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Some harmonic aggregation operators for N-valued neutrosophic trapezoidal numbers and their application to multi-criteria decision-making

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Abstract: As an extension of the both trapezoidal fuzzy numbers and neutrosophic trapezoidal numbers, the N-valued neutrosophic trapezoidal numbers, which are special neutrosophic multi-sets on subset of real numbers. Harmonic mean is a conservative average, which is widely used to aggregate central tendency data. In the existing literature, the harmonic mean is generally considered as a fusion technique of numerical data information. In this paper, we investigate a method for the situations in which the input data are expressed in neutrosophic values. Therefore, we propose two aggregations are called harmonic aggregation operators and weighted harmonic mean operators on N-valued neutrosophic trapezoidal numbers. We also proved some desired properties such as idempotency, monotonicity, commutativity and boundedness of the developed operators. Moreover, we developed an algorithm by defining a score function under N-valued neutrosophic trapezoidal numbers to compare the N-valued neutrosophic trapezoidal numbers. Finally, we gave an illustrative example, using the proposed aggregation operators to rank the alternatives with N-valued neutrosophic trapezoidal numbers.

Keywords: Neutrosophic sets; neutrosophic multi-sets; N-valued neutrosophic trapezoidal numbers; score degrees; accuracy degrees; multi-criteria decision-making.

1. Introduction

Many different theories put forward to deal with problems involving uncertainty such as are interval mathematics, probability theory, fuzzy set theory, intuitionistic fuzzy set theory, neutrosophic set theory in our daily lives. Among these theories, current and widely applied one is the fuzzy set theory, developed by Zadeh [1], which is constructed with the help of a membership function in [0,1]. Then, Atanassov [2] proposed intuitionistic fuzzy set theory by adding a non-membership membership function to the membership function such that the sum of the values of the membership function and the non-membership function for each element of the universal set X always falls within the range [0,1]. This limitation of the membership function and non-membership function creates modeling many difficulties for problems which contain uncertain informations. To overcome this situation Smarandache [3] presented a new set theory called neutrosophic set theory, which generalization of fuzzy set and intuitionistic fuzzy set theory which have independent truth membership function, indeterminacy membership function and membership on the interval [0,1]. We can find more study examples given about the theory in Wang et al. [4], Karaaslan [5], Khatter [6], Tamilarasi and Paulraj [7], Pramanik et al. [8].

Since multi-sets constructed by Yager [9] are an important theory in classical mathematics, Miyamoto [10] defined with help of membership series to use in uncertain structures. In addition, intuitionistic fuzzy multi-sets were defined by Rajarajeswari and Uma [11] to model some different problems by using the non-membership degree series. Finally, for some situations that could not model with multi-sets and intuitionistic fuzzy multi-sets, neutrosophic multi-sets were proposed by Ye and Ye [12] and Smarandache [13] based on the indeterminacy degree series in addition to the membership degree and non-membership degree series in intuitionistic fuzzy multi-sets. These theories have been applied to various fields over time, for example, Rajarajeswari and Uma [14] intuitionistic fuzzy multi similarity measure based on cotangent function, Rajarajeswari and Uma [15] correlation measure for intuitionistic fuzzy multi sets, Shinoj and John [16-18] on intuitionistic fuzzy multisets , Ejegwa and Awolola [19] some algebraic structures of intuitionistic fuzzy multisets, Ejegwa [20] on new operations on intuitionistic fuzzy multisets, Sebastian and Ramakrishnan [21-23] on multi-fuzzy sets, Syropoulos [24] on generalized fuzzy multisets, Broumi and Deli [25] correlation measure for neutrosophic refined sets, Chatterjee et al. [26] son single valued neutrosophic multisets, Deli [27] refined neutrosophic sets and refined neutrosophic soft sets, Deli et al. [28] on neutrosophic refined sets, Mondal et al. [29] on linguistic refined neutrosophic strategy, Sahin et al. [30,31] on neutrosophic multi-sets, Ye and Smarandache [32] on refined single-valued neutrosophic sets, Ulucay [33] are some of the studies.

Ulucay et al. [34] developed trapezoidal fuzzy multi-numbers on real numbers set R based on the repeated occurrences of any element trapezoidal fuzzy multi-numbers have studied many authors in Deli and Keleş [35] on N-valued fuzzy numbers, Ulucay et al. [36] worked a new hybrid distance-based similarity measure for refined neutrosophic sets, Ulucay et al. [37] MCDM based on intuitionistic trapezoidal fuzzy multi-numbers, Kesen and Deli [38] developed trapezoidal fuzzy multi aggregation operator based on archimedean norms, Deli and Kesen [39] defined bonferroni geometric mean operator of trapezoidal fuzzy multi numbers. To generalization the concept of neutrosophic multi-sets. Deli et al. [40] put forward the concept of N-valued neutrosophic trapezoidal numbers (NVNT-numbers). Then, Kesen and Deli [41], developed a method for NVNT-numbers under multi-criteria decision-making problems based on centroid point of NVNT-numbers. In decision- making process, aggregation operators are the most effective way to make inferences by obtaining a value from many values. Therefore, Harmonic aggregation operator studied different structures which contain uncertain except for NVNT-numbers in Shit et al. [42], Zhao et al. [43], Xu [44] Aydin et al. [45], Ulucay and Sahin [46] and Bakbak and Ulucay [47], Başer and Uluçay [48] developed energy of a neutrosophic soft set, Mohamed et al. [49] on neutrosophic approach for evaluating possible industry 5.0 enablers in consumer electronics, Mohamed et al.[50] developed MCDM approaches integrated with MEREC and RAM on single-valued neutrosophic, Uluçay and Şahin [51] on intuitionistic fuzzy soft expert graphs, Kané et al. [52] developed a new algorithm for fuzzy transportation problems with trapezoidal fuzzy numbers under fuzzy circumstances, Sezgin and Yavuz [53] defined a new type of operation for soft sets, Ghaforiyan et al. [54] developed a method on identifying and prioritizing antifragile tourism strategies in a neutrosophic environment, Adak et al. [55] developed some new operations on pythagorean fuzzy sets.

The originality of this paper is that it is the first study related to some harmonic aggregation operators for N-valued neutrosophic trapezoidal numbers. Because of the ability of harmonic operators in catching the average value of data and gives less importance to outliers we proposed harmonic mean operators for N-valued neutrosophic trapezoidal numbers. Hence, integrating harmonic mean operators with N-valued neutrosophic trapezoidal numbers provides many advantages to the solutions of multi-criteria decision making (MCDM) problems. For this, in this study we give harmonic aggregation operators and weighted harmonic mean operators on N-valued neutrosophic trapezoidal numbers with some desired properties such as idempotency, monotonicity, commutativity and boundedness of the developed operators. Finally, we gave an algorithm and illustrative example, using the proposed aggregation operators in multi-criteria decision-making problems.

2. Preliminaries

This section firstly introduces several the known definitions and propositions that would be helpful for better study of this paper.

Definition 2.1 [48] A t-norm is a function $t:[0,1]\times[0,1]\to[0,1]$ which satisfies the following properties:

i. $t(0,0)=0$ and $t(\mu_{X_1}(x),1)=t(1,\mu_{X_1}(x))=\mu_{X_1}(x)$, $x\in E$

ii. If $\mu_{X_1}(x)\le\mu_{X_3}(x)$ and $\mu_{X_2}(x)\le\mu_{X_4}(x)$, then

$t(\mu_{X_1}(x),\mu_{X_2}(x))\le t(\mu_{X_3}(x),\mu_{X_4}(x))$

iii. $t(\mu_{X_1}(x),\mu_{X_2}(x))=t(\mu_{X_2}(x),\mu_{X_1}(x))$

iv. $t(\mu_{X_1}(x),t(\mu_{X_2}(x),\mu_{X_3}(x)))=t(t(\mu_{X_1}(x),\mu_{X_2}(x),\mu_{X_3}(x))$

Definition 2.2 [48] An s-norm is a function $s:[0,1]\times[0,1]\to[0,1]$ which satisfies the following properties:

i. $s(1,1)=1$ and $s(\mu_{X_1}(x),0)=s(0,\mu_{X_1}(x))=\mu_{X_1}(x), x\in E$

ii. if $\mu_{X_1}(x)\le\mu_{X_3}(x)$ and $\mu_{X_2}(x)\le\mu_{X_4}(x)$, then

$s(\mu_{X_1}(x),\mu_{X_2}(x))\le s(\mu_{X_3}(x),\mu_{X_4}(x))$

iii. $s(\mu_{X_1}(x),\mu_{X_2}(x))=s(\mu_{X_2}(x),\mu_{X_1}(x))$

iv. $s(\mu_{X_1}(x),s(\mu_{X_2}(x),\mu_{X_3}(x)))=s(s(\mu_{X_1}(x),\mu_{X_2})(x),\mu_{X_3}(x))$

Definition 2.3 [4] Assume that $\mathcal{H}$ is the universe. Then, a single-valued neutrosophic set (N-set) $\mathcal{L}$ in $\mathcal{H}$ defined as

$$\mathcal{L}=\{<x,\mu_{\mathcal{L}}(x),\eta_{\mathcal{L}}(x),\vartheta_{\mathcal{L}}(x)>:x\in\mathcal{H}\} \quad (1)$$

where $\mu_{\mathcal{L}}(x),\eta_{\mathcal{L}}(x),\vartheta_{\mathcal{L}}(x)\in[0,1]$ for each point x in such that $0\le\mu_{\mathcal{L}}(x)+\eta_{\mathcal{L}}(x)+\vartheta_{\mathcal{L}}(x)\le 3$.

