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Weighted Statistical Convergence in Neutrosophic Normed Linear Spaces

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Abstract:. In this research, the authors provide a definition among the Neutrosophic norm(NN)$(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ in terms of the summability property $(\bar{N}, \mathfrak{q}_n)$. We generalize the idea among Statistical Convergence $(\mathfrak{SC})$ in terms of NN by introducing the notion of $(\bar{N}, \mathfrak{q}_n)$-summability with respect to NN(μ, ν, ω). Here, the novel strategy is referred to by the term neutrosophic weighted statistical convergence. We also investigated its dependence on the NN$(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ in terms of statistical convergence and $(\bar{N}, \mathfrak{q}_n)$-summability.

Keywords: Neutrosophic normed linear space, weighted statistical convergence, statistical convergence, strong summability.

1. **Introduction**

Zadeh [1] first developed the concept among a set in 1965. Since then, scholars have extended the idea among sets to many other areas, including computing [2], quantum mechanics [6], demographics [5], systems of non-linear dynamic [4] as well as chaos controls [3]. Generalized the sets by introducing the idea of neutrosophic sets by Atanassov [7]. The neutrosophic metric spaces were first established by Park [8] using the concept of Neutrosophic sets, while $\mathfrak{NNLS}$ were first presented by Saadati and Park [9].

To improve the convergence sequence, Fast [11] described the idea of $\mathfrak{SC}$ along with Steinhaus [10]. Many scholars have examined $\mathfrak{SC}$ within the theory of ergodic theory, number theory, and Fourier analysis over the years. In addition, research has been conducted on the theory of summation of $\mathfrak{SC}$. Schoenberg [12] established a link between computability theory in addition $\mathfrak{SC}$. Salat, Connor, Fridy, along with Miller ([13], [14], [15], and [16]) investigated $\mathfrak{SC}$ as a summary technique.

Kirisci and Simsek [17] define $\mathfrak{NNLS}$ and $\mathfrak{SC}$, examining statistically Cauchy sequences and the concept of statistically completeness in this specialized mathematical context. Various investigations [18] have been carried out, expanding this idea. Sharma et. al., [19,28] discussed various types of summability in neutrosophic normed space.

Moricz along with Orhan [21] introduced statistical summability $(\bar{N}, \mathfrak{q}_n)$ as follows:
Let $\mathfrak{q} = (\mathfrak{q}_\iota)_{\iota=0}^{\infty}$ represent a sequence of non-negative numbers in which $\mathfrak{q}_0 > 0$ as well as $\mathfrak{Q}_n = \sum_{\iota=0}^{n} \mathfrak{q}_\iota \to \infty$ as $n \to \infty$. Demonstrate that $\varpi_n = \frac{1}{\mathfrak{Q}_n} \sum_{\iota=0}^{n} \mathfrak{q}_\iota x_\iota, n = 0, 1, 2, ...$
$\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is statistically summable towards L using the weighted mean technique based on the sequence $(\mathfrak{q}_\iota)$ or statistically summable $(\bar{N}, \mathfrak{q}_n)$ in a brief statistical manner if $S-\lim_{n\to\infty} \varpi_n = L$.

Thus, we put $\bar{N}(S) - \lim \hat{\mathfrak{u}} = L$. $\bar{N}(S)$ denotes the set among all statistically summable $(\bar{N}, \mathfrak{q}_n)$ sequences. Conversely, if $\lim_{n\to\infty} \dfrac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \mathfrak{q}_\iota |\hat{\mathfrak{u}}_\iota - L| = 0$ then the sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is called as strongly $(\bar{N}, \mathfrak{q}_n)$ - summable to L. Here we write $|\bar{N}, \mathfrak{q}_n| - \lim \hat{\mathfrak{u}} = L$.

Karkaya together with Chishti [22] applied $(\bar{N}, \mathfrak{q}_n)$-compatibility to apply the concept about $\mathfrak{SC}$ generically and it also known as new approach $\mathfrak{WSC}$. Mursaleen together with others [23] modified the definition of $\mathfrak{WSC}$ and alos $(\bar{N}, \mathfrak{q}_n)$ - noted how they relate to the idea of computability. Additionally, it was shown why the definition given below should be applied:

