NSS Neutrosophic Sets and Systems, Vol. 79, 2025

University of New Mexico

Analytical Solutions of Heat Transfer Model in Two-Dimensional Case of Neutrosophic Fredholm Integro-Differential Equations

Amer Darweesh[1,*], Kamel Al- Khaled[1], Marwan Alquran[1], Adel Almalki[2] and Sohad Al-Omari[1]

[1]Department of Mathematics and Stat., Jordan University of Science and Technology, Irbid 22110, Jordan; ahdarweesh@just.edu.jo; kamel@just.edu.jo: marwan04@just.edu.jo; szalomary15@sci.just.edu.jo.

[2]Department of Mathematics, Al-Gunfudah University College, Umm Al-Qura University, Mecca 21955, Saudi Arabia; aaamalki@uqu.edu.sa.

[*]Correspondence: ahdarweesh@just.edu.jo

Abstract. This article uses the fractional residual power series (FRPS) method to solve a linear neutrosophic fractional integro-differential equation in two dimensions. In what context does the term "fractional derivative" appeared, we presented the modified fractional power series method, a new technique that uses fractional power series expansion to approximate neutrosophic fractional integro-differential equations. A modified new method has been formulated, which is an improvement on the RPS, named as Modified Fractional Power Series Method (MFPSM), to solve the same problem under investigation. Novel results associated with the rate of convergent and error order of the (MFPSM) was examined, and some findings—along with detailed proof—were documented as theories. Several numerical examples are used to describe and test the validity and applicability of preset approaches. We investigate a semi-infinite rod using the solution of our model, where heat transfer is influenced by both the memory of past states and the current temperature distribution. The fractional derivative of order α is used to represent memory effects in heat transfer processes. To demonstrate the precision and efficacy of the two approaches, the results are shown in terms of tables and graphs. The modified fractional power series approach proved to be more effective, efficient, and straightforward for solving the neutrosophic two-dimensional integro-differential equations than the residual power series method, while also generating less error and computing time.

Keywords: Fractional residue power series method; numerical solutions; Heat Transfer; Neutrosophic Integral Equations; Fredholm integro-differential equation in two dimensions.

---

1. Introduction

In many fields, including science, engineering, and modeling, integral and integro-differential equations are used to represent issues. The intricacy of most of these equations makes it difficult to acquire analytical solutions; consequently, it is crucial to develop numerical methods to obtain approximate answers. Significant advancements have been made in the last fifty years in the development of analytical and numerical solutions for various types of integral equations; both linear and nonlinear examples have been taken into consideration. Few numerical techniques have been used to approximate the solutions of two-dimensional problems, despite the fact that there are numerous approaches for one-dimensional integral and integro-differential equations. Given the numerous uses of two-dimensional integral and integro-differential equations in physics, mechanics, modeling, engineering, and other applied sciences, methods for handling these equations also need further attention. Neutrosophic integro-differential equations are used in many different domains, such as engineering and medicine, proving their adaptability in simulating dynamic systems with inherent uncertainties. As a result, neutrosophic calculus enhances the theory of integral differential equations and provides a more thorough method for handling real-world complexity than interval computations.

Using an approximating subspace, a unique space of spline functions, [34] and [31] examined the numerical solution of two-dimensional Fredholm integral equations. [36] used artificial neutral networks for modeling and simulating complex real-world problems in two-dimensional Volterra-type fractional integro-differential equations. In [3, 17], it is demonstrated that the non-linear Fredholm integro-differential equation in two dimensions exists and has a unique solution.

The basic concepts of fractional calculus were first presented by the Marquis de Hopital and G.W. Leibniz in 1695, as is well known. Since then, a number of other people have defined fractional operators [29]. Due to their applicability in a variety of scientific fields, such as physics, chemistry, engineering, and others, fractional-order integro-differential equations have been the focus of more discussion in the literature. As a result, alternative numerical methods are being given much more consideration.

Abu Arqub was the first who proposed the residual power series method (RPSM) in 2013 [11]. He also introduced a new method to obtain a numerical solution for linear and non-linear fuzzy problems. Additionally, this method was used to obtain a series solution to the non-linear time fractional reaction diffusion equation [43]. Authors in [43] found that the method is effective and powerful in solving that equation. In [10], Alquran developed a new study of finding the analytic solution to the time fractional two-component evolutionary system of order 2 using RPSM. The time-fractional foam drainage equation was also introduced by [9].

The time fractional Zakharov-Kuznetsov equation was approximated by Alquran in [37] using RPSM. Also, they provided the RPSM convergence study and demonstrated the method's competitiveness, strength, dependability, and ease of use. The RPSM was used in [19] to obtain an approximation solution and an analytical solution for the fractional Susceptible-Infected-Recovered (SIR) epidemic model in the form of a convergent power series. The time fractional Korteweg de Vries was numerically solved by Senol and Ayse [38], and the RPSM is used to generate a modified version of the solution. The multi-pantograph delay differential equations system was approximated by the authors of [21] using RPSM. Similarly, using the RPSM, the authors of [22] discovered an analytic solution to the system of 1-dimensional Fredholm integral equations. Furthermore, Bayrak and Demir found numerical solutions to space-time fractional partial differential equations using the RPSM [12].

Numerous physical phenomena, including heat conduction in memory-containing materials and diffusion processes, are modeled by fractional integro-differential equations. In [27], Nawaz used the variational iteration method and the homotopy perturbation method to approximate solutions to the nonlinear boundary value problems for 4^{th}-order fractional integro-differential equations. While, for the linear Volterra integro-fractional differential equation, Ahmed and Salh in [2] employed the generalized Taylor matrix approach. Also, the homotopy perturbation method was used by Saeedi and Samimi [35] to approximate non-linear Fredholm integro-fractional differential equations. The Chebyshev-Legendre spectral approach for fractional Fredholm integro-differential equations was later proposed by the authors of [44]. Rashed [32] addressed a particular class of integro-differential equations. Additionally, Hu [20] proposed the interpolation collocation method to solve the Fredholm linear integro-differential equations analytically. Heris used the modified Laplace Adomian decomposition approach to find approximate solutions for integro-differential equations. However, no one has yet used RPSM to solve fractional integro-differential equations in two dimensions. Evaluating the function coefficients that show up in the definite integral term is the difficult part. Fractional integro-differential equation in two-dimensions has numerical solutions obtained in [15] using the Haar wavelet method.

Integro-differential equations are a special and intriguing area of mathematics. The theory of heat conduction is typically the source of problems involving time as an independent variable. For example, its solution yields the temperature at a distance x following a temperature distribution of t seconds. In practice, integro-differential equations representing dynamical systems frequently provide ambiguous or insufficient information, particularly when it comes to heat transfer equations. The initial conditions, boundary conditions, and other components of integro-differential equations can all exhibit this uncertainty. Therefore, in order to get more accurate results than those based on real conditions, we solve integro-differential equations based on neutrosophic conditions. In [26], the authors give a scientific summary of neutrosophists.

An equation with an unknown neutrosophic function appearing under one or more integral signs is known as a neutrosophic integral equation, which has the following form

$$u(x, I) = f(x, I) + \lambda \int_a^b K(x, t, I)u(t, I)dt,$$

where $u(x, I)$ is the unknown neutrosophic function, I is an indeterminate number, while $f(x, I)$ and $K(x, t, I)$ are known functions, and λ, a, b are constants, for more details see [13, 40–42]. In [39], a suggested neutrosophic Laplace transform technique that makes it possible to solve integral equations involving non-linear neutrosophic numbers effectively is presented. A modified iterative approach in a neutrosophic setting has been used in [25] to determine the numerical solution of the second-kind Fredholm integral equation. Using certain theorems, it has been shown that the iterative method converges in a neutrosophic environment. For further studies of these results, these tools can be developed by linking them with other concepts that can be found in the following works [1, 4–8, 23].

In this article, we will examine two novel methods for finding exact or numerical solutions to the two-dimensional neutrosophic fractional integro-differential equations of the general form:

$$D_t^\alpha u(x, t, I) = f(x, t, I) + g(x, I) \int_0^a \int_0^b K(t, y, \tau, I)u(y, \tau, I)d\tau dy, \tag{1}$$

where $D_t^\alpha u(x, t, I)$ is the Caputo derivative of $u(x, t, I)$ with respect to t, and $0 < \alpha \leq 1$ denotes the fractional derivative of order α. We solve equation (1) subject to the neutrosophic initial condition $u(x, 0, I) = F(x, I)$.