Definition 2.4 [13] Let $\mathcal{H}$ be a universe. $\mathcal{L}$ neutrosophic multi-set set $\mathcal{L}$ on $\mathcal{H}$ can be defined as follows:

$\mathcal{L}$={<x,($\mu_{\mathcal{L}}^1(x),\mu_{\mathcal{L}}^2(x),\dots,\mu_{\mathcal{L}}^P(x)),(\eta_{\mathcal{L}}^1(x),\eta_{\mathcal{L}}^2(x),\dots,\eta_{\mathcal{L}}^P(x)),(\vartheta_{\mathcal{L}}^1(x),\vartheta_{\mathcal{L}}^2(x),\dots,\vartheta_{\mathcal{L}}^P(x))>:x\in E\}$,

where

$$\mu_{\mathcal{L}}^1(x),\mu_{\mathcal{L}}^2(x),\dots,\mu_{\mathcal{L}}^P(x),\eta_{\mathcal{L}}^1(x),\eta_{\mathcal{L}}^2(x),\dots,\eta_{\mathcal{L}}^P(x),\vartheta_{\mathcal{L}}^1(x),\vartheta_{\mathcal{L}}^2(x),\dots,\vartheta_{\mathcal{L}}^P(x):\mathcal{H}\to[0,1]$$

such that $0\le\sup\mu_{\mathcal{L}}^i(x)+\sup\eta_{\mathcal{L}}^i(x)+\sup\vartheta_{\mathcal{L}}^i(x)\le 3$ (i=1,2,…,P) for any $x\in\mathcal{H}$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively.

Definition 2.5 [40] Let

$A_1 = \langle (a_1, b_1, c_1, d_1); (\eta^1_{A_1}, \eta^2_{A_1}, \dots, \eta^P_{A_1}), (\vartheta^1_{A_1}, \vartheta^2_{A_1}, \dots, \vartheta^P_{A_1}), (\theta^1_{A_1}, \theta^2_{A_1}, \dots, \theta^P_{A_1}) \rangle$ be a NVNT-number. If A_1 is not normalized NVTN-number $(a_1, b_1, c_1, d_1 \notin [0,1])$, the normalized NVTN-number of A_1, denoted by $\overline{A}_1$ is given by;

$$\overline{A}_1 = \langle \left[\frac{a_1}{a_1+b_1+c_1+d_1}, \frac{b_1}{a_1+b_1+c_1+d_1}, \frac{c_1}{a_1+b_1+c_1+d_1}, \frac{d_1}{a_1+b_1+c_1+d_1}\right]; \left(\eta^1_{A_1}, \eta^2_{A_1}, \dots, \eta^P_{A_1}\right), \left(\vartheta^1_{A_1}, \vartheta^2_{A_1}, \dots, \vartheta^P_{A_1}\right), (\theta^1_{A_1}, \theta^2_{A_1}, \dots, \theta^P_{A_1}) \rangle. \quad (2)$$

Definition 2.6 [40] Let

$A_1 = \langle [a_1, b_1, c_1, d_1]; (\eta^1_{A_1}, \eta^2_{A_1}, \dots, \eta^P_{A_1}), (\vartheta^1_{A_1}, \vartheta^2_{A_1}, \dots, \vartheta^P_{A_1}), (\theta^1_{A_1}, \theta^2_{A_1}, \dots, \theta^P_{A_1}) \rangle$, $A_2 = \langle [a_2, b_2, c_2, d_2]; (\eta^1_{A_2}, \eta^2_{A_2}, \dots, \eta^P_{A_2}), (\vartheta^1_{A_2}, \vartheta^2_{A_2}, \dots, \vartheta^P_{A_2}), (\theta^1_{A_2}, \theta^2_{A_2}, \dots, \theta^P_{A_2}) \rangle \in \Lambda$ and $\gamma \neq 0$ be any real number. Then,

i. $$A_1 + A_2 = \langle [a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2]; \left(s(\eta^1_{A_1}, \eta^1_{A_2}),, \dots, s(\eta^p_{A_1}, \eta^p_{A_2})\right) \left(t(\vartheta^1_{A_1}, \vartheta^1_{A_2}), \dots, t(\vartheta^p_{A_1}, \vartheta^p_{A_2})\right), (t(\theta^1_{A_1}, \theta^1_{A_2}), \dots, t(\theta^p_{A_1}, \theta^p_{A_2})) \rangle \quad (3)$$

ii. $$A_1 . A_2 = \begin{cases} \langle [a_1a_2, b_1b_2, c_1c_2, d_1d_2]; \left(t(\eta^1_{A_1}, \eta^1_{A_2}), \dots, t(\eta^p_{A_1}, \eta^p_{A_2})\right), \\ \quad \left(s(\vartheta^1_{A_1}, \vartheta^1_{A_2}), \dots, s(\vartheta^p_{A_1}, \vartheta^p_{A_2})\right), (s(\theta^1_{A_1}, \theta^1_{A_2}), \dots, s(\theta^p_{A_1}, \theta^p_{A_2}))\rangle, (d_1 > 0, d_2 > 0) \\ \langle [a_1d_2, b_1c_2, c_1b_2, d_1a_2]; \left(t(\eta^1_{A_1}, \eta^1_{A_2}), \dots, t(\eta^p_{A_1}, \eta^p_{A_2})\right), \\ \quad \left(s(\vartheta^1_{A_1}, \vartheta^1_{A_2}), \dots, s(\vartheta^p_{A_1}, \vartheta^p_{A_2})\right), (s(\theta^1_{A_1}, \theta^1_{A_2}), \dots, s(\theta^p_{A_1}, \theta^p_{A_2}))\rangle, (d_1 < 0, d_2 > 0) \\ \langle [d_1d_2, c_1c_2, b_1b_2, a_1a_2]; \left(t(\eta^1_{A_1}, \eta^1_{A_2}), \dots, t(\eta^p_{A_1}, \eta^p_{A_2})\right), \\ \quad \left(s(\vartheta^1_{A_1}, \vartheta^1_{A_2}), \dots, s(\vartheta^p_{A_1}, \vartheta^p_{A_2})\right), (s(\theta^1_{A_1}, \theta^1_{A_2}), \dots, s(\theta^p_{A_1}, \theta^p_{A_2}))\rangle, (d_1 < 0, d_2 < 0) \end{cases} \quad (4)$$

iii. $$\gamma A_1 = \langle [\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1]; (1-(1-\eta^1_{A_1})^\gamma, \dots, 1-(1-\eta^p_{A_1})^\gamma), ((\vartheta^1_{A_1})^\gamma, \dots, (\vartheta^p_{A_1})^\gamma) \rangle \ (\gamma \geq 0). \quad (5)$$

iv. $$A_1{}^\gamma = \langle [a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma]; ((\eta^1_{A_1})^\gamma, \dots, (\eta^p_{A_1})^\gamma), (1-(1-\vartheta^1_{A_1})^\gamma, \dots, 1-(1-\vartheta^p_{A_1})^\gamma) \rangle, \ (\gamma \geq 0). \quad (6)$$

v. $$A_1 \leq A_2 \Leftrightarrow a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, \eta^1_{A_1} \leq \eta^1_{A_2}, \eta^2_{A_1} \leq \eta^2_{A_2}, \dots, \eta^P_{A_1} \leq \eta^P_{A_2},$$
$$\vartheta^1_{A_1} \geq \vartheta^1_{A_2}, \vartheta^2_{A_1} \geq \vartheta^2_{A_2}, \dots, \vartheta^P_{A_1} \geq \vartheta^P_{A_2}, \theta^1_{A_1} \geq \theta^1_{A_2}, \theta^2_{A_1} \geq \theta^2_{A_2}, \dots, \theta^P_{A_1} \geq \theta^P_{A_2}. \quad (7)$$

Definition 2.7 [44] Let $x_1, x_2, x_3, \dots, x_n$ be n real numbers. Then, harmonic mean operator

$$M_{harmonic}(x_1, x_2, x_3, \dots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}} = \frac{n}{\sum_{j=1}^{n} \frac{1}{x_j}} \quad (8)$$

Definition 2.8 [44] Let $x_1, x_2, x_3, \dots, x_n$ be n real numbers. Then, weighted harmonic mean operator

$$M_{weighted\ harmonic}(x_1, x_2, x_3, \dots, x_n) = \frac{n}{\frac{w_1}{x_1} + \frac{w_2}{x_2} + \frac{w_3}{x_3} + \cdots + \frac{w_n}{x_n}} = \frac{\boldsymbol{n}}{\sum_{\boldsymbol{j=1}}^{\boldsymbol{n}} \frac{\boldsymbol{w_j}}{\boldsymbol{x_j}}} \quad (9)$$

where $w = (w_1, w_2, w_3, \dots, w_n)^T$ is a weight vector of x_j $(j = 1,2,3, \dots, n), w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1.$

