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Development of Neutrosophic Gamma Distribution for Modeling Neonatal Mortality Data

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Abstract: This study explores and investigates the key characteristics of the neutrosophic gamma model (NGM) to analyze neonatal mortality data. The proposed model has the ability to handle imprecision, vagueness and uncertainty in data which often exist in health statistics. The key characteristics of the proposed model such as the probability density function, cumulative distribution function, statistical moments and some basic shape coefficients are discussed to clarify its difference from the classical model. Air pollution mortality data are commonly encountered imprecise and incomplete information due to factors such as missing values, measurement errors, reporting inconsistencies. The proposed NGM has the inherent ability to model such ambiguous data as robust tool for addressing these challenges. Through a detailed statistical analysis of mortality data linked to air pollution in Saudi Arabia, we demonstrate that the NGM outperforms traditional models in managing uncertainty and providing more accurate mortality analysis. This study not only enhances the theoretical structure of the NGM but also provides practical implications for policy formulation and healthcare management.

Keywords: Neutrosophic probability; gamma function; estimation; simulation

1. Introduction

Probability distributions and random variables are not only a core concept in probability theory and statistics, but they also serve a critical function in helping model uncertainty, as well as in aiding data-driven decisions in different fields [1]. A random variable is a variable that takes numerical values on the outcomes of a random phenomenon [2]. These can be variable either discrete, it will take specific values, or continuous that can take the value in a range. The probability distribution, meanwhile, specifies the probability of each of the possible outcomes of a random variable. Probabilistic models can be used to understand and predict various systems in the real world, including the weather, the stock market, healthcare data, and more. Probability distributions are critical in conducting hypothesis tests, making risk assessments and optimizing models since they provide the framework for uncertainty analysis [3]. Thorough understanding of random variables and probability distributions is of utmost importance in domains in different applications such as engineering, economics and machine learning and artificial intelligence [4]. One of the most common continuous distributions is gamma distribution [5]. The gamma distribution is one of the most flexible that is used in many statistical applications and plays a unique role among the hundreds of models that we use in many applications. The gamma distribution is a more generalized form of exponential distribution [6]. It has many applications in survival and reliability studies. The gamma distribution is useful for instruments or equipment that needs to be calibrated [7]. Its adaptability to different shapes and scales makes it a vital tool in fields including finance, environmental science, and healthcare. In addition to being prominent in statistical literature, the gamma distribution has been extensively used in statistical methodologies; it is essentially used as conjugate prior for rate parameters in Bayesian analysis, an additional point of its importance [8].

Among various models widely applied, the gamma distribution is an unparalleled one with diverse modelling capabilities in real-world phenomena, making it the most relevant statistical tool [9]. It is vital when dealing with such data, including wait time, survival analysis, and reliability studies [10]. The flexibility of the shape and scale makes the gamma distribution a great model for situations that can be found in finance, environmental science, and health care [11]. It shows how popular it is in the modern-day analytics data applications. Random uncertainty, representable with random variable, is common in the way of a stochastic environment; however fuzzy uncertainty appears when the information is not random but vague or unclear. Fuzzy uncertainty refers to situations in which precise values cannot be ascertained, and where the boundaries between possible outcomes are not clearly defined [12]. Fuzzy set theory is employed to model this kind of uncertainty; its underlying principle is that it is more appropriate to describe elements by degrees of membership rather than a definitive classification [13]. For example: The temperature is high. In classical probability we might define a random variable with respect to temperature values whereas in fuzzy logic, we might simply refer to the concept of "high temperature", which we would now express as a fuzzy set in which temperature is high to different extents depending on context. Other examples would be customer satisfaction ratings (where satisfaction could be "low," "medium," or "high," with overlapping meanings) or the age of a person (where an age could be "young" or "old," but the switch between the categories is gradual and fuzzy). Fuzzy uncertainty is applied when exact measurement is infeasible, such as in expert/knowledge-based systems, controllers, decision-making processes, etc., where human perception and intangibles are translated into quantitative measures.