Examine a semi-infinite rod in which the memory of previous states and the current temperature distribution both affect heat transfer. Equation (1) provides a two-dimensional fractional Fredholm integro-differential equation that models this phenomenon. Memory effects in heat transfer processes are represented by the fractional derivative of order α, the kernel function characterizing the memory effect is $K(t, y, \tau, I)$, and the temperature distribution at position x and time t is $u(x, t, I)$. Here, the fractional-order rate of temperature change is captured by $D_t^\alpha u(x, t, I)$, which takes into consideration memory effects that are common in materials with thermal hysteresis. The cumulative effect of previous temperatures, weighted by the kernel K, is represented by the expression $g(x, I) \int_0^a \int_0^b K(t, y, \tau, I)u(y, \tau, I), d\tau, dy$. Applications of this model can be found in materials with unusual heat conduction characteristics, such as composite materials, biological tissues, or polymers, where thermal response is dependent on both the current and previous states (see [24, 30]).

2. Preliminaries and Basic Concepts

2.1. *Caputo's Fractional Derivative*

In this work, the fractional derivatives are considered in the sense of Caputo, which is defined in the following definition.

Definition 2.1 (Caputo derivative)**.** [16] Let $\alpha > 0$ such that $n-1 < \alpha < n$ for some $n \in \mathbb{N}$. The Caputo fractional derivative of $f(x)$ is defined by

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x f^{(n)}(t)(x-t)^{n-\alpha-1} dt. \tag{2}$$

It is not hard to check that Caputo derivative is linear operator. The following formulas are helpful in order to compute Caputo derivative of common functions (see [14, 16, 33]).

(1) $D_t^\alpha (t^p) = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} \quad , & n-1 < \alpha < n,\ p > n-1,\ p \in \mathbb{R} \\ 0 \quad , & n-1 < \alpha < n,\ p \leq n-1,\ p \in \mathbb{N}_0. \end{cases}$

(2) The Caputo fractional derivative of the exponential function has the following form:

$$D_t^\alpha e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)}.$$

2.2. *Fractional Power series*

Definition 2.2. [18] A fractional power series (FBS) about c is a formal sum of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^{n\alpha} = a_0 + a_1(x-c)^\alpha + a_2(x-c)^{2\alpha} + \cdots ,$$

where α>0 $x \geq c$, c is a constant, x is a variable, and $a_n^{\prime} s$ are the coefficients of the series.

As in the case of ordinary power series, the fractional power series has interval of convergence and radius of convergence, which we state in the following two theorems (see [18]).

Theorem 2.3. *We have two cases for the FPS* $\sum_{n=0}^{\infty} a_n x^{n\alpha}$, $x \geq 0$*:*

(1) If the FPS converges when $x = b > 0$*, then it converges whenever* $0 \leq x < b$*,*
(2) If the FPS diverges when $x = d > 0$*, then it diverges whenever* $x > d$*.*

Theorem 2.4. *The PS at* $c = 0$ *and* $-\infty < x < \infty$ *has radius of convergence* R *if and only if the FPS at* $c = 0$ *and* $x \geq 0$ *has radius of convergence* $R^{\frac{1}{\alpha}}$*.*

3. Residual Power Series Method (RPSM)

3.1. *Methodology of RPSM*

In this part, we solve 2-dimensional fractional Fredholm integro-differential equations using the residual power series approach. How to handle the evaluation of the coefficient functions that show up in the integral term is our contribution. To the best of our knowledge, we are the first to solve these kinds of problems using RPSM. Consider the following: Integro-differential equation of Fredholm

$$D_t^{\alpha} u(x,t,I) = f(x,t,I) + g(x,I) \int_0^a \int_0^b K(t,y,\tau,I) u(y,\tau,I) d\tau dy, \tag{3}$$

based on the condition

$$u(x,0,I) = C_0(x,I) \tag{4}$$

where $0 < \alpha \leq 1$ and $(x,t,I) \in [0,a] \times [0,b]$.
It is assumed by the residual power series approach [28] that the fractional power series form of the solution of (3) can be expanded

$$u(x,t) = \sum_{n=0}^{\infty} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

and defines the N^{th} approximation of $u(x,t)$ by

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{5}$$

Moreover, the k^{th} residual function for (3) is defined by

$$\text{Res}_{u,k}(x,t) = D_t^{\alpha} u_k(x,t) - f(x,t) - g(x) \int_0^a \int_0^b K(t,y,\tau) u_N(y,\tau) d\tau dy. \tag{6}$$

Note that the fact that Caputo derivative of a constant is 0 implies that

$$D_t^{(k-1)\alpha} \text{Res}_{u,k}(x,t) \Big|_{t=0} = 0, \ \texttt{for} \ k = 1, 2, \ldots .$$

We must compute the function coefficients $C_1(x), C_2(x), ..., C_N(x)$ in order to get an approximate solution of N^{th}, where $C_0(x) = u(x,0)$ is already provided. Therefore, we build the system.

$$D_t^{(k-1)\alpha} \text{Res}_{u,k}(x,t) \Big|_{t=0} = 0, \ \texttt{for} \ k = 1, 2, ...N. \tag{7}$$

We simplify (7) by computing

$$D_t^{(k-1)\alpha}\mathrm{Res}_{u,k}(x,t)\Big|_{t=0} = D_t^{(k-1)\alpha}\left(D_t^{\alpha}\left(\sum_{n=0}^{k} C_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right) - f(x,t) - g(x)\int_0^a\int_0^b K(t,y,\tau)\left(\sum_{n=0}^{k} C_n(y)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}\right)d\tau dy\right)\Big|_{t=0}$$

$$= \sum_{n=0}^{k} C_n(x)\frac{D_t^{k\alpha}[t^{n\alpha}]}{\Gamma(n\alpha+1)} - D_t^{(k-1)\alpha}f(x,t) - g(x)\int_0^a\int_0^b D_t^{(k-1)\alpha}[K(t,y,\tau)]\left(\sum_{n=0}^{k} C_n(y)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}\right)d\tau dy\Big|_{t=0}$$

$$= C_k(x) - D_t^{(k-1)\alpha}f(x,0) - g(x)\int_0^a\int_0^b D_t^{(k-1)\alpha}K(0,y,\tau)\left(\sum_{n=0}^{k} C_n(y)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}\right)d\tau dy$$

$$= C_k(x) - D_t^{(k-1)\alpha}f(x,0) - g(x)\sum_{n=0}^{k}\int_0^a C_n(y)\left(\int_0^b D_t^{(k-1)\alpha}K(0,y,\tau)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}d\tau\right)dy.$$

If we set

$$h_{k,n}(y) = \int_0^b D_t^{(k-1)\alpha}K(0,y,\tau)\frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)}d\tau, \quad \text{for } n = 0,1,2,...,k, \tag{8}$$

and

$$\eta_{k,n} = \int_0^a C_n(y)h_{k,n}(y)dy, \quad \text{for } n = 0,1,2,...,k. \tag{9}$$

Then, for $k = 1,2,3,...,N$, we have

$$D_t^{(k-1)\alpha}\mathrm{Res}_{u,k}(x,t)\Big|_{t=0} = C_k(x) - D_t^{(k-1)\alpha}f(x,0) - g(x)\sum_{n=0}^{k}\eta_{k,n} = 0,$$

or equivalently,

$$C_k(x) = D_t^{(k-1)\alpha}f(x,0) + g(x)\sum_{n=0}^{k}\eta_{k,n} = 0, \tag{10}$$

Our next step is to calculate $\eta_{k,n}$ for $n = 0,1,2,...,k$ and $k = 1,2,3,...,N$. The process is recursive. In Equation (10), enter $k = 1$ to obtain

$$C_1(x) = f(x,0) + g(x)\eta_{1,0} + g(x)\eta_{1,1}. \tag{11}$$

Observe that

$$\eta_{1,0} = \int_0^a\int_0^b K(0,y,\tau)C_0(y)d\tau dy.$$

Thus, $\eta_{1,1}$ is found in terms of $\eta_{1,0}$. Multiplying Equation (11) by $h_{1,1}(x)$ and integrating over $[0,a]$ will do this.

$$\eta_{1,1} - \int_0^a h_{1,1}(x)\left(f(x,0)+\eta_{1,0}g(x)\right)dx - \eta_{1,1}\int_0^a h_{1,1}(x)g(x)dx = 0.$$

It follows that

$$\eta_{1,1} = \frac{\int_0^a h_{1,1}(x)\left(f(x,0)+\eta_{1,0}g(x)\right)dx}{1-\int_0^a h_{1,1}(x)g(x)dx},$$

and finally,

$$C_1(x) = f(x,0) + g(x)\left(\eta_{1,0}+\eta_{1,1}\right).$$

Now, plug $k=2$ in Equation 10:

$$C_2(x) = D_t^{\alpha} f(x,0) + g(x)\eta_{2,0} + g(x)\eta_{2,1} + g(x)\eta_{2,2} \tag{12}$$