3. Some weight harmonic mean operators for NVNT-numbers

Definition 3.1 Let $\mathcal{L}_r = \langle [a_r, b_r, c_r, d_r]; (\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^P_{\mathcal{L}_r}), (\eta^1_{\mathcal{L}_r}, \eta^2_{\mathcal{L}_r}, \dots, \eta^P_{\mathcal{L}_r}), (\vartheta^1_{\mathcal{L}_r}, \vartheta^2_{\mathcal{L}_r}, \dots, \vartheta^P_{\mathcal{L}_r}) \rangle$ be a collection of NVNT-numbers for $(r = 1,2,3, \dots, n)$. A mapping $f^w_{\mathrm{NVNTNWHM}}: \mathcal{L}_r{}^n \to \mathcal{L}$ is called N-valued neutrosophic trapezoidal numbers weighted harmonic mean (NVNTNWHM) operator if it satisfies:

$$\mathrm{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)=\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathcal{L}_r}} \tag{10}$$

where $w=(w_1,w_2,w_3,\dots,w_n)^T$ is the associated weight vector of $\mathcal{L}_r$ for $r=1,2,3,\dots,n$ and

$$\sum_{r=1}^{n} w_r = 1.$$

Theorem 3.2 Let $\mathcal{L}_r=\langle[\mathrm{a_r},\mathrm{b_r},\mathrm{c_r},\mathrm{d_r}];(\mu^1_{\mathcal{L}_r},\mu^2_{\mathcal{L}_r},\dots,\mu^{\mathrm{p}}_{\mathcal{L}_r}),(\eta^1_{\mathcal{L}_r},\eta^2_{\mathcal{L}_r},\dots,\eta^{\mathrm{p}}_{\mathcal{L}_r}),(\vartheta^1_{\mathcal{L}_r},\vartheta^2_{\mathcal{L}_r},\dots,\vartheta^{\mathrm{p}}_{\mathcal{L}_r})\rangle$ be a collection of NVNT-numbers for $r=1,2,3,\dots,n$, $k=1,2.3,\dots p$ and the associated weight vector of $\mathcal{L}_r$ is $w=(w_1,w_2,w_3,\dots,w_n)^T$ for $\sum_{r=1}^{n} w_r=1$ then

$$\mathrm{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)=\frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}+\dots+\frac{w_n}{\mathcal{L}_n}}$$

$$\begin{aligned}=\Bigg\langle&\left[\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{d}_r}}\right];\Bigg(\frac{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}-\prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}}\\
&\frac{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}-\prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}},\dots,\frac{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}-\prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}}\Bigg)\\
&\Bigg(\frac{2\prod_{\mathrm{r=1}}^{\mathrm{n}}(\eta^1_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(2-\eta^1_{\mathcal{L}_r})^{\mathrm{w_r}}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(\eta^1_{\mathcal{L}_r})^{\mathrm{w_r}}},\frac{2\prod_{\mathrm{r=1}}^{\mathrm{n}}(\eta^2_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(2-\eta^2_{\mathcal{L}_r})^{\mathrm{w_r}}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(\eta^2_{\mathcal{L}_r})^{\mathrm{w_r}}},\dots,\\
&\frac{2\prod_{\mathrm{r=1}}^{\mathrm{n}}(\eta^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(2-\eta^2_{\mathcal{L}_r})^{\mathrm{w_r}}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(\eta^2_{\mathcal{L}_r})^{\mathrm{w_r}}}\Bigg),\Bigg(\frac{2\prod_{\mathrm{r=1}}^{\mathrm{n}}(\vartheta^1_{\mathcal{L}_r})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(2-\vartheta^1_{\mathcal{L}_r})^{w_r}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(\vartheta^1_{\mathcal{L}_r})^{w_r}},\\
&\frac{2\prod_{\mathrm{r=1}}^{\mathrm{n}}(\vartheta^2_{\mathcal{L}_r})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(2-\vartheta^2_{\mathcal{L}_r})^{w_r}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(\vartheta^2_{\mathcal{L}_r})^{w_r}},\dots,\frac{2\prod_{\mathrm{r=1}}^{\mathrm{n}}(\vartheta^{\mathrm{p}}_{\mathcal{L}_r})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(2-\vartheta^{\mathrm{p}}_{\mathcal{L}_r})^{w_r}+\prod_{\mathrm{r=1}}^{\mathrm{n}}(\vartheta^{\mathrm{p}}_{\mathcal{L}_r})^{w_r}}\Bigg)\Bigg\rangle\end{aligned} \tag{11}$$

Proof When n=2, then $\mathrm{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2)$ is calculated as follows:

$$\mathrm{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2)=\frac{1}{\sum_{r=1}^{2}\frac{w_r}{\mathcal{L}_r}}=\frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}}$$

$$= \frac{1}{\frac{w_1}{\langle [\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1, \mathrm{d}_1]; \left(\mu^1_{\mathcal{L}_1}, \mu^2_{\mathcal{L}_1}, \dots, \mu^{\mathrm{P}}_{\mathcal{L}_1}\right), \left(\eta^1_{\mathcal{L}_1}, \eta^2_{\mathcal{L}_1}, \dots, \eta^{\mathrm{P}}_{\mathcal{L}_1}\right), \left(\vartheta^1_{\mathcal{L}_1}, \vartheta^2_{\mathcal{L}_1}, \dots, \vartheta^{\mathrm{P}}_{\mathcal{L}_1}\right)\rangle} + \frac{w_2}{\langle [\mathrm{a}_2, \mathrm{b}_2, \mathrm{c}_2, \mathrm{d}_2]; \left(\mu^1_{\mathcal{L}_2}, \mu^2_{\mathcal{L}_2}, \dots, \mu^{\mathrm{P}}_{\mathcal{L}_2}\right), \left(\eta^1_{\mathcal{L}_2}, \eta^2_{\mathcal{L}_2}, \dots, \eta^{\mathrm{P}}_{\mathcal{L}_2}\right), \left(\vartheta^1_{\mathcal{L}_2}, \vartheta^2_{\mathcal{L}_2}, \dots, \vartheta^{\mathrm{P}}_{\mathcal{L}_2}\right)\rangle}}$$

$$= \frac{1}{w_1 \frac{1}{\left\langle \left[\frac{1}{\mathrm{d}_1}, \frac{1}{\mathrm{c}_1}, \frac{1}{\mathrm{b}_1}, \frac{1}{\mathrm{a}_1}\right]; \left(\mu^1_{\mathcal{L}_1}, \mu^2_{\mathcal{L}_1}, \dots, \mu^{\mathrm{P}}_{\mathcal{L}_1}\right), \left(\eta^1_{\mathcal{L}_1}, \eta^2_{\mathcal{L}_1}, \dots, \eta^{\mathrm{P}}_{\mathcal{L}_1}\right), \left(\vartheta^1_{\mathcal{L}_1}, \vartheta^2_{\mathcal{L}_1}, \dots, \vartheta^{\mathrm{P}}_{\mathcal{L}_1}\right)\right\rangle} + w_2 \frac{1}{\left\langle \left[\frac{1}{\mathrm{d}_2}, \frac{1}{\mathrm{c}_2}, \frac{1}{\mathrm{b}_2}, \frac{1}{\mathrm{a}_2}\right]; \left(\mu^1_{\mathcal{L}_2}, \mu^2_{\mathcal{L}_2}, \dots, \mu^{\mathrm{P}}_{\mathcal{L}_2}\right), \left(\eta^1_{\mathcal{L}_2}, \eta^2_{\mathcal{L}_2}, \dots, \eta^{\mathrm{P}}_{\mathcal{L}_2}\right), \left(\vartheta^1_{\mathcal{L}_2}, \vartheta^2_{\mathcal{L}_2}, \dots, \vartheta^{\mathrm{P}}_{\mathcal{L}_2}\right)\right\rangle}}$$

$$= \frac{1}{\frac{1}{\left\langle \left[\frac{w_1}{\mathrm{d}_1}, \frac{w_1}{\mathrm{c}_1}, \frac{w_1}{\mathrm{b}_1}, \frac{w_1}{\mathrm{a}_1}\right]; \left(\frac{\left(1+\mu^1_{\mathcal{L}_1}\right)^{w_1} - \left(1-\mu^1_{\mathcal{L}_1}\right)^{w_1}}{\left(1+\mu^1_{\mathcal{L}_1}\right)^{w_1} + \left(1-\mu^1_{\mathcal{L}_1}\right)^{w_1}}, \frac{\left(1+\mu^2_{\mathcal{L}_1}\right)^{w_1} - \left(1-\mu^2_{\mathcal{L}_1}\right)^{w_1}}{\left(1+\mu^2_{\mathcal{L}_1}\right)^{w_1} + \left(1-\mu^2_{\mathcal{L}_1}\right)^{w_1}}\right), \left(\frac{2\left(\eta^1_{\mathcal{L}_1}\right)^{w_1}}{\left(2-\eta^1_{\mathcal{L}_1}\right)^{w_1} + \left(\eta^1_{\mathcal{L}_1}\right)^{w_1}}, \frac{2\left(\eta^2_{\mathcal{L}_1}\right)^{w_2}}{\left(2-\eta^2_{\mathcal{L}_1}\right)^{w_2} + \left(\eta^2_{\mathcal{L}_1}\right)^{w_2}}\right), \left(\frac{2\left(\vartheta^1_{\mathcal{L}_1}\right)^{w_1}}{\left(2-\vartheta^1_{\mathcal{L}_1}\right)^{w_1} + \left(\vartheta^1_{\mathcal{L}_1}\right)^{w_1}}, \frac{2\left(\vartheta^2_{\mathcal{L}_1}\right)^{w_2}}{\left(2-\vartheta^2_{\mathcal{L}_1}\right)^{w_2} + \left(\vartheta^2_{\mathcal{L}_1}\right)^{w_2}}\right)\right\rangle} + \frac{1}{\left\langle \left[\frac{w_2}{\mathrm{d}_2}, \frac{w_2}{\mathrm{c}_2}, \frac{w_2}{\mathrm{b}_2}, \frac{w_2}{\mathrm{a}_2}\right]; \left(\frac{\left(1+\mu^1_{\mathcal{L}_1}\right)^{w_1} - \left(1-\mu^1_{\mathcal{L}_1}\right)^{w_1}}{\left(1+\mu^1_{\mathcal{L}_1}\right)^{w_1} + \left(1-\mu^1_{\mathcal{L}_1}\right)^{w_1}}, \frac{\left(1+\mu^2_{\mathcal{L}_1}\right)^{w_1} - \left(1-\mu^2_{\mathcal{L}_1}\right)^{w_1}}{\left(1+\mu^2_{\mathcal{L}_1}\right)^{w_1} + \left(1-\mu^2_{\mathcal{L}_1}\right)^{w_1}}\right), \left(\frac{2\left(\eta^1_{\mathcal{L}_1}\right)^{w_1}}{\left(2-\eta^1_{\mathcal{L}_1}\right)^{w_1} + \left(\eta^1_{\mathcal{L}_1}\right)^{w_1}}, \frac{2\left(\eta^2_{\mathcal{L}_1}\right)^{w_2}}{\left(2-\eta^2_{\mathcal{L}_1}\right)^{w_2} + \left(\eta^2_{\mathcal{L}_1}\right)^{w_2}}\right), \left(\frac{2\left(\vartheta^1_{\mathcal{L}_1}\right)^{w_1}}{\left(2-\vartheta^1_{\mathcal{L}_1}\right)^{w_1} + \left(\vartheta^1_{\mathcal{L}_1}\right)^{w_1}}, \frac{2\left(\vartheta^2_{\mathcal{L}_1}\right)^{w_2}}{\left(2-\vartheta^2_{\mathcal{L}_1}\right)^{w_2} + \left(\vartheta^2_{\mathcal{L}_1}\right)^{w_2}}\right)\right\rangle}}$$