If the limit exists, then $\delta_{\bar{N}}(K) = \lim_{n\to\infty} \dfrac{1}{\mathfrak{Q}_n} |K_{\mathfrak{Q}_n}|$ defines the weighted density on $K \subseteq N$. A sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ represent weighted statistically convergent (or SN-convergent) towards $\tilde{\mathfrak{P}}$ when, for each $\acute{\varepsilon} > 0$,

$$\delta_{\bar{N}}(\{\iota \in \mathbb{N} : \mathfrak{q}_\iota |\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}| \geq \acute{\varepsilon}\}) = 0, \text{ or similarly } \lim_{n\to\infty} \frac{1}{\mathfrak{Q}} |\{\iota \leq \mathfrak{Q}_n : \mathfrak{q}_\iota |\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}| \geq \acute{\varepsilon}\}| = 0.$$

Thus, we put $S_{\bar{N}} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. Here, we present a new concept among $\mathfrak{SC}$ it is defined as $\mathfrak{WSC}$ within $\mathfrak{NNLS}$. Few links between this concept and $(\bar{N}, q_n)$- summability within $\mathfrak{NNLS}$ are shown.

2. Basic definitions

Definition 2.1. The 7-tuple $(\hat{\Im}, \dot{\mathcal{J}}, \ddot{\mathcal{G}}, \dddot{\mathcal{H}}, \breve{*}, \breve{\odot}, \breve{\Diamond})$ be referred as $\mathcal{NNS}$ when $\hat{\Im}$ represent a linear space, $\breve{*}$ denote a continuous ϖ-norm, $\breve{\odot}$ as well as $\breve{\Diamond}$ indicate continuous ϖ-co-norm, $\dot{\mathcal{J}}, \ddot{\mathcal{G}}$ and $\dddot{\mathcal{H}}$ represent fuzzy sets on $\hat{\Im} \times (0, \infty)$ fulfills the below conditions:
For each $\lceil, \dagger \in \hat{\Im}$ and $\int, \varpi > 0$;

(a) $0 \leq \dot{\mathcal{J}}(\lceil, \varpi) \leq 1; 0 \leq \ddot{\mathcal{G}}(\lceil, \varpi) \leq 1; 0 \leq \dddot{\mathcal{H}}(\lceil, \varpi) \leq 1$,
(b) $\dot{\mathcal{J}}(\lceil, \varpi) + \ddot{\mathcal{G}}(\lceil, \varpi) + \dddot{\mathcal{H}}(\lceil, \varpi) \leq 3$,
(c) $\dot{\mathcal{J}}(\lceil, \varpi) > 0$,
(d) $\dot{\mathcal{J}}(\lceil, \varpi) = 1$ if and only if $\lceil = 0$,
(e) $\dot{\mathcal{J}}(\acute{\alpha}\lceil, \varpi) = \dot{\mathcal{J}}\left(\lceil, \frac{\varpi}{|\acute{\alpha}|}\right)$ for each $\acute{\alpha} \neq 0$,
(f) $\dot{\mathcal{J}}(\lceil, \varpi)\breve{*}\dot{\mathcal{J}}(\dagger, \int) \leq \dot{\mathcal{J}}(\lceil + \dagger, \varpi + \int)$,
(g) $\dot{\mathcal{J}}(\lceil, \varpi) : (0, \infty) \to [0, 1]$ is continuous,
(h) $\lim_{\varpi \to \infty} \dot{\mathcal{J}}(\lceil, \varpi) = 1$ and $\lim_{\varpi \to 0} \dot{\mathcal{J}}(\lceil, \varpi) = 0$,
(i) $\ddot{\mathcal{G}}(\lceil, \varpi) < 1$,
(j) $\ddot{\mathcal{G}}(\lceil, \varpi) = 0$ if and only if $\lceil = 0$,
(k) $\ddot{\mathcal{G}}(\acute{\alpha}\lceil, \varpi) = \ddot{\mathcal{G}}\left(\lceil, \frac{\varpi}{|\acute{\alpha}|}\right)$ for each $\acute{\alpha} \neq 0$,
(l) $\ddot{\mathcal{G}}(\lceil, \varpi)\breve{\odot}\ddot{\mathcal{G}}(\dagger, \int) \geq \ddot{\mathcal{G}}(\lceil + \dagger, \varpi + \int)$,
(m) $\ddot{\mathcal{G}}(\lceil, \varpi) : (0, \infty) \to [0, 1]$ is continuous,
(n) $\lim_{\varpi \to \infty} \ddot{\mathcal{G}}(\lceil, \varpi) = 0$ and $\lim_{\varpi \to 0} \ddot{\mathcal{G}}(\lceil, \varpi) = 1$,
(o) $\dddot{\mathcal{H}}(\lceil, \varpi) < 1$,
(p) $\dddot{\mathcal{H}}(\lceil, \varpi) = 0$ if and only if $\lceil = 0$,
(q) $\dddot{\mathcal{H}}(\acute{\alpha}\lceil, \varpi) = \dddot{\mathcal{H}}\left(\lceil, \frac{\varpi}{|\acute{\alpha}|}\right)$ for each $\acute{\alpha} \neq 0$,
(r) $\dddot{\mathcal{H}}(\lceil, \varpi)\breve{\Diamond}\ddot{\mathcal{G}}(\dagger, \int) \geq \dddot{\mathcal{H}}(\lceil + \dagger, \varpi + \int)$
(s) $\dddot{\mathcal{H}}(\lceil, \varpi) : (0, \infty) \to [0, 1]$ is continuous,
(t) $\lim_{\varpi \to \infty} \dddot{\mathcal{H}}(\lceil, \varpi) = 0$ and $\lim_{\varpi \to 0} \dddot{\mathcal{H}}(\lceil, \varpi) = 1$.

