On Consistent and Weak Transitive Neutrosophic Fuzzy Matrices

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Abstract: This paper delves into the properties of two specific types of Neutrosophic Fuzzy Matrices (NFM) namely consistent and weakly transitive NFM. A significant focus is placed on nilpotent and transitive NFM, highlighting their critical role in the analysis. It is demonstrated that these two types of matrices are controllable, and a formula is derived for determining the canonical form of a weakly transitive NFM. To support and clarify the findings, counterexamples are provided throughout the discussion. We introduce an operation on NFM, referred to as the Gödel implication operator. Utilizing this operator, we establish several significant results for NFM, with a particular emphasis on properties related to pre-orders. Our analysis focuses primarily on reflexive and transitive NFM, enabling us to derive meaningful insights. Additionally, we demonstrate a method for constructing an idempotent NFM from any given matrices.

Keywords: Weakly transitive, Gödel implication operator, Consistent, Idempotent NFM.

1. Introduction

The concept of fuzzy sets, presented by Zadeh [1], has significantly influenced research in modeling uncertainty and imprecision. This foundational work laid the groundwork for various extensions, including intuitionistic fuzzy sets by Atanasov [15, 33], which incorporate membership, non-membership, and hesitation degrees, offering a more generalized framework for representing imprecise information. These advancements have spurred the development of fuzzy and intuitionistic fuzzy matrices, essential tools for solving complex mathematical problems and decision-making challenges. The study of fuzzy matrices began with Thomason [2], who examined their convergence properties, and Kim and Roush [3, 41], who introduced generalized fuzzy matrices. Research by Hashimoto [8, 35,38-39] and Chenggong [7] further advanced the field by exploring canonical forms and transitivity properties of fuzzy matrices. Additionally, Emam and Fendh [6, 35] investigated max-min and min-max compositions, providing new insights into fuzzy relations and their applications. In the context of intuitionistic fuzzy matrices, Khan and Pal [22, 44] explored generalized inverses, while Muthuraji et al. [24, 24, 27] focused on decomposition techniques and methods to reduce intuitionistic fuzzy matrices to standard fuzzy matrices. These contributions have expanded the utility of intuitionistic fuzzy matrices in areas such as preference modeling [14, 32], aggregation operations [28], and transitivity analysis [29].

Smarandache [52] introduced neutrosophic sets, which extend fuzzy and intuitionistic fuzzy sets by integrating a third component for indeterminacy. This innovation has inspired researchers to explore neutrosophic fuzzy matrices (NFM) as a framework for analyzing uncertainty in data. Despite significant progress, challenges remain. For instance, while properties like transitivity [9, 27] and idempotence [10, 45] have been extensively studied, their extension to newer constructs such as neutrosophic fuzzy matrices requires further exploration. Anandhkumar et al [4–5] have studied pseudo Similarity of Neutrosophic Fuzzy matrices and On various Inverse of Neutrosophic Fuzzy Matrices. Anandhkumar et al [50] have presented Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices. Moreover, relationships between various transitivity-related concepts and their practical implications warrant deeper investigation [29, 47]. Sudha et al [54-56] have studied A new technique for using Symmetric pentagonal Intuitionistic fuzzy numbers to solve Assignment problem, An Approach for solving fuzzy sequencing problem with Triangular fuzzy Neutrosophic numbers using Ranking function and A Method for finding Critical path problem with Triangular Neutrosophic Numbers by utilizing numerous methods for Ranking.

Anandhkumar et al [42-43] have presented Secondary k-column symmetric Neutrosophic Fuzzy Matrices and Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices. This paper builds on the foundational work of Zadeh [1], Atanasov [15, 33], and others, extending the analysis of intuitionistic fuzzy and neutrosophic fuzzy matrices. We focus on reflexive and transitive properties and propose methods for constructing idempotent matrices, addressing existing gaps and advancing the understanding of these mathematical constructs. This article delves into the properties of two specific types of Neutrosophic Fuzzy Matrices (NFM) namely consistent and weakly transitive NFM. A significant focus is placed on nilpotent and transitive NFM, highlighting their critical role in the analysis. It is demonstrated that these two types of matrices are controllable, and a formula is derived for determining the canonical form of a weakly transitive NFM. To support and clarify the findings, counterexamples are provided throughout the discussion. We introduce an operation on NFM, referred to as the Gödel implication operator. Utilizing this operator, we establish several significant results for NFM, with a particular emphasis on properties related to pre-orders. Our analysis focuses primarily on reflexive and transitive NFM, enabling us to derive meaningful insights. Additionally, we demonstrate a method for constructing an idempotent NFM from any given matrices.

1.1 Abbreviations

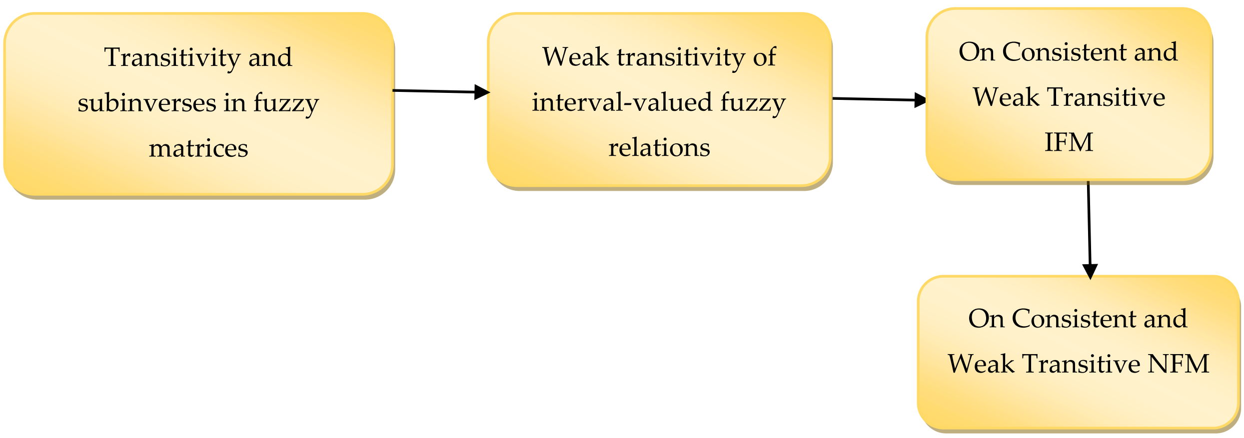
IFM = Intuitionistic Fuzzy Matrices
NFM = Neutrosophic Fuzzy Matrices
NFPM = Neutrosophic Fuzzy Permutation Matrices
TNFM = Transitive Neutrosophic Fuzzy Permutation Matrices

1.2 Research gap

Mishref, Emam and Xu YJ, Wang H M, Yu DJ [45,33,53] introduced the concepts of Transitivity and subinverses in fuzzy matrices, Weak transitivity of interval-valued fuzzy relations, On Consistent and Weak Transitive IFM. Building on their work, we have applied these principles to NFM. In this study, we examine several results and extend these concepts to NFMs.

Table:1 An overview of the advancements in extending NFM.

References	**Extension of NFM.**	**Year**
E.G. Emam [45]	Transitivity and subinverses in fuzzy matrices	1992
Xu YJ et al [32]	Weak transitivity of interval-valued fuzzy relations	2014
E. G. Emam [53]	On Consistent and Weak Transitive IFM	2022
Proposed	On Consistent and Weak Transitive NFM	2024



Flowchart for the research gap

1.3 Structure of the Article

This article is structured to provide a logical and detailed presentation of our research on Neutrosophic Fuzzy Matrices (NFM). It begins with a Literature Review [2], where we analyze existing studies, identifying key advancements and limitations while establishing the relevance of NFM in addressing contemporary challenges. Following this, the section on Novelty [4] and Contribution of Our Work [5] highlights the unique aspects of our approach, showcasing how our research fills critical gaps in the existing body of knowledge. In Preliminaries [5] , we introduce the foundational concepts, definitions, and notations that form the basis for the proposed methodologies. This ensures that readers are equipped with the necessary tools to engage with the subsequent sections. The Theorems and Results section [6] presents the core findings of the study, including the primary theorems, supported by rigorous proofs and illustrative examples.

The article further delves into the Reflexivity and Transitivity [7] of Neutrosophic Fuzzy Matrices, exploring these essential properties in depth and discussing their implications for theoretical and practical applications. Finally, the Conclusion and Future [8] Work section summarizes the key outcomes of our research and provides insights into potential avenues for future exploration in this domain. This structured approach ensures clarity and coherence, guiding the reader through the progression of our work.

1.4 Comparison of Neutrosophic Fuzzy Matrices with Existing Models

To comprehensively understand the significance and advantages of Neutrosophic Fuzzy Matrices (NFM), it is essential to compare them with existing models such as fuzzy matrices and intuitionistic fuzzy matrices.

(i) **Fuzzy Matrices**

Fuzzy matrices are widely used for modeling uncertainty, where each element of the matrix is represented by a value in the interval [0,1][0,1][0,1], reflecting the degree of membership of an element to a set. However, fuzzy matrices are limited in addressing situations with incomplete or conflicting information. They operate solely within the framework of membership values and lack the ability to quantify non-membership or indeterminate states.

(ii) **Intuitionistic Fuzzy Matrices (IFM)**

Intuitionistic fuzzy matrices extend fuzzy matrices by incorporating both membership μ and non-membership ν values, ensuring that $\mu+\nu \leq 1$ While this model improves the representation of uncertainty, it still assumes that the sum of the membership and non-membership values accounts for all possibilities, neglecting the explicit representation of indeterminacy. This limitation makes intuitionistic fuzzy matrices less effective in scenarios involving a high degree of vagueness or incomplete knowledge.