$$= 1 \Big/ \left\langle \left[\sum_{r=1}^{2} \frac{w_r}{\mathrm{d}_r}, \sum_{r=1}^{2} \frac{w_r}{\mathrm{c}_r}, \sum_{r=1}^{2} \frac{w_r}{\mathrm{b}_r}, \sum_{r=1}^{2} \frac{w_r}{\mathrm{a}_r}\right]; \left(\frac{\prod_{\mathrm{r}=1}^{2}(1+\mu^1_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r}=1}^{2}(1-\mu^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{2}(1+\mu^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}} + \prod_{\mathrm{r}=1}^{2}(1-\mu^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}, \frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu^2_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}} + \prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}\right), \left(\frac{2\prod_{\mathrm{r}=1}^{2}(\eta^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{2}(2-\eta^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}} + \prod_{\mathrm{r}=1}^{2}(\eta^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}, \frac{2\prod_{\mathrm{r}=1}^{2}(\eta^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{2}(2-\eta^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}} + \prod_{\mathrm{r}=1}^{2}(\eta^2_{\mathcal{L}_r})^{\mathrm{w}_r}}\right), \left(\frac{2\prod_{\mathrm{r}=1}^{2}(\vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{2}(2-\vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r} + \prod_{\mathrm{r}=1}^{2}(\vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r}}, \frac{2\prod_{\mathrm{r}=1}^{2}(\vartheta^2_{\mathcal{L}_r})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{2}(2-\vartheta^2_{\mathcal{L}_r})^{\mathrm{w}_r} + \prod_{\mathrm{r}=1}^{2}(\vartheta^2_{\mathcal{L}_r})^{\mathrm{w}_r}}\right)\right\rangle$$

$$= 1 \Big/ \left\langle \left[\frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathrm{d}_r}}\right]; \left(\frac{\prod_{\mathrm{r}=1}^{2}(1+\mu^1_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r}=1}^{2}(1-\mu^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{2}(1+\mu^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}} + \prod_{\mathrm{r}=1}^{2}(1-\mu^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}, \frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu^2_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}} + \prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}\right), \left(\frac{2\prod_{\mathrm{r}=1}^{2}(\eta^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{2}(2-\eta^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}} + \prod_{\mathrm{r}=1}^{2}(\eta^1_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}, \frac{2\prod_{\mathrm{r}=1}^{2}(\eta^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}}}{\prod_{\mathrm{r}=1}^{2}(2-\eta^2_{\mathcal{L}_r})^{\mathrm{w}_\mathrm{r}} + \prod_{\mathrm{r}=1}^{2}(\eta^2_{\mathcal{L}_r})^{\mathrm{w}_r}}\right), \left(\frac{2\prod_{\mathrm{r}=1}^{2}(\vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{2}(2-\vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r} + \prod_{\mathrm{r}=1}^{2}(\vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r}}, \frac{2\prod_{\mathrm{r}=1}^{2}(\vartheta^2_{\mathcal{L}_r})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{2}(2-\vartheta^2_{\mathcal{L}_r})^{\mathrm{w}_r} + \prod_{\mathrm{r}=1}^{2}(\vartheta^2_{\mathcal{L}_r})^{\mathrm{w}_r}}\right)\right\rangle$$

Suppose that Equation 12 holds for $n = k$, i.e.,

$$\mathrm{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_k) = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2} + \cdots + \frac{w_k}{\mathcal{L}_k}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{d}_r}}\right]; \left(\frac{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{1})^{w_r} - \prod_{\mathrm{r=1}}^{k}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}\right.\right.$$

$$\left.\frac{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{2})^{w_r} - \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r} - \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}\right)$$

$$\left(\frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}, \frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \dots,\right.$$

$$\left.\frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}\right), \left(\frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}},\right.$$

$$\left.\left.\frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}, \dots, \frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}\right)\right\rangle$$

For $n = k + 1$, using above expression and operational laws, we have

$$\mathrm{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_k, \mathcal{L}_{k+1}) =$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{d}_r}}\right]; \left(\frac{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{1})^{w_r} - \prod_{\mathrm{r=1}}^{k}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}\right.\right.$$

$$\left.\frac{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{2})^{w_r} - \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r} - \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}\right)$$

$$\left(\frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}, \frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \dots,\right.$$

$$\left.\frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\eta_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}\right), \left(\frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}},\right.$$

$$\left.\left.\frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}, \dots, \frac{2\prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}{\prod_{\mathrm{r=1}}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r} + \prod_{\mathrm{r=1}}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}\right)\right\rangle +$$

$$= \left\langle \left[\frac{1}{\frac{w_{k+1}}{\mathrm{a}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{b}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{c}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{d}_{k+1}}}\right]; \left(\frac{(1+\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w}_{k+1}} - (1-\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w}_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w}_{k+1}} + (1-\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w}_{k+1}}}\right.\right.$$

$$\left.\frac{(1+\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w}_{k+1}} - (1-\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w}_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w}_{k+1}} + (1-\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w}_{k+1}}}, \dots, \frac{(1+\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w}_{k+1}} - (1-\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w}_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w}_{k+1}} + (1-\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w}_{k+1}}}\right)$$

$$\left(\frac{2(\eta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}}{(2-\eta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}} + (\eta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}}, \frac{2(\eta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}{(2-\eta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}} + (\eta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}, \dots,\right.$$

$$\left.\frac{2(\eta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}}{(2-\eta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}} + (\eta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}}\right), \left(\frac{2(\vartheta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}} + (\vartheta_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w_{k+1}}}},\right.$$

$$\left.\left.\frac{2(\vartheta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}} + (\vartheta_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w_{k+1}}}}, \dots, \frac{2(\vartheta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}} + (\vartheta_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w_{k+1}}}}\right)\right\rangle$$

$$=\left\langle\left[\frac{1}{\sum_{r=1}^{k+1}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{k+1}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{k+1}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{k+1}\frac{w_r}{\mathrm{d}_r}}\right];\left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1+\mu_{\mathcal{L}_r}^1)^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{k+1}(1-\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1+\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1-\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}\right.\right.$$
$$\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1+\mu_{\mathcal{L}_r}^2)^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1-\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1+\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1-\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}},\ldots,\left.\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}\right)$$
$$\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(2-\eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}},\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(2-\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}},\ldots,\right.$$
$$\left.\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\eta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(2-\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}\right),\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\vartheta_{\mathcal{L}_r}^1)^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(2-\vartheta_{\mathcal{L}_r}^1)^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\vartheta_{\mathcal{L}_r}^1)^{\mathrm{w}_r}},\right.$$
$$\left.\left.\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\vartheta_{\mathcal{L}_r}^2)^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(2-\vartheta_{\mathcal{L}_r}^2)^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\vartheta_{\mathcal{L}_r}^2)^{\mathrm{w}_r}},\ldots,\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{k}+1}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}}\right)\right\rangle$$

So, the proof is complete.