Neutrosophic theory is an extension of fuzzy set theory which helps in dealing with indeterminate and inconsistent information where it fails [14]. Fuzzy set theory is used when elements are assigned membership degrees in range of 0 to 1 point; neutrosophic theory, on the other hand, considers truth, indeterminacy, and falsity, presenting a further granulated means of dealing with uncertainty [15]. While in fuzzy sets, the elements can only have a certain degree of truth or membership, neutrosophic sets can contain information that is indeterminate, as truth, indeterminacy, and falsity which are independent components and take values between 0 and 1. This was a major shortcoming of fuzzy set theory as it failed to model situations where the information is conflicting or uncertain and cannot be represented by a single membership function [16]-[18]. In fuzzy set theory, for instance, an answer to the question “Is the weather cold?” would be a fuzzily defined concept, and could be denoted with a membership value, for example, 0.7 for "cold." In neutrosophic theory, however, the answer might have a truth degree of 0.7 (e.g., "cold"), falsity degree of 0.2 (e.g., "not cold"), and indeterminacy degree of 0.5 (e.g., "uncertain") at the same time, and this represents a more realistic and multi-dimensional representation of uncertainty. The neutrosophic theory is, therefore, a very convenient opportunity in decision-making, artificial intelligence, finance, business, engineering, healthcare and many real field applications where vagueness and contradictory information commonly exist [19]-[21]. Due to neutrosophic logic being able to manage truth, falsity, and indeterminacy separately, its use is widespread in all fields of science and engineering, particularly where uncertainty, vagueness, and incomplete information are concerned. In the domain of neutrosophic statistics, a wide range of indecisive data cannot be modeled using exact statistics and can rather induce a neutrosophic logic in decision making process [22]. Neutrosophic statistics generalizes classical and traditional statistical techniques by introducing uncertainty and considering the uncertain and contradictory nature of information and data, which is potentially most valuable in many complex scenarios, decision-making, risk analysis, and pattern recognition [23]. For example, a medical diagnostic in which signs and test results may be conflicting or uncertain; neutrosophic statistics can evaluate more accurately the probability of diagnosis because it correlates indeterminate and conflicting evidence. Neutrosophic logic can also be utilized in machine learning and artificial intelligence to process incomplete or ambiguous data, enhancing model accuracy and enabling improved decision-making processes [24]. Neutrosophic statistics can further find application in finance for risk assessment and also in engineering for fault detection and in environmental studies for modeling certain uncertain and fluctuating data due to the capability and effectiveness to signifying degrees of truth, falsity and uncertainty separately. Under the neutrosophic logic many extensions of the classical probability distributions have been developed in literature [25]-[29]. These extensions are important because such distributions exhibit properties that are well-suited to modeling types of information that involve vagueness (incomplete, inconsistent, or unclear information). Neutrosophic probability distributions utilize neutrosophic logic, providing a flexible framework for statistical modeling in difficult real-life situations. Their versatility renders them best proven for applications in domains where classical probabilistic methods are challenged in areas including social sciences, engineering studies, survival analysis, health care studies, and environmental studies and much more. Although many neutrosophic structures have been developed so far, there is still a need to provide real-life applications for many developed neutrosophic probability distributions, particularly for commonly used models.

In this paper, we explore the properties of the NGM and its application to air pollution mortality rates. Incorporating neutrosophic parameters in the standard gamma distribution allows the neutrosophic gamma distribution to effectively handle uncertainties within the data. This allows for a more accurate and nuanced understanding of the data behind the mortality rates associated with air pollution in relation to other forms of pollution, capturing both the unknowns in terms of environmental and health conditions.

This work is organized as follows: Section 2 provides some useful properties of classical gamma model. In Section 3 the proposed model with essential properties is discussed. In Section 4, estimation methodology of the neutrosophic version of the gamma model is demonstrated. Section 5 relates the applications of the suggested model to air pollution mortality rate data. Eventually, the findings of the work are concluded in Section 6.

2. Gamma Distribution

In this section we will discuss the classical structure of gamma distribution. The gamma distribution has its unique role in statistical analysis. A random variable $T > 0$ is said to follow gamma distribution with scale parameter θ and shape parameter ρ if it has the probability density function (pdf) given below:

$$f_T(t;\rho,\theta) = \frac{t^{\rho-1}e^{-\frac{t}{\theta}}}{\Gamma(\rho)\,\theta^{\rho}}, \quad t > 0 \tag{1}$$

where $\Gamma(\rho)$ represent the gamma function with the defined value:

$$\Gamma(\rho) = \int_0^{\infty} z^{\rho-1}e^{-z}\,dz$$

A pdf is central to statistics and refers to the probability of a continuous random variable taking on the value or range of values. The distribution of the data refers to the definition of the relative frequency of occurrence which specifies a complete characterization of (the) sampled data. In probability, the area under the curve of the pdf over any interval corresponds to the probability that the random variable falls within that interval, making it fundamental for probability calculation. The pdf additionally elucidates important aspects of the data, including central tendencies, spread, and whether skewness or multimodality is present. In reliability analysis, for example, the pdf of a lifetime distribution illustrates how failure rates are age-dependent, which in turn, helps with decision-making. It connects theory and practice and facilitates modeling and prediction in engineering, economics, and the natural sciences. The shape of the gamma pdf is shown in Figure 1.

