBM (see

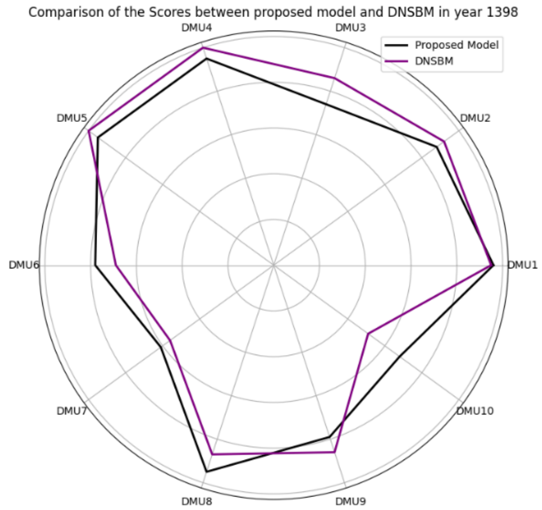
Figure 3(h)). However, for specific periods and DMUs, such as DMU6, DMU8, and DMU10 in Figure 3(c), and DMU10, DMU7, and DMU6 in Figure 3(a), the period scores of the DNSBM are lower than those of the proposed model. This difference likely arises from the distinct assumptions regarding the allocation of carryovers. It should be noted that the discretionary assumption for carryover variables in the proposed model provides flexibility, allowing the model to minimize overall efficiency rather than period efficiency. Therefore, these results are consistent with the objectives of the proposed model.



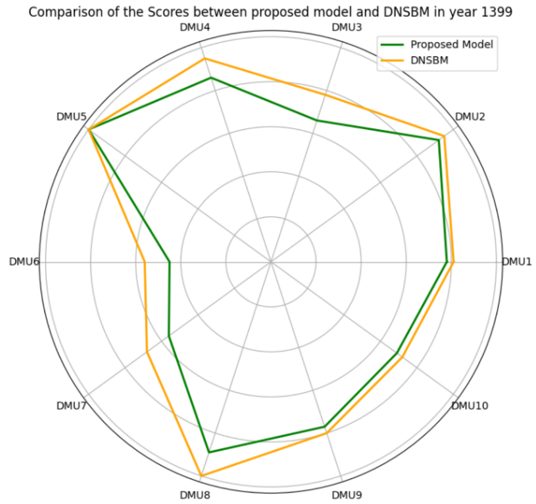
(a)



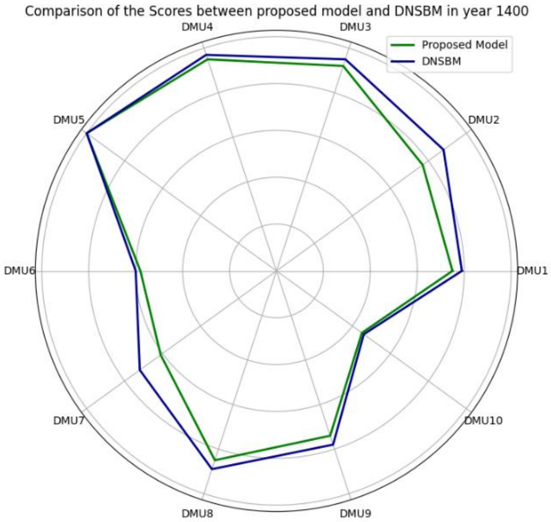
(b)



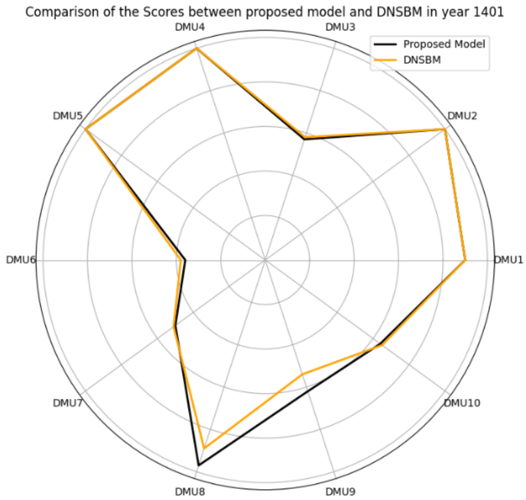
(c)