(iii) **Neutrosophic Fuzzy Matrices (NFM)**

Neutrosophic fuzzy matrices overcome the limitations of both fuzzy and intuitionistic fuzzy matrices by introducing three independent components: membership (T), non-membership (F), and indeterminacy (I), where $T,F,I\in[0,1]$ and $T+F+I \leq 3$. This three-component structure provides a more flexible and comprehensive framework for representing uncertainty, conflict, and indeterminacy simultaneously. Consequently, NFM is particularly suitable for complex systems where multiple sources of uncertainty coexist.

Comparative Insights

While fuzzy matrices and intuitionistic fuzzy matrices have been effective in handling uncertainty to varying degrees, NFM offers superior expressiveness and adaptability. By explicitly accommodating indeterminacy, NFM allows for more nuanced modeling and analysis of uncertain systems, making it a powerful tool for solving problems in areas like decision-making, systems analysis, and artificial intelligence.

2. Literature Review

Fuzzy set theory, presented by Zadeh [1], has laid a robust foundation for handling uncertainty and imprecision, influencing a wide array of fields, including matrix theory. The study of fuzzy matrices began with Thomason [2], who explored their convergence properties, and was further developed by Kim and Roush [3, 41], who generalized the concept of FM. Punithavalli et al [12] have presented Kernel and k-Kernel Symmetric Intuitionistic Fuzzy Matrices**.** Anandhkumar et al [13] have discussed Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices. These works emphasized the role of FMs in modeling dynamic systems and their structural properties. Hashimoto [8, 38–40] contributed significantly to understanding the canonical forms and decomposition of fuzzy matrices, providing key insights into their structural analysis. Chenggong [7] focused on strongly transitive matrices, developing canonical forms that enhance their applicability in system analysis. Emam and Fendh [6, 35] investigated the max-min and min-max compositions of bifuzzy matrices, revealing novel relationships between fuzzy relations and operations. Anandhkumar et al [19] have studied Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices. Punithavalli et al [20] have focussed on Reverse Sharp and Left-T Right-T Partial Ordering On Intuitionistic Fuzzy Matrices.

In the domain of IFMs, Atanasov [15, 33] extended the fuzzy set theory by incorporating membership, non-membership, and hesitation degrees. Pal et al. [18, 46] and Khan and Pal [22, 44] explored intuitionistic fuzzy matrices, focusing on generalized inverses and algebraic structures. Muthuraji et al. [23,24,27] advanced this research by examining decomposition methods, representation techniques, and reduction strategies, enabling the transformation of intuitionistic fuzzy matrices into fuzzy matrices while preserving critical algebraic properties. The properties of transitivity and idempotence in fuzzy and intuitionistic fuzzy matrices have also been widely studied. Hashimoto [39] and Jian-Xin [10, 11] analyzed. Anandhkumar et al [25] have presented Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices. Radhika et al [26] have characterized On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices.

3. Novelty

This research introduces several innovative contributions to the study of neutrosophic fuzzy matrices, addressing key gaps in the literature. One significant advancement is the extension of the Gödel implication operator to NFM, enabling a deeper exploration of pre-order relations, particularly in reflexive and transitive matrices. Radhika et al [30] have studied Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making. Prathab et al [31] presented Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses. Anandhkumar et al [49] have studied Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems. While earlier studies [4, 25] focused on basic operations in fuzzy matrices, this work broadens their scope by applying them to NFM. Another notable contribution is the derivation of idempotent neutrosophic fuzzy matrices (NFM) from any given matrix, a topic that remains largely unexplored in existing research [9, 27]. Anandhkumar et al [36-37] have studied Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic fuzzy matrices and Generalized Symmetric Neutrosophic Fuzzy Matrices.

Furthermore, this study introduces canonical forms for weakly transitive NFM, building upon prior work on strongly transitive matrices [6, 38] and extending their applicability to cases involving weaker transitivity conditions. Additionally, by integrating neutrosophic set theory [52] into fuzzy matrix analysis, the research bridges the gap between classical fuzzy matrices and their neutrosophic counterparts, allowing for the modeling of indeterminacy alongside membership and non-membership values. These contributions collectively enhance the theoretical understanding and practical utility of neutrosophic fuzzy matrices, offering new tools and methodologies for applications in decision-making, systems analysis, and beyond.

4. Contribution of Our Work

(i) **Analysis of Consistent and Weakly Transitive NFM**:

This study examines two specific types of neutrosophic fuzzy matrices (NFM): consistent and weakly transitive NFM. We demonstrate their controllability and derive a formula for the canonical form of weakly transitive NFM, providing a novel analytical perspective.

(ii) **Significance of Nilpotent and Transitive NFM**:

A significant focus is placed on the properties of nilpotent and transitive NFM, emphasizing their critical role in matrix analysis and contributing to the broader understanding of their applications in uncertainty modeling.

(iii) **Gödel Implication Operator on NFM**:
We introduce the Gödel implication operator as a novel operation on NFM, extending its applicability and enabling the derivation of significant results. This operator plays a key role in analyzing pre-orders and their implications in reflexive and transitive NFM.

(iv) **Construction of Idempotent NFM**:
A method for constructing idempotent NFM from any given matrix is presented, addressing an underexplored area in existing literature. This contribution provides a new approach for generating matrices with desirable algebraic properties.

(v) **Counterexamples for Clarity**:
To enhance the understanding and practical applicability of the results, counterexamples are provided throughout the discussion, clarifying the theoretical findings.

5. Preliminaries

In this part, we introduce operations for NFMs. For two NFMs P and Q, we define the subsequent operations $U \vee V$, $U \wedge V$.

$$U \vee V = \left[u_{ij} \vee v_{ij}\right] = \left[\max < u_{ij}^{T}, v_{ij}^{T} >, \max < u_{ij}^{I}, u_{ij}^{I} >, \min < u_{ij}^{F}, u_{ij}^{F} >\right]$$

$$U \wedge V = \left[u_{ij} \wedge v_{ij}\right] = \left[\min < u_{ij}^{T}, v_{ij}^{T} >, \min < u_{ij}^{I}, v_{ij}^{I} >, \max < u_{ij}^{F}, v_{ij}^{F} >\right]$$

Definition: 5.1 A NFSs P on the universe of discourse Y is well-defined as

$U = \left\{< y, u^{T}(y), u^{I}(y), u^{F}(y) >, y \in Y\right\}$, everywhere $u^{T}, u^{I}, u^{F} : Y \rightarrow]^{-}0,1^{+}[$ also

$0 \leq u^{T} + u^{I} + u^{F} \leq 3.$

Definition:5.2 A neutrosophic Fuzzy Matrices U is less then or equal to V (or U and V are comparable)

That is $U \leq V$ if $\left(u_{ij}^{T}, u_{ij}^{I}, u_{ij}^{F}\right) \leq \left(v_{ij}^{T}, v_{ij}^{I}, v_{ij}^{F}\right)$ means $u_{ij}^{T} \leq v_{ij}^{T}, u_{ij}^{I} \leq v_{ij}^{I}, u_{ij}^{F} \geq v_{ij}^{F}$.

Example: 5.1 Let us consider 3x3 NFM

$$U = \begin{bmatrix} <0.5,0.5,0.3> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \\ <0.8,0.1,0.3> & <0.8,0.2,0.3> & <0.8,0.2,0.3> \\ <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.2> \end{bmatrix}$$

$$V=\begin{bmatrix} <0.9,0.6,0.1> & <0.7,0.5,0.2> & <0.9,0.3,0.1> \\ <0.8,0.7,0.2> & <1,0.7,0.2> & <0.9,0.8,0.2> \\ <0.8,0.4,0.1> & <1,0.6,0.1> & <0.9,0.3,0.1> \end{bmatrix}$$

$$(0.5,0.5,0.3)\leq(0.9,0.6,0.1)$$

$$0.5\leq 0.9, 0.5\leq 0.6, 0.3\geq 0.1$$

Therefore, all the above elements satisfy the same manner

Definition 5.3 . A NFM is considered null if all its elements are (0,0,0). This type of matrix is denoted by$N_{(0,0,0)}$. On the other hand, an NFM is defined as zero if all its elements are (0,0,1) and it is represented by O.

Definition 5.4 A square NFM is referred to as a Neutrosophic Fuzzy Permutation Matrix (NFPM) if each row and each column contains exactly one element with a value of (1,1,0) while all other entries are (0,0,1).