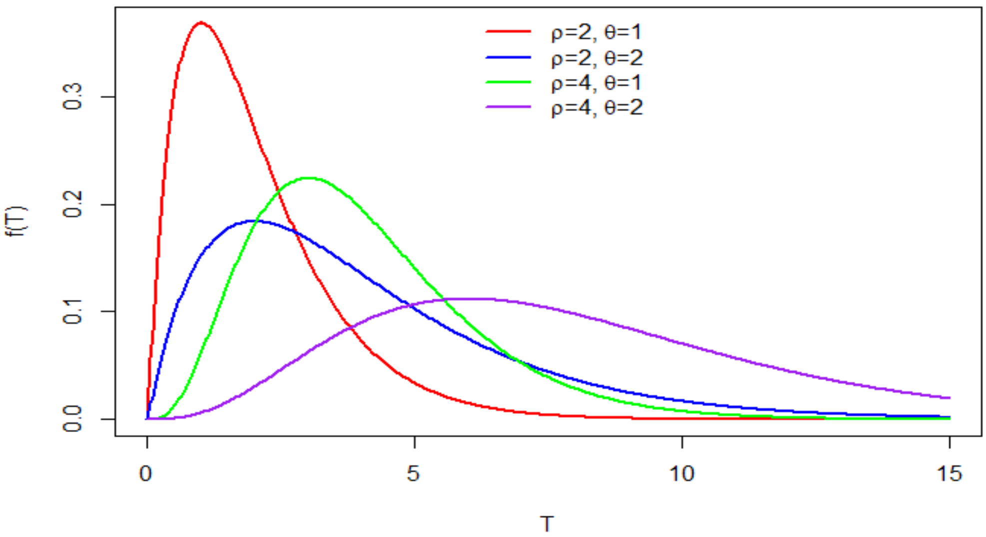

Figure 1. The pdf of the gamma distribution with different parameter settings

Figure 1 shows the curves of the pdf given in Eq (1) for different values of shape and scale parameters. It can be noticed that shape turns to symmetry for lager value of the shape parameter. Similarly the cumulative function (cdf) is another key concept related to pdf of any distribution. The cdf of the gamma distribution can be written as:

$$F_T(t;\rho,\theta) = \frac{1}{\Gamma(\rho)}\int_0^{\frac{t}{\theta}} x^{\rho-1}e^{-x}\,dx, \quad t > 0 \tag{2}$$

The cdf in Eq (2) shows that it involves incomplete gamma fuction and not in closed form. However, due to existing software we can easily sketch plots of the cdf as shown in Figure 2.

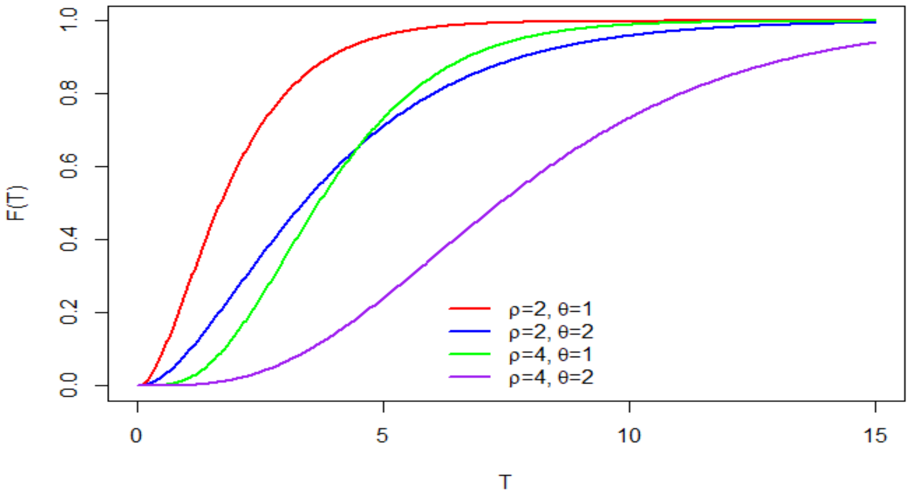

Figure 2. The cdf plot of the gamma distribution with different parameter settings.

The cdf is a key concept in probability theory, providing the cumulative probability that a random variable T is less than or equal to a specific value. While the pdf demonstrates the probability of certain values. The cdf is a full portrayal of the distribution and displays probabilities over intervals. This is crucial for reading probability and quantiles and percentiles. The cdf can be used in a reliability analysis to find the probability of failure over a period, for instance. Its meaning is intuitive: for any value t it provides us the proportion of the population or data less than tt which gives way to real-world applications like risk assessment, decision-making, and hypothesis testing.
The moment generating function of the gamma distribution can be written as:

$$M_T(s) = (1 - \theta s)^{-\rho}, \quad s < \frac{1}{\theta} \tag{3}$$

The moments about mean can easily be established using Eq (3) that may help to write its other properties like mean, variance, skewness and kurtosis coefficients.