Theorem 3.3 (Idempotency)

Let $\mathcal{L}_r=\langle[\mathrm{a_r},\mathrm{b_r},\mathrm{c_r},\mathrm{d_r}];(\mu_{\mathcal{L}_r}^1,\mu_{\mathcal{L}_r}^2,\ldots,\mu_{\mathcal{L}_r}^{\mathrm{p}}),(\eta_{\mathcal{L}_r}^1,\eta_{\mathcal{L}_r}^2,\ldots,\eta_{\mathcal{L}_r}^{\mathrm{p}}),(\vartheta_{\mathcal{L}_r}^1,\vartheta_{\mathcal{L}_r}^2,\ldots,\vartheta_{\mathcal{L}_r}^{\mathrm{p}})\rangle$ be a collection of NVNT-numbers for $r=1,2,3,\ldots,n$. If $\mathcal{L}_n=\mathcal{L}$ for all r that is all are identical then,

$$\mathrm{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\ldots,\mathcal{L}_n)=\mathcal{L}. \tag{12}$$

Proof We know that

$$\mathrm{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\ldots,\mathcal{L}_n)=\frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}+\cdots+\frac{w_n}{\mathcal{L}_n}}$$
$$=\left\langle\left[\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{d}_r}}\right];\left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^1)^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}\right.\right.$$
$$\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^2)^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}},\ldots,\left.\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}\right)$$
$$\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^1)^{\mathrm{w_r}}},\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}},\ldots,\right.$$
$$\left.\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\eta_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}\right),\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^1)^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^1)^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^1)^{\mathrm{w}_r}},\right.$$
$$\left.\left.\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^2)^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^2)^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^2)^{\mathrm{w}_r}},\ldots,\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}}\right)\right\rangle$$
$$=\left\langle\left[\frac{1}{\frac{\sum_{r=1}^{n}w_r}{\mathrm{a}}},\frac{1}{\frac{\sum_{r=1}^{n}w_r}{\mathrm{b}}},\frac{1}{\frac{\sum_{r=1}^{n}w_r}{\mathrm{c}}},\frac{1}{\frac{\sum_{r=1}^{n}w_r}{\mathrm{d}}}\right];\left(\frac{(1+\mu_{\mathcal{L}}^1)^{\sum_{r=1}^{n}w_r}-(1-\mu_{\mathcal{L}}^1)^{\sum_{r=1}^{n}w_r}}{(1+\mu_{\mathcal{L}}^1)^{\sum_{r=1}^{n}w_r}+(1-\mu_{\mathcal{L}}^1)^{\sum_{r=1}^{n}w_r}}\right.\right.$$
$$\frac{(1+\mu_{\mathcal{L}}^2)^{\sum_{r=1}^{n}w_r}-(1-\mu_{\mathcal{L}}^2)^{\sum_{r=1}^{n}w_r}}{(1+\mu_{\mathcal{L}}^2)^{\sum_{r=1}^{n}w_r}+(1-\mu_{\mathcal{L}}^2)^{\sum_{r=1}^{n}w_r}},\ldots,\left.\frac{(1+\mu_{\mathcal{L}}^p)^{\sum_{r=1}^{n}w_r}-(1-\mu_{\mathcal{L}}^p)^{\sum_{r=1}^{n}w_r}}{(1+\mu_{\mathcal{L}}^p)^{\sum_{r=1}^{n}w_r}+(1-\mu_{\mathcal{L}}^p)^{\sum_{r=1}^{n}w_r}}\right)$$

$$\left(\frac{2(\eta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n} w_r}}{(2-\eta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n} w_r}+(\eta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n} w_r}},\frac{2(\eta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n} w_r}}{(2-\eta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n} w_r}+(\eta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n} w_r}},\dots,\right.$$
$$\left.\frac{2(\eta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n} w_r}}{(2-\eta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n} w_r}+(\eta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n} w_r}}\right),\left(\frac{2(\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n} w_r}}{(2-\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n} w_r}+(\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n} w_r}},\right.$$
$$\left.\left.\frac{2(\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n} w_r}}{(2-\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n} w_r}+(\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n} w_r}},\dots,\frac{2(\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n} w_r}}{(2-\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n} w_r}+(\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n} w_r}}\right)\right\rangle$$

$$=\left\langle\left[\frac{1}{\frac{1}{\mathrm{a}}},\frac{1}{\frac{1}{\mathrm{b}}},\frac{1}{\frac{1}{\mathrm{c}}},\frac{1}{\frac{1}{\mathrm{d}}}\right];\left(\frac{(1+\mu_{\mathcal{L}}^{1})-(1-\mu_{\mathcal{L}}^{1})}{(1+\mu_{\mathcal{L}}^{1})+(1-\mu_{\mathcal{L}}^{1})},\frac{(1+\mu_{\mathcal{L}}^{2})-(1-\mu_{\mathcal{L}}^{2})}{(1+\mu_{\mathcal{L}}^{2})+(1-\mu_{\mathcal{L}}^{2})},\dots,\frac{(1+\mu_{\mathcal{L}}^{p})-(1-\mu_{\mathcal{L}}^{p})}{(1+\mu_{\mathcal{L}}^{p})+(1-\mu_{\mathcal{L}}^{p})}\right.\right.$$
$$\left(\frac{2(\eta_{\mathcal{L}}^{1})}{(2-\eta_{\mathcal{L}}^{1})+(\eta_{\mathcal{L}}^{1})},\frac{2(\eta_{\mathcal{L}}^{2})}{(2-\eta_{\mathcal{L}}^{2})+(\eta_{\mathcal{L}}^{2})},\dots,\frac{2(\eta_{\mathcal{L}}^{p})}{(2-\eta_{\mathcal{L}}^{p})+(\eta_{\mathcal{L}}^{p})}\right.$$
$$\left.\left(\frac{2(\vartheta_{\mathcal{L}}^{1})}{(2-\vartheta_{\mathcal{L}}^{1})+(\vartheta_{\mathcal{L}}^{1})},\frac{2(\vartheta_{\mathcal{L}}^{2})}{(2-\vartheta_{\mathcal{L}}^{2})+(\vartheta_{\mathcal{L}}^{2})},\dots,\frac{2(\vartheta_{\mathcal{L}}^{p})}{(2-\vartheta_{\mathcal{L}}^{p})+(\vartheta_{\mathcal{L}}^{p})}\right)\right\rangle=\mathcal{L}.$$

Theorem 3.4 (Monotoniticy property):

Let $\mathcal{L}_r=\langle[\mathrm{a_r},\mathrm{b_r},\mathrm{c_r},\mathrm{d_r}];(\mu_{\mathcal{L}_r}^{1},\mu_{\mathcal{L}_r}^{2},\dots,\mu_{\mathcal{L}_r}^{\mathrm{P}}),(\eta_{\mathcal{L}_r}^{1},\eta_{\mathcal{L}_r}^{2},\dots,\eta_{\mathcal{L}_r}^{\mathrm{P}}),(\vartheta_{\mathcal{L}_r}^{1},\vartheta_{\mathcal{L}_r}^{2},\dots,\vartheta_{\mathcal{L}_r}^{\mathrm{P}})\rangle$ and $\mathcal{L}_r'=\langle[\mathrm{a}_r',\mathrm{b}_r',\mathrm{c}_r',\mathrm{d}_r'];\left((\mu_{\mathcal{L}_r}')^{1},(\mu_{\mathcal{L}_r}')^{2},\dots,(\mu_{\mathcal{L}_r}')^{p}\right),\left((\eta_{\mathcal{L}_r}')^{1},(\eta_{\mathcal{L}_r}')^{2},\dots,(\eta_{\mathcal{L}_r}')^{p}\right),\left((\vartheta_{\mathcal{L}_r}')^{1},(\vartheta_{\mathcal{L}_r}')^{2},\dots,(\vartheta_{\mathcal{L}_r}')^{p}\right)\rangle$

be two collections of NVNT-numbers. If $\mathrm{a_r}\le\mathrm{a}_r'$, $\mathrm{b_r}\le\mathrm{b}_r'$, $\mathrm{c_r}\le\mathrm{c}_r'$, $\mathrm{d_r}\le\mathrm{d}_r'$, $\mu_{\mathcal{L}_r}^{1}\le(\mu_{\mathcal{L}_r}')^{1}$,

$\mu_{\mathcal{L}_r}^{2}\le(\mu_{\mathcal{L}_r}')^{2},\dots,\mu_{\mathcal{L}_r}^{\mathrm{p}}\le(\mu_{\mathcal{L}_r}')^{p},\eta_{\mathcal{L}_r}^{1}\ge(\eta_{\mathcal{L}_r}')^{1},\eta_{\mathcal{L}_r}^{2}\ge(\eta_{\mathcal{L}_r}')^{2},\dots,\eta_{\mathcal{L}_r}^{\mathrm{p}}\ge(\eta_{\mathcal{L}_r}')^{p}$ and

$\vartheta_{\mathcal{L}_r}^{1}\ge(\vartheta_{\mathcal{L}_r}')^{1},\vartheta_{\mathcal{L}_r}^{2}\ge(\vartheta_{\mathcal{L}_r}')^{2},\dots,\vartheta_{\mathcal{L}_r}^{\mathrm{p}}\ge(\vartheta_{\mathcal{L}_r}')^{p}$ then

$$\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)\le\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\dots,\mathcal{L}_n'). \tag{13}$$

Proof. Since $\mathrm{a_r}\le\mathrm{a}_r'$ and $\varphi_{\mathrm{r}}\ge 0$ for all r.

$$\frac{1}{\mathrm{a_r}}\ge\frac{1}{\mathrm{a}_r'}\Rightarrow\frac{\varphi_{\mathrm{r}}}{\mathrm{a_r}}\ge\frac{\varphi_{\mathrm{r}}}{\mathrm{a}_r'}\Rightarrow\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a_r}}\ge\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a}_r'}\Rightarrow\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a_r}}}\le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a}_r'}}$$

the other calculations are calculated as follows:

$$\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{b_r}}}\le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{b}_r'}},\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{c_r}}}\le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{c}_r'}},\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{d_r}}}\le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{d}_r'}}$$

$$\mu_{\mathcal{L}_r}^{1}\le(\mu_{\mathcal{L}_r}')^{1}\Rightarrow-\mu_{\mathcal{L}_r}^{1}\ge-(\mu_{\mathcal{L}_r}')^{1}\Rightarrow(1-\mu_{\mathcal{L}_r}^{1})\ge\left(1-(\mu_{\mathcal{L}_r}')^{1}\right)\Rightarrow(1-\mu_{\mathcal{L}_r}^{1})^{\varphi_{\mathrm{r}}}\ge\left(1-(\mu_{\mathcal{L}_r}')^{1}\right)^{\varphi_{\mathrm{r}}}$$

$$\Rightarrow \prod_{r=1}^{n}\left(1-\mu_{\mathcal{L}_r}^{1}\right)^{\varphi_r} \geq \prod_{r=1}^{n}\left(1-\left(\mu_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_r} \Rightarrow 1-\prod_{r=1}^{n}\left(1-\mu_{\mathcal{L}_r}^{1}\right)^{\varphi_r} \leq 1-\prod_{r=1}^{n}\left(1-\left(\mu_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_r},$$

similarly

$$\eta_{\mathcal{L}_r}^{1} \geq \left(\eta_{\mathcal{L}_r}'\right)^{1} \Rightarrow \left(\eta_{\mathcal{L}_r}^{1}\right)^{\varphi_r} \geq \left(\left(\eta_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_r} \Rightarrow \prod_{r=1}^{n}\left(\eta_{\mathcal{L}_r}^{1}\right)^{\varphi_r} \geq \prod_{r=1}^{n}\left(\left(\eta_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_r}$$

and

$$\vartheta_{\mathcal{L}_r}^{1} \geq \left(\vartheta_{\mathcal{L}_r}'\right)^{1} \Rightarrow \left(\vartheta_{\mathcal{L}_r}^{1}\right)^{\varphi_r} \geq \left(\left(\vartheta_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_r} \Rightarrow \prod_{r=1}^{n}\left(\vartheta_{\mathcal{L}_r}^{1}\right)^{\varphi_r} \geq \prod_{r=1}^{n}\left(\left(\vartheta_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_r}$$

therefore

$$\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n) \leq \mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\dots,\mathcal{L}_n').$$