We find only $\eta_{2,2}$ because, once more, $\eta_{2,0}$ and $\eta_{2,1}$ are known. As in $\eta_{1,1}$, we solve for $\eta_{2,2}$ by multiplying Equation (12) by $h_{2,2}(x)$, integrating over $[0,a]$.

$$\eta_{2,2} = \frac{\int_0^a h_{2,2}(x)\left(D_t^{\alpha} f(x,0)+\eta_{2,0}g(x)+\eta_{2,1}g(x)\right)dx}{1-\int_0^a h_{2,2}(x)g(x)dx}.$$

Inductively, we have

$$C_k(x) = D_t^{(k-1)\alpha} f(x,0) + g(x)\sum_{n=0}^{k}\eta_{k,n},$$

where

$$\eta_{k,n} = \int_0^a\int_0^b \frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)} D_t^{(k-1)\alpha} K(0,y,\tau)C_n(y)d\tau dy, \quad \text{for } n=0,1,2,...,k-1, \tag{13}$$

and

$$\eta_{k,k} = \frac{\int_0^a h_{k,k}(x)D_t^{(k-1)\alpha} f(x,0)dx + \int_0^a g(x)h_{k,k}(x)dx\sum_{n=0}^{k-1}\eta_{k,n}}{1-\int_0^a h_{k,k}(x)g(x)dx}. \tag{14}$$

4. **Applications of FPSM to Linear Fractional Problem**

Here, we go over two cases to illustrate the process of the suggested approach. Following a review of the findings, we will offer some suggestions and remarks.

Example 4.1. Consider the two-dimensional linear fractional Fredholm integro-differential equation:

$$D_t^{0.5}u(x,t) = -\frac{4}{675} + \frac{3}{4}\sqrt{\pi}\,t\,x^2 + \int_0^1\int_0^1 (y^2-\tau^2)u(y,\tau)\,d\tau\,dy, \tag{15}$$

subject to the initial condition $u(x,0)=4$. The exact solution is given by: $u(x,t) = x^2\,t^{3/2}+4$. and $f(x,t) = -\frac{4}{675}+\frac{3}{4}\sqrt{\pi}\,t\,x^2$ and $g(x)=1$ provide the source term in this equation. According to the technique, we assume that the approximate solution takes the following form:

$$u(x,t) = \sum_{n=0}^{4} C_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

In this case, we employ $\alpha = 0.5$ and the first four fractional powers of the fractional power series expansion. To calculate $C_0(x)$, we simply utilize the initial condition, which yields:

$$C_0(x) = u(x, 0) = 4.$$

While, the coefficient $C_1(x)$ is given by:

$$\begin{aligned} C_1(x) &= f(x, 0) + g(x)\left(\eta_{1,0} + \eta_{1,1}\right) \\ &= -\frac{4}{675} + \eta_{1,0} + \eta_{1,1} = \frac{32}{5400 + 42525\sqrt{\pi}} - \frac{4}{675}. \end{aligned}$$

Therefore, the first residual power series approximate solution is:

$$u_1(x, t) = 4 - \frac{56\sqrt{t}}{75(8 + 63\sqrt{\pi})}.$$

Next, in order to compute $C_2(x)$, it is to be noted that the coefficient $C_2(x)$ is given by:

$$\begin{aligned} C_2(x) &= D_t^{\frac{1}{2}} f(x, 0) + g(x)\left(\eta_{2,0} + \eta_{2,1} + \eta_{2,2}\right) \\ &= \eta_{2,0} + \eta_{2,1} + \eta_{2,2} = 0. \end{aligned}$$

As a result, the second approximate residual power series solution is:

$$u_2(x, t) = 4 - \frac{56\sqrt{t}}{75(8 + 63\sqrt{\pi})}.$$

Finally, we compute $C_3(x)$. The coefficient $C_3(x)$ is given by:

$$C_3(x) = D_t^{(2)\frac{1}{2}} f(x, 0) + g(x)\sum_{n=0}^{3} \eta_{3,n} = \frac{3\sqrt{\pi}x^2}{4}.$$

Consequently, the following is the estimated third residual power series solution:

$$u_3(x, t) = 4 - \frac{56\sqrt{t}}{75(8 + 63\sqrt{\pi})} + t^{\frac{3}{2}}x^2.$$

We provide Table 1, which displays the absolute errors at chosen sites for particular values of x and t, to demonstrate the accuracy of the approximate solutions. Additionally, Figure (1) shows the comparison of the approximate and exact results.

Example 4.2. Consider the fractional Fredholm integro-differential equation (1) as

$$D_t^{\alpha} u(x, t) = f(x, t) + g(x)\int_0^1 \int_0^1 K(t, y, \tau)u(y, \tau)\, d\tau\, dy, \tag{16}$$

subject to the initial condition:

$$u(x, 0) = C_0(x), \tag{17}$$

where:

$$f(x, t) = -(2/3) + e^t\,\mathrm{Erf}(\sqrt{t}), \quad K(t, y, \tau) = y\tau, \quad g(x) = 1, \quad C_0(x) = 1 + x, \quad \text{and } \alpha = 0.5.$$

t	x	$u(x,t)$	$u_3(x,t)$	$\lvert u(x,t) - u_3(x,t)\rvert$
0.1	0.1	4.00032	3.999125	1.19×10^{-3}
	0.2	4.00127	3.99956	1.71×10^{-3}
	0.3	4.00255	4.00168	1.04×10^{-3}
	0.4	4.00506	4.00475	8.70×10^{-4}
	0.5	4.00791	4.00854	6.31×10^{-4}
	0.6	4.01139	4.01285	1.46×10^{-3}
	0.7	4.01549	4.01767	2.18×10^{-3}
	0.8	4.02020	4.02918	2.91×10^{-3}
	0.9	4.02554	4.02918	3.63×10^{-3}
0.3	0.1	4.10207	4.10254	4.70×10^{-4}
	0.2	4.05578	4.05879	3.01×10^{-3}
	0.3	4.08795	4.08145	6.50×10^{-3}
	0.4	4.25454	4.25987	5.33×10^{-3}
	0.5	4.38511	4.38125	3.86×10^{-3}
	0.6	4.41325	4.41456	1.31×10^{-3}
	0.7	4.65421	4.65897	4.76×10^{-3}
	0.8	4.81250	4.81453	2.03×10^{-3}
	0.9	4.85652	4.85132	5.20×10^{-3}
0.5	0.1	4.32315	4.32520	2.05×10^{-3}
	0.2	4.15864	4.15352	5.12×10^{-3}
	0.3	4.30195	4.30315	1.20×10^{-3}
	0.4	4.36862	4.36121	7.41×10^{-3}
	0.5	4.55385	4.55982	5.97×10^{-3}
	0.6	4.81148	4.81671	5.23×10^{-3}
	0.7	4.10541	4.10662	1.21×10^{-3}
	0.8	4.44472	4.44802	3.30×10^{-3}
	0.9	4.82362	4.82555	1.93×10^{-3}

TABLE 1. Absolute Error Analysis for the Approximate Solutions at Selected Points for Example 4.1

The exact solution is $u(x,t) = e^t + x$. It is assumed that the RPSM solution can be expressed as follows in order to determine the problem's approximate solution:

$$u(x,t) = \sum_{n=0}^{\infty} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}, \tag{18}$$

where the N-th approximation is:

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{19}$$

Our next objective is to determine the coefficients and residual function: The residual function for that is:

$$\mathrm{Res}_{u,k}(x,t) = D_t^{\alpha} u_k(x,t) - f(x,t) - g(x) \int_0^a \int_0^b K(t,y,\tau) u_N(y,\tau)\, d\tau\, dy. \tag{20}$$

Recursively, the coefficients $C_k(x)$ are calculated using:

$$C_k(x) = D_t^{(k-1)\alpha} f(x,0) + g(x) \sum_{n=0}^{k} \eta_{k,n}, \tag{21}$$

$$\eta_{k,n} = \int_0^a C_n(y) \int_0^b \frac{\tau^{n\alpha}}{\Gamma(n\alpha+1)} D_t^{(k-1)\alpha} K(0,y,\tau)\, d\tau\, dy. \tag{22}$$

For $k = 1$, we compute:

$$C_1(x) = f(x,0) + g(x)(\eta_{1,0} + \eta_{1,1}),$$

where:

$$\eta_{1,0} = \int_0^a \int_0^b K(0,y,\tau) C_0(y)\, d\tau\, dy.$$

We now determine the recursive coefficient computation. When the coefficients are greater $(k \geq 2)$:

$$C_k(x) = D_t^{(k-1)\alpha} f(x,0) + g(x) \sum_{n=0}^{k} \eta_{k,n}, \tag{23}$$

$$\eta_{k,k} = \frac{\int_0^a h_{k,k}(x) D_t^{(k-1)\alpha} f(x,0) dx + \int_0^a g(x) h_{k,k}(x) \sum_{n=0}^{k-1} \eta_{k,n} dx}{1 - \int_0^a h_{k,k}(x) g(x) dx}. \tag{24}$$