$$\text{Mean: } E[T] = \rho\theta \tag{4}$$

$$\text{Variance: } \mathrm{Var}(T) = \rho\theta^2 \tag{5}$$

$$\text{Skewness} = \frac{2}{\sqrt{\rho}} \tag{6}$$

$$\text{Excess Kurtosis} = \frac{6}{\rho} \tag{7}$$

Two important coefficients in statistics that show us something about the distribution of a probability distribution, are skewness and kurtosis. Skewness refers to the imbalance of the distribution, where the data could be skewed to the left side or right of the mean. A positive skew means the tail on the right side is longer, whereas a negative skew means the left tail is longer. Kurtosis measures the sharpness of the distribution. High kurtosis indicates a distribution that is more outlier-prone than the normal distribution, while low kurtosis reveals less outliers than the normal distribution. Those two coefficients together give us an insight to general shape of the distribution considering higher moment values, that is why they are useful for task overview for model selection, data analysis and risk assessment, particularly in finance, engineering and natural sciences. Now in the next Section we illustrate the characteristics of the proposed NGM with essential functions.

3. Neutrosophic Gamma Model

In this section, neutrosophic structure of the gamma model has been discussed in connection with the classical model. By contrast, a NGM uses neutrosophic logic to model uncertainty, indeterminacy, and impreciseness in the data. The NGM is an extension of the classical distance principle with

neutrosophic parameters as opposed to classical numbers; Its numeric parameters can be expressed in ranges, rather than fixed numbers. This augmented framework is capable of accounting for richer uncertainty layers than the classical gamma distribution, which is ideal considering the ambiguity of the data.
A gamma random variable T follows the NGM with the following pdf:

$$f_n(t) = \frac{1}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}}, \quad t > 0 \tag{8}$$

where incomplete neutrosophic function is given by: $\Gamma\rho_n = \int_0^\infty z^{\rho_n - 1} e^{-z} dz$.
The distributional parameters θ_n and p_n and θ_n are the scale and shape parameter respectively of the NGM. By assuming different values to shape and scale parameters , pdf of the suggested model can be depicted in Figure 3

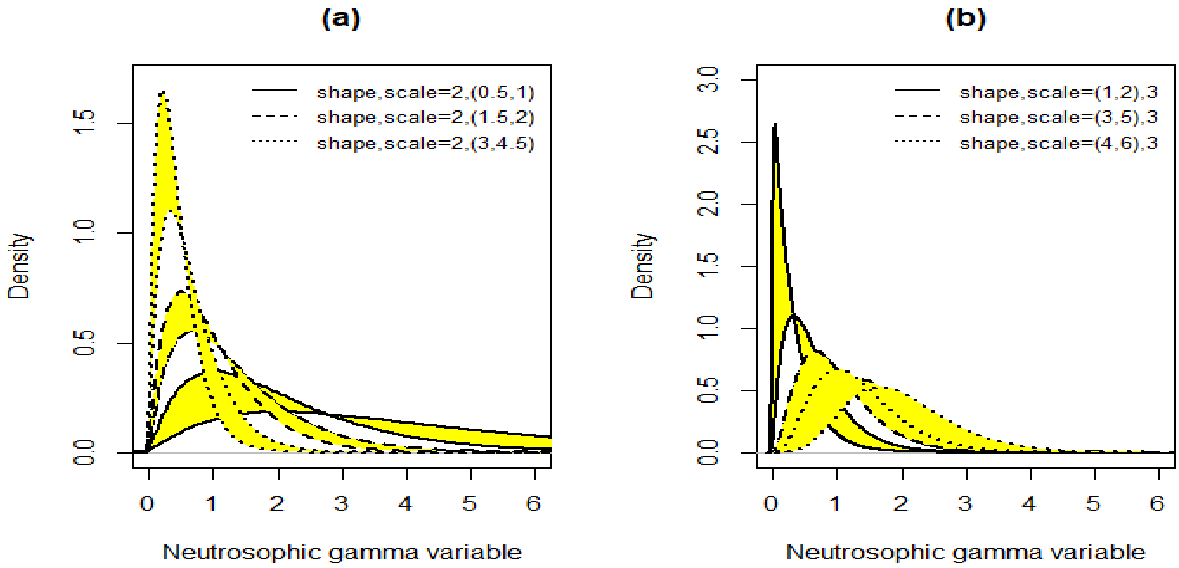

Figure 3. Shapes of the pdf of NGM with varying imprecise parameter setting

Figure 3 shows the sturdy curves of the NGM for varying imprecise parameter settings. In Figure 3 (a), scale parameter θ_n varies to three different values but shape parameter is taken as a crisp value, i.e., $\rho_n = [8, 8]$. In Figure 3(b) we assume shape parameter as an imprecise value and varies but scale parameter is set to fixed [6,6]. The pdf of the NGM in either case has the same properties as it exists in case of classical gamma distribution curve. Although we have limited our curves to some fixed values of shape and scale parameters, however; one can sketch these curves easily by using the “neutrostat” R library [22].
Some basic properties of the proposed model can be determined under the neutrosophic algebra. For example the kth moment about origin of the proposed model can be established as follows:
The kth moment of NGM can be defined as:

$$\vartheta'_{kn} = \int_o^\infty \frac{1}{\Gamma\rho_n \theta_n^{\rho_n}} t^{\rho_n - 1} e^{-\frac{t}{\theta_n}} t^k \; dt \tag{9}$$