Theorem 3.5 (Commutativity Property):

Let $\mathcal{L}_r = \langle [\mathrm{a_r},\mathrm{b_r},\mathrm{c_r},\mathrm{d_r}];\left(\mu_{\mathcal{L}_r}^{1},\mu_{\mathcal{L}_r}^{2},\dots,\mu_{\mathcal{L}_r}^{\mathrm{p}}\right),\left(\eta_{\mathcal{L}_r}^{1},\eta_{\mathcal{L}_r}^{2},\dots,\eta_{\mathcal{L}_r}^{\mathrm{p}}\right),\left(\vartheta_{\mathcal{L}_r}^{1},\vartheta_{\mathcal{L}_r}^{2},\dots,\vartheta_{\mathcal{L}_r}^{\mathrm{p}}\right)\rangle$

be a collection of positive NVNT-numbers and $w=(w_1,w_2,w_3,\dots,w_n)^T$ be an associated weight vector where $w_r \in [0,1]$, $\sum_{r=1}^{n} w_r = 1$.

$$\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n) = \mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\dots,\mathcal{L}_n'). \tag{14}$$

where $\mathcal{L}_n'$ is any permutation of $\mathcal{L}_n$ for $r = 1,2,3,\dots,n$.

Proof. We get from Equation (11). Since $\mathcal{L}_n'$ is any permutation of $\mathcal{L}_n$. Therefore

$$\mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n) = \frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}+\dots+\frac{w_n}{\mathcal{L}_n}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{d}_r}}\right];\left(\frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}-\prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}+\prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{1})^{w_r}}\right.\right.$$

$$\left.\frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}-\prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{2})^{w_r}},\dots,\frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}-\prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}+\prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}\right)$$

$$\left(\frac{2\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(2-\eta_{\mathcal{L}_r}^{1})^{w_r}+\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^{1})^{w_r}},\frac{2\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-\eta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^{2})^{w_r}},\dots,\right.$$

$$\left.\frac{2\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}{\prod_{r=1}^{n}(2-\eta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{n}(\eta_{\mathcal{L}_r}^{2})^{w_r}}\right),\left(\frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r}+\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}},\right.$$

$$\left.\left.\frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}},\dots,\frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}+\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}\right)\right\rangle$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n} \frac{w_r}{a'_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{b'_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{c'_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{d'_r}} \right]; \left(\frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{1})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{1})^{w_r}} \right.\right.$$

$$\left. \frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{2})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{2})^{w_r}}, \ldots, \frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{p})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{p})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{p})^{w_r}} \right)$$

$$\left(\frac{2\prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(2-(\eta')_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{1})^{w_r}}, \frac{2\prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-(\eta')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{2})^{w_r}}, \ldots, \right.$$

$$\left. \frac{2\prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(2-(\eta')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}((\eta')_{\mathcal{L}_r}^{2})^{w_r}} \right), \left(\frac{2\prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{1})^{w_r}}, \right.$$

$$\left.\left. \frac{2\prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{2})^{w_r}}, \ldots, \frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{p})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{p})^{w_r}} \right) \right\rangle$$

$$= \frac{1}{\frac{w_1}{\mathcal{L}'_1} + \frac{w_2}{\mathcal{L}'_2} + \cdots + \frac{w_n}{\mathcal{L}'_n}}$$

$$= \mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}'_1, \mathcal{L}'_2, \mathcal{L}'_3, \ldots, \mathcal{L}'_n).$$

Hence the proof completed.

Theorem 3.6 (Boundedness Property):

Let $\mathcal{L}_r = \langle [a_r, b_r, c_r, d_r]; (\mu_{\mathcal{L}_r}^{1}, \mu_{\mathcal{L}_r}^{2}, \ldots, \mu_{\mathcal{L}_r}^{p}), (\eta_{\mathcal{L}_r}^{1}, \eta_{\mathcal{L}_r}^{2}, \ldots, \eta_{\mathcal{L}_r}^{p}), (\vartheta_{\mathcal{L}_r}^{1}, \vartheta_{\mathcal{L}_r}^{2}, \ldots, \vartheta_{\mathcal{L}_r}^{p}) \rangle$

be a collection of positive NVNT-numbers and let,

$$\mathcal{L}_r^{+} = \left\langle \left[\max_r\{a_r\}, \max_r\{b_r\}, \max_r\{c_r\}, \max_r\{d_r\} \right]; \left(\max_r\{\mu_{\mathcal{L}_r}^{1}\}, \max_r\{\mu_{\mathcal{L}_r}^{2}\}, \ldots, \max_r\{\mu_{\mathcal{L}_r}^{p}\} \right), \right.$$

$$\left. \left(\min_r\{\eta_{\mathcal{L}_r}^{1}\}, \min_r\{\eta_{\mathcal{L}_r}^{2}\}, \ldots, \min_r\{\eta_{\mathcal{L}_r}^{p}\} \right), \left(\min_r\{\vartheta_{\mathcal{L}_r}^{1}\}, \min_r\{\vartheta_{\mathcal{L}_r}^{2}\}, \ldots, \min_r\{\vartheta_{\mathcal{L}_r}^{p}\} \right) \right\rangle$$

$$\mathcal{L}_r^{-} = \left\langle \left[\min_r\{a_r\}, \min_r\{b_r\}, \min_r\{c_r\}, \min_r\{d_r\} \right]; \left(\min_r\{\mu_{\mathcal{L}_r}^{1}\}, \min_r\{\mu_{\mathcal{L}_r}^{2}\}, \ldots, \min_r\{\mu_{\mathcal{L}_r}^{p}\} \right), \right.$$

$$\left. \left(\max_r\{\eta_{\mathcal{L}_r}^{1}\}, \max_r\{\eta_{\mathcal{L}_r}^{2}\}, \ldots, \max_r\{\eta_{\mathcal{L}_r}^{p}\} \right), \left(\max_r\{\vartheta_{\mathcal{L}_r}^{1}\}, \max_r\{\vartheta_{\mathcal{L}_r}^{2}\}, \ldots, \max_r\{\vartheta_{\mathcal{L}_r}^{p}\} \right) \right\rangle$$

Then,

$$\mathcal{L}_r^{-} \leq \mathrm{NVNTNWHM}^{\varphi}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_n) \leq \mathcal{L}_r^{+}. \tag{15}$$

Proof. Since $\min_r\{a_r\} \leq a_r \leq \max_r\{a_r\}, \forall r$

$$\frac{\varphi_r}{\min_r\{a_r\}} \geq \frac{\varphi_r}{a_r} \geq \frac{\varphi_r}{\max_r\{a_r\}} \Rightarrow \sum_{r=1}^{n} \frac{\varphi_r}{\min_r\{a_r\}} \geq \sum_{r=1}^{n} \frac{\varphi_r}{a_r} \geq \sum_{r=1}^{n} \frac{\varphi_r}{\max_r\{a_r\}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\min\limits_{r}\{\mathrm{a_r}\}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a_r}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\max\limits_{r}\{\mathrm{a_r}\}}}$$

In the same way,

$$\Rightarrow \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\min\limits_{r}\{\mathrm{b_r}\}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{b_r}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\max\limits_{r}\{\mathrm{b_r}\}}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\min\limits_{r}\{\mathrm{c_r}\}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{c_r}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\max\limits_{r}\{\mathrm{c_r}\}}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\min\limits_{r}\{\mathrm{d_r}\}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{d_r}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\max\limits_{r}\{\mathrm{d_r}\}}}$$

and,

$$\min_{r}\{\mu_{\mathrm{r}}\} \leq \mu_{\mathrm{r}} \leq \max_{r}\{\mu_{\mathrm{r}}\} \Rightarrow \left(1-\min_{r}\{\mu_{\mathrm{r}}\}\right) \geq (1-\mu_{\mathrm{r}}) \geq \left(1-\max_{r}\{\mu_{\mathrm{r}}\}\right)$$

$$\Rightarrow \left(1-\min_{r}\{\mu_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}} \geq (1-\mu_{\mathrm{r}})^{\varphi_{\mathrm{r}}} \geq \left(1-\max_{r}\{\mu_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}}, \varphi_{\mathrm{r}} \geq 0, \forall \mathrm{r}.$$

$$= \prod_{r=1}^{n}\left(1-\min_{r}\{\mu_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}} \geq \prod_{r=1}^{n}(1-\mu_{\mathrm{r}})^{\varphi_{\mathrm{r}}} \geq \prod_{r=1}^{n}\left(1-\max_{r}\{\mu_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}}$$

$$= 1-\prod_{r=1}^{n}\left(1-\min_{r}\{\mu_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}} \leq 1-\prod_{r=1}^{n}(1-\mu_{\mathrm{r}})^{\varphi_{\mathrm{r}}} \leq 1-\prod_{r=1}^{n}\left(1-\max_{r}\{\mu_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}}$$

