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Operators for Uncertain Over/Under/Off-Sets/-Logics/-Probabilities/-Statistics

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Abstract: In this paper we propose for the first time the operators (intersection/conjunction and union/disjunction) for the Uncertain Over/Under/Off-Sets/-Logics/-Probabilities-Statistics. Practical applications of Fuzzy Over/Under/Off-Sets/-Logics/-Probabilities-Statistics are presented. Similar applications can be designed for any fuzzy-extensions Over/Under/Off-Sets/-Logics/-Probabilities-Statistics.

Keywords: Norm, OverNorm, UnderNorm, Offnorm; Conorm, OverConorm, UnderConorm, OffConorm.

1. Introduction

The Neutrosophic Set was extended for the first time by Smarandache [2-6] in the year 2007 to **Neutrosophic OverSet** (when some neutrosophic component is > 1), since we observed that, for example, an employee working overtime deserves a degree of membership > 1, with respect to an employee that only works regular full-time and whose degree of membership = 1; and to **Neutrosophic UnderSet** (when some neutrosophic component is < 0), since, for example, an employee making more damage than benefit to his company deserves a degree of membership < 0, with respect to an employee that produces benefit to the company and has the degree of membership > 0; and to the **Neutrosophic OffSet** (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and other neutrosophic component < 0).
Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively **Neutrosophic Over-/Under-/Off- Logic / Measure / Probability / Statistics** etc.

They resulted from practical applications when such degrees [φ, ψ] are outside the box (outside the interval [0, 1]), or $\varphi \leq 0 < 1 \leq \psi$.

"Over" means degree > 1, "Under" means degree < 0, while "Off" means both types of degrees > 1 and < 0.

But "uncertain" sets/logics/measures/probabilities/statistics we mean, in general, all kinds of Fuzzy, and fuzzy-extensions (Neutrosophic, Plithogenic, Intuitionistic Fuzzy, Inconsistent Intuitionistic Fuzzy, Picture Fuzzy, Ternary Fuzzy, Pythagorean Fuzzy, Atanassov's Intuitionistic Fuzzy, Fermatean Fuzzy, Spherical Fuzzy, n-HyperSpherical Neutrosophic, q-Rung Orthopair Fuzzy, Refined Neutrosophic, and so on).

In this paper we extend the t-norm to the over/under/off-norm, and similarly we extend the t-conorm to the over/under/off-conorm.

2. Annotation

Unlike the previous versions of the Uncertain Over/Under/Off-Sets/-Logics/-Probabilities papers and books [2 – 6], where the non-classical interval was denoted by [ψ, φ], we now propose to use [φ, ψ], since the letter φ is ahead of the letter ψ in the Greek alphabet.

All other ideas remain the same.

3. Classical Triangular Norms and Conorms

By classical triangular norms and conorms [1] we mean those norms and conorms whose domains and codomains are equal to [0, 1], as for example in fuzzy theories.

While, by **Over/Under/Off Triangular Norms and Conorms** [2 – 6], we mean those norms and conforms whose domains and codomains are different from [0, 1].

3.1. Definition of Classical Triangular Norm

A **triangular norm** (abbreviation **t-norm**) is a binary operation T on the interval [0, 1] satisfying the following conditions:

$T: [0,1] \times [0,1] \rightarrow [0,1]$.

For any $(x, y) \in [0,1] \times [0,1]$, one has:

$T(x, y) = T(y, x)$ (commutativity)

$T(x, T(y, z)) = T(T(x, y), z)$ (associativity)

$y \leq z \Rightarrow T(x, y) \leq T(x, z)$ (monotonicity)

$T(x, 1) = x$ (neutral element 1)

3.1.1. Examples of t-norms

$T_M(x, y) = \min(x, y)$ (minimum or Gödel t-norm)

$T_P(x, y) = x \cdot y$ (product t-norm)

$T_L(x, y) = max(x + y - 1, 0)$ (Lukasiewicz t-norm)

3.2. Definition of Classical Triangular Conorm

A dual notion to the triangular norm is the **triangular conorm** (abbreviation **t-conorm**, also **s-norm**).

$S: [0,1] \times [0,1] \rightarrow [0,1]$.

And for all $(x, y) \in [0,1] \times [0,1]$, one has:

$S(x, y) = S(y, x)$ (commutativity)

$S(x, S(y, z)) = S(S(x, y), z)$ (associativity)

$y \leq z \Rightarrow S(x, y) \leq S(x, z)$ (monotonicity)

$S(x, 0) = x$ (neutral element 0)

S has 0 (zero) as neutral element, instead of 1, but all other conditions remain unchanged.