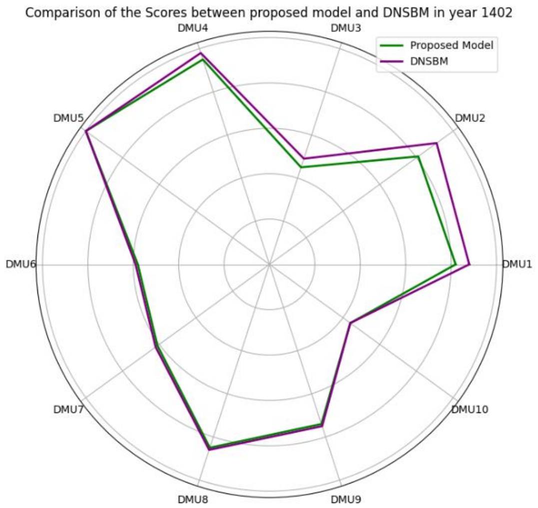
(d)



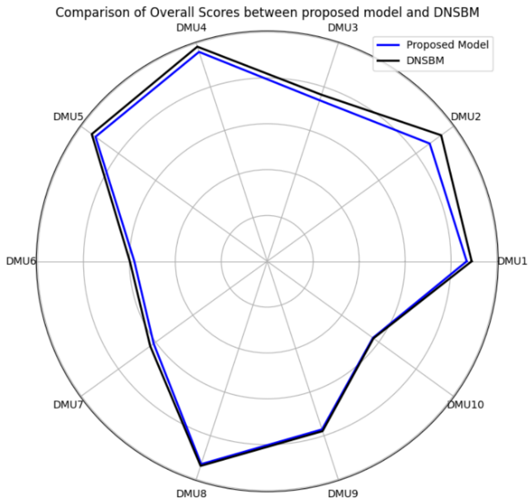
(e)



(f)



(g)



(h)

Figure 3. Comparison of the overall and period scores obtained by the proposed model and DNSBM.

Figure 4 demonstrates the trend of average period and overall scores obtained by the proposed model and DNSBM across the years 1396 to 1402 at $\varphi_{(0,1.0,1.0)}$. The trends are roughly similar, but there exist some gaps between the scores, likely caused by different assumptions on allocating carry-over variables. However, as discussed earlier, the overall scores obtained from the proposed model are lower than those of DNSBM. Notably, the lowest average score in both models is observed in 1402, while the highest average score is recorded in 1396.

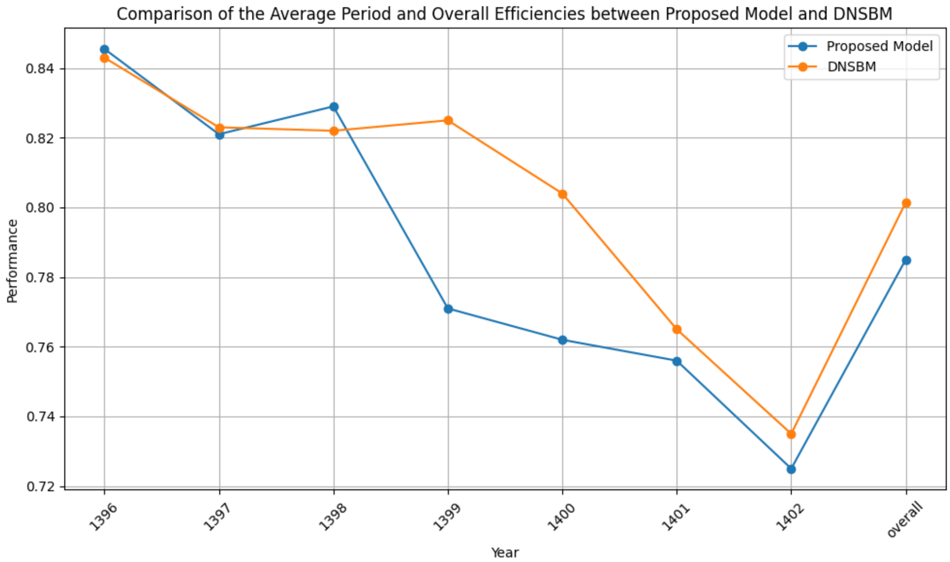


Figure 4. Comparisons of the average scores between the proposed and DNSBM models.