Example 2.2. [25] Let $(\hat{\mathfrak{A}}, |\cdot|)$ denote a normed space, in addition let $\acute{\alpha}\breve{*}\acute{\gamma} = \acute{\alpha}\gamma$ as well as $\acute{\alpha}\breve{\odot}\acute{\gamma} = \min\{\acute{\alpha} + \gamma, 1\}$ for every $\acute{\alpha}, \acute{\gamma} \in [0, 1]$. For all $\hat{\mathfrak{u}} \in \hat{\mathfrak{A}}$ and every $\hat{\varphi} > 0$, take $\dot{\xi}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{\varpi}{\varpi + \|\hat{\mathfrak{u}}\|}, \ddot{\zeta}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{\|\hat{\mathfrak{u}}\|}{\varpi + \|\hat{\mathfrak{u}}\|}$ as well as $\dddot{\varrho}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{\|\hat{\mathfrak{u}}\|}{\|\varpi + \hat{\mathfrak{u}}\|}$. Then $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \breve{*}, \breve{\odot}, \breve{\Diamond})$ is an $\mathfrak{NNLS}$.

Definition 2.3. [25] Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \breve{*}, \breve{\odot}, \breve{\Diamond})$ represent an $\mathfrak{NNLS}$. A $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ sequence within $\hat{\mathfrak{A}}$ is convergent towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$ in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ when, for any $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$, that $\iota_0 \in \mathbb{N}$ exists which means $\dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) > 1 - \acute{\varepsilon}, \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) < \acute{\varepsilon}$ as well as $\dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) < \acute{\varepsilon}$ for everyone $\iota \geq \iota_0$ where $\iota \in \mathbb{N}$. It's indicated by $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho}) - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

Theorem 2.4. *[23] Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \circledast, \breve{\odot}, \breve{\Diamond})$ represent a $\mathfrak{NNLS}$. Then, $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ sequence within $\hat{\mathfrak{A}}$ which is convergent towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$ in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ if and only if $\lim_{\iota\to\infty} \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) = 1$, $\lim_{\iota\to\infty} \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) = 0$ as well as $\lim_{\iota\to\infty} \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) = 0$.*

3. Weighted statistical convergence in $\mathfrak{NNLS}$

Weighted Statistical Convergent ($\mathfrak{WSC}$) related the new idea NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ we're going to developed in the below subsection. Here, in addition examine few connections between this description and $\mathfrak{SC}$ as well as $(\bar{N}, \mathfrak{q}_n)$- summability within $\mathfrak{NNLS}$.

Definition 3.1. Let a sequence $\mathfrak{q} = (\mathfrak{q}_\iota)_{\iota=0}^{\infty}$ among nonnegative numbers where $\mathfrak{q}_0 > 0$ along with $\mathfrak{Q}_n = \sum_{\iota=0}^{n} \mathfrak{q}_\iota \to \infty$ as $n \to \infty$ in addition $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \circledast, \breve{\odot}, \breve{\Diamond})$ be an $\mathfrak{NNLS}$. A sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ in $\hat{\mathfrak{A}}$ is known as weighted statistically convergent towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$ with respect to the NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ (or $S_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$ - convergent to $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$) when, for any $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$,

$$\delta_{\bar{N}}\left(\left\{\begin{array}{c} \iota \in \mathbb{N} : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}\right) = 0, \tag{1}$$

or similarly

$$\lim_{n\to\infty} \frac{1}{\mathfrak{Q}_n}\left|\left\{\begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}\right| = 0.$$

Thus, we put $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. Weighted statistically convergence can be simplified to statistically convergence in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ described by Karakus et al. [17], if we assume that $\mathfrak{q}_\iota = 1$ for all $\iota \in \mathbb{N}$ and thus, we put $\mathfrak{S}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

Using equality (1) and the weighted density's properties, we may rapidly arrive at the following lemma.