3.2.1. Examples of t-conorms

$S_M(x, y) = max(x, y)$ (maximum, or Gödel t-conorm)

$S_P(x, y) = x + y - x \cdot y$ (product t-conorm, or probabilistic sum)

$S_L(x, y) = max(x + y, 1)$ (Lukasiewicz t-conorm, or bounded sum)

4. Over/Under/Off-Norm and Over/Under/Off-Conorm

We extend the classical triangular norm and conorm, whose domain and codomain is [0, 1], to:

— overnorm and overconorm, when their domain and codomain are [0, ψ], with $\psi > 1$;

— undernorm and underconorm, when their domain and codomain are [φ, 1], with φ < 0;

— offnorm and offconorm, when their domain and codomain are [φ, ψ], where $\varphi < 0 < 1 < \psi$.

4.1. Definition of Triangular OffNorm

A **Triangular OffNorm** (abbreviation **t-offnorm**) is a binary operation
$T_{OffNorm}: [\varphi, \psi] \times [\varphi, \psi] \rightarrow [\varphi, \psi]$, $\varphi < 0 < 1 < \psi$,
so that, for all $(x, y) \in [\varphi, \psi] \times [\varphi, \psi]$ one has:
$T_{OffNorm}(x, y) = T_{OffNorm}(y, x)$, commutativity;
$T_{OffNorm}\left(x, T_{OffNorm}(y, z)\right) = T_{OffNorm}\left(T_{OffNorm}(x, y), z\right)$, associativity;
If $y \leq z$, then $T_{OffNorm}(x, y) \leq T_{OffNorm}(x, z)$, monotonicity;
$T_{OffNorm}(x, \psi) = x$, or the neutral element is ψ (the largest one).

4.1.1. Examples of Triangular OffNorm

$T_{OffNorm}(x, y) = min(x, y)$, which is the extended Gödel t-norm from the unitary interval [0, 1] to the non-unitary interval [φ, ψ], for $\varphi < 0 < 1 < \psi$.

Proof

The commutativity, associativity, and monotonicity of the *min* operator are obvious on any real interval.

While the neutral element is the largest element in the given interval, in this case the neutral element is ψ (the biggest one).

4.2. Definition of Triangular OverNorm

It is like the Triangular OffNorm, but on the interval [0, ψ], ψ>1.
$T_{OverNorm}: [0, \psi] \times [0, \psi] \rightarrow [0, \psi]$, where $\psi > 1$.
The axioms of commutativity, associativity, monotonicity, and neutral element are verified:
$T_{OverNorm}(x, y) = T_{OverNorm}(y, x)$, commutativity;
$T_{OverNorm}\left(x, T_{OverNorm}(y, z)\right) = T_{OverNorm}(T_{OverNorm}(x, y), z)$, associativity;
If $y \leq z$, then $T_{OverNorm}(x, y) \leq T_{OverNorm}(x, z)$, monotonicity;
$T_{OverNorm}(x, \psi) = x$, or the neutral element is ψ.

4.2.1. Examples of Triangular OverNorm

$T_{OverNorm}(x, y) = min\ (x, y)$

4.3. Definition of Triangular UnderNorm

It is similar to the Triangular OffNorm, but on the interval [φ, 1], φ < 0.
$T_{UnderNorm}: [\varphi, 1] \times [\varphi, 1] \rightarrow [\varphi, 1]$, where $\varphi > 0$.
For any $(x, y) \in [\varphi, 1] \times [\varphi, 1]$,
the axioms of commutativity, associativity, monotonicity, and neutral element are verified.
$T_{UnderNorm}(x, y) = T_{UnderNorm}(y, x)$, commutativity;
$T_{UnderNorm}\left(x, T_{UnderNorm}(y, z)\right) = T_{UnderNorm}(T_{UnderNorm}(x, y), z)$, associativity;
If $y \leq z$, then $T_{UnderNorm}(x, y) \leq T_{UnderNorm}(x, z)$, monotonicity;
$T_{UnderNorm}(x, 1) = x$, or the neutral element is 1.

4.3.1. Examples of Triangular UnderNorm

$T_{UnderNorm}(x, y) = min\ (x, y)$

4.4. Definition of Triangular OffConorm

Similarly to the classical Triangular Conorm, the Triangular OffConorm is the dual of the Triangular OffNorm.