Table 9 provides insights into the overall lower and upper scores of the DMUs under evaluation at different levels of truth, indeterminacy, and falsity, as calculated by the DNSBM model. Similar to the results of the proposed model, DMU4 and DMU5 consistently achieve perfect scores, indicating optimal performance across all scenarios. On the other hand, DMUs like DMU6 and DMU10 exhibit significant inefficiencies, with lower scores dropping as low as 0.68 and 0.64, respectively, under specific conditions. These variations suggest that while some DMUs perform at a near-optimal level, others face challenges in maintaining efficiency, particularly when subject to uncertainty. The gap between the lower and upper scores in several DMUs, such as DMU6 and DMU10, highlights the potential for improvement in resource allocation and decision-making processes.

Table 9. Scores of the DMUs by the DNSBM model at different levels of truth, indeterminacy and falsity.

DMU	Overall lower Score at $\boldsymbol{\varphi 1} = \boldsymbol{\varphi}_{(0.2,0.7,0.9)}$	Overall upper Score at $\boldsymbol{\varphi 1} = \boldsymbol{\varphi}_{(0.2,0.7,0.9)}$	Overall lower Score at $\boldsymbol{\varphi 2} = \boldsymbol{\varphi}_{(0.5,0.8,0.9)}$	Overall upper Score at $\boldsymbol{\varphi 2} = \boldsymbol{\varphi}_{(0.5,0.8,0.9)}$
DMU1	0.92	1.00	1.00	1.00
DMU2	0.91	1.00	1.00	1.00
DMU3	0.84	0.94	0.84	1.00
DMU4	1.00	1.00	1.00	1.00
DMU5	1.00	1.00	1.00	1.00
DMU6	0.68	0.92	0.91	1.00
DMU7	0.81	1.00	0.89	1.00
DMU8	1.00	1.00	1.00	1.00
DMU9	1.00	1.00	0.87	1.00
DMU10	0.76	0.91	0.64	0.84
average	0.892	0.977	0.915	0.984

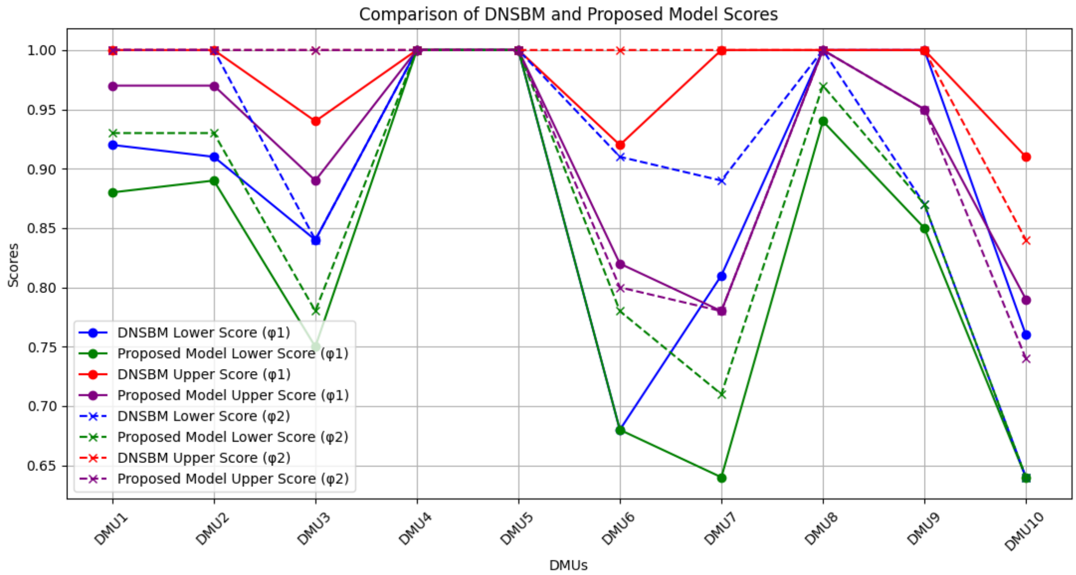


Figure 5. No caption.