Eq (9) further can be extended as:

$$= \left[\int_0^\infty \frac{1}{\Gamma\rho_l \theta_l^{\rho_l}} t^{\rho_l - 1} e^{-\frac{t}{\theta_l}} t^k \; dt, \; \int_0^\infty \frac{1}{\Gamma\rho_u \theta_u^{\rho_u}} t^{\rho_u - 1} e^{-\frac{t}{\theta_u}} t^k \; dt \right] \tag{10}$$

Considering the transformation $y = \frac{t}{\theta_n}$ in Eq (10) and simplification yielded:

$$\vartheta'_{kn} = \left[\frac{\theta_l^k}{\Gamma\rho_l} (\Gamma k + \rho_l), \; \frac{\theta_u^k}{\Gamma\rho_u} (\Gamma k + \rho_u) \right] = \frac{\theta_n^k}{\Gamma\rho_n} \; ; \quad k = 1, 2, \ldots \tag{11}$$

Assigning different values to k in Eq (11) generates moments of origin which can be extended as:

$\vartheta_{1n} = \theta_n$ (First moment of origin)
$\vartheta_{2n} = \theta_n$ (Second moment of origin
$\vartheta_{3n} = 2\theta_n$ (Third moment of origin)
$\vartheta_{4n} = 6\theta_n - 3\theta_n^2$ (Fourth moment of origin)
The other important function of the proposed model is cumulative distribution function(cdf) which can be defined asL

$$F_n(x) = \int_0^x f_n(t)\ dt. \tag{12}$$

$$\mathrm{F_n(t)} = \int_0^x \frac{1}{\Gamma(\rho_n)\,\theta_n^{\rho_n}}\, t^{\rho_n - 1} e^{-\frac{t}{\theta_n}}\ dx. \tag{13}$$

Simplification of Eq (13) provided:

$$F_n(x) = \frac{\gamma\left(\rho_n, \frac{x}{\theta_n}\right)}{\Gamma(\rho_n)} \tag{14}$$

where $\gamma\left(\rho_n, \frac{x}{\theta_n}\right)$ is incomplete gamma function which can easily be found by using function in R library. The construction of the Eq (14) can be done by using the function of neutrostat package. The cdf of the proposed model for some specific values of shape and scale parameter is shown in Figure 4.
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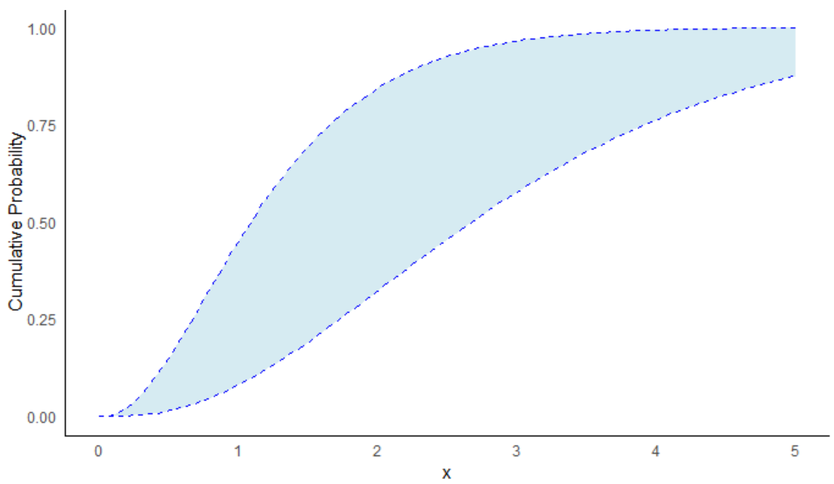

Figure 4. The cdf curve of the NGM at $\theta_n = [2, 3.5]$, and $\rho_n = [0\ .5, 1]$

Figure 4 shows the cdf curve of the proposed model when both parameters of the distribution are not exactly defined. Different cdf curves can be depicted with different parameter settings using the neutrostat package in R.
By using the definition of expected value, mean of the proposed NGM can be derived as:

$$E(T) = \int_0^\infty \frac{t}{\Gamma\rho_n\theta_n^{\rho_n}} t^{\rho_n-1} e^{-\frac{t}{\theta_n}} dt \tag{15}$$