$T_{OffConorm}: [\varphi, \psi] \times [\varphi, \psi] \rightarrow [\varphi, \psi]$, where $\varphi < 0 < 1 < \psi$,

and for any $(x, y) \in [\varphi, \psi] \times [\varphi, \psi]$ one has:

$S_{OffConorm}(x, y) = S_{OffConorm}(y, x)$, commutativity;

$S_{OffConorm}\left(x, S_{OffConorm}(y, z)\right) = S_{OffConorm}\left(S_{OffConorm}(x, y), z\right)$, associativity;

If $y \leq z$, then $S_{OffConorm}(x, y) \leq S_{OffConorm}(x, z)$, monotonicity;

$S_{OffConorm}(x, \varphi) = x$, the neutral element is the smallest element φ.

4.4.1. Examples of Triangular OffConorm

$S_{OffConorm}(x, y) = max\ (x, y)$, which is another extended Gödel t-conorm.

Commutativity, associativity, and monotonicity of the max operator are valid on any real interval.

But as neutral element one has now φ (the smallest element of the real interval [φ, ψ], instead of 0 (zero) as in the classical S_{Conorm}.

4.5. Definition of Triangular OverConorm

It is like the Triangular OffConorm, but on the interval [0, ψ], ψ>1.

$T_{OverConorm}: [0, \psi] \times [0, \psi] \rightarrow [0, \psi]$, where $\psi > 1$.

and for any $(x, y) \in [0, \psi] \times [0, \psi]$,

the axioms of commutativity, associativity, monotonicity, and neutral element are verified:

$S_{OverConorm}(x, y) = S_{OverConorm}(y, x)$, commutativity

$S_{OverConorm}\left(x, S_{OverConorm}(y, z)\right) = S_{OverConorm}(S_{OverConorm}(x, y), z)$, associativity;

If $y \leq z$, then $S_{OverConorm}(x, y) \leq S_{OverConorm}(x, z)$, monotonicity;

$S_{OverConorm}(x, 0) = x$, or the neutral element is 0 (zero).

4.5.1. Example of Triangular OverConorm

$S_{OverConorm}(x, y) = max\ (x, y)$

4.6. Definition of Triangular UnderConorm

It is alike the Triangular OffConorm, but on the interval [φ, 1], φ<0.

$T_{UnderConorm}: [\varphi, 1] \times [\varphi, 1] \rightarrow [\varphi, 1],\ \varphi > 0$

and for any $(x, y) \in [\varphi, 1] \times [\varphi, 1]$,

the same axioms are verified:

$S_{UnderConorm}(x, y) = S_{UnderConorm}(y, x)$, commutativity;

$S_{UnderConorm}\left(x, S_{UnderConorm}(y, z)\right) = S_{UnderConorm}(S_{UnderConorm}(x, y), z)$, associativity;

If $y \leq z$, then $S_{UnderConorm}(x, y) \leq S_{UnderConorm}(x, z)$, monotonicity;

$S_{UnderConorm}(x, \varphi) = x$, or the neutral element is φ (the smallest one).

4.6.1. Example of Triangular UnderConorm

$S_{UnderConorm}(x, y) = max(x, y)$

5. Practical Applications of OffNorm and OffConorm

5.1. Application of the Fuzzy OverSet

In a math class, the professor gives a test to the students with 10 mandatory questions. For each question answered correctly, 10 points are awarded.

However, considering that there are several very gifted students who participate in regional Mathematics Olympiads, the teacher adds an additional difficult question, which is optional.

Correctly solving the first ten mathematical problems, a student gets $10 \times 10 = 100$ points, equivalent to the grade A.

We recall that the grades are the following, in ascending order:

F < D < C < B < A,

or

{F, D, C, B, A}

They mean, failing if a student gets F or D, and passing if he gets C, B, or A.

That are respectively corresponding numerically approximately to:

{0, 0.25, 0.50, 0.75, 1}, or F(0), D(0.25), C(0.50), B(0.75), A(1),

where, of course, 0 = 0% meaning 0 points out of 100 points, 0.25 = 25% or 25 points out of 100 points, etc.

But for students who also solve the 11^{th} problem, the grade A^+ is awarded, which is higher than A, since:

11 questions × 10 points = 110 points = A^+, which is > 100 points = A.

110 points out 100 required points is 110/100 =1.1 degree corresponding to A^+.

Therefore, the grading scale was extended from the classical unit interval [0, 1] to the non-unit interval [0, 1.1].

Therefore, the degree 1.1 of A^+ is bigger than the interval [0, 1], that's why we have an OverSet.

Application of the Fuzzy OverLogic

Let be the proposition:

P = "The student Marcella gets a passing grade."