For this example:

$$C_1(x) = -(1/4) - 1/(-4 + 10\sqrt{\pi}), \quad C_2(x) = 1, \quad C_3(x) = 0, \quad C_4(x) = 1, \ldots$$

Finally, the approximate solution is:

$$u_{\mathrm{app}}(x,t) = C_0(x) + \sum_{n=1}^{N} C_n(x) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{25}$$

where:

$$u_{\mathrm{app}}(x,t) = 1 + x + \frac{2\left(-(1/4) - 1/(-4 + 10\sqrt{\pi})\right)\sqrt{t}}{\sqrt{\pi}} + t + \frac{t^2}{2} + \frac{t^3}{6}.$$

In order to study the error analysis, we refer to table 2 that present the absolute error for various values of x and t. This illustration shows how to approximate solutions for fractional integro-differential equations using the Residual Power Series Method (RPSM).

t	x	Approximate Solution	Absolute Error
0.1	0.0	0.989961	0.104246
	0.1	1.089960	0.095596
	0.2	1.189960	0.088272
	0.3	1.289960	0.081990
	0.4	1.389960	0.076543
	0.5	1.489960	0.071774
0.3	0.0	1.149960	0.148090
	0.1	1.249960	0.137876
	0.2	1.349960	0.128980
	0.3	1.449960	0.121162
	0.4	1.549960	0.114238
	0.5	1.649960	0.108062
0.5	0.0	1.388230	0.157998
	0.1	1.488230	0.148963
	0.2	1.588230	0.140905
	0.3	1.688230	0.133675
	0.4	1.788230	0.127150
	0.5	1.888230	0.121232

Table 2. Absolute Error Analysis for the Approximate Solutions at Selected Points for Example 4.2

5. **Modified Fractional Power Series Method**

In order to address the same problem, the power series solution approach is modified to fractional order in this section's work.

5.1. *Methodology of Modified Fractional Power Series Method (MFPSM)*

In this section we present a new method using fractional series to solve fractional integro-differential equations of two dimensions. More precisely, we consider the following fractional integro-differential equation

$$D_t^{\alpha} u(x,t) = f(x,t) + \int_0^a \int_0^b K(y,\tau)u(y,\tau)dyd\tau, \tag{26}$$

subject to the initial condition

$$u(x,0) = C_0(x),$$

where $0 < \alpha \leq 1$ and $(x,t) \in [0,a] \times [0,b]$.

We assume the solution has the fractional series expansion

$$u(x,t) = \sum_{n=0}^{\infty} C_n(x) t^{n\alpha}. \tag{27}$$

Next, we approximate $u(x,t)$ by

$$u(x,t) \approx u_N(x,t) = \sum_{n=0}^{N} C_n(x) t^{n\alpha}.$$

Where $C_n(x)$ are unknown function coefficients to determined. Using the notation $\beta! = \Gamma(\beta+1)$, the fact that

$$D_t^{\alpha}\left(t^{n\alpha}\right) = \frac{(n\alpha)!}{((n-1)\alpha)!} t^{(n-1)\alpha},$$

and substituting $u_N(x,t)$ in (26) to get

$$\begin{aligned} D_t^{\alpha}(C_0(x) + C_1(x)t^{\alpha} + C_2(x)t^{2\alpha} + \cdots + C_N(x)t^{N\alpha}) &= f(x,t) \\ &+ \int_0^a \int_0^b K(y,\tau)\left(C_0(y) + \cdots + C_N(y)t^{N\alpha}\right) d\tau dy. \end{aligned}$$

Define

$$A = \int_0^a \int_0^b K(y,\tau)\left(C_0(y) + \cdots + C_N(y)t^{N\alpha}\right) d\tau dy, \tag{28}$$

then we have

$$C_1(x)\alpha! + C_2(x)\frac{2\alpha!}{\alpha!}t^{\alpha} + \cdots + C_N(x)\frac{(N\alpha)!}{((N-1)\alpha)!}t^{(N-1)\alpha} = f(x,t) + A. \tag{29}$$

Now, evaluate Equation (29) at $t = 0$ to obtain

$$C_1(x)\alpha! = f(x,0) + A.$$

Thus

$$C_1(x) = \frac{f(x,0) + A}{\alpha!}, \tag{30}$$

where A still unknown constant. Next, by differentiating Equation (29) α-fractional derivative and evaluating at $t = 0$, one can have

$$C_2(x)(2\alpha)! = D_t^{\alpha} f(x,0),$$

thus

$$C_2(x) = \frac{D_t^{\alpha} f(x,0)}{(2\alpha)!}.$$

Inductively, differentiate Equation (29) n α-fractional derivatives and evaluate at $t = 0$, to get

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}, \qquad n = 2, 3, 4, \cdots, N. \tag{31}$$

Note that the coefficients $C_n(x)$ are known now, except for $n = 1$. Now, to evaluate $C_1(x)$ we need to compute the constant A. Go back to

$$A = \sum_{n=0}^{N} \int_0^a \int_0^b K(y,\tau) C_n(y) \tau^{n\alpha} d\tau dy,$$

and set

$$K_n = \int_0^a \int_0^b K(y,\tau) C_n(y) \tau^{n\alpha} d\tau dy.$$

Then,

$$\begin{aligned}
K_0 &= \int_0^a \int_0^b K(y,\tau) u(y,0) d\tau dy, \\
K_1 &= \int_0^a \int_0^b K(y,\tau) \frac{f(y,0)+A}{\alpha!} \tau^{\alpha} d\tau dy, \text{ and} \\
K_n &= \int_0^a \int_0^b K(y,\tau) \frac{D_t^{(n-1)\alpha} f(y,0) \tau^{n\alpha}}{(n\alpha)!} d\tau dy, \text{ for} n = 2,3,4,...,N.
\end{aligned}$$

Hence,

$$\begin{aligned}
A &= \sum_{n=0}^{N} K_n \\
&= (K_0 + K_2 + \cdots + K_N) + K_1 \\
&= (K_0 + K_2 + \cdots + K_N) + \int_0^a \int_0^b K(y,\tau) \frac{f(y,0)}{\alpha!} \tau^{\alpha} d\tau dy + A \int_0^a \int_0^b \frac{K(y,\tau)\tau^{\alpha}}{\alpha!} d\tau dy.
\end{aligned}$$

If we solve for A, we find that

$$A = \frac{K_0 + \sum_{n=2}^{N} K_n + \int_0^a \int_0^b K(y,\tau) \frac{f(y,0)}{\alpha!} \tau^{\alpha} d\tau dy}{1 - \int_0^a \int_0^b \frac{K(y,\tau)\tau^{\alpha}}{\alpha!} d\tau dy}. \tag{32}$$

Finally, we summaries the algorithm of the modified fractional power series method in the following steps. **Step 1.** Assume that the approximate solution of the form

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x) t^{\alpha n}.$$

Step 2. Compute the coefficient functions

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}, \qquad n = 2,3,4,\cdots,N.$$

Step 3. Compute the constant

$$A = \frac{\sum_{n=0}^{N} A_n}{1 - \int_0^a \int_0^b \frac{K(y,\tau)\tau^{\alpha}}{\alpha!} d\tau dy},$$

where,

$$A_0 = \int_0^a \int_0^b K(y,\tau)u(y,0)d\tau dy,$$
$$A_n = \int_0^a \int_0^b K(y,\tau)\frac{D_t^{(n-1)\alpha} f(y,0)\tau^{n\alpha}}{(n\alpha)!}d\tau dy, \quad \text{for } n = 1,2,3,4,...,N.$$

Step 4. Finally, compute

$$C_1(x) = \frac{f(x,0)+A}{\alpha!}.$$

Therefore,

$$u(x,t) \approx \sum_{n=0}^{N} C_n(x)t^{\alpha n}$$

is now determined.

6. Convergence and Error Analysis of (MFPSM)

Novel results associated with the convergent of the (MFPSM) are studies in this section. The analysis show a fast convergence rate and small calculation error.

6.1. *Convergence of MFPSM*

In this section, we give some details analysis for the convergence of the modified approache.