Extend form Eq (15) can be written as;

$$E(T) = \int_0^\infty \left[\frac{t}{\Gamma\rho_n\theta_n^{\rho_n}} t^{\rho_n-1} e^{-\frac{t}{\theta_n}},\ \frac{t}{\Gamma\rho_n\theta_n^{\rho_n}} t^{\rho_n-1} e^{-\frac{t}{\theta_n}}\right] dz \tag{16}$$

Considering the transformation $y = \frac{t}{\theta_n}$ provided:

$$E(T) = [\rho_l\theta_l,\ \rho_u\theta_u] = \rho_n\theta_n \tag{17}$$

Now if we define the variance, we can write

$$var(T) = E(T^2) - [E(T)]^2 \tag{18}$$
where $E(T^2)$ can be obtained as:
$$E(Z^2) = \int_0^\infty \frac{t^2}{\Gamma\rho_n\theta_n^{\rho_n}} t^{\rho_n-1} e^{-\frac{t}{\theta_n}}\, dt$$
$$= \int_0^\infty \left[\frac{t^2}{\Gamma\rho_n\theta_n^{\rho_n}} t^{\rho_n-1} e^{-\frac{t}{\theta_n}},\ \frac{t^2}{\Gamma\rho_n\theta_n^{\rho_n}} t^{\rho_n-1} e^{-\frac{t}{\theta_n}}\right] dt \tag{19}$$
Simplification provided:
$$E(T^2) = [\theta_l^2\rho_l(\rho_l+1),\ \theta_u^2\rho_u(\rho_u+1)] \tag{20}$$
Thus Eq (18) becomes
$$var\ (T) = [\rho_l\theta_l^2,\ \rho_u\theta_u^2] = \rho_n\theta_n^2 \tag{21}$$
The other important measures of any distribution are the shape coefficients. The neutrosophic coefficient of skewness and the neutrosophic coefficient of kurtosis can be successfully used to study asymmetry and tail behavior, offering a solid framework for uncertainty and imprecision in data analysis. While skewness quantifies the skew of a traditional probability distribution, neutrosophic skewness reflects the interval (when the data components are not exactly defined) or amount of skew in a neutrosophic probability distribution. Likewise, kurtosis measures the pointiness or flatness of the peak of a distribution, while neutrosophic kurtosis makes this measure applicable to uncertain environments, quantifying the degree of sharpness as an interval or identifying indeterminacy. This flexibility of interpretation enables these measures to be especially beneficial when the data is affected by incompleteness and vagueness.
The skewness coefficient of the proposed model can be defined as
$$\beta_{1n} = \frac{\vartheta_{3n}}{(\vartheta_{2n})^{3/2}} \tag{22}$$
where the values of ϑ_{2n} and ϑ_{3n} can be determined from Eq (11).
Thus Eq (22) becomes
$$\beta_{1n} = \frac{2}{\sqrt{\theta_n}}$$
where $\beta_{1n} = [\beta_{1l}, \beta_{1u}]$.
Similarly kutorsis or access kurtosis coefficient of the NGM is defined as:

$$\beta_{2n} = \frac{\vartheta_{4n}}{(\vartheta_{2n})^2} \tag{23}$$
Eq (11) can be utilized to simplify Eq (23)
$\beta_{2n} = \frac{6}{\theta_n}$, which is the required result.
where $\beta_{2n} = [\beta_{2l}, \beta_{2u}]$ is neutrosophic measure of the proposed model.

4 Random Sample Generation and Estimation

In this section, we discuss the random generation and estimation procedure of the proposed model. The inverse cdf or the inverse transform method is a widely used technique to generate random samples from a specified probability distribution. It uses that fact that for any continuous random variable X with some F(x) there exists a way to sample from it, by drawing a uniform random variable and then applying the inverse cdf. This approach will work because the cdf "transforms" the probability range [0, 1] to the values of the random variable. First, uniform random values are generated and then, the inverse cdf is applied to transform these random values into the target distribution.
The cdf of the proposed function is defined in Eq (14) so the quantile function is given by:
$$Q(p; \rho_n, \theta_n) = F^{-1}(p; \rho_n, \theta_n) = U \tag{24}$$
where $\boldsymbol{p}$ follows the uniform distribution.

Now assigning different values to scale and shape parameters, we can draw random samples from the NGM. Assume that $\theta_n = [2, 3.5]$, and $\rho_n = [0\ .5, 1]$ we draw the random sample of 40 values the NGM, neutrosophic characteristics of simulated data are shown results Table 1.