Example: 5.2 Let us consider 3x3 NFPM

$$P=\begin{bmatrix} (0,0,1) & (1,1,0) & (0,0,1) \\ (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$$

Definition 5.5 For identity NFM of order nx n is represented by I_n and is well-defined by

$$\left(\delta_{ij}^{\ T},\delta_{ij}^{\ I},\delta_{ij}^{\ F}\right)=\begin{cases}(1,1,0)\ if\ \ i=j\\(0,0,1)\ if\ \ i\neq j\end{cases}.$$

Example: 5.3 Let us consider 3x3 INFM

$$P=\begin{bmatrix} (1,1,0) & (0,0,1) & (0,0,1) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (0,0,1) & (0,0,1) & (1,1,0) \end{bmatrix}$$

We now present the following operations for NFMs $U=\left(u^T,u^I,u^F\right)$ and $V=\left(v^T,v^I,v^F\right)$ us defined the binary operation

(i) $U*V=\left[u_{ij}*v_{ij}\right].$

(ii) $U\Theta V=\left[u_{ij}\Theta v_{ij}\right].$

(iii) $U\times V=\left[\bigcup_{k=1}^{n}\left(u_{ik}\wedge v_{kj}\right)\right].$

(iv) $U^{k+1} = U^k \times U, (k = 1, 2, 3, \ldots)$

(v) $U^T = \left[u_{ji}{}^{\alpha}, u_{ji}{}^{\beta}, u_{ji}{}^{\gamma} \right]$ (the transpose of U)

(vi) $\Delta U = U \Theta U^T$

(vii) $\nabla U = U \wedge U^T$

(viii) $U^2 = U$ (U is idempotent)

(ix) $U \wedge I_n = 0$ (U is irreflexive)

(x) $U^T = U$ (U is SNFM)

(xi) $U \wedge U^T \le I_n$ (P is antisymmetric)

(xii) $$u \Theta v = \begin{cases} (0,0,1) & \text{if } u^T \le v^T,\ u^I \le v^I \text{ and } u^F \ge v^F \\ (0,0,u^F) & \text{if } u^T \le v^T, u^I \le v^I \text{ and } u^F < v^F \\ (u^T, u^I, u^F) & \text{if } u^T > v^T, u^I > v^I \text{ and } (u^F \ge v^F \text{ or } u^F < v^F) \end{cases}$$

$u \le v$ iff $u^T \le v^T, u^I \le v^I, u^F \ge v^F$

Definition 5.6 Let $i, j, k \le n$ and let $U = \left[u_{ij} = \left(u_{ij}{}^T, u_{ij}{}^I, u_{ij}{}^F \right) \right]$ be an NFM. Then U is called

Transitive iff $U^2 = UU \le U$ *i.e.,* $u_{ik}{}^T \wedge u_{kj}{}^T \le u_{ij}{}^T, u_{ik}{}^I \wedge u_{kj}{}^I \le u_{ij}{}^I$ and $u_{ik}{}^F \vee u_{kj}{}^F \ge u_{ij}{}^F$ for every $i, j, k \le n.$

Nilpotent iff $U^n = UU \ldots U$ (n-times) = 0

consistent iff $u_{ij} \ge u_{ji}$ and $u_{jk} \ge u_{kj}$ implies $u_{ik} \ge u_{ki}$.

Weak transitive if and only if $u_{ij} > u_{ji}$ and $u_{jk} > u_{kj}$ implies $u_{ik} > u_{ki}$

Example: 5.4 Let assume that 3x3 Weak transitive NFM

$$\text{NFM} U = \begin{bmatrix} <0.8, 0.1, 0.1> & <0.1, 0.2, 0.3> & <0.4, 0.5, 0.1> \\ <0.2, 0.3, 0.4> & <0.8, 0.2, 0.3> & <0.8, 0.2, 0.3> \\ <0.5, 0.6, 0.2> & <0.8, 0.2, 0.3> & <0.8, 0.2, 0.1> \end{bmatrix}$$

Definition:5.7 (Idempotent NFM) An idempotent neutrosophic fuzzy matrix U satisfies the condition: $U^2 = U$ where U^2 denotes the matrix product of U with itself. This means that when you multiply the matrix by itself, it results in the same matrix.

Example: 5.5 Let assume that 3x3 Idempotent NFM

$$U = \begin{bmatrix} <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \\ <0.8,0.1,0.3> & <0.8,0.2,0.3> & <0.8,0.2,0.3> \\ <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \end{bmatrix}$$

$$U^2 = \begin{bmatrix} <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \\ <0.8,0.1,0.3> & <0.8,0.2,0.3> & <0.8,0.2,0.3> \\ <0.8,0.1,0.1> & <0.8,0.2,0.3> & <0.8,0.2,0.1> \end{bmatrix}$$

$\Rightarrow U^2 = U$

Definition 5.8 . A NFM is considered null if all its elements are (0,0,0). This type of matrix is denoted by $N_{(0,0,0)}$. On the other hand, an NFM is defined as zero if all its elements are (0,0,1) and it is represented by O.

Definition 5.9 The transpose of a NFM is an operation that flips the matrix over its diagonal. In other words, the rows of the original matrix become the columns in the transposed matrix, and vice versa. i.e., $U^T = \left[u_{ji}^{\alpha}, u_{ji}^{\beta}, u_{ji}^{\gamma} \right]$

Example: 5.6 Let assume that 3x3 NFM

$$U = \begin{bmatrix} <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0.8,0.2,0.1> & <0,0,1> \end{bmatrix}$$

$$U^T = \begin{bmatrix} <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0.8,0.2,0.1> \\ <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix}$$

Definition 5.10 An n × n NFM U is said to be controllable if there exists an Neutrosophic Fuzzy permutation matrix P such that $F = PRP^T$ satiesfies $f_{ij} \geq f_{ji}$ for every $i > j$. The matrix F here is called the canonical form of U.

6. Theorems and Results

In this section, we explore and analyze the properties of **consistency** and **weak transitivity** in **NFMs**. These properties are employed to develop a **canonical form** of an NFM that is either consistent or weakly transitive.Throughout this section, let $U = \left[u_{ij} = \left(u_{ij}^{T}, u_{ij}^{I}, u_{ij}^{F} \right) \right]$ represent an n×n NFM, We begin by presenting the following proposition to establish foundational concepts:

Proposition 6.1: If U is consistent (weak transitive) iff ΔU is consistent (weak transitive).

Proof: We provide a proof for the case of consistency, and the proof for weak transitivity can be established in a similar fashion. Let U be a consistent NFM and let $V = \Delta U$. Assume that $v_{ij} \geq v_{ji}$ and $v_{jk} \geq v_{kj}$. i.e., Suppose that $\left(u_{ij}\Theta u_{ji}\right) \geq \left(u_{ji}\Theta u_{ij}\right)$ and $\left(u_{ik}\Theta u_{ki}\right) \geq \left(u_{kj}\Theta u_{jk}\right)$.

Now we have several cases for the inequality $\left(u_{ij}\Theta u_{ji}\right) \geq \left(u_{ji}\Theta u_{ij}\right)$

Case(i) If $u_{ij}{}^{T} < u_{ji}{}^{T}, u_{ij}{}^{I} < u_{ji}{}^{I}, u_{ij}{}^{F} > u_{ji}{}^{F}$ then

i.e., if $u_{ij} < u_{ji}$ then $\left(u_{ij}\Theta u_{ji}\right) = \left(0,0,1\right)$ and $\left(u_{ji}\Theta u_{ij}\right) = \left(u_{ji}{}^{T}, u_{ji}{}^{I}, u_{ji}{}^{F}\right)$

which is a contradiction since we have that $\left(u_{ij}\Theta u_{ji}\right) \geq \left(u_{ji}\Theta u_{ij}\right)$.

Case(ii) If $u_{ij}{}^{T} < u_{ji}{}^{T}, u_{ij}{}^{I} < u_{ji}{}^{I}, u_{ij}{}^{F} < u_{ji}{}^{F}$, then

$\left(u_{ij}\Theta u_{ji}\right) = \left(0,0,u_{ij}{}^{F}\right)$ but $\left(u_{ji}\Theta u_{ij}\right) = \left(u_{ji}{}^{T}, u_{ji}{}^{I}, u_{ji}{}^{F}\right)$ which is a contradiction since $u_{ij}{}^{F} < u_{ji}{}^{F}$.

Case(iii) If $u_{ij}{}^{T} < u_{ji}{}^{T}, u_{ij}{}^{I} < u_{ji}{}^{I}, u_{ij}{}^{F} = u_{ji}{}^{F}$,

then $\left(u_{ij}\Theta u_{ji}\right) = \left(0,0,1\right)$ and $\left(u_{ji}\Theta u_{ij}\right) = \left(u_{ji}{}^{T}, u_{ji}{}^{I}, u_{ji}{}^{F}\right)$ and this is also a contradiction.

Case(iv) If $u_{ij}{}^{T} > u_{ji}{}^{T}, u_{ij}{}^{I} > u_{ji}{}^{I}, u_{ij}{}^{F} > u_{ji}{}^{F}$,then $\left(u_{ij}\Theta u_{ji}\right) \geq \left(u_{ji}\Theta u_{ij}\right)$ and this is a contradiction since $u_{ij}{}^{F} > u_{ji}{}^{F}$.

Case(v) If $u_{ij}{}^{T} = u_{ji}{}^{T}, u_{ij}{}^{I} = u_{ji}{}^{I}, u_{ij}{}^{F} > u_{ji}{}^{F}$, then $\left(u_{ij}\Theta u_{ji}\right) = \left(0,0,1\right)$ and $\left(u_{ji}\Theta u_{ij}\right) = \left(0,0,u_{ji}{}^{F}\right)$. Again,this is a contradiction since and $u_{ij}{}^{F} > u_{ji}{}^{F}$.

Case(vi) If $u_{ij}{}^{T} \geq u_{ji}{}^{T}, u_{ij}{}^{I} \geq u_{ji}{}^{I}, u_{ij}{}^{F} \leq u_{ji}{}^{F}$ i.e., if, then

$\left(u_{ij}\Theta u_{ji}\right) = \left(u_{ij}{}^{T}, u_{ij}{}^{I}, u_{ij}{}^{F}\right)$ and $\left(u_{ji}\Theta u_{ij}\right) = \left(0,0,1\right)$.

From the above cases, we conclude that if $\left(u_{ij}\Theta u_{ji}\right) \geq \left(u_{ji}\Theta u_{ij}\right)$, then $u_{ij} \geq u_{ji}$. Similarly, ifwe have $\left(u_{jk}\Theta u_{kj}\right) \geq \left(u_{kj}\Theta u_{jk}\right)$, then $u_{jk} \geq u_{kj}$. But since we have that R is consistent, we get $u_{ik} \geq u_{ki}$ which implies $\left(u_{ik}\Theta u_{ki}\right) \geq \left(u_{ki}\Theta u_{ik}\right)$. That is $v_{ik} \geq v_{ki}$ and $V = \Delta U$ is thus consistent. The reverse implication is straightforward.

Hence the Theorem.

It is observed that U iff U satisfies weak transitive and $u_{ij} = u_{ji}$ and $u_{jk} = u_{kj}$ implies $u_{ik} = u_{ki}$.