What may it be the OverTruth-value of this proposition?

Passing means getting a score between 50% and 110%, or in between [0.5, 1.1].

Thus, the OverTruth-value is the whole interval [0.5, 1.1], while in the classical logic the truth-value is smaller, in the whole interval [0.5, 1].

Application of the Fuzzy OverProbability

What is the OverProbability that Marcella passes the test (getting A^+ is included)?

Another one: what is the OverProbability that a student gets an A^+ (degree > 1)?

Passing means getting a score between 50% and 110%, or in between [0.5, 1.1] out of [0, 1.1], hence: $\frac{1.1-0.5}{1.1-0} = \frac{0.6}{1.1} \approx 0.55$.

It is worth noting that in the classic case (no-OverProbability, or no A^+ on the grading scale), when the domain and codomain are just [0, 1], the classical probability result is in general smaller:

$$\frac{1-0.5}{1-0} = \frac{0.5}{1} = 0.50.$$

Application of the Fuzzy OverStatistics

The OverStatistics, in addition to the classical statistics, takes into account the OverProbability events as well. For example, the number of students that get over 100% to the test.

5.1.1. Numerical Fuzzy Examples

$$T_{OverNorm} : [0,1.1]\times[0,1.1]\rightarrow[0,1.1]$$

$$T_{OverNorm}(0.4,1.1) = \min\{0.4,1.1\} = 0.4$$

$$S_{OverConorm} : [0,1.1]\times[0,1.1]\rightarrow[0,1.1]$$

$$S_{OverConorm}(0.4,1.1) = \max\{0.4,1.1\} = 1.1$$

5.2. Application of the Fuzzy UnderSet

In an entrance exam at the University X, students must pass several tests. On each test, they score between 0% - 100% points, or a degree between [0, 1]. The lowest value is 0 (zero) point. Even if a student gets a zero on a test, he is not eliminated, he still has a chance of being admitted if he gets higher marks on the other tests and, of course, other students get lower marks.

Student Sean is caught *cheating* and is then *expelled* from the competition, therefore he has no more chance. So, the verdict of "expelled from the competition" should have a degree less than zero.

Therefore, we now deal with the subset $\varphi\cup[0,1]$, where $\varphi<0$, φ represents the degree of being expelled. As a numerical example, $\varphi=-1$, and the subset is $\{-1\}\cup[0,1]$.

$$T_{UnderNorm} : (\{-1\}\cup[0,1])\times(\{-1\}\cup[0,1])\rightarrow(\{-1\}\cup[0,1])$$

$$S_{UnderNorm} : (\{-1\}\cup[0,1])\times(\{-1\}\cup[0,1])\rightarrow(\{-1\}\cup[0,1])$$

Application of the Fuzzy UnderLogic

Similarly, let be the proposition:

P = "The student Marcella gets a passing grade."

What may it be the Fuzzy UnderTruth-value of this proposition?

Passing means getting a score between 50% and 100%, or in between [0.5, 1].

Thus, the Fuzzy OverTruth-value is the interval [0.5, 1], the same as in the classical logic.

Application of the Fuzzy UnderProbability

In the classical probability, what is the probability that the student John gets 23 points out of 100 total points? It is: one favorite possibility (23 points), out of one hundred and one total possible possibilities:

{0, 1, 2, 3, 4, 5, …, 23, 24, …, 100}

i.e. $\frac{1}{101} = 0.00990099...$

In the Fuzzy UnderProbability, the probability that the student John gets 23 points out of 100 points is:

one favorite possibility (23 points), but out of one hundred two total possible possibilities:

{-1, 0, 1, 2, 3, 4, 5, …, 23, 24, …, 100}

i.e. $\frac{1}{102} = 0.00980392...$

which is smaller than the classical probability.

What is the Fuzzy UnderProbability that the student Raymond gets zero points, or is expelled from the competition (i.e. the student gets the degree -1)? Two favorable cases out of 102 total possible cases:

it is: $\frac{2}{102} = 0.01960784...$

Application of the Fuzzy UnderStatistics

The Fuzzy UnderStatistics, in addition to the classical statistics, considers the UnderProbability events as well. For example, the number of students that get below 0% points (expelled) from the test, etc.

5.3. Application of the Fuzzy OffSet

We retake the previous practical examples and combine them:

In a math class, the professor gives a test to the students with 10 mandatory questions. For each question answered correctly, 10 points are awarded.

However, considering that there are several very gifted students who participate in regional Mathematical Olympiads, the examiners add an additional difficult question, which is optional, and assign 10 points as well for correctly solving it.