Table 1 Random samples from the proposed NGM

[0.268, 1.425]	[0.7, 2.78]	[2.875, 8.214]	[1.575, 5.097]	[0.48, 2.125]
[2.097, 6.376]	[1.721, 5.459]	[1.121, 3.934]	[0.645, 2.622]	[0.958, 3.499]
[1.13, 3.958]	[0.564, 2.38]	[1.08, 3.826]	[0.727, 2.858]	[1.003, 3.62]
[0.549, 2.335]	[0.962, 3.508]	[1.218, 4.188]	[0.159, 1.009]	[2.939, 8.362]
[1.279, 4.346]	[1.241, 4.247]	[0.093, 0.714]	[0.726, 2.855]	[1.1, 3.878]
[0.939, 3.446]	[1.124, 3.94]	[0.684, 2.735]	[1.301, 4.401]	[0.32, 1.606]
[1.513, 4.94]	[0.264, 1.41]	[1.473, 4.842]	[1.998, 6.139]	[1.183, 4.096]
[1.778, 5.6]	[0.832, 3.154]	[0.709, 2.807]	[2.441, 7.197]	[0.924, 3.406]

Table 1 presents the random samples generated at specific shape and scale parameter values. Note that these values are randomly drawn using a function from the nutrostat package. Different seed settings in the program can yield different random samples, even for the same parameter settings. To study the different statistical characteristics of the proposed model , 10000 random samples are drawn at the same parameters setting and results are shown in Table 2.

Table 2 Characteristics of the simulated data generated from the NGM

Characteristics	Estimated value
Mean	[1.001, 3.501]
Variance	[0.713, 1.888]
Median	[0.836, 3.164]
Skewness	[1.058, 1.396]
Kurtosis	[4.572, 5.758]

Table 2 shows the neutrosophic characteristics of the NGM. It can be seen from the results that all characteristics are in interval values due to existence of indeterminacy in the shape and scale parameters of the proposed model. If this indeterminacy becomes zero, results of the proposed model match with the classical model.

Now we discuss the maximum likelihood (ML) approach for estimating the parameters of the NGM under the uncertain environment. ML is a common statistical approach used to estimate the parameters of a probabilistic model. This means finding the parameter values that maximize the likelihood function, a measure of how well the model accounts for the observed data. This approach works on the principle of maximizing the probability of the observed data, conditional on the model being used to describe the data. It is important because of its versatility and theoretical properties. The estimators obtained from ML approach are consistent and usually efficient (i.e. they converge to the true parameters with the lowest variance among biased estimators under regular conditions. Let $t_1\ , t_2, \ldots . t_n$ are sample of randomly values taken from the NGM, the log likelihood function is defined as

$$\vartheta(\rho_n, \theta_n\ |T) = -n\rho_n log(\theta_n) + (\rho_n - 1)\sum_{i=1}^{n} ln(t_i) - nln(\Gamma\rho_n) - \frac{\sum_{i=1}^{n} t_i}{\theta_n} \quad (24)$$

Maximizing (24) with unknown θ_n and ρ_n provided:

$$\frac{d\vartheta(\rho_n, \theta_n\ |T)}{d\theta_n} = n\left[-ln(\theta_n) - \frac{dln(\rho_n)}{d\rho_n} + \frac{\sum_{i=1}^{n} ln(t_i)}{n}\right] \quad (25)$$

$$\frac{d\vartheta(\rho_n, \theta_n\ |T)}{dd_n} = -\frac{n\rho_n}{\theta_n} + \frac{\sum_{i=1}^{n} t_i}{{\theta_n}^2} \quad (26)$$

Equating the expressions (25) and (26) to zero yielded:

$$\theta_n = \frac{\bar{t}}{\rho_n} \quad (27)$$

$$\overline{ln(t)} - ln(\bar{t}) + ln(\rho_n) - \frac{dln(\rho_n)}{d\rho_n} = 0 \quad (28)$$

The structures of Eq (27) and Eq (28) show that closed form does not exist so some iterative algorithm is required to achieved these estimates. Numerical set up to solve these unclosed forms can be achieved easily in R. In addition, the reliability study of these estimators can be assessed using the following metrics [26]:

$$Bias = \frac{\sum_{i=1}^{N}(\hat{\vartheta}_{ni} - \vartheta_n)}{N} \quad (29)$$

$$RSME = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\vartheta}_{ni} - \vartheta_n)^2}{N}} \quad (30)$$

where $RSME$ is formula used to find the root mean squared error of the estimator calculated from the sample. Bias and RMSE (Root Mean Square Error) are two important metrics in (non-)theoretically measuring the performance of your estimators. Bias (as a measure of systematic error) quantifies the expected difference between the value of the estimator and the true value of the parameter. A low bias in an estimator means that it is accurate on average, which is one reason why it is an important aspect of making predictions with confidence. In contrast, RMSE is a single value that incorporates both bias and variability; it is the square root of the average of the squared difference between the estimated and true values. This metric offers a comprehensive assessment of the accuracy of an estimator, as it accounts for both systematic and random errors. To assess the accuracy of the ML estimator, a simulation study has been conducted with $\theta_n = [0.5, 2]$ and $\rho_n = [3, 3]$ and results are shown in Table 3.