Proposition 6.2: Let U and V be two m×n NFM such that U≺ V. Then for any l×m NFM W we have WU≺WV.

Proof: Similarly, if $v_{tj}{}^{F} = 1$, then $u_{tj}{}^{F} = 1$ and so $x_{ij}{}^{F} = 1$.Also, if $w_{it} = 1$, then $x_{ij}{}^{F} = 1$.Thus $f_{ij}{}^{F} = 1$ implies $x_{ij}{}^{F} = 1$. Therefore, $f_{ij} = 0$ implies $x_{ij} = 0$ and hence WU≺WV.

It is observed that the relation ≺ is reflexive and transitive NFM. The following example illustrates this proposition.

Example 6.1: Let us assume that the NFM

$$U = \begin{bmatrix} <0,0,1> & <0.3,0.3,0.4> & <0.2,0.3,0.7> \\ <0.7,0.3,0.3> & <0.5,0.3,0.4> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0.4,0.3,0.6> \end{bmatrix}$$

$$V = \begin{bmatrix} <0,0,1> & <0.3,0.3,0.4> & <0.2,0.3,0.7> \\ <0.8,0.3,0.2> & <0.6,0.3,0.4> & <0,0,1> \\ <0.2,0.3,0.8> & <0,0,1> & <0.9,0.3,0.1> \end{bmatrix}$$

and $W = \begin{bmatrix} <0.5,0.3,0.4> & <0,0,1> & <0.3,0.3,0.7> \\ <0.5,0.3,0.5> & <0,0,1> & <0,0,1> \\ <0.7,0.3,0.3> & <0,0,1> & <0,0,1> \end{bmatrix}$

Then it is clear that U≺V.

Also, $WU = \begin{bmatrix} <0,0,1> & <0.3,0.3,0.4> & <0.3,0.3,0.7> \\ <0,0,1> & <0.3,0.3,0.5> & <0.2,0.3,0.5> \\ <0,0,1> & <0.3,0.3,0.4> & <0.2,0.3,0.3> \end{bmatrix}$

and $WV = \begin{bmatrix} <0.2,0.3,0.8> & <0.3,0.3,0.4> & <0.3,0.3,0.7> \\ <0,0,1> & <0.3,0.3,0.5> & <0.2,0.3,0.7> \\ <0,0,1> & <0.3,0.3,0.4> & <0.2,0.3,0.7> \end{bmatrix}$

and it is clear also that CU≺CV

Proposition6.3: If U is weak transitive if and only if $(\Delta U)^2 \prec \Delta U$.

Proof: Conversely, suppose that $S^2 \prec S$ and let $u_{ij}{}^{T} > u_{ji}{}^{T}, u_{jk}{}^{T} > u_{jk}{}^{T}$, $u_{ij}{}^{I} > u_{ji}{}^{I}, u_{jk}{}^{I} > u_{jk}{}^{I}$

And let also ${u_{ij}}^F < {u_{ji}}^F, {u_{jk}}^F < {u_{jk}}^F$.Then ${u_{ij}}^T \wedge {u_{jk}}^T > {u_{ji}}^T \wedge {u_{kj}}^T$, ${u_{ij}}^I \wedge {u_{jk}}^I > {u_{ji}}^I \wedge {u_{kj}}^I$ and ${u_{ij}}^F \vee {u_{jk}}^F < {u_{ji}}^F \wedge {u_{kj}}^F$.Now let ${u_{ik}}^T \le {u_{ki}}^T, {u_{ik}}^I > {u_{ki}}^I$ and ${u_{ik}}^F \ge {u_{ki}}^F$. i.e., let $s_{ik} = 0$. Since $S^2 \prec S$ we get ${s_{ik}}^{(2)} = 0$. That is $s_{it} \wedge s_{tk} = 0$ for every $t \ge n$ and so $s_{ij} \wedge s_{jk} = 0.$ If $s_{ij} = 0$, then ${u_{ij}}^T \le {u_{ji}}^T, {u_{ij}}^I \le {u_{ji}}^I$ and ${u_{ij}}^F \ge {u_{ji}}^F$. Also, if $s_{jk} = 0,$ then ${u_{jk}}^T \le {u_{kj}}^T, {u_{jk}}^I \le {u_{kj}}^I$ and ${u_{jk}}^F \ge {u_{kj}}^F$.

Therefore, ${u_{ij}}^T \wedge {u_{jk}}^T \le {u_{ji}}^T \wedge {u_{kj}}^T, {u_{ij}}^I \wedge {u_{jk}}^I \le {u_{ji}}^I \wedge {u_{kj}}^I$ and ${u_{ij}}^F \vee {u_{jk}}^F < {u_{ji}}^F \vee {u_{kj}}^F$ which is a contradiction. Hence ${u_{ik}}^T > {u_{ki}}^T, {u_{ik}}^I > {u_{ki}}^I$ and ${u_{ik}}^F < {u_{ki}}^F$. Thus, U is weak transitive.

Remark6.1: Consistency is evidently a stronger property than weak transitivity. Specifically, if U is consistent, then it is weakly transitive; however, the converse does not necessarily hold. Furthermore, it is observed that transitivity implies weak transitivity.

Proposition6.4: If U is weak transitive, then ΔU is nilpotent.

Proof: By Proposition 6.3, we obtain $(\Delta U)^2 \prec \Delta U.$ and also by Proposition3.2,we can see that $(\Delta U)^3 \prec (\Delta U)^2$ and soon. But by the transitivity of the relation $\prec$, we get

$(\Delta U)^k \prec (\Delta U)^{k-1}$ for every $k \le n.$ Now, let $S = \Delta U.$

Then $s_{ii} = 0$ and hereafter ${s_{ii}}^{(k)} = 0$ for all $k, i \in I_n.$

Therefore by, we conclude $S = \Delta U$ is nilpotent.

Proposition 6.5: If U is controllable if and only if ΔU is nilpotent. By Proposition since we obtain ΔU is nilpotent for a weak transitive NFM U, we conclude by Definition that U is controllable and so, there exists a PNFM P such that F = PRP^T satisfies $f_{ij} \ge f_{ji}$ for every $i > j$ which is th ecanonical form of the weak transitive NFM U.

Let $E_{(i,j)}$) represents the matrix derived from the identity NFM I_n by exchanging the ith row by jth row for i, j ≤ n. We can now present the following proposition.

Proposition6.6: If U is consistent (weak transitive) iff $E_{(i,j)}$ U $E_{(i,j)}$ is consistent (weak transitive).

Proof: Let U be consistent and let V = $E_{(i,j)}$U $E_{(i,j)}$.

The elements of E(i,j) can be expressed as follows:

$$e_{kl}=\begin{cases} a_{kl} & if\ \ k\neq i\neq j\ or\ l\neq i\neq j, \\ 1 & if\ \ k=i\ and\ l=j\ \ or\ if\ \ k=i\ and\ i=l, \\ 0 & otherwise \end{cases}$$

The elements of the NFM are derived from the elements of U as follows

$$v_{kl}=\begin{cases} u_{rs} & if\ \ r\neq s\neq i\neq j, \\ u_{rs} & if\ \ r=i\ and\ v=j\ \ or\ if\ \ u=j\ and\ v=i, \\ u_{ri} & if\ \ s=j, \\ u_{rj} & if\ \ s=i, \\ u_{is} & if\ \ r=j, \\ u_{is} & if\ \ r=i, \\ u_{ii} & if\ \ r=s=j, \\ u_{jj} & if\ \ r=s=i, \end{cases}$$

Next, we demonstrate that V is consistent, relying on the consistency of U. This requires evaluating several cases. To perform these computations, $v_{hk}\geq v_{kh}$ and $v_{kl}\geq v_{lk}$ .We will show that $v_{hl}\geq v_{lh}$ in all the following cases where $h\neq k\neq l$ everywhere.

Case(i) If $j\neq i\neq h\neq k\neq l,$ then $v_{hk}\geq v_{kh}$ and $v_{kl}\geq v_{lk}$ implies $u_{hk}\geq u_{kh}$ and $u_{kl}\geq u_{lk}$ .Thus $u_{hl}\geq u_{lh}$ and hence $v_{hl}\geq v_{lh}.$

Case(ii) If $h=i$ and $k=j$ or if $h=j$ and $k=i$, then $v_{hk}\geq v_{kh}$ and $v_{kl}\geq v_{lk}$ implies $u_{hk}\geq u_{kh}$ and $u_{il}\geq u_{li}$ .Thus $u_{kh}\geq u_{hk}$ and $v_{hl}\geq v_{lh}$. Hence $u_{kl}\geq u_{lk}$ by the consistency of U. That is $u_{jl}\geq u_{lj}$ and therefore $v_{hl}\geq v_{lh}$.

Case(iii) If $k=i$ and $l=j$ or if $k=j$ and $l=i$, then $v_{hk}\geq v_{kh}$ and $v_{kl}\geq v_{lk}$ implies $u_{hj}\geq u_{jh}$ and $u_{lk}\geq u_{kl}$ .Thus $u_{hj}\geq u_{jh}$ and $u_{ji}\geq u_{ij}$ .Hence $u_{hi}\geq u_{ih}$ and $v_{hl}\geq v_{lh}.$

Case(iv) If $k=i$ and $l\neq j$ then $v_{hk}\geq v_{kh}$ and $v_{kl}\geq v_{lk}$ implies $u_{hj}\geq u_{jh}$ and $u_{jl}\geq u_{lj}$ .Thus $u_{hl}\geq u_{lh}$ and so $v_{hl}\geq v_{lh}.$

Case(v) If $k = j$ and $i \neq l$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hi} \geq u_{ih}$ and $u_{il} \geq u_{li}$ .Thus $u_{hl} \geq u_{lh}$ and so $v_{hl} \geq v_{lh}$.