Lemma 3.2. *Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \circledast, \breve{\odot}, \breve{\Diamond})$ being a $\mathfrak{NNLS}$ along with $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ within $\hat{\mathfrak{A}}$. After that for all $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$, the given statements are identical:*

(i) $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

(ii) $\delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dot{\xi}(q_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon}\}) = \delta_{\bar{N}}(\{\iota \in \mathbb{N} : \ddot{\zeta}(q_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}\}) = \delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dddot{\varrho}(q_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}\}) = 0$.

(iii) $\delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > 1 - \acute{\varepsilon}, \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon} \text{ and } \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}\}) = 1$.

(iv) $\delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > 1 - \acute{\varepsilon}\}) = \delta_{\bar{N}}(\{\iota \in \mathbb{N} : \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}\}) = \delta_{\bar{N}}(\{\iota \in \mathbb{N} : \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}\}) = 1$.

Definition 3.3. Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \circledast, \breve{\odot}, \breve{\Diamond})$ being a $\mathfrak{NNLS}$. A sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ within $\hat{\mathfrak{A}}$ is referred to as $(\bar{N}, \mathfrak{q}_n)$ - summable towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$ in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ (or $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$-

summable towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$) when, for all $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$, that $n_0 \in \mathbb{N}$ exists which means

$$\frac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > 1 - \acute{\varepsilon},$$

$$\frac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}\text{and}$$

$$\frac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}.$$

for any a$n \geq n_0$. Therefore, we have, $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. For any $\iota \in \mathbb{N}$, if we assume that $\mathfrak{q}_\iota = 1$, then $(\bar{N}, \mathfrak{q}_n)$- summability in terms of the NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ is simplified towards $(C, 1)$-summability in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$ and thus we put $(C, 1)^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

The first theorem establishes the link between $\mathfrak{SC}$ and $\mathfrak{WSC}$ in $\mathfrak{NNLS}$.

Theorem 3.4. *Let* $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \circledast, \odot)$ *being a* $\mathfrak{NNLS}$ *and in addition* $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ *represent a sequence in* $\hat{\mathfrak{A}}$*. The following conditions are true:*

(i) *If* $\mathfrak{q}_\iota \geq 1$ *for each* $\iota \in \mathbb{N}$, $\lim_{n\to\infty} \sup \frac{\mathfrak{Q}_n}{n} < \infty$ *along with* $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$*, then* $\mathfrak{S}(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho}) - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

(ii) *If* $\mathfrak{q}_\iota \leq 1$ *for all* $\iota \in \mathbb{N}$, $\lim_{n\to\infty} \inf \frac{\mathfrak{Q}_n}{n} > 0$ *and in addition* $\mathfrak{S}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$*, after that* $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

Proof. (i) Assume that $\mathfrak{q}_\iota \geq 1$ for each $\iota \in \mathbb{N}$, $\lim_{n\to\infty} \sup \frac{\tilde{\mathfrak{P}}_n}{n} < \infty$. Given, that positive constant $\mathfrak{K}$ exists which means $1 \leq \frac{\tilde{\mathfrak{P}}_n}{n} \leq \mathfrak{K}$. Furthermore, $\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$, $\ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$ and $\dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$ given that $\mathfrak{q}_\iota \geq 1$ for each $\iota \in \mathbb{N}$ along with $\dot{\xi}(\hat{\mathfrak{u}}, \hat{\varphi})$ indicate increasing function of $\hat{\varphi} \in \mathbb{R}^+$, $\ddot{\zeta}(\hat{\mathfrak{u}}, \hat{\varphi}), \dddot{\varrho}(\hat{\mathfrak{u}}, \hat{\varphi})$ represent decreasing function of $\hat{\varphi} \in \mathbb{R}^+$ based on the Definition (2.1) provided in [**?**]. After that, we obtain for each $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$

$$\frac{1}{n} \left| \left\{ \begin{array}{c} \iota \leq n : \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \geq \acute{\varepsilon}, \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\} \right| \leq \frac{1}{n} \left| \left\{ \begin{array}{c} \iota \leq n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\} \right|$$

$$\leq \frac{\mathfrak{K}}{\mathfrak{Q}_n} \left| \left\{ \begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\} \right|.$$

Given that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$, as a result $\mathfrak{S}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

(ii) Assume that $\mathfrak{q}_\iota \leq 1$ for each $\iota \in \mathbb{N}$ together with $\lim_{n\to\infty} \inf \frac{\mathfrak{Q}_n}{n} > 0$. Hence, that $\delta > 0$ exists where $\delta \leq \frac{\mathfrak{Q}_n}{n} \leq 1$. Furthermore,