Again,

$$\min_{r}\{\eta_{\mathrm{r}}\} \leq \eta_{\mathrm{r}} \leq \max_{r}\{\eta_{\mathrm{r}}\} \Rightarrow \left(\min_{r}\{\eta_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}} \leq (\eta_{\mathrm{r}})^{\varphi_{\mathrm{r}}} \leq \left(\max_{r}\{\eta_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}}, \varphi_{\mathrm{r}} \geq 0, \forall \mathrm{r}$$

$$= \prod_{r=1}^{n}\left(\min_{r}\{\eta_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}} \leq \prod_{r=1}^{n}(\eta_{\mathrm{r}})^{\varphi_{\mathrm{r}}} \leq \prod_{r=1}^{n}\left(\max_{r}\{\eta_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}}$$

In the same way,

$$\min_{r}\{\vartheta_{\mathrm{r}}\} \leq \vartheta_{\mathrm{r}} \leq \max_{r}\{\vartheta_{\mathrm{r}}\} \Rightarrow \left(\min_{r}\{\vartheta_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}} \leq (\vartheta_{\mathrm{r}})^{\varphi_{\mathrm{r}}} \leq \left(\max_{r}\{\vartheta_{\mathrm{r}}\}\right)^{\varphi_{\mathrm{r}}}, \varphi_{\mathrm{r}} \geq 0, \forall \mathrm{r}$$

$$= \prod_{r=1}^{n} \left(\min_{r}\{\vartheta_{r}\}\right)^{\varphi_{r}} \le \prod_{r=1}^{n} (\vartheta_{r})^{\varphi_{r}} \le \prod_{r=1}^{n} \left(\max_{r}\{\vartheta_{r}\}\right)^{\varphi_{r}}.$$

Definition 3.2 Let $\mathcal{L}_r = \langle [a_r, b_r, c_r, d_r]; \left(\mu_{\mathcal{L}_r}^{1}, \mu_{\mathcal{L}_r}^{2}, \dots, \mu_{\mathcal{L}_r}^{P}\right), \left(\eta_{\mathcal{L}_r}^{1}, \eta_{\mathcal{L}_r}^{2}, \dots, \eta_{\mathcal{L}_r}^{P}\right), \left(\vartheta_{\mathcal{L}_r}^{1}, \vartheta_{\mathcal{L}_r}^{2}, \dots, \vartheta_{\mathcal{L}_r}^{P}\right)\rangle$ be a collection of positive NVNT-number, then

$$S(\mathcal{L}_r) = \frac{1}{4p}[a + b + c + d] \times \left(2p + \sum_{r=1}^{p} \mu_{\mathcal{L}_r}^{P} - \sum_{r=1}^{p} \eta_{\mathcal{L}_r}^{P} - \sum_{r=1}^{p} \vartheta_{\mathcal{L}_r}^{P}\right)$$

and

$$A(\mathcal{L}_r) = \frac{1}{4p}[a + b + c + d] \times \left(2p + \sum_{r=1}^{p} \mu_{\mathcal{L}_r}^{P} - \sum_{r=1}^{p} \eta_{\mathcal{L}_r}^{P} + \sum_{r=1}^{p} \vartheta_{\mathcal{L}_r}^{P}\right)$$

is called the score and accuracy degrees of $\mathcal{L}_r$, respectively.

Example 3.2 : Let $\mathcal{L} = \langle [2,3,5,7]; (0.2,0.5,0.6,0.8), (0.3,0.4,0.5,0.1), (0.2,0.3,0.4,0.8)\rangle$ be NVNT-number then,

$$S(\mathcal{L}) = \frac{1}{4.4}[2 + 3 + 5 + 7] \times \left(8 + (0.2 + 0.5 + 0.6 + 0.8) - (1.3) - (1.7)\right) = 7.54$$

$$A(\mathcal{L}) = \frac{1}{4.4}[2 + 3 + 5 + 7] \times \left(8 + (0.2 + 0.5 + 0.6 + 0.8) - (1.3) + (1.7)\right) = 11.16.$$

Definition 3.4 Let $\mathcal{L}_r^1$ and $\mathcal{L}_r^2$ be two NVNT-numbers;

a. If $S(\mathcal{L}_r^1) < S(\mathcal{L}_r^2)$, then $\mathcal{L}_r^1$ is smaller than $\mathcal{L}_r^2$, denoted by $\mathcal{L}_r^1 < \mathcal{L}_r^2$.

b. If $S(\mathcal{L}_r^1) = S(\mathcal{L}_r^2)$;

   i. If $A(\mathcal{L}_r^1) < A(\mathcal{L}_r^2)$, then $\mathcal{L}_r^1$ is smaller than $\mathcal{L}_r^2$, denoted by $\mathcal{L}_r^1 < \mathcal{L}_r^2$.

   ii. If $A(\mathcal{L}_r^1) = A(\mathcal{L}_r^2)$, then $\mathcal{L}_r^1$ and $\mathcal{L}_r^2$ are the same, denoted by $\mathcal{L}_r^1 = \mathcal{L}_r^2$.

4. An algorithm for proposed work

In this section, we shall present a multi-criteria decision-making problem with normalized NVNT-numbers under neutrosophic information using NVNTNWHM operator.

Assume that $K = \{K_1, K_2, \dots, K_m\}$ be the set of altenatives and $G = \{g_1, g_2, \dots, g_n\}$ be the set of criterias. In Deli et al. (2021), the normalized NVNT-numbers decision matrix is given as;

$$(\mathrm{K_{kj}})_{\mathrm{mxn}} = \begin{pmatrix} \mathrm{K_{11}} & \mathrm{K_{12}} & \cdots & \mathrm{K_{1n}} \\ \mathrm{K_{21}} & \mathrm{K_{22}} & \cdots & \mathrm{K_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{K_{m1}} & \mathrm{K_{m2}} & \cdots & \mathrm{K_{mn}} \end{pmatrix}$$

such that

$\mathrm{K_{kj}} = \langle[\mathrm{a_{kj}}, \mathrm{b_{kj}}, \mathrm{c_{kj}}, \mathrm{d_{kj}}], \left(\eta^1_{\mathrm{kj}}, \eta^2_{\mathrm{kj}}, \eta^3_{\mathrm{kj}}, \dots, \eta^{\mathrm{p}}_{\mathrm{kj}}\right), \left(\vartheta^1_{\mathrm{kj}}, \vartheta^2_{\mathrm{kj}}, \vartheta^3_{\mathrm{kj}}, \dots, \vartheta^{\mathrm{p}}_{\mathrm{kj}}\right), \left(\theta^1_{\mathrm{kj}}, \theta^2_{\mathrm{kj}}, \theta^3_{\mathrm{kj}}, \dots, \theta^{\mathrm{p}}_{\mathrm{kj}}\right)\rangle$, (k=1,2,…,m) and (j=1,2,…,n).

It is carried out the following algorithm to get best choice:

Algorithm:

Step 1: Identify and determine the criterias and alternatives and then construct decision matrices,

$$(\mathrm{K_{kj}})_{\mathrm{mxn}}, \text{ (k=1,2,…,m; j=1,2,…,n).}$$

Step 2: Get preferable for $\mathrm{K_1}, \mathrm{K_2}, \dots, \mathrm{K_m}$ based on V_i $(i = 1,2,3, \dots, m)$ to aggregate the normalized NVNT-numbers $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n$ as;

$$V_i = \mathrm{NVNTNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n).$$

Step 3: Calculate score value whose formula is given in Definition 3.2 for each V_i to rank alternatives.

Step 4: Rank all score value of V_i according to descending order.

For more convenience, we have illustrated the proposed algorithm by Flowcard given in Figure-1.

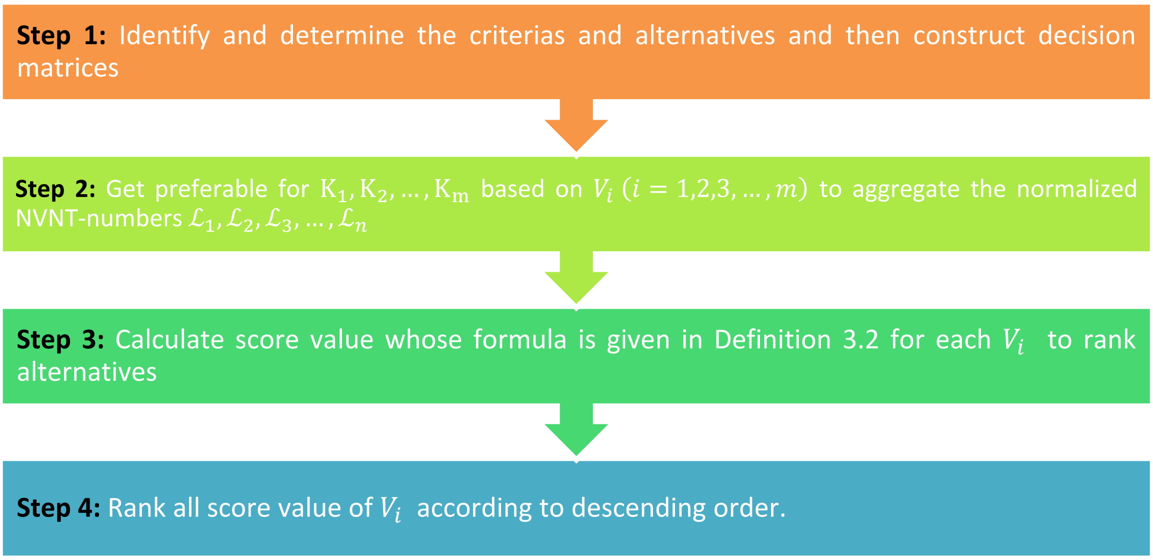

Figure-1: Flowcard for Proposed Algorthm

5. Application of the proposed method

In this section, an explanatory example is given to view the strength of the presented work. A firewall is a computer network security system that restricts internet traffic into, out of, or inside a private network. This software or dedicated hardware-software unit works by selectively blocking or allowing data packets. It generally aims to help prevent malicious activity and prevent anyone inside or outside a private network from getting involved in unauthorized web activities.