Case(vi) If $j \neq l$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{jk} \geq u_{kj}$ and $u_{kl} \geq u_{lk}$ .Thus $u_{jl} \geq u_{lj}$ and so $v_{hl} \geq v_{lh}$.

Case(vii) If $h = j$ and $i \neq l$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{ik} \geq u_{ki}$ and $v_{kl} \geq v_{lk}$ and $u_{ik} \geq u_{ki}$ .Thus $u_{il} \geq u_{lj}$ and so $v_{hl} \geq v_{lh}$.

Case(viii) If $l = i$ and $l \neq j$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hk} \geq u_{kh}$ and $u_{kj} \geq u_{jk}$ .Thus $u_{hj} \geq u_{jh}$ and so $v_{hl} \geq v_{lh}$..

Case(ix) If $l \neq i$ and $l = j$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{hk} \geq u_{kh}$ and $u_{ki} \geq u_{ik}$ .Thus $u_{hi} \geq u_{ih}$ and so $v_{hl} \geq v_{lh}$.

Case(x) If $h = i$ and $l = j$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{jk} \geq u_{kj}$ and $u_{ki} \geq u_{ik}$ .Thus $u_{lh} \geq u_{hl}$ and so $v_{hl} \geq v_{lh}$.

Case(xi) If $h = j$ and $l = i$ then $v_{hk} \geq v_{kh}$ and $v_{kl} \geq v_{lk}$ implies $u_{ik} \geq u_{ki}$ and $u_{kj} \geq u_{jk}$ .Thus $u_{ij} \geq u_{ji}$ that is $u_{lh} \geq u_{hl}$ and so $v_{hl} \geq v_{lh}$. **Conversely,** suppose $V = E_{(i,j)} U E_{(i,j)}$ is consistent. Then $E_{(i,j)} V E_{(i,j)}$ is consistent. But $E_{(i,j)} V E_{(i,j)} = E_{(i,j)} E_{(i,j)} U E_{(i,j)} U E_{(i,j)} = IUI = U$ Thus U is consistent. This concludes the proof. Similarly, the case for weak transitivity can be demonstrated in an analogous manner.

Example 6.2: Let us assume that the NFM

$$U = \begin{bmatrix} <0.6,0.3,0.4> & <0.7,0.3,0.2> & <0.5,0.3,0.5> & <0.4,0.3,0.5> \\ <0.6,0.3,0.3> & <0.9,0.3,0.1> & <0.6,0.3,0.3> & <0.5,0.3,0.5> \\ <0.4,0.3,0.6> & <0.7,0.3,0.2> & <0.8,0.3,0.2> & <0.9,0.3,0.0> \\ <0.3,0.3,0.6> & <0.4,0.3,0.6> & <0.6,0.3,0.4> & <0.7,0.3,0.1> \end{bmatrix}$$

By the definition of weak transitivity, it is evident that U is weakly transitive. For instance $u_{13} > u_{31}$

and $u_{32} > u_{23}$ implies $u_{12} > u_{21}$. Also, $u_{32} > u_{23}$ and $u_{24} > u_{42}$ implies $u_{34} > u_{43}$ and so on for all $i \neq j \neq k \leq 4$.

Now $$\Delta U = U\Theta U^T = \begin{bmatrix} <0,0,1> & <0.7,0.3,0.2> & <0.5,0.3,0.5> & <0.4,0.3,0.5> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0.5,0.3,0.5> \\ <0,0,1> & <0.7,0.3,0.2> & <0,0,1> & <0.9,0.3,0.0> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix}$$

Whichisweaktransitiveandmoreover

$$(\Delta U)^2 = \begin{bmatrix} <0,0,1> & <0.5,0.3,0.5> & <0,0,1> & <0.4,0.3,0.5> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0.5,0.3,0.5> \\ <0,0,1> & <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix} \prec \Delta U$$

and $(\Delta U)^3 = 0$ (thezeromatrix) which means that ΔU is nilpotent. Let

$$P = \begin{bmatrix} <0,0,1> & <0,0,1> & <0,0,1> & <1,1,0> \\ <0,0,1> & <1,1,0> & <0,0,1> & <0,0,1> \\ <0,0,1> & <0,0,1> & <1,1,0> & <0,0,1> \\ <1,1,0> & <0,0,1> & <0,0,1> & <0,0,1> \end{bmatrix}$$

Be a permutation NFM.Then

$$F = PUP^T = \begin{bmatrix} <0.7,0.3,0.1> & <0.4,0.3,0.6> & <0.6,0.3,0.4> & <0.3,0.3,0.6> \\ <0.5,0.3,0.5> & <0.9,0.3,0.1> & <0.6,0.3,0.3> & <0.6,0.3,0.3> \\ <0.9,0.3,0.0> & <0.7,0.3,0.2> & <0.8,0.3,0.2> & <0.4,0.3,0.6> \\ <0.4,0.3,0.5> & <0.7,0.3,0.2> & <0.5,0.3,0.5> & <0.6,0.3,0.4> \end{bmatrix}$$

With $f_{ij} \geq f_{ji}$ for every $i > j$ and U is thus controllable. Observe that F is weakly transitive. This example illustrates the results derived in Propositions 6.3, 6.4, 6.5, and 6.6. Additionally, it demonstrates that the weakly transitive NFM U is controllable.

7.Reflexivity and Transitivity o f NFM

Definition:7.1 For $U = (u^T, u^I, u^F)$ and $V = (v^T, v^I, v^F) \in (NFM)$, we define as:

$$u \to v = \begin{cases} (1,1,0) & if\ u^T \leq v^T,\ u^I \leq v^I, \\ (v^T,0,0) & if\ u^T > v^T, u^F \geq v^F, \\ (v^I,0,0) & if\ u^I > v^I \text{ and } u^F \geq v^F, \\ (u^T,u^I,u^F) & if\ u^T > v^T, u^I > v^I\ and\ u^F < v^F \end{cases}$$

However, theoperator is the Godel implication operator which is well known in many branches of fuzzy mathematics.

The min $\rightarrow$ composition of two NFM $U=\left[u_{ij}=\left(u^{T}{}_{ij},u^{I}{}_{ij},u^{F}{}_{ij}\right)\right]$ and $W=\left[w_{ij}=\left(w^{T}{}_{ij},w^{I}{}_{ij},w^{F}{}_{ij}\right)\right]$ is denoted by $X=U\rightarrow W$ and is defined as

$x_{ij}=\bigcap_{k=1}^{n}\left(u_{ik}\rightarrow w_{kj}\right)$.

In this section, we explore various properties of the operations defined earlier. Additionally, we briefly examine some properties of NFMs representing Neutrosophic fuzzy pre-orders using these operations $\rightarrow$. Our focus will be on the reflexive and transitive IFM. Now, let us highlight some key properties of these operations $\rightarrow$

Lemma 7.1 For $u=\left(u^{T},u^{I},u^{F}\right),v=\left(v^{T},v^{I},v^{F}\right),w=\left(w^{T},w^{I},w^{F}\right)\in\left(NFM\right)$, we have $\left(u\rightarrow v\right)\leftarrow w=u\rightarrow\left(v\leftarrow w\right)=v\leftarrow\left(u\wedge w\right)$.

Proof. Based on the definition of the operation, we identify the following four cases, each of which contains four subcases. The cases are as follows

Csse(i) If $u^{T}\leq v^{T}$, $u^{I}\leq v^{I}$

Csse (ii) If $u^{T}>v^{T},u^{F}\geq v^{F}$,

Csse (iii) If $u^{I}>v^{I}$ and $u^{F}\geq v^{F}$

Csse (iv) If $u^{T}>v^{T},u^{I}>v^{I}\ and\ u^{F}<v^{F}$

The subcases of each case are:

Csse(i) If $w^{T}\leq v^{T}$, $w^{I}\leq v^{I}$

Csse (ii) If $w^{T}>v^{T},w^{F}\geq v^{F}$,

Csse (iii) If $w^{I}>v^{I}$ and $w^{F}\geq v^{F}$

Csse (iv) If $w^{T}>v^{T},w^{I}>v^{I}\ and\ w^{F}<v^{F}$

We prove one case,namely, Case(ii) and (iii) and the proofs of the other two cases are similar. To do that, suppose $w^{T}>v^{T},w^{F}\geq v^{F}$, $w^{I}>v^{I}$ and $w^{F}\geq v^{F}$

(i)If $w^T \le v^T$, $w^I \le v^I$, then

$$\left[\left(u^T,u^I,u^F\right)\to\left(v^T,v^I,v^F\right)\leftarrow\left(w^T,w^I,w^F\right)\right] =\left(v^T,0,0\right)\leftarrow\left(w^T,w^I,w^F\right)=\left(1,1,0\right)$$

and $\left(u^T,u^I,u^F\right)\to\left[\left(v^T,v^I,v^F\right)\leftarrow\left(w^T,w^I,w^F\right)\right]=\left(u^T,u^I,u^F\right)\to\left(1,1,0\right)=\left(1,1,0\right)$

Since $u^T \wedge u^F \le u^I$,

we get $\left(v^T,v^I,v^F\right)\leftarrow\left[\left(u^T,u^I,u^F\right)\wedge\left(w^T,w^I,w^F\right)\right]=\left(1,1,0\right)$

(ii) If $w^T > v^T, w^I > v^I$ and $w^F < v^F$,then

$$\left[\left(u^T,u^I,u^F\right)\to\left(v^T,v^I,v^F\right)\right]\leftarrow\left(w^T,w^I,w^F\right)=\left(v^T,0,0\right)\leftarrow\left(w^T,w^I,w^F\right)=\left(v^T,0,0\right)$$

and $\left(u^T,u^I,u^F\right)\to\left[\left(v^T,v^I,v^F\right)\leftarrow\left(w^T,w^I,w^F\right)\right]=\left(u^T,u^I,u^F\right)\to\left(v^T,v^I,v^F\right)=\left(v^T,0,0\right)$