$\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}), \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$ and $\dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi})$

because $\mathfrak{q}_\iota \leq 1$ for any $\iota \in \mathbb{N}$ along with $\dot{\xi}(\hat{\mathfrak{u}}, \hat{\varphi})$ is an increasing function among $\hat{\varphi} \in \mathbb{R}^+, \ddot{\zeta}(\hat{\mathfrak{u}}, \hat{\varphi}), \dddot{\varrho}(\hat{\mathfrak{u}}, \hat{\varphi})$ indicate decreasing function among $\hat{\varphi} \in \mathbb{R}^+$. Consequently, we have

$$\frac{1}{n}\left|\left\{\begin{array}{c} \iota \leq n : \dot{\xi}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}, \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}\right| \geq \frac{1}{n}\left|\left\{\begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}\right|$$

$$\geq \frac{\delta}{\mathfrak{Q}_n}\left|\left\{\begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_k(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}\right|.$$

The concept implies that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. □

The below conditions demonstrate that the converse among theorem (3.4) (ii) denote not typically true.

Example 3.5. The symbol $(\mathbb{R}, |\cdot|)$ represents the space of real number with the ordinary norm along with for each $\acute{\alpha}, \acute{\beta} \in [0, 1]$, let $\acute{\alpha} \breve{*} \acute{\beta} = \acute{\alpha}\acute{\beta}$ as well as $\acute{\alpha} \breve{\odot} \acute{\beta} = \min\{\acute{\alpha} + \acute{\beta}, 1\}$. All $x \in \mathbb{R}$ in addition any of the $\hat{\varphi} > 0$, take that $\dot{\xi}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{\hat{\varphi}}{\hat{\varphi} + |\hat{\mathfrak{u}}|}, \ddot{\zeta}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{|\hat{\mathfrak{u}}|}{\hat{\varphi} + |\hat{\mathfrak{u}}|}$ along with $\dddot{\varrho}(\hat{\mathfrak{u}}, \hat{\varphi}) := \frac{|\hat{\mathfrak{u}}|}{|\hat{\varphi} + \hat{\mathfrak{u}}|}$.

Since, $(\mathbb{R}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \breve{*}, \breve{\odot}, \breve{\Diamond})$ represent an $\mathfrak{NNLS}$. Let describe the sequences $\tilde{\mathfrak{P}}_\iota = 3^{\iota - 1}$ for any $\iota \in \mathbb{N}$ alongwith $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ which means $m \in \mathbb{N}$ as well as $\hat{\mathfrak{u}}_\iota = \begin{cases} 1, & \text{first } \left[\sqrt{3}^{m-1}\right] \text{ integers in} (\mathfrak{Q}_{m-1}, \mathfrak{Q}_m], \\ 0, & \text{otherwise.} \end{cases}$ We have, for every $0 < \acute{\varepsilon} < 1$ as well as $\hat{\varphi} > 0$, notify

$$\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) := \left\{\begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}_0(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \leq 1 - \acute{\varepsilon}, \\ \ddot{\zeta}_0(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon}, \text{and} \\ \dddot{\varrho}_0(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon} \end{array}\right\}.$$

Given that

$$\begin{aligned} \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) &= \left\{\iota \leq \mathfrak{Q}_n : \frac{\hat{\varphi}}{\hat{\varphi} + 3^{\iota-1}|\hat{\mathfrak{u}}_\iota|} \leq 1 - \acute{\varepsilon} yada \frac{3^{\iota-1}|\hat{\mathfrak{u}}_\iota|}{\hat{\varphi} + 3^{\iota-1}|\hat{\mathfrak{u}}_\iota|} \geq \acute{\varepsilon}\right\} \\ &= \left\{\iota \leq \mathfrak{Q}_n : |\hat{\mathfrak{u}}_\iota| \geq \frac{\acute{\varepsilon}\hat{\varphi}}{3^{\iota-1}(1 - \acute{\varepsilon})}\right\} \\ &= \{\iota \leq \mathfrak{Q}_n : \hat{\mathfrak{u}}_\iota = 1\}, \end{aligned}$$