Example 5.1 Suppose Kilis 7 Aralık University wants to select the best firewall software for the security and safety of their information. Let there are initially three firewall software that is to be considered and set of these firewall software is given as: $K = \{k_1 = \text{Software Firewall Systems},\ k_2 = \text{Hardware Firewall Systems},\ k_3 = \text{Hybrid Firewalls},\ k_4 = \text{Circuit Level Firewalls},\ k_5 = \text{Static Packet Filter Firewalls}\}$ and according to three criteria determined $G = \{g_1 = \text{full protection},\ g_2 = \text{price},\ g_3 = \text{usefulness}\}$. Thent we try to choose and rank all alternatives K_k for all k=1,2,…,5 by using the following algorithm.

Algorithm:

Step 1: The evaluation matrix$(K_{kj})_{5x3}$ is given by an expert as;

$$
(K_{kj})_{5x3} = \begin{array}{c} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{pmatrix}
\langle [0.32,0.44,0.51,0.69]; (0.3,0.5,0.7,0.8), (0.1,0.3,0.2,0.3), (0.6,0.3,0.5,0.6) \rangle \\
\langle [0.23,0.25,0.41,0.45]; (0.4,0.2,0.3,0.5), (0.2,0.5,0.7,0.6), (0.7,0.5,0.6,0.8) \rangle \\
\langle [0.66,0.72,0.79,0.83]; (0.7,0.6,0.4,0.8), (0.5,0.5,0.5,0.6), (0.4,0.3,0.4,0.5) \rangle \\
\langle [0.52,0.63,0.76,0.91]; (0.8,0.7,0.5,0.6), (0.4,0.3,0.7,0.5), (0.1,0.5,0.7,0.7) \rangle \\
\langle [0.13,0.35,0.41,0.58]; (0.7,0.8,0.9,0.9), (0.2,0.7,0.5,0.6), (0.1,0.7,0.8,0.4) \rangle \\
\langle [0.28,0.32,0.38,0.43]; (0.5,0.3,0.4,0.6), (0.2,0.1,0.5,0.4), (0.4,0.6,0.5,0.7) \rangle \\
\langle [0.12,0.15,0.18,0.23]; (0.3,0.7,0.9,0.9), (0.1,0.2,0.3,0.7), (0.3,0.4,0.7,0.5) \rangle \\
\langle [0.65,0.66,0.72,0.75]; (0.6,0.8,0.9,0.8), (0.2,0.5,0.4,0.3), (0.2,0.3,0.6,0.6) \rangle \\
\langle [0.08,0.15,0.27,0.37]; (0.3,0.9,0.8,0.4), (0.8,0.7,0.6,0.5), (0.1,0.1,0.4,0.3) \rangle \\
\langle [0.09,0.13,0.19,0.69]; (0.2,0.5,0.7,0.9), (0.1,0.3,0.5,0.4), (0.6,0.7,0.8,0.8) \rangle \\
\langle [0.12,0.27,0.60,0.65]; (0.2,0.7,0.8,0.9), (0.4,0.3,0.8,0.5), (0.2,0.5,0.6,0.4) \rangle \\
\langle [0.22,0.43,0.48,0.73]; (0.3,0.8,0.9,0.7), (0.5,0.7,0.7,0.6), (0.1,0.4,0.8,0.6) \rangle \\
\langle [0.14,0.33,0.43,0.83]; (0.1,0.6,0.9,0.5), (0.8,0.5,0.6,0.7), (0.1,0.3,0.5,0.8) \rangle \\
\langle [0.63,0.73,0.83,0.93]; (0.4,0.5,0.7,0.6), (0.2,0.6,0.8,0.3), (0.3,0.6,0.7,0.2) \rangle \\
\langle [0.41,0.43,0.68,0.74]; (0.5,0.7,0.8,0.3), (0.1,0.2,0.6,0.4), (0.4,0.2,0.9,0.7) \rangle
\end{pmatrix}
$$

Step 2: Calculated for $K_1, K_2, \ldots, K_m$ based on V_i $(i = 1,2,3,\ldots,m)$ to aggregate the normalized NVNT-numbers $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_n$ follow as;

$$V_1 = \langle [0.19,0.32,0.48,0.56]; (0.001,0.004,0.01,0.03), (0.3,0.28,0.55,0.47), (0.4,0.53,0.58,0.58) \rangle$$

$V_2 = \langle[0.17,0.23,0.3,0.38]; (0.001,0.006,0.02,0.02), (0.3,0.48,0.57,0.66), (0.3,0.48,0.73,0.64)\rangle$

$V_3 = \langle[0.26,0.48,0.58,0.8]; (0.002,0.017,0.03,0.02), (0.5,0.55,0.55,0.55), (0.25,0.37,0.55,0.68)\rangle$

$V_4 = \langle[0.18,0.31,0.48,0.6]; (0.004,0.022,0.02,0.007), (0.5,0.58,0.72,0.47), (0.23,0.38,0.62,0.38)\rangle$

$V_5 = \langle[0.15,0.23,0.33,0.68]; (0.003,0.017,0.04,0.02), (0.2,0.38,0.58,0.5), (0.4,0.48,0.85,0.67)\rangle$

Step 3: The calculated score value whose formula is given in Definition 3.2 for each V to rank alternatives;

$$S(V_1) = \frac{1}{4.4}[0.19 + 0.32 + 0.48 + 0.56] \times \big(8 + (0.001 + 0.004 + 0.01 + 0.003) - (0.3 + 0.28 + 0.55 + 0.47) - (0.4 + 0.53 + 0.58 + 0.58)\big) = 0.423$$

Like

$$S(V_2) = 0.262,\ S(V_3) = 0.539,\ S(V_4) = 0.414,\ S(V_5) = 0.345$$

Step 4: Based on the score values $\mathrm{S}(\mathrm{V_i})(\mathrm{i} = 1,2,\ldots,5)$ the ranking of alternatives $\mathrm{k_k}(\mathrm{k} = 1,2,\ldots,5)$ are shown in Figure 1 and given as;

$$\mathrm{k_3} > \mathrm{k_1} > \mathrm{k_4} > \mathrm{k_5} > \mathrm{k_2}.$$

Finally the best alternative is $\mathrm{k_3}$.

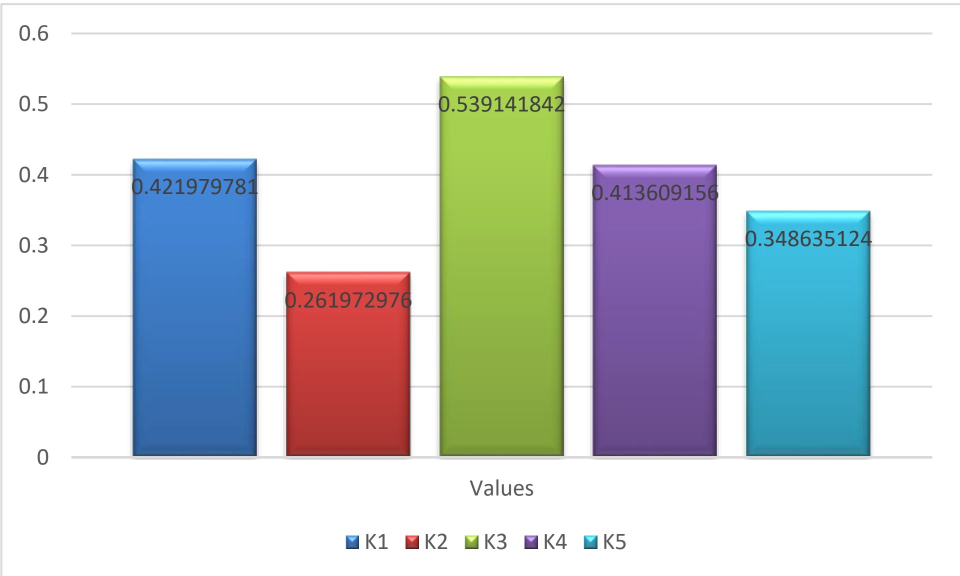

Figure 2: The ranking of alternatives $\mathrm{K_k}$ $(\mathrm{k} = 1,2,\ldots,5)$

6. Results and Discussion

In this paper, we have proposed two aggregations are called harmonic aggregation operators and weighted harmonic mean operations on N-valued neutrosophic trapezoidal numbers. We have discussed a variety of special cases of proposed operators. We define some desired properties such as idempotency, monotoniticy, commutativity and boundedness of the developed operators. To demonstrate the applicability of the proposed methodology, gave an illustrative example, using the proposed aggregation operators to rank the alternatives in multi-criteria decision-making problems. In the future work, the developed operators can be proposed with extensions neutrosophic trapezoidal numbers and neutrosophic sets.

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