Since $u^T \wedge u^F > u^I$ and $v^T \vee v^F \ge v^I$ we get

$\left(v^T,v^I,v^F\right)\leftarrow\left[\left(u^T,u^I,u^F\right)\wedge\left(w^T,w^I,w^F\right)\right]=\left(v^T,0,0\right)$

(iii)If $w^T > v^T$ and $w^F \ge v^F$, then

$$\left[\left(u^T,u^I,u^F\right)\to\left(v^T,v^I,v^F\right)\right]\leftarrow\left(w^T,w^I,w^F\right)=\left(v^T,0,0\right)\leftarrow\left(w^T,w^I,w^F\right)=\left(v^T,0,0\right)$$

and $\left(u^T,u^I,u^F\right)\to\left[\left(v^T,v^I,v^F\right)\leftarrow\left(w^T,w^I,w^F\right)\right]=\left(u^T,u^I,u^F\right)\to\left(v^T,0,0\right)=\left(v^T,0,0\right)$

Last, $u^T \wedge u^F > u^I$ and $v^T \vee v^F \ge v^I$ implies

$\left(v^T,v^I,v^F\right)\leftarrow\left[\left(u^T,u^I,u^F\right)\wedge\left(w^T,w^I,w^F\right)\right]=\left(v^T,0,0\right)$.

Hence from all the above cases we conclude

$\left(u\to v\right)\leftarrow w=u\to\left(v\leftarrow w\right)=v\leftarrow\left(u\wedge w\right)$ for every $u,v,w\in(NFM)$.

Proposition 7.1 Let $U=\left[u_{ij}\right]_{m\times p}, V=\left[v_{ij}\right]_{p\times g}$ and $W=\left[w_{ij}\right]_{g\times n}$ be three NFM. Then

$\left(U\to V\right)\leftarrow V=U\to\left(V\leftarrow V\right)$.

Proof. Let $X=\left(U\to V\right)\leftarrow V$ and $Y=U\to\left(V\leftarrow V\right)$

Then by Lemma 7.1,

$$x_{ij}=\bigcap_{l=1}^{g}\left[\bigcap_{k=1}^{p}\left(u_{ik}\to v_{kl}\right)\leftarrow w_{lj}\right]=\bigcap_{l=1}^{g}\bigcap_{k=1}^{p}\left(u_{ik}\to v_{kl}\leftarrow w_{lj}\right)$$

Also, $y_{ij}=\bigcap_{k=1}^{p}\left[u_{ik}\rightarrow\bigcap_{k=1}^{p}\left(v_{kl}\leftarrow w_{lj}\right)\right]=\bigcap_{k=1}^{p}\bigcap_{l=1}^{g}\left(u_{ik}\rightarrow v_{kl}\leftarrow w_{lj}\right)$

$x_{ij}=y_{ij}$.

Proposition 7.2 If an NFM U is reflexive and transitive, then U is idempotent.

Proof. Since U is transitive, it is sufficient to prove that $U^2\geq U$

Now, let U be of order nxn and let $S=U^2$.

Then $s_{ij}=\left(s_{ij}{}^{T},s_{ij}{}^{I},s_{ij}{}^{F}\right)=\left[\bigcup_{k=1}^{n}\left(u_{ik}{}^{T}\wedge u_{kj}{}^{T}\right),\bigcup_{k=1}^{n}\left(u_{ik}{}^{I}\wedge u_{kj}{}^{I}\right),\bigcap_{k=1}^{n}\left(u_{ik}{}^{F}\vee u_{kj}{}^{F}\right)\right]$

$\geq\left(u_{ij}{}^{T}\wedge u_{ij}{}^{T},u_{ij}{}^{I}\wedge u_{ij}{}^{I},u_{ij}{}^{F}\vee u_{ij}{}^{F}\right)$

$=\left(u_{ij}{}^{T}\wedge 1,u_{ij}{}^{I}\wedge 1,u_{ij}{}^{F}\vee 0\right)$

$=\left(u_{ij}{}^{T},u_{ij}{}^{I},u_{ij}{}^{F}\right)$

$=u_{ij}$ (since U is reflexive).

i.e., $U^2\geq U$ and U is so idempotent.

Theorem 7.1 Let U and V be two mxn NFM such that $U\leq V$. Then $U\leftarrow V^t$ and $V^t\rightarrow U$ are transitive.

Proof. Let $W=U\leftarrow V^t$. To show that W is transitive, we must prove that

$w_{ij}\geq w_{il}\wedge w_{li}$ for all $l\leq n.$ $w_{il}\wedge w_{li}=x>0.$ then

$w_{il}=\bigcap_{k=1}^{n}\left(u_{ik}\leftarrow v_{ik}\right)\geq x$ and $w_{lj}=\bigcap_{k=1}^{n}\left(u_{lk}\leftarrow v_{jk}\right)\geq x$

If $w_{ij}=\left(w_{ij}{}^{T},w_{ij}{}^{I},w_{ij}{}^{F}\right)$ be such that $w_{ij}{}^{T}<x^{T}$, $w_{ij}{}^{I}<x^{I}$ and $w_{ij}{}^{F}>x^{F}$

Where $x=\left(x^{T},x^{I},x^{F}\right)$

That is, if $\left(u_{ih}{}^{T},u_{ih}{}^{I},u_{ih}{}^{F}\right)\leftarrow\left(v_{jh}{}^{T},v_{jh}{}^{I},v_{jh}{}^{F}\right)<\left(x^{T},x^{I},x^{F}\right)$,

Then $\left(u_{ih}{}^{T},u_{ih}{}^{I},u_{ih}{}^{F}\right)\leftarrow\left(v_{jh}{}^{T},v_{jh}{}^{I},v_{jh}{}^{F}\right)\neq(1,1,0)$

Since $0\leq x^{F}$, So that $\left(u_{ih}{}^{T},u_{ih}{}^{I},u_{ih}{}^{F}\right)\leftarrow\left(v_{jh}{}^{T},v_{jh}{}^{I},v_{jh}{}^{F}\right)=\left(u_{ih}{}^{T},u_{ih}{}^{I},u_{ih}{}^{F}\right)<\left(x^{T},x^{I},x^{F}\right)$.

Thus, $u_{ih}{}^{T}<x^{T},u_{ih}{}^{I}<x^{I}$ and $u_{ih}{}^{F}>x^{F}$ and in this case, $u_{ih}{}^{T}<v_{jh}{}^{T},u_{ih}{}^{I}<v_{jh}{}^{I}$ and $u_{ih}{}^{F}>v_{jh}{}^{F}$

By the definitionof $\leftarrow$

Since $w_{il} \geq x,$ we have $\left(u_{ih}{}^{T}, u_{ih}{}^{I}, u_{ih}{}^{F}\right) \leftarrow \left(v_{jh}{}^{T}, v_{jh}{}^{I}, v_{jh}{}^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right)$

and so $\left(u_{ih}{}^{T}, u_{ih}{}^{I}, u_{ih}{}^{F}\right) \leftarrow \left(v_{jh}{}^{T}, v_{jh}{}^{I}, v_{jh}{}^{F}\right) \neq \left(u_{ih}{}^{T}, 0, 0\right), \left(u_{ih}{}^{T}, u_{ih}{}^{I}, u_{ih}{}^{F}\right).$

Therefore, $\left(u_{ih}{}^{T}, u_{ih}{}^{I}, u_{ih}{}^{F}\right) \leftarrow \left(v_{jh}{}^{T}, v_{jh}{}^{I}, v_{jh}{}^{F}\right) = (1,1,0)$.

But $u_{ih}{}^{T} < v_{jh}{}^{T}, u_{ih}{}^{I} < v_{jh}{}^{I}$ and $u_{ih}{}^{F} > v_{jh}{}^{F}$.

Then $u_{ih}{}^{T} \geq v_{jh}{}^{T}, u_{ih}{}^{I} \geq v_{jh}{}^{I}$ and $u_{ih}{}^{F} \leq v_{jh}{}^{F}$

Also, $w_{lj} \geq x$, we have that $\left(u_{lh}{}^{T}, u_{lh}{}^{I}, u_{lh}{}^{F}\right) \leftarrow \left(v_{jh}{}^{T}, v_{jh}{}^{I}, v_{jh}{}^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right)$

and so that $u_{lh}{}^{T} \geq v_{jh}{}^{T}, u_{lh}{}^{I} \geq v_{jh}{}^{I}$ and $u_{lh}{}^{F} \leq v_{jh}{}^{F}$

Since $U \leq V$, we have $x^{T} > u_{ih}{}^{T} \geq v_{lh}{}^{T} \geq u_{lh}{}^{T} \geq v_{jh}{}^{T}, x^{I} > u_{ih}{}^{I} \geq v_{lh}{}^{I} \geq u_{lh}{}^{I} \geq v_{jh}{}^{I}$

However, this contradicts that $u_{ih}{}^{T} < v_{jh}{}^{T}, u_{ih}{}^{I} < v_{jh}{}^{I}$. So that $x_{ij}{}^{T} \geq x^{I}$.

Aiso, since $U \leq V$, We have $x^{F} < u_{ih}{}^{F} \leq v_{lh}{}^{T} \leq u_{lh}{}^{T} \leq v_{jh}{}^{T}$.

However, this contradicts that $x^{F} \geq u_{ih}{}^{F}$.So that $w \leq x^{F}$

Therefore, $w_{ij} = \left(w^{T}, w^{I}, w^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right) = x$ and W is transitive.

The transitivity of the matrix $V^{t} \rightarrow V$ can be obtained in a similar manner.