we get

$$\begin{aligned}\frac{1}{\mathfrak{Q}_n}|\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})| &= \frac{2}{3^n-1}\left|\left\{\iota \le \frac{3^n-1}{2} : \hat{\mathfrak{u}}_\iota = 1\right\}\right| \\ &= \frac{2}{3^n-1}\left(1+\left[\sqrt{3}\right]+\left[\sqrt{3}^2\right]+\cdots+\left[\sqrt{3}^{n-1}\right]\right) \\ &\le \frac{2}{3^n-1}\left(1+(1.8)+(1.8)^2+\cdots+(1.8)^{n-1}\right) \\ &\le \frac{5}{2}\left(\frac{1.8}{3}\right)^n \frac{\left(1-\left(\frac{1}{1.8}\right)^n\right)}{\left(1-\left(\frac{1}{3}\right)^n\right)},\end{aligned}$$

it attains $\lim_{n\to\infty}\frac{1}{\mathfrak{Q}_n}|\mathfrak{K}\mathfrak{Q}_n(\acute{\varepsilon})| = 0$.
A positive integer m like that $\mathfrak{Q}_{m-1} < n \le \mathfrak{Q}_m$ exist given a large enough number n. After that, for every $0 < \acute{\varepsilon} < 1$ along with $\hat{\varphi} > 0$, let
$\mathfrak{K}_n(\acute{\varepsilon}) := \left\{\iota \le n : \dot{\xi}_0(\hat{\mathfrak{u}}_\iota, \hat{\varphi}) \le 1 - \acute{\varepsilon}\ddot{\zeta}_0(\hat{\mathfrak{u}}_\iota, \hat{\varphi}) \ge \acute{\varepsilon}, \dddot{\varrho}_0(\hat{\mathfrak{u}}_\iota, \hat{\varphi}) \ge \acute{\varepsilon}\right\}$
Given that

$$\begin{aligned}\mathfrak{K}_n(\acute{\varepsilon}) &= \left\{\iota \le n : \frac{\hat{\varphi}}{\hat{\varphi}+|\hat{\mathfrak{u}}_\iota|} \le 1-\acute{\varepsilon} yada \frac{|\hat{\mathfrak{u}}_\iota|}{\hat{\varphi}+|\hat{\mathfrak{u}}_\iota|} \ge \acute{\varepsilon}\right\} \\ &= \left\{\iota \le n : |\hat{\mathfrak{u}}_\iota| \ge \frac{\acute{\varepsilon}\hat{\varphi}}{(1-\acute{\varepsilon})}\right\} \\ &= \left\{\iota \le n : \hat{\mathfrak{u}}_\iota = 1\right\},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{1}{|\mathfrak{K}_n(\acute{\varepsilon})|} &= \frac{1}{n}\left|\left\{\iota \le n : \hat{\mathfrak{u}}_\iota = 1\right\}\right| \\ &> \frac{1}{n}\left\{1+(1.2)+(1.2)^2+\cdots+(1.2)^{m-1}\right\} \\ &> \frac{1}{n}(1.2)^m - \frac{1}{n} \\ &> \frac{1}{n}(1.2)^{\frac{n}{\log 2}} - \frac{1}{n} \text{ since } m > \frac{n}{\log 2},\end{aligned}$$

in which $(\hat{\mathfrak{u}}_n)$ is $\mathfrak{WSC}$ to 0 with respect towards the NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$; however, $\mathfrak{SC}$ not with respect towards the NN.

In the following theorems, we prove the link between $\mathfrak{WSC}$ and $(\bar{N}, \mathfrak{q}_n)$ - summability within $\mathfrak{NNLS}$.

Theorem 3.6. *Let $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \circledast, \odot, \Diamond)$ be an $\mathfrak{NNLS}$ and in addition $\frac{\mathfrak{Q}_n}{n} \ge 1$ for each $n \in \mathbb{N}$. If $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ within $\hat{\mathfrak{A}}$ is $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$- summable towards $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$, after that $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$ - convergent to $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$.*

Proof. Assume that $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})}$ - summable to $\tilde{\mathfrak{P}} \in \hat{\mathfrak{A}}$. For all $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$, consider

$$\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) = \left\{ \begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\} \text{ and}$$

$$\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon}) = \left\{ \begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > 1 - \acute{\varepsilon} \text{ and} \\ \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \acute{\varepsilon} \end{array} \right\}$$

After that,

$$\begin{aligned} \frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) &= \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})}} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) + \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \\ &\geq \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > \frac{1}{\mathfrak{Q}_n} \left|\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})\right| (1 - \acute{\varepsilon}). \end{aligned} \tag{2}$$

Resulting from inequity (2), we get $\lim_{n\to\infty} \frac{1}{\mathfrak{Q}_n} \left|\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})\right| = 1$. Similarly, for every $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$,

$$\begin{aligned} \frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) &= \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})}} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) + \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \\ &\leq \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1, \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) < \frac{1}{\mathfrak{Q}_n} \left|\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})\right| \acute{\varepsilon}. \end{aligned} \tag{3}$$

By inequality (3), we have $\lim_{n\to\infty} \frac{1}{\mathfrak{Q}_n} |\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})| = 0$. Hence proved. □

Theorem (3.4) is not often true, as demonstrated by the case that follows.