Theorem 6.1. *Let the fractional integro-differential equation be given by*

$$D_t^{\alpha} u(x,t) = f(x,t) + \int_0^a \int_0^b K(x,t)u(x,t)\, dt\, dx,$$

with initial condition $u(x,0) = C_0(x)$*, where* $0 < \alpha \leq 1$ *and* $(x,t) \in [0,a] \times [0,b]$*. The series solution*

$$u(x,t) = \sum_{n=0}^{\infty} C_n(x)t^{n\alpha}$$

converges to the exact solution, and the truncated approximation

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x)t^{n\alpha}$$

converges uniformly to $u(x,t)$ *as* $N \to \infty$*, where the coefficients* $C_n(x)$ *satisfy:*

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}.$$

Proof. To demonstrate that the truncated series $u_N(x,t)$ converges to the exact solution $u(x,t)$.

$$E_N(x,t) = u(x,t) - u_N(x,t) = \sum_{n=N+1}^{\infty} C_n(x) t^{n\alpha}.$$

The coefficients $C_n(x)$ are determined as:

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}.$$

Assuming the fractional derivative $D_t^{(n-1)\alpha} f(x,0)$ is bounded for all n, there is a constant M_n according to which

$$|C_n(x)| \leq \frac{M_n}{(n\alpha)!}.$$

Therefore, the error term can be bounded by:

$$|E_N(x,t)| \leq \sum_{n=N+1}^{\infty} \frac{M_n |t|^{n\alpha}}{(n\alpha)!}.$$

We use the Weierstrass M-test to demonstrate uniform convergence. Let:

$$M_n = \sup_{x\in[0,a]} |D_t^{(n-1)\alpha} f(x,0)|.$$

Then:

$$\sum_{n=N+1}^{\infty} \frac{M_n |t|^{n\alpha}}{(n\alpha)!}$$

The factorial term $(n\alpha)!$ in the denominator rapidly grows, causing the series to converge. Thus, the series $\sum_{n=0}^{\infty} C_n(x) t^{n\alpha}$ converges uniformly on any compact subset of $[0,a] \times [0,b]$, implying

$$\lim_{N\to\infty} u_N(x,t) = u(x,t).$$

As a result, the MFPSM has achieved convergence. □

6.2. *Error Order of MFPSM*

This subsection discusses novel results on the rate error order of the MFPSM.

Theorem 6.2. *For the approximate solution given by:*

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x) t^{n\alpha},$$

the truncation error is of the order:

$$|E_N(x,t)| = |u(x,t) - u_N(x,t)| = \mathcal{O}\left(\frac{|t|^{(N+1)\alpha}}{((N+1)\alpha)!}\right).$$

Proof. Consider the error term:

$$E_N(x,t) = \sum_{n=N+1}^{\infty} C_n(x)t^{n\alpha}.$$

Recall that

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}.$$

Thus, the error can be bounded as:

$$|E_N(x,t)| \leq \sum_{n=N+1}^{\infty} \frac{|D_t^{(n-1)\alpha} f(x,0)||t|^{n\alpha}}{(n\alpha)!}.$$

Using Stirling's approximation for the factorial term $(n\alpha)!$ for large n.

$$(n\alpha)! \approx \sqrt{2\pi n\alpha}\left(\frac{n\alpha}{e}\right)^{n\alpha}.$$

As n increases, the terms $C_n(x)t^{n\alpha}$ decrease rapidly. For large N, the dominant term in the error sum is given by:

$$|E_N(x,t)| \approx \frac{|D_t^{N\alpha} f(x,0)||t|^{(N+1)\alpha}}{((N+1)\alpha)!}.$$

Consequently, the error is of the order:

$$|E_N(x,t)| = \mathcal{O}\left(\frac{|t|^{(N+1)\alpha}}{((N+1)\alpha)!}\right).$$

The factorial growth in the denominator causes a rapid decrease in error as N increases, confirming the method's high-order convergence. □

7. **Solving Equation (1) by MFPSM**

Here we will present some examples to study the effectiveness of the newly developed method (MFPSM) and its use in approximating solutions to the fractional integro-differential equations under study.

Example 7.1. Consider the two-dimensional linear fractional Fredholm integro-differential equation provided by:

$$D_t^{0.5}u(x,t) = \frac{\sqrt{\pi}}{2}e^x - \frac{2(e-1)t}{3} + \int_0^1\int_0^1 u(y,\tau)t\,d\tau\,dy. \tag{33}$$

Subject to the initial condition $u(x,0) = 0$. The exact solution is given by $u(x,t) = \sqrt{t}e^x$. To solve the given equation, we use the Modified Fractional Power Series Method (MFPSM). First, we assume an approximate solution, where $\alpha = 0.5$, to be in the following form:

$$u_N(x,t) = \sum_{n=0}^{N} C_n(x)t^{n/2}, \tag{34}$$

Next, we compute the coefficient functions $C_n(x)$ given by:

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!}, \quad n = 2,3,4,\ldots,N. \tag{35}$$

Given that $D_t^{0.5} u(x,t) = \frac{\sqrt{\pi}}{2} e^x - \frac{2(e-1)t}{3}$, we first focus on the constant A using the integral terms. The constant A is given by:

$$A = \frac{\sum_{n=0}^{N} A_n}{1 - \int_0^a \int_0^b \frac{K(y,\tau)\tau^\alpha}{\alpha!} \, d\tau \, dy}, \tag{36}$$

where:

$$A_0 = \int_0^a \int_0^b K(y,\tau) u(y,0) \, d\tau \, dy,$$

$$A_n = \int_0^a \int_0^b K(y,\tau) \frac{D_t^{(n-1)\alpha} f(y,0) \tau^{n\alpha}}{(n\alpha)!} \, d\tau \, dy, \quad \text{for } n = 1,2,3,\ldots,N.$$

In the above integral equation, the kernel function is $K(y,\tau) = \tau$. Since $u(y,0) = 0$, we have:

$$A_0 = \int_0^1 \int_0^1 \tau \cdot u(y,0) \, d\tau \, dy = 0. \tag{37}$$

For $n \geq 1$:

$$A_n = \int_0^1 \int_0^1 \tau \cdot \frac{D_t^{(n-1)\alpha} f(y,0) \cdot \tau^{n\alpha}}{(n\alpha)!} \, d\tau \, dy. \tag{38}$$

Since $\alpha = 0.5$, we have $n\alpha = 0.5n$. Thus:

$$A_n = \int_0^1 \int_0^1 \frac{\tau^{1+0.5n}}{(n/2)!} \cdot D_t^{(n-1)\alpha} f(y,0) \, d\tau \, dy. \tag{39}$$

Given that the source term $f(x,t) = \frac{\sqrt{\pi}}{2} e^x - \frac{2(e-1)t}{3}$ yields zero higher-order derivatives at $t = 0$ for $n \geq 2$, we have:

$$D_t^{(n-1)\alpha} f(y,0) = 0 \quad \text{for } n \geq 2 \implies A_n = 0 \quad \text{for } n \geq 2. \tag{40}$$

To compute A_1, we have

$$A_1 = \int_0^1 \int_0^1 \frac{\tau \cdot D_t^0 f(y,0) \cdot \tau^{0.5}}{(0.5)!} \, d\tau \, dy = \int_0^1 \int_0^1 \frac{\tau^{1.5} \cdot \frac{\sqrt{\pi}}{2} e^y}{\Gamma(1.5)} \, d\tau \, dy. \tag{41}$$

Using $\Gamma(1.5) = \frac{\sqrt{\pi}}{2}$, we arrive at:

$$A_1 = \int_0^1 \int_0^1 e^y \cdot \tau^{1.5} \, d\tau \, dy. \tag{42}$$

Computing simple integrals, we obtain

$$A_1 = \int_0^1 e^y \cdot \frac{2}{5} \, dy = \frac{2}{5} \left[e^y\right]_0^1 = \frac{2}{5}(e-1). \tag{43}$$

That leads to the values of

$$A = \frac{\frac{2}{5}(e-1)}{1 - \frac{2}{5}} = \frac{\frac{2}{5}(e-1)}{\frac{3}{5}} = \frac{2}{3}(e-1). \tag{44}$$

and,

$$C_1(x) = \frac{f(x,0)+A}{\alpha!} = \frac{\frac{\sqrt{\pi}}{2}e^x + \frac{2}{3}(e-1)}{\frac{\sqrt{\pi}}{2}} = e^x. \tag{45}$$

For Higher-Order Terms $C_n(x)$, when $n \geq 2$ we have

$$C_n(x) = \frac{D_t^{(n-1)\alpha} f(x,0)}{(n\alpha)!} = 0 \quad \text{(since higher derivatives vanish).} \tag{46}$$

Finally, the approximate solution, which matches the exact solution, and confirming the accuracy of the MFPSM is provided by

$$u_N(x,t) = \sum_{n=0}^{1} C_n(x) t^{n/2} = C_0(x) + C_1(x)t^{0.5} = 0 + e^x \cdot t^{0.5} = \sqrt{t}e^x. \tag{47}$$