If a student is caught cheating, he is expelled from the school. Similarly, expelled means less than 0 (zero) points.

The domain and range are now, for example, [-1.2, 1.1], where the degree -1.2 means expelled and the degree 1.1 means having solved the 11^{th} bonus problem as well.

The lower (φ) and upper (ψ) margins of the Fuzzy OffSet/OffLogic/OffProbability/OffStatistics domain and codomain [φ , ψ] are established by the experts according to the application where they are used in.

Application of the Fuzzy OffLogic

Find the Fuzzy OffTruth-value of the proposition, P = "Miguel does get an A^+".

It means, Miguel gets between (1.0, 1.1] to the test, out of [-1.2, 1.1],

or $\frac{1.1-1.0}{1.1-(-1.2)} = \frac{0.1}{2.3} = 0.043...$

Application of the Fuzzy OffProbability

What is the Fuzzy OffProbability that Marcella gets expelled?

It means, Marcella gets between [-1.2, 0) to the test, out of [-1.2, 1.1],

$$\frac{|0-(-1.2)|}{1.1-(-1.2)} = \frac{1.2}{2.3} = 0.521...$$

Application of the Fuzzy OffStatistics

The Fuzzy OffStatistics, in addition to the classical statistics, consider the Fuzzy OffProbability events as well. For example, the statistics about the students that get over 100% or less than 0% to the test.

6. Future Research

Better designs are need for the Fuzzy (and other fuzzy-extensions) Over/Under/Off-Norms and Over/Under/Off-Conorms, possibly using continuous functions, or extensions of the classical Product t-norm / t-conorm and Lukasiewicz t-norm/t-conorm, satisfying of course the required axioms (commutativity, associativity, monotonicity, and the existence of a neutral element). And introducing more operators, for example: complement/negation, inclusion/implication, and equality/equivalence.

The readers are welcome to contribute to developing this new field of study about fuzzy/intuitionistic_fuzzy/neutrosophic/plithogenic and other fuzzy-extensions membership/truth, nonmembership/falsehood, indeterminacy/neutrality etc. degrees that are outside the interval [0, 1].

Practical applications of *Fuzzy* Over/Under/Off-Sets/-Logics/-Probability/-Statistics have been presented in this paper, while applications for the *Neutrosophic* Over/Under/Off-Sets/-Logics/-Probability/-Statistics have been previously published in the years 2007 - 2016 in articles and books [2-6].

Similar applications can be designed for any fuzzy-extension Over/Under/Off-Sets/-Logics/-Probabilities-Statistics and the readers are invited to contribute them.

7. Conclusions

For the first time we introduced now the operators (intersection, union) of the Fuzzy Off/Over/Under Sets/Logics and used them in real life applications.

References

[1] Alsina, Claudi; Frank, Maurice J.; and Schweizer Berthold (2006). Associative Functions: Triangular Norms and Copulas. *World Scientific*. DOI: 10.1142/6036

[2] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic set, Neutrosophic Probability and Statistics (sixth edition), ProQuest Information & Learning, Ann Arbor, Michigan, USA, 2007, https://fs.unm.edu/eBook-Neutrosophics6.pdf

[3] Smarandache, Florentin, Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly, for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics. Brussels: Pons Editions, 167 p., 2016, https://fs.unm.edu/NeutrosophicOversetUndersetOffset.pdf https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf

[4] Florentin Smarandache, Operators on Single-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets, *Journal of Mathematics and Informatics*, Vol. 5, 63-67, 2016, https://fs.unm.edu/SVNeutrosophicOverset-JMI.pdf

[5] F. Smarandache, Interval-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets, *International Journal of Science and Engineering Investigations*, vol. 5, issue 54, July 2016, https://fs.unm.edu/IV-Neutrosophic-Overset-Underset-Offset.pdf

[6] F. Smarandache, Degrees of Membership > 1 and < 0 of the Elements with Respect to a Neutrosophic OffSet, *Neutrosophic Sets and Systems*, Vol. 12, 3-8, 2016, https://fs.unm.edu/NSS/DegreesOf-Over-Under-Off-Membership.pdf

[6] Smarandache, Florentin - editor (2016). New types of Neutrosophic Set/Logic/Probability, Neutrosophic Over-/Under-/Off-Set, Neutrosophic Refined Set, and their Extension to Plithogenic Set/Logic/Probability, with Applications. *Symmetry* - Special Issue. Basel: MDPI, 685 p., https://fs.unm.edu/Symmetry-neutro-SI-2019.pdf

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