Corollary 7.1 For any mxn NFM U, the NFM $V \rightarrow V^{t}$ and $V^{t} \rightarrow V$ are idempotent.

Proof. It is easy to see that U $V \rightarrow V^{t}$ and $V^{t} \rightarrow V$ are reflexive. Then by Theorem 7.1 and Proposition 7.2 they are idempotent.

Theorem3.2 Let U be any nxn NFM. Then U is reflexive and transitive if and only if $V \leftarrow V^{t} = V$ (also if and only if $V^{t} \rightarrow V = V$.)

Proof. Suppose U is reflexive and transitive and let $W = V \leftarrow V^{t}$

$$i.e., w_{ij} = \bigcap_{k=1}^{n} \left[\left(u_{ik}{}^{T}, u_{ik}{}^{I}, u_{ik}{}^{F}\right) \leftarrow \left(u_{jk}{}^{T}, u_{jk}{}^{I}, u_{jk}{}^{F}\right) \right].$$

Suppose $w_{ij} = \left(w_{ij}{}^{T}, w_{ij}{}^{I}, w_{ij}{}^{F}\right) = \left(x^{T}, x^{I}, x^{F}\right) > (0,0,1)$.

Then $\left(u_{ij}{}^{T}, u_{ij}{}^{I}, u_{ij}{}^{F}\right) \leftarrow \left(u_{jj}{}^{T}, u_{jj}{}^{I}, u_{jj}{}^{F}\right) \geq \left(x^{T}, x^{I}, x^{F}\right).$

That is $\left(u_{ij}{}^{T},u_{ij}{}^{I},u_{ij}{}^{F}\right)\geq(0,0,1)\geq\left(x^{T},x^{I},x^{F}\right)$ (by the reflexivity of U)which yielding

$u_{ij}=\left(u_{ij}{}^{T},u_{ij}{}^{I},u_{ij}{}^{F}\right)\geq\left(x^{T},x^{I},x^{F}\right)=w_{ij}$ and so $U\geq U\leftarrow U^{t}$. On the other hand we will show that $U\geq U\leftarrow U^{t}$. using the transitivity of U. .

Suppose that $u_{ij}=\left(u_{ij}{}^{T},u_{ij}{}^{I},u_{ij}{}^{F}\right)=\left(x^{T},x^{I},x^{F}\right)>(0,0,1)$

and $w_{ij}=\left(w_{ij}{}^{T},w_{ij}{}^{I},w_{ij}{}^{F}\right)=\left(u_{il}{}^{T},u_{il}{}^{I},u_{il}{}^{F}\right)\leftarrow\left(u_{jl}{}^{T},u_{jl}{}^{I},u_{jl}{}^{F}\right)$

For a given $l\leq n.$ According to the definition of the operation, the following three cases arise:
$W\geq U.$

Case(i) $\left(u_{il}{}^{T},u_{il}{}^{I},u_{il}{}^{F}\right)\leftarrow\left(u_{jl}{}^{T},u_{jl}{}^{I},u_{jl}{}^{F}\right)=(0,0,1)$ then it is clear that $w_{ij}\geq x=u_{ij}$. Thus

Case(ii) If $\left(u_{il}{}^{T},u_{il}{}^{I},u_{il}{}^{F}\right)\leftarrow\left(u_{jl}{}^{T},u_{jl}{}^{I},u_{jl}{}^{F}\right)=\left(u_{jl}{}^{T},u_{jl}{}^{I},u_{jl}{}^{F}\right)<\left(x^{T},x^{I},x^{F}\right),$

Then $u_{il}{}^{T}<x^{T},u_{il}{}^{I}<x_{ij}{}^{I}$ and $u_{il}{}^{F}>x^{F}$

$i.e.,u_{il}{}^{T}<x^{T}\wedge u_{jl}{}^{T},u_{il}{}^{I}<x^{I}\wedge u_{jl}{}^{I}$ and $u_{il}{}^{F}>x^{F}\vee u_{jl}{}^{F}$.

Now since we have that U is transitive, we get $\left(u_{ij}{}^{T},u_{ij}{}^{I},u_{ij}{}^{F}\right)\wedge\left(u_{jl}{}^{T},u_{jl}{}^{I},u_{jl}{}^{F}\right)\leq\left(u_{il}{}^{T},u_{il}{}^{I},u_{il}{}^{F}\right)$.

Thus $\left(x^{T},x^{I},x^{F}\right)\wedge\left(u_{jl}{}^{T},u_{jl}{}^{I},u_{jl}{}^{F}\right)\leq\left(u_{il}{}^{T},u_{il}{}^{I},u_{il}{}^{F}\right).$

Therefore, $u_{il}{}^{T}<x^{T}\wedge u_{jl}{}^{T}\leq u_{il}{}^{T},u_{il}{}^{I}<x^{I}\wedge u_{jl}{}^{I}\leq u_{il}{}^{I}$ and $u_{il}{}^{F}>x^{F}\vee u_{jl}{}^{F}\geq u_{il}{}^{F}$. However, these are contradictions and so $w_{jl}{}^{T}\geq x^{T},w_{jl}{}^{I}\geq x^{I}$ and $w_{jl}{}^{F}\leq x^{F}$. Thus $w_{ij}\geq x$ = u_{ij} and so
$W\geq U.$

Case(iii) If $w_{ij}=\left(u_{il}{}^{T},0,0\right)$, Then $u_{il}{}^{T}<u_{jl}{}^{T}$. Since we have that $0<x^{F}$

it is enough to show that $u_{il}{}^{T}\geq x^{T}$. Suppose that $u_{il}{}^{T}<x^{I}$.

Then $u_{il}{}^{T}<x^{T}\wedge u_{jl}{}^{T}=u_{ij}{}^{T}\wedge u_{jl}{}^{T}\leq u_{il}{}^{T}$ (Again by the transitivity of U) and also we have a contradiction and so $u_{il}{}^{T}\geq x^{T}$ and $w_{ij}\geq x.$ Thus $w_{ij}\geq x=u_{ij}$ $i.e.,W\geq U.$

Conversely, if $U\leftarrow U^{t}=U$ then by the proof of Corollary 310, U is reflexive and transitive so it is idempotent.
Similarly, we can prove that $U^{t}\rightarrow U=U$.

Theorem 3.2 shows interesting properties of preorders. Thus U is a matrix representing a preorder iff $U \leftarrow U^t = U$ (or $U^t \rightarrow U = U$). However, since $U \leftarrow U^t$ is obtained by using U if we multiply $U \leftarrow U^t$ by U, any information is no tadded to U. That is, the product $\left(U \leftarrow U^t\right)U$ is equal to U.

Example: 7.1 Let us consider the NFM

$$U = \begin{bmatrix} <0.5,0.3,0.4> & <0.7,0.3,0.3> \\ <0.6,0.3,0.3> & <0.8,0.3,0.2> \\ <0.9,0.3,0.0> & <0.4,0.3,0.6> \end{bmatrix}$$

Then $U \leftarrow U^t = \begin{bmatrix} <0.5,0.3,0.4> & <0.7,0.3,0.3> \\ <0.6,0.3,0.3> & <0.8,0.3,0.2> \\ <0.9,0.3,0.0> & <0.4,0.3,0.6> \end{bmatrix}$

$$\leftarrow \begin{bmatrix} <0.5,0.3,0.4> & <0.6,0.3,0.3> & <0.9,0.3,0.0> \\ <0.7,0.3,0.3> & <0.8,0.3,0.2> & <0.4,0.3,0.6> \end{bmatrix}$$

$$= \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix}$$

It is clear that $U \leftarrow U^t$ is reflexive and

$$\left(U \leftarrow U^t\right)^2 = \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix} = U \leftarrow U^t$$

That is is reflexive and transitive and so idempotent. Moreover, if we let $V = U \leftarrow U^t$ .Then

$$V \leftarrow V^t = \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix}$$

$$\leftarrow \begin{bmatrix} <1,1,0> & <1,1,0> & <0.4,0.3,0.6> \\ <0.5,0.3,0.4> & <1,1,0> & <0.4,0.3,0.6> \\ <0.5,0.3,0.4> & <0.6,0.3,0.3> & <1,1,0> \end{bmatrix}$$

$$= \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix} = V$$

$$\left(V \leftarrow V^t\right)V = \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix}$$

$$\begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix}$$

$$= \begin{bmatrix} <1,1,0> & <0.5,0.3,0.4> & <0.5,0.3,0.4> \\ <1,1,0> & <1,1,0> & <0.6,0.3,0.3> \\ <0.4,0.3,0.6> & <0.4,0.3,0.6> & <1,1,0> \end{bmatrix} = V .$$

8. Conclusion and Future Work

This paper has thoroughly investigated the properties of consistent and weakly transitive Neutrosophic Fuzzy Matrices (NFM), emphasizing their controllability and the derivation of a canonical form for weakly transitive NFM. The study has highlighted the significance of nilpotent and transitive NFM in matrix analysis, underscoring their importance in uncertainty modeling. By introducing the Gödel implication operator as a novel operation on NFM, we extended the theoretical framework to explore properties related to pre-orders, particularly in reflexive and transitive matrices. Additionally, a method for constructing idempotent NFM from any given matrix was presented, contributing a practical tool for further applications. The use of counterexamples throughout the discussion has provided clarity and validated the theoretical findings, ensuring their robustness and practical relevance.

Building on these findings, future research could focus on extending the analysis to higher-order properties, such as symmetry and anti-symmetry, to enhance the theoretical understanding of NFM. Applications of consistent and weakly transitive NFM in real-world decision-making problems, such as multi-criteria decision analysis and preference modeling, offer promising avenues for exploration. Developing efficient algorithms for constructing idempotent and transitive NFM could improve their computational usability. Additionally, integrating NFM with other fuzzy systems, such as intuitionistic and bipolar fuzzy systems, could address more complex problems requiring multi-dimensional uncertainty modeling. Finally, empirical studies validating these theoretical contributions in practical domains, including network analysis, machine learning, and knowledge representation, could further demonstrate their applicability and impact.