Example 7.2. Consider the fractional Fredholm integro-differential equation:

$$D_t^\alpha u(x,t) = f(x,t) + g(x) \int_0^1 \int_0^1 K(t,y,\tau) u(y,\tau)\, d\tau\, dy, \tag{48}$$

subject to the initial condition:

$$u(x,0) = C_0(x), \tag{49}$$

where,

$$f(x,t) = -(2/3) + e^t\, \mathrm{Erf}(\sqrt{t}), \quad K(t,y,\tau) = y\tau, \quad g(x) = 1, \quad C_0(x) = 1+x, \quad \text{and } \alpha = 0.5.$$

The exact solution is:

$$u(x,t) = e^t + x.$$

Define auxiliary function $f_1(x,t)$ by

$$f_1(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha}\, d\tau - \int_0^1 \int_0^1 K(x,t) u(x,t)\, dx\, dt. \tag{50}$$

Simplified, this becomes:

$$f(x,t) = -\frac{2}{3} + e^t\, \mathrm{Erf}(\sqrt{t}).$$

The coefficients $C_n(x)$ are calculated as:

$$C_n(x) = \left. \frac{\partial^{(n-1)\alpha} f(x,t)}{\Gamma(n\alpha+1)} \right|_{t=0}. \tag{51}$$

This generates the values

$$C_2 = 1,\, C_3 = 0,\, C_4 = \frac{1}{2},\, C_5 = 0,\, C_6 = \frac{1}{6},\, C_7 = 0,\, C_8 = \frac{1}{24}, \ldots$$

While, the constant A is computed as:

$$A = \frac{\sum_{n=2}^{M} K_n + K_0 + \int_0^1 \int_0^1 K(x,t) \frac{f_1(x,0)}{\Gamma(\alpha+1)} t^\alpha\, dt\, dx}{1 - \int_0^1 \int_0^1 K(x,t) \frac{1}{\Gamma(\alpha+1)} t^\alpha\, dt\, dx}. \tag{52}$$

Here:

$$K_n = \int_0^1 \int_0^1 K(x,t) C_n(x) t^{n\alpha} \, dt \, dx,$$
$$K_0 = \int_0^1 \int_0^1 K(x,t) C_0(x) \, dt \, dx.$$

Upon passing simple calculations, we arrive at the value $A \approx 0.666667$. The approximate solution is given by:

$$u_{\text{app}}(x,t) = C_0(x) + C_1(x) t^{\alpha} + \sum_{n=2}^{M} C_n(x) t^{n\alpha}. \tag{53}$$

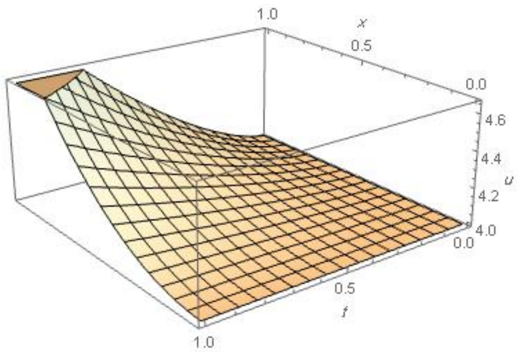
Substituting the coefficients, the solution becomes:

$$u_{\text{app}}(x,t) = 1 + x - 1.52113 \times 10^{-9} \sqrt{t} + t + 0.5t^2 + 0.166667t^3 + \dots$$

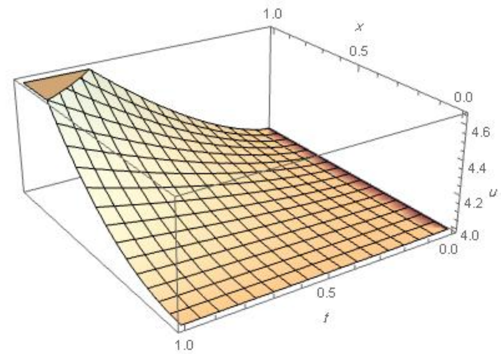
In regrads to the error analysis, Table (3) present the absolute error for various values of x and t. The efficiency of the Modified Fractional Power Series Method (MFPSM) in approximating solutions to fractional integro-differential equations is illustrated by this example.

t	x	Absolute Error
0.1	0.0	8.5223×10^{-8}
	0.1	8.5223×10^{-8}
	0.2	8.5223×10^{-8}
	0.3	8.5223×10^{-8}
	0.4	8.5223×10^{-8}
	0.5	8.5223×10^{-8}
0.3	0.0	2.1308×10^{-5}
	0.1	2.1308×10^{-5}
	0.2	2.1308×10^{-5}
	0.3	2.1308×10^{-5}
	0.4	2.1308×10^{-5}
	0.5	2.1308×10^{-5}
0.5	0.0	2.8377×10^{-4}
	0.1	2.8377×10^{-4}
	0.2	2.8377×10^{-4}
	0.3	2.8377×10^{-4}
	0.4	2.8377×10^{-4}
	0.5	2.8377×10^{-4}

TABLE 3. Absolute Error for the Approximate Solutions at Selected Points for Example 7.2



(A) Exact solution.



(B) The third RPS approximation.

FIGURE 1. The behavior of the third RPS approximation together with exact solution for Example 4.1.

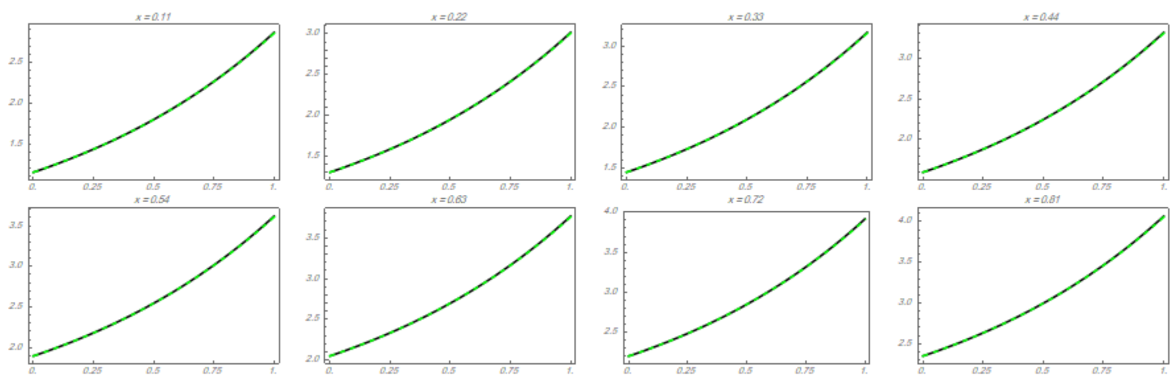


FIGURE 2. Approximate solution for different values of x in Example 7.2.

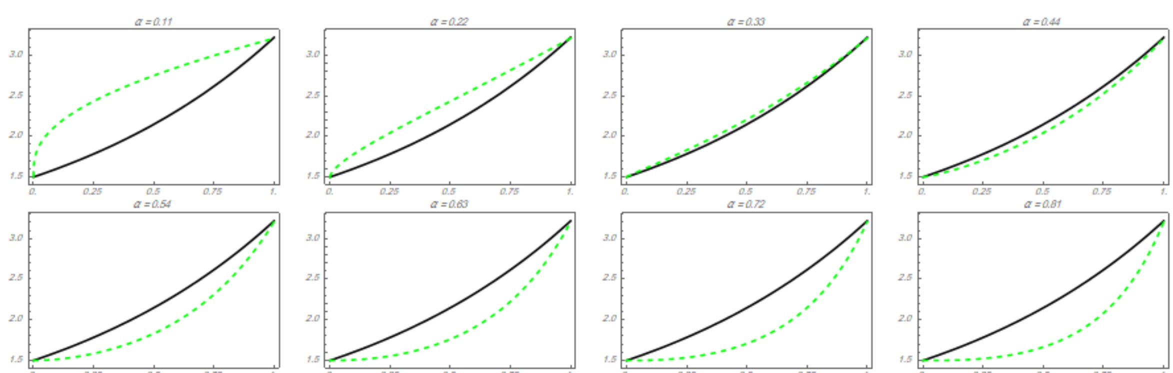


FIGURE 3. Approximate solution for different values of α in Example 7.2.