Example 3.7. Let $(\mathbb{R}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \circledast, \breve{\odot}, \breve{\Diamond})$ be as shown in Example (3.5). Take that $\mathfrak{q}_\iota = \frac{1}{\iota+1}$ for each $\iota \in \mathbb{N}$ and create a sequence $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ which is defined by

$$\hat{\mathfrak{u}}_\iota = \begin{cases} \iota, & \text{if } \iota = m^2 (m \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Then, for every $0 < \acute{\varepsilon} < 1$ along with each $\hat{\varphi} > 0$, consider

$$\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) = \left\{ \begin{array}{c} \iota \leq \mathfrak{Q}_n : \dot{\xi}(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \leq 1 - \acute{\varepsilon} \text{ or} \\ \ddot{\zeta}(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon}, \\ \dddot{\varrho}(\mathfrak{q}_\iota \hat{\mathfrak{u}}_\iota, \hat{\varphi}) \geq \acute{\varepsilon} \end{array} \right\}.$$

Given that
$\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon}) = \left\{ \iota \leq \mathfrak{Q}_n : \frac{\hat{\varphi}}{\hat{\varphi}+|\hat{\mathfrak{u}}_\iota|} \leq 1 - \acute{\varepsilon} \text{ or } \frac{\frac{|\hat{\mathfrak{u}}_\iota|}{\iota+1}}{\hat{\varphi}+\frac{|\hat{\mathfrak{u}}_\iota|}{\iota+1}} \geq \acute{\varepsilon} \right\} = \left\{ \iota \leq \mathfrak{Q}_n : |\hat{\mathfrak{u}}_\iota| \geq \frac{(\iota+1)\acute{\varepsilon}\hat{\varphi}}{(1-\acute{\varepsilon})} \right\}$
$\subseteq \{\iota \leq \mathfrak{Q}_n : \hat{\mathfrak{u}}_\iota = \iota\}$, we get $\frac{1}{\mathfrak{Q}_n} |\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})| \leq \frac{\sqrt{1+\frac{1}{2}+\cdots+\frac{1}{n}}}{1+\frac{1}{2}+\cdots+\frac{1}{n}}$

it yields that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = 0$.

Therefore,

$$\begin{aligned}
&\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dot{\xi}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi}) \\
&= \frac{1}{\mathfrak{Q}_{m^2}+s}\sum_{\iota=1}^{m^2+s}\dot{\xi}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi}) \\
&= \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{m^2+s}}\left(\begin{array}{c} \frac{\hat{\varphi}}{\hat{\varphi}+1}\underbrace{1+1}_{2}+\frac{4\hat{\varphi}}{4\hat{\varphi}+4}+\underbrace{1+1+1+1}_{4} \\ +\frac{9\hat{\varphi}}{9\hat{\varphi}+9}+\cdots+\underbrace{1+1+1+1}_{2(m-1)}+\frac{m^2\hat{\varphi}}{m^2\hat{\varphi}+m^2}+\underbrace{1+1+1+1}_{2m}\end{array}\right) \\
&= \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{m^2+s}}\left(\frac{mt}{t+1}+m^2+m\right) \text{ and} \\
&\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\ddot{\zeta}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi}) = \frac{1}{\mathfrak{Q}_{m^2+s}}\sum_{\iota=1}^{m^2+s}\ddot{\zeta}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi}) \\
&= \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{m^2+s}}\left(\frac{\hat{\varphi}}{\hat{\varphi}+1}+\frac{4\hat{\varphi}}{4\hat{\varphi}+4}+\frac{9\hat{\varphi}}{9\hat{\varphi}+9}+\cdots+\frac{m^2\hat{\varphi}}{m^2\hat{\varphi}+m^2}\right) \\
&= \frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{m^2+s}}\frac{m\hat{\varphi}}{\hat{\varphi}+1}
\end{aligned}$$

about $m, s \in \mathbb{N}$. Therefore, $\hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ is not $(\bar{N}, \mathfrak{q}_n)$ - summable towards 0 in terms of NN $(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})$, because to the statement that

$$\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dot{\xi}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi}) \to \infty, \frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\ddot{\zeta}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi}) \to \infty \text{ and } \frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dddot{\varrho}(\mathfrak{q}_\iota\hat{\mathfrak{u}}_\iota,\hat{\varphi}) \to \infty$$

as $n \to \infty$.