Figure 5 compares the results obtained by the proposed model and DNSBM model at different variation of truth, indeterminacy and falsity. As can be seen from the figure the scores of the proposed model are not greater than those of DNSBM at the corresponding level of φ. When comparing the lower scores between the DNSBM and the proposed model across all DMUs, several differences are evident. For DMUs such as DMU1, DMU2, and DMU8, the proposed model slightly underperforms compared to DNSBM in both levels of $\boldsymbol{\varphi}$. For instance, DMU1's lower score in the proposed model is 0.88, while in DNSBM it is 0.92. This trend is consistent across DMUs where the proposed model generally yields lower scores, indicating more inefficiencies at the lower bound. Particularly, DMU3, DMU6, DMU7, and DMU10 exhibit significant differences. For example, DMU7 has a lower score of 0.64 in the proposed model compared to 0.81 in DNSBM, highlighting a pronounced inefficiency in the proposed model's performance under these conditions. Overall, lower scores tend to show that the proposed model introduces greater inefficiencies compared to DNSBM. In contrast, the comparison of upper scores shows more alignment between the two models. For many DMUs, such as DMU4, DMU5, and DMU8, both DNSBM and the proposed model achieve the highest possible score of 1.00 across both conditions, indicating optimal performance. However, in a few cases, such as DMU6 and DMU7, there are notable gaps in the upper scores. For example, DMU6's upper score in the proposed model is 0.82, compared to 0.92 in DNSBM under the first condition, and 0.80 compared to 1.00 under the second condition. While these discrepancies are not as large as those observed in the lower scores, they still suggest that the proposed model can provide lower scores in certain scenarios, particularly in terms of upper-bound efficiency under uncertainty. In summary, the results show that the scores of the proposed model exhibit more pronounced inefficiencies compared to those of the DNSBM model, while also demonstrating greater discriminatory power.

6 Conclusions

This paper has explored the dynamic allocation of carry-over variables within the context of resource management using a novel DNDEA model. Through a comprehensive review of existing literature, we have shed light on the significance of allocating discretionary carry-over activities in enhancing organizational efficiency over time periods. the proposed dynamic network DEA model not only computes the overall, divisional, and period efficiencies of DMUs with network structure but also adeptly incorporates inefficiencies regarding carry-over activities, especially in the presence of neutrosophic data. By strategically distributing carry-over variables across time periods, our proposed model optimizes overall efficiency scores. This dynamic allocation not only enhances organizations' adaptability and responsiveness to changing market conditions but also strengthens their competitive edge. To clarify the idea of the proposed approach and illustrate its capability, an empirical study for performance evaluation of 10 Iranian bank branches across years 1396 to 1402 is conducted in this paper. The results obtained from the model reveal the inefficient units in time periods and identify their sources of inefficiencies. Comparing the results obtained from the proposed model with those from DNSBM reveals that the proposed model exhibits greater discriminative power and can better identify the sources of inefficiency. Moreover, it is able to reveal the optimal allocation of resources, thereby providing valuable insights for enhancing organizational efficiency. The proposed model measures the scores lower than those of DNSBM. The feasible region of the proposed model encompasses that of DNSBM and same objective function with DNSBM.

Future research could focus on extending the application of the proposed DNDEA model to other sectors, such as manufacturing, healthcare, or education, to evaluate its generalizability across different industries. Additionally, further studies could integrate other forms of uncertainty, such as stochastic data or interval data, to enhance the model's robustness. Exploring advanced optimization techniques or algorithms may also improve the model's scalability and computational efficiency, particularly when applied to larger and more complex datasets. Finally, incorporating real-time data or multi-objective optimization approaches could provide deeper insights into dynamic decision-making and resource allocation in rapidly changing environments.

Statements & Declarations

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The authors have no relevant financial or non-financial interests to disclose.

All authors contributed to the study conception and design. Material preparation performed by Maryam Heydar and data collection and analysis were performed by Hadi Bagerzadeh valami and Maryam Heydar. The first draft of the manuscript was written by Maryam Heydar and Hadi Bagerzadeh valami commented on previous versions of the manuscript. All authors read and approved the final manuscript.

The datasets generated during the current study are available from the corresponding author on reasonable request.

Compliance with Ethical Standards

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of interest:

Maryam Heydar declares that she has no conflict of interest.

Hadi Bagherzadeh valami declares that he has no conflict of interest.\

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