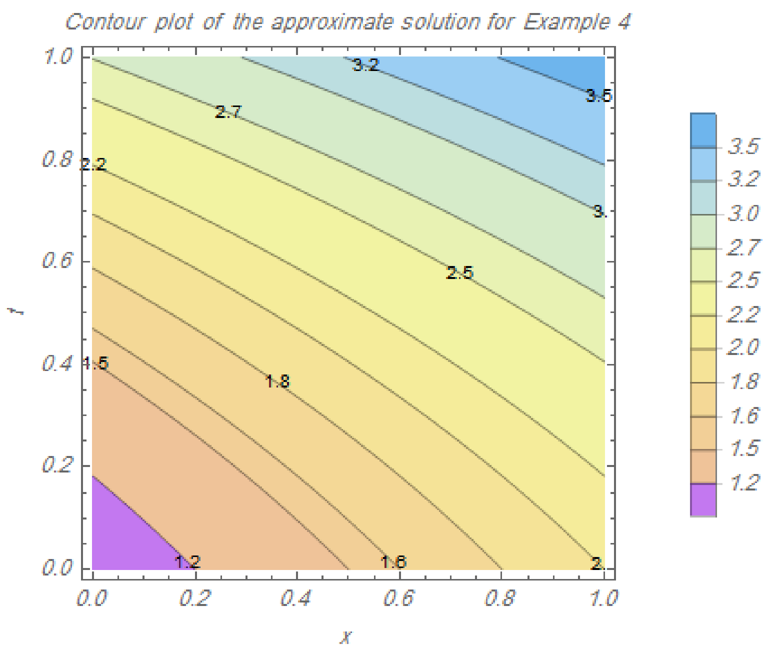


FIGURE 4. The approximate solution in Example 7.2 .

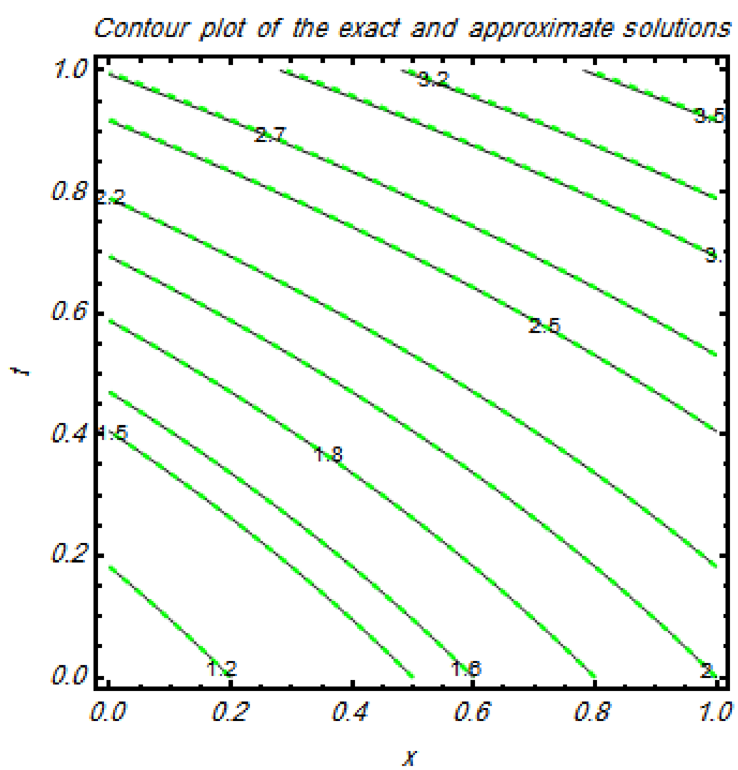


FIGURE 5. Values for both exact (Black) and approximate (Green) solutions in Example 7.2.

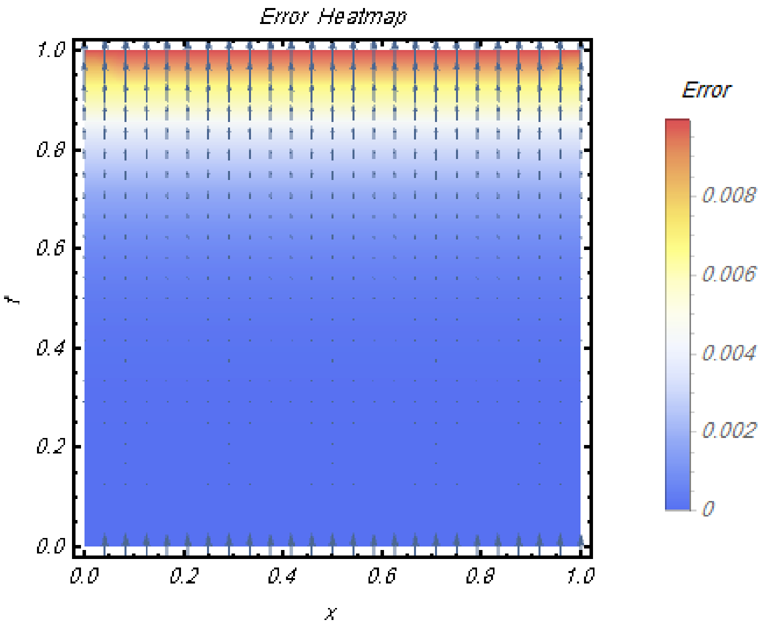


FIGURE 6. The error between the exact and approximate solution, where the blue color indicates a small error in example 7.2.

A quick look at Figure 2, when the value of $\alpha = 0.5$ (where the solution is exact) is present and we plot the approximate solution for several different values of the variable x, we found that the approximate solution matched the exact solution. In contrast, looking at Figure 3, where we draw the approximate solution for several different α, we notice from the figure that the closer the value of α to 0.5, the more the approximate solution is identical to the exact solution, and vice versa, that the further the value of α is from 0.5, the farther the approximate solution is from the exact solution. Figure 4 represents a contour graph of the approximate solution when $\alpha = 0.5$ with values of $u(x,t)$ labeled along the contour lines. In Figure 5 we plot both the exact solution (black) and the approximate one (green) together with the values of $u(x,t)$ labeled along the contour lines, showing that the approximate solution agrees with the exact. Figure 6 show the absolute error for the obtained approximate solution when $\alpha = 0.5$, and both x, t are between 0 and 1, we observe that the dark blue color indicates the area where the solution is more accurate.

For comparison purposes between RPSM and MFPSM, we introduce the following example.

Example 7.3. The same fractional Fredholm integro-differential equation (1) is solved using the Modified Residual Power Series Method (MRPSM) and the Residual Power Series Method (RPSM), with

$$f(x,t) = -(2/3) + e^t \operatorname{Erf}(\sqrt{t}), \quad K(t,y,\tau) = y\tau, \quad g(x) = 1, \quad u(x,0) = 1 + x, \quad \alpha = 0.5.$$

The exact solution is:

$$u(x,t) = e^t + x.$$

In the table 4, we provide an overview of the absolute errors for the Residual Power Series Method (RPSM) and the Modified Residual Power Series Method (MRPSM) over a range of x and t values.

t	x	Error (MRPSM)	Error (RPSM)
0.1	0.0	8.5223×10^{-8}	0.104246
	0.1	8.5223×10^{-8}	0.095596
	0.2	8.5223×10^{-8}	0.088272
	0.3	8.5223×10^{-8}	0.081990
	0.4	8.5223×10^{-8}	0.076543
	0.5	8.5223×10^{-8}	0.071774
0.3	0.0	2.1308×10^{-5}	0.148090
	0.1	2.1308×10^{-5}	0.137876
	0.2	2.1308×10^{-5}	0.128980
	0.3	2.1308×10^{-5}	0.121162
	0.4	2.1308×10^{-5}	0.114238
	0.5	2.1308×10^{-4}	0.108062
0.5	0.0	2.8377×10^{-4}	0.157998
	0.1	2.8377×10^{-4}	0.148963
	0.2	2.8377×10^{-4}	0.140905
	0.3	2.8377×10^{-4}	0.133675
	0.4	2.8377×10^{-4}	0.127150
	0.5	2.8377×10^{-4}	0.121232

TABLE 4. Absolute Error Analysis using both MRPSM and RPSM for for Example 7.3

The following are our accuracy findings: For all x and t values, the MRPSM provides significantly less absolute errors than RPSM. Additionally, the errors using MRPSM are orders of magnitude reduced, particularly at smaller values of t. In terms of error consistency, we find that MRPSM errors are consistent across different x values at the same t, indicating the robustness of the solution approach. However, RPSM exhibits more errors and is more volatile. In terms of appropriateness, MRPSM performs better in situations requiring a high level of precision, such as scientific computations or problems where accuracy is essential. Nonetheless, the RPSM provides a less complex and computationally intensive method, which is advantageous for exploratory studies or circumstances in which approximations suffice. Lastly, we find that MRPSM requires more computing work to handle higher-order modifications, but that the precision of MRPSM outweighs the expense in high-stakes scenarios. However, RPSM is useful for fast estimations since it strikes a mix between convenience and reasonable accuracy.