Theorem 3.8. *Let* $(\hat{\mathfrak{A}}, \dot{\xi}, \ddot{\zeta}, \dddot{\varrho}, \circledast, \odot, \check{\Diamond})$ *being a* $\mathfrak{NNLS}, \hat{\mathfrak{u}} = (\hat{\mathfrak{u}}_\iota)$ *in* $\hat{\mathfrak{A}}$ *and* $\frac{\mathfrak{Q}_n}{n} \geq 1$ *for all* $n \in \mathbb{N}$. *When* $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}, \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq 1 - M, \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq M$ *and in addition* $\dddot{\varrho}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \leq M$ *regards to* $M \in (0, 1)$ *and* $\iota \in \mathbb{N}$, *after that* $(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$.

Proof. Assume that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi},\ddot{\zeta},\dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$. After that, for each $\acute{\varepsilon} > 0$ as well as $\hat{\varphi} > 0$,

$$\begin{aligned}
\frac{1}{\mathfrak{Q}_n}\sum_{\iota=1}^{n}\dot{\xi}(\mathfrak{q}_\iota(_\iota - \tilde{\mathfrak{P}}),\hat{\varphi}) &= \frac{1}{\mathfrak{Q}_n}\sum_{\substack{\iota=1\\ \iota\in\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n}\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}),\hat{\varphi}) + \frac{1}{\mathfrak{Q}_n}\sum_{\substack{\iota=1\\ \iota\in\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n}\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}),\hat{\varphi}) \\
&= \mathfrak{S}_1(n) + \mathfrak{S}_2(n)
\end{aligned}$$

in which

$$\mathfrak{S}_1(n) = \frac{1}{\mathfrak{Q}_n}\sum_{\substack{\iota=1\\ \iota\in\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n}\dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}),\hat{\varphi}) \tag{4}$$

as well as

$$\mathfrak{S}_2(n) = \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1 \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \tag{5}$$

When $\iota \in \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})$, after that

$$\mathfrak{S}_1(n) = \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1 \\ \iota \in \mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) \geq \frac{|\mathfrak{K}_{\mathfrak{Q}_n}(\acute{\varepsilon})|}{\mathfrak{Q}_n}(1 - M). \tag{6}$$

Given that $\mathfrak{S}_{\bar{N}}^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}$,

$$\lim_{n \to \infty} \mathfrak{S}_1(n) \geq 0. \tag{7}$$

If $\iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})$, then we obtain

$$\mathfrak{S}_2(n) = \frac{1}{\mathfrak{Q}_n} \sum_{\substack{\iota=1 \\ \iota \in \mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})}}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) > \frac{|\mathfrak{K}^c_{\mathfrak{Q}_n}(\acute{\varepsilon})|}{\mathfrak{Q}_n}(1 - \acute{\varepsilon}) \tag{8}$$

which gives that

$$\lim_{n \to \infty} \mathfrak{S}_2(n) > (1 - \acute{\varepsilon}). \tag{9}$$

Using equality (4)-(5) and inequalities (6)-(9), we obtain

$$\lim_{n \to \infty} \frac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \dot{\xi}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) = 1. \tag{10}$$

Similarly, we obtain

$$\lim_{n \to \infty} \frac{1}{\mathfrak{Q}_n} \sum_{\iota=1}^{n} \ddot{\zeta}(\mathfrak{q}_\iota(\hat{\mathfrak{u}}_\iota - \tilde{\mathfrak{P}}), \hat{\varphi}) = 0. \tag{11}$$

As a result from equality (10) and (11), we get

$$(\bar{N}, \mathfrak{q}_n)^{(\dot{\xi}, \ddot{\zeta}, \dddot{\varrho})} - \lim \hat{\mathfrak{u}} = \tilde{\mathfrak{P}}. \tag{12}$$

□

4. **Conclusion**

We present a new method among summability within $\mathfrak{NNLS}$ in this paper, represented by the notation $(\bar{N}, \mathfrak{q}_n)$ -summability, as well as apply that summability to develope a new kind of $\mathfrak{SC}$ in $\mathfrak{NNLS}$, denoted by the notation $\mathfrak{WSC}$. The connections among these ideas are also explored. The derived findings were greater in scope compared to the associated findings for normed linear spaces because any crispness norm may generate a NN. While this paper's findings do overlap with those of previous studies on the subject, the arguments presented here were proven using an alternative methodology for the largest part. The authors of this research want to allow space for follow-up studies.

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