References

[1]. L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338-353.

[2]. M. G. Thomason, Convergence of powers of a fuzzy matrix, Journal of Mathematical Analysis and Applications, 57 (1977) 476-480.

[3]. Kim KH, Roush FW. Generalized fuzzy matrices. Fuzzy Sets Syst. 1980; 4: 293–315.

[4]. Anandhkumar, M., Kamalakannan, V., Chithra, S.M., Said, B., "Pseudo Similarity of Neutrosophic Fuzzy matrices", International Journal of Neutrosophic Science, Vol. 20, No. 04, PP. 191-196, 2023.

[5]. Anandhkumar, M., Kanimozhi, B., Chithra, S.M., Kamalakannan, V., Said, B., "On various Inverse of Neutrosophic Fuzzy Matrices", International Journal of Neutrosophic Science, Vol. 21, No. 02, PP. 20-31, 2023.

[6]. Emam EG ,Fendh MA. Some results associated with the max-min and min-max compositions of bifuzzy matrices. J Egypt Math Soc. 2016;24:515–521.

[7]. Chenggong H. Canonical form of strongly transitive matrices over lattices. Fuzzy Sets Syst. 1992;45:219–222.

[8]. Hashimoto H. Canonical form of a transitive fuzzy matrix. Fuzzy Sets Syst. 1983;11:157–162.

[9]. Kolodziejczyk W. Canonical form of a strongly transitive fuzzy matrix. Fuzzy Sets Syst. 1987;22:297–302.

[10]. Jian-Xin L. Controllable fuzzy matrices. Fuzzy Sets Syst. 1992;45:313–319.

[11]. Jian-XinL.Convergence of powers of controllable fuzzy matrices. Fuzzy Sets Syst.1994; 62: 83–88.

[12]. G. Punithavalli, M. Anandhkumar, Kernel and K-Kernel Symmetric Intuitionistic Fuzzy Matrices,TWMS J. App. and Eng. Math. V.14, N.3, 2024, pp. 1231-1240.

[13]. M. Anandhkumar, T. Harikrishnan, S. M. Chithra, V. Kamalakannan, B. Kanimozhi, Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices, International Journal of Neutrosophic Science, Vol. 23, no. 2, 2024, pp. 286-295.

[14]. Xu YJ, Li MQ, Cabrerizo FJ, et al. Algorithms to detect and rectify multiplicative and ordinal inconsistencies of fuzzy preference relations. IEEE Trans Syst Man Cybern Syst. doi:101109/TSMC.2019.2031536.

[15]. Atanasov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986;20:87–96.

[16]. Desechijver G, Kerre EE. On the relationship between some extensions of fuzzy set theory. Fuzzy Sets Syst. 2003;133:227–235.

[17]. Desechijver G, Kerre EE. On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision. Inf Sci. 2007;177:1860–1866.

[18]. Pal M, Khan SK, Shyamal AK. Intuitionistic fuzzy matrices. Notes Intuitionistic Fuzzy Sets. 2002;8:51–62.

[19]. M. Anandhkumar, H. Prathab, S. M. Chithra, A. S. Prakaash, A. Bobin, Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, vol. 23, no. 4, 2024, pp. 23-28.

[20]. G. Punithavalli , M. Anandhkumar, Reverse Sharp and Left-T Right-T Partial Ordering On Intuitionistic Fuzzy Matrices, TWMS J. App. and Eng. Math. V.14, N.4, 2024, pp. 1772-1783.

[21]. Dudziak U, Pekala B. Equivalent bipolar fuzzy relations. Fuzzy Sets Syst. 2010;161:234–253.

[22]. Khan SK, Pal A. The Generalized inverses of intuitionistic fuzzy matrices. JPhysSci. 2007; 11: 62–67.

[23]. Muthuraji T, Srirarm S, Murugadas P. Decomposition of intuitionistic fuzzy matrices. Fuzzy Inf Eng. 2016;8:345–354.

[24]. Muthuraji T, Srirarm S. Representation and Decomposition of intuitionistic fuzzy matrices using some(á,á) cuts. Appl Appl Math. 2017;12(1):241–258.

[25]. Anandhkumar, M., Harikrishnan, T., Chithra, S.M., Kanimozhi, B., Said, B. "Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices"International Journal of Neutrosophic Science, 2023, 21(4), pp. 135–145.

[26]. K. Radhika, T.Harikrishnan, R. Ambrose Prabhu, P.Tharaniya, M.John peter, M.Anandhkumar, On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems, Vol. 78, 2025.

[27]. Muthuraji T, Srirarm S. Reduction of an intuitionistic fuzzy matrix to fuzzy matrix with some algebraic properties. Ann Fuzzy Math Inform. 2017;13(4):475–483.

[28]. Drewniak J, Dudziak U. Aggregations preserving classes of fuzzy relations. Kybernetika. 2005;41:265–284.

[29]. Wang X. An investigation into relations between some transitivity-related concepts. Fuzzy Sets Syst. 1997;89:257–262.

[30]. K. Radhika, S. Senthil, N. Kavitha, R.Jegan, M.Anandhkumar, A. Bobin, Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making, Neutrosophic Sets and Systems, Vol. 78, 2025.

[31]. H. Prathab, N. Ramalingam, E. Janaki, A. Bobin, V. Kamalakannan and M. Anandhkumar, Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses, IAENG International Journal of Computer Science, Volume 51, Issue 12, December 2024, Pages 2051-2066.

[32]. Xu YJ, Wang HM, Yu DJ. Weak transitivity of interval-valued fuzzy relations. Knowl Based Syst. 2014;63:24–32.

[33]. K. Atanasov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986) 87-96.

[34]. U. Dudziak, B. Pekala, Equivalent bipolar fuzzy relations, Fuzzy Sets and Systems, 161 (2010) 234-253.

[35]. E. G. Emam, M. A. Fendh, Some results associated with the max min and min max compositions of bifuzzy matrices, Journal of the Egyptian Mathematical Society, 24(2016),515-521.

[36]. M.Anandhkumar, B.Kanimozhi, S.M. Chithra, V.Kamalakannan, Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic fuzzy matrices, Mathematical Modelling of Engineering Problems, 2023, 10(4), pp. 1427–1432.

[37]. M.Anandhkumar; G.Punithavalli; T.Soupramanien; Said Broumi, Generalized Symmetric Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems, Vol. 57,2023, 57, pp. 114–127.

[38]. H. Hashimoto, Canonical form of a transitive fuzzy matrix, Fuzzy Sets and Systems, 11 (1983) 157-162.

[39]. H. Hashimoto, Subinverses of Fuzzy Matrices, Fuzzy Sets and Systems, 12 (1984) 155-168.

[40]. H. Hashimoto, Decomposition of Fuzzy Matrices, Siam J. Alg. Disc. Math.,6 (1985) 32-38.

[41]. K. Hang Kim, F. W. Roush, Generalized fuzzy matrices, Fuzzy Sets and Systems, 4 (1980) 293-315.

[42]. Anandhkumar, M.; G. Punithavalli; and E. Janaki. "Secondary k-column symmetric Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 64, 1 (2024).

[43]. Anandhkumar, M.; G. Punithavalli; R. Jegan; and Said Broumi. "Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 61, 1 (2024).

[44]. S. K. Khan, A. Pal, The Generalized Inverses of Intuitionistic Fuzzy Matrices, Journal of Physical Sciences, 11 (2007) 62-67.

[45]. M. A. Mishref, E.G. Emam, Transitivity and subinverses in fuzzy matrices, Fuzzy Sets and Systems, 52 (1992) 337-343.

[46]. M. Pal, S. K. Khan, A. K. Shymal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets, 8 (2002) 51-62.

[47]. M. Z. Ragab, E.G.Emam On the min max compositions of fuzzy matrices, Fuzzy Sets and Systems, 75 (1995) 83-92.

[48]. E. Sanchez, Resolution of composite fuzzy relation equations, Information and Control, 30 (1976) 38-48.

[49]. M.Anandhkumar, S. Prathap, R. Ambrose Prabhu, P.Tharaniya, K. Thirumalai, & B. Kanimozhi. (2024). Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems. Neutrosophic Sets and Systems, 79, 234-265.

[50]. Anandhkumar, M.; A. Bobin; S. M. Chithra; and V. Kamalakannan. "Generalized Symmetric Fermatean Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 70, 1 (2024).

[51]. B. M. Schein, Regular elements of the semigroup of all binary relations, Semigroup Forum, 13 (1976) 95-102.

[52]. Smarandache,F, Neutrosophic set, a generalization of the intuitionistic fuzzy set. Int J Pure Appl Math.; (2005), 24(3): 287–297.

[53]. E. G. Emam , On Consistent and Weak Transitive Intuitionistic Fuzzy Matrices, Fuzzy Information and Engineering_Vol 14, 2022.

[54]. G. Sudha, A new technique for using Symmetric pentagonal Intuitionistic fuzzy numbers to solve Assignment problem, Indian Journal of Natural Sciences, 15(84), 2024, pp 74499 -- 74504.

[55]. G. Sudha, An Approach for solving fuzzy sequencing problem with Triangular fuzzy Neutrosophic numbers using Ranking function, Advances in Nonlinear Variational Inequalities, 27(2), 2024, pp 23- 31.

[56]. G.Sudha, A Method for finding Critical path problem with Triangular Neutrosophic Numbers by utilizing numerous methods for Ranking, Advances in Mechanics, 12(1), 2024, pp 356- 365.

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