Conclusion

In this paper, we applied the residual power series method to the two dimensional neutrosophic fractional integro differential equations with smooth kernel. All the previous works did not involve the Fredholm type using the residual power series method, the problem is to find the coefficients $C_1, C_2, \ldots, C_n$. We introduced a technique to find these coefficients. We found analytic solution and numerical solution for some of two-dimensional fractional- Fredholm integro differential equations to show the accuracy of this method. To obtain higher accuracy, we suggested a new residual power series method and we applied this method to some examples. The examples show that this modified Residual power series method is efficient and accurate more than the ordinary Residual power series method. Most the time we obtain an exact solution.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Majid Mohammed Abed, Nasruddin Hassan, and Faisal Al-Sharqi. On neutrosophic multiplication module. *Neutrosophic Sets and Systems*, 49(1):198–208, 2022.
2. ShazadShawki Ahmed and Shokhan Ahmed Hama Salh. Generalized taylor matrix method for solving linear integro-fractional differential equations of volterra type. *Applied Mathematical Sciences*, 5(33-36):1765–1780, 2011.
3. Abeer M. Al-Bugami. Two-dimensional fredholm integro-differential equation with singular kernel and its numerical solutions. *Advances in Mathematical Physics*, 2022(Article ID 2501947), 2022.
4. Y. Al-Qudah and F. Al-Sharqi. Algorithm for decision-making based on similarity measures of possibility interval-valued neutrosophic soft settings. *Inter. J. Neutrosophic Sci.*, 22(3):69–83, 2023.
5. Y. Al-Qudah, F. Al-Sharqi, M. Mishilish, and M. M. Rasheed. Hybrid integrated decision-making algorithm based on AO of possibility interval-valued neutrosophic soft settings. *Inter. J. Neutrosophic Sci.*, 22(3):84–98, 2023.
6. Yousef Al-Qudah, Abdulqader O. Hamadameen, Nadia Abdalla Kh, and Faisal Al-Sharqi. A new generalization of interval-valued q-neutrosophic soft mztrix and its applications. *Inter. J. Neutrosophic Sci.*, 25(3):242–257, 2025.
7. Faisal Al-Sharqi, Yousef Al-Qudah, and Naif Alotaibi. Decision-making techniques based on similarity measures of possibility neutrosophic soft expert sets. *Neutrosophic Sets and Systems*, 55(1):22, 2023.
8. Faisal Ghazi Al-Sharqi, Majd Mohammed Abed, and Ali A Mhassin. On polish groups and their applications. *Journal of Engineering and Applied Sciences*, 13(18):7533–7536, 2018.

9. Marwan Alquran. Analytical solutions of fractional foam drainage equation by residual power series method. *Mathematical sciences*, 8(4):153–160, 2014.
10. Marwan Alquran. Analytical solution of time-fractional two-component evolutionary system of order 2 by residual power series method. *J. Appl. Anal. Comput*, 5(4):589–599, 2015.
11. O Abu Arqub. Series solution of fuzzy differential equations under strongly generalized differentiability. *Journal of Advanced Research in Applied Mathematics*, 5(1):31–52, 2013.
12. Mine Aylin Bayrak and Ali Demir. A new approach for space-time fractional partial differential equations by residual power series method. *Applied Mathematics and Computation*, 336:215–230, 2018.
13. Prasen Boro and Bhimraj Basumatary. Application of neutrosophic fourier transform in solving heat equation and integral equation. *Neutrosophic Sets and Systems*, 73, 2024.
14. Michele Caputo. Linear models of dissipation whose q is almost frequency independent—ii. *Geophysical Journal International*, 13(5):529–539, 1967.
15. Amer Darweesh, Marwan Alquran, and Khawla Aghzawi. New numerical treatment for a family of two-dimensional fractional fredholm integro-differential equations. *Algorithms*, 13(37), 2020-doi:10.3390/a13020037.
16. Kai Diethelm and Neville J Ford. Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 265(2):229–248, 2002.
17. Tahereh Eftekhar and Jalil Rashidinia. an investigation on existence uniqueness and approximate solutions for two-dimensional nonlinear fractional integro-differential equations. *Mathematics*, 11(824), 2023.
18. Ahmad El-Ajou, Omar Arqub, Zeyad Zhour, and Shaher Momani. New results on fractional power series: theories and applications. *Entropy*, 15(12):5305–5323, 2013.
19. Shatha Hasan, Ayat Alzoubi, Asad Freihat, Mohammed Al-Smad, and Shaher Momani. Solution of fractional sir epidemic model using residual power series method. *Applied Mathematics and Information Sciences*, 13:153–161, 03 2019.
20. Qiya Hu. Interpolation correction for collocation solutions of fredholm integro-differential equations. *Mathematics of Computation of the American Mathematical Society*, 67(223):987–999, 1998.
21. Iryna Komashynska, Mohammed Al-Smadi, Abdallah Al-Habahbeh, and Ali Ateiwi. Analytical approximate solutions of systems of multi-pantograph delay differential equations using residual power-series method. *arXiv preprint arXiv:1611.05485*, 2016.
22. Iryna Komashynska, Mohammed Al-Smadi, Ali Ateiwi, and Sadoon Al-Obaidy. Approximate analytical solution by residual power series method for system of fredholm integral equations. *Appl. Math. Inf. Sci*, 10:1–11, 2016.
23. Abed M. M, Al-Sharqi F. G, and Mhassin A. Study fractional ideals over some domains. *AIP conference Proceedings*, 2138(1), 2019.
24. Francesco Mainardi. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. World Scientific, Singapore, 2010.
25. S. Moi, Biswas, S., and S.P. Sarkar. An efficient method for solving neutrosophic fredholm integral equations of second kind. *Granul. Comput.*, 8:1–22, 2023.
26. Gugu Narzary, Bhimraj Basumatary, and Giorgio Nordo. A neutrosophic solution of heat equation by neutrosophic laplace transform. *Neutrosophic Sets and Systems*, 73:606–614, 2024.
27. Yasir Nawaz. Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations. *Computers & Mathematics with Applications*, 61(8):2330–2341, 2011.
28. Juan Nieto, Alireza Khastan, and K Ivaz. Numerical solution of fuzzy differential equations under generalized differentiability. *Nonlinear Analysis: Hybrid Systems*, 3:700–707, 11 2009.
29. Keith Oldham and Jerome Spanier. *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, volume 111. Elsevier, 1974.

30. Igor Podlubny. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. Academic Press, San Diego, 1999.
31. Prapart Pue-on. The fractional residual power series method for solving a system of linear fractional fredholm integro-differential equations. *Mathematics and Statistics*, 9(5), 2021.
32. MT Rashed. Numerical solution of a special type of integro-differential equations. *Applied Mathematics and computation*, 143(1):73–88, 2003.
33. Herrmann Richard. *Fractional calculus: an introduction for physicists*. World Scientific, 2014.
34. Yaser Rostami and Khosrow Maleknejad. The solution of the nonlinear mixed partial integro-differential equation via two-dimensional hybrid functions. *Mediterr. J. Math.*, 19(89):1–20, 2022.
35. H Saeedi and F Samimi. He's homotopy perturbation method for nonlinear ferdholm integrodifferential equations of fractional order. *International Journal of Engineering Research and Applications*, 2(5):52–56, 2012.
36. R. Saneifard, A. Jafarian, N. Ghalami, and S. Measoomy Nia. Extended artificial neural networks approach for solving two-dimensional fractional-order volterra-type integro-differential equations. *Infromation Sciences*, 612:887–897, 2022.
37. Mehmet Şenol, Marwan Alquran, and Hamed Daei Kasmaei. On the comparison of perturbation-iteration algorithm and residual power series method to solve fractional zakharov-kuznetsov equation. *Results in Physics*, 9:321–327, 2018.
38. Mehmet Şenol and ATA Ayşe. Approximate solution of time-fractional kdv equations by residual power series method. *Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, 20(1):430–439, 2018.
39. A. Mohammed shapique and E. Mathivadhana. Modified non-linear triangular neutrosophic numbers: Theory and applications in integral equation. *Neutrosophic Sets and Systems*, 72, 2024.
40. F. Smarandache. *Neutrosophy-Neutrosophic Probability, Set and Logic*. American Research Press, Rehoboth, USA, 1998.
41. F. Smarandache. *Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics*. University of New Mexico, Gallup, NM 87301, USA, 2002.
42. F. Smarandache. *Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability*. Sitech-Education Publisher, Craiova-Columbus, 2013.
43. Fairouz Tchier, Mustafa Inc, Zeliha S Korpinar, and Dumitru Baleanu. Solutions of the time fractional reaction–diffusion equations with residual power series method. *Advances in Mechanical Engineering*, 8(10):1687814016670867, 2016.
44. A Yousefi, Shahnam Javadi, Esmail Babolian, and Eslam Moradi. Convergence analysis of the chebyshev–legendre spectral method for a class of fredholm fractional integro-differential equations. *Journal of Computational and Applied Mathematics*, 358:97–110, 2019.

Received: Sep 25, 2024. Accepted: Dec 30, 2024