Table 3. Reliability assessment of ML estimator of the NGM

Sample	Bias	RMSE
10	[0.0011, 0.0046]	[6.8920, 10.2065]
25	[0.0005, 0.0019]	[6.8893, 10.1882]
50	[0.0003, 0.0011]	[6.8891, 10.1852]
75	[0.0000, 0.0000]	[6.8893, 10.1849]

Table 3 provides the accuracy of the ML estimators crossing different sample sizes. The Bias and RMSE assign a numerical value to the accuracy of an estimation, but these numbers are hard to interpret without fixed values. That is why estimation accuracy is assessed using the baseline values $\theta_n = [0.5, 2]$ and $\rho_n = [3, 3]$. Numerical results show that Bias and RMSE decrease with increasing sample size. This shows that as sample sizes increase, neutrosophic estimators provide better estimates. Consequently, the differences between the baseline values and the estimated values diminish as the sample size grows.

5 Real Data Example

In this section, we provide the real-data application of the proposed model. The proposed model is applied to neonatal mortality rate specific to Saudi Arabia. Neonatal mortality is the death of a newborn within the neonatal period which is the first 28 days of life [31]. This period is decisive for infant survival and establishes the basis for lifelong health. Neonatal mortality is a key indicator of the quality of maternal and newborn health care and, more broadly, of the society. Since early neonatal deaths can be prevented by timely and effective intervention including attendance of skilled/ trained personnel at birth, effective neonatal care and access to basic medical facilities. Intervening in this hugely impactful situation is therefore of urgent concern: scaling up interventions to reduce neonatal mortality is critical to saving lives for healthy societies and to achieving global health goals. Neonatal mortality data have significant attributes of health statistics under which they served to ascertain quality of a country in terms of health facilities, maternal and child health services and socio-economic status. Understanding this allows policymakers and healthcare providers to recognize gaps, prioritize resources, and design targeted interventions to improve newborn survival rates. Neonatal mortality has dramatically decreased in Saudi Arabia over the years as a result of improved healthcare infrastructure, maternal care, and access to neonatal intensive care units. But the access to health care is not equal in all parts of the region. Consequently, projects within Vision 2030 have been enacted by the Saudi government as part of a strategy to tackle neonatal mortality in newborns through the expansion of health services, training of health professionals, and improved awareness around prenatal and postnatal care. This way, they can help strengthen safeguards for newborns, leading hopefully to a healthier next generation. The upper and lower values of the neonatal mortality from the source are given [32] in Table 4.

Table 4. Neonatal mortality rate for Saudi Arabia for the year 2001-2020

[5.97, 9.60]	[5.25, 8.56]	[4.63, 7.52]	[3.43, 5.73]	[2.47, 5.09]
[8.71, 13.26]	[7.68, 11.91]	[6.75, 10.82]	[4.90, 8.05]	[4.34, 7.01]
[2.77, 5.22]	[7.26, 11.35]	[6.37, 10.20]	[4.05, 6.53]	[8.17, 12.58]
[5.60, 9.08]	[3.76, 6.07]	[3.11, 5.44]	[2.26, 4.99]	[2.11, 4.89]

It can be seen from Table 4 that classical gamma model is not suitable for data given in interval forms. To analysis that we use the proposed model. Results of the proposed model are shown in Table 5.
Table 5 Statistical characteristics of the neonatal mortality rate using NGM

Table 5. Statistical characteristics of the neonatal mortality data

Characteristics	Estimated values
Shape	[5.91, 9.17]
Scale	[0.65, 0.72
Mean	[3.82, 6.61]
Median	[3.61, 6.37]
Skewness	[0.66, 0.82]
Kurtosis	[3.65, 4.01]

Table 5 indicates that all sample statistics are in interval form. The proposed method effectively analyzes data in interval forms and is versatile enough to handle cases where zero indeterminacy exists. This adaptability ensures its applicability to a wide range of datasets, providing robust analysis under varying conditions of indeterminacy.

6 Conclusions

This work explores the utility of the proposed neutrosophic gamma Model (NGM) in demonstrating the effectiveness of neutrosophic probabilistic models for healthcare datasets. The statistical properties of the NGM are discussed to differentiate it from the classical gamma distribution. The NGM is specifically designed to analyze data involving imprecise and vague information. The quantile function of the proposed model is examined to facilitate the generation of random samples. The maximum likelihood (ML) approach under the neutrosophic framework has been developed, and the reliability of these estimates is evaluated through bias and mean squared error, using a simulation approach. Simulated results show that imprecise data with a larger sample size efficiently estimate the unknown neutrosophic parameters. A practical example using a real dataset illustrates the utility of the proposed model. The results show that the NGM offers a significant improvement over the conventional gamma model. Specifically, its application to neonatal mortality rates demonstrates that the classical gamma model fails to account for the interval nature of the data, whereas the NGM provides valid and consistent interval-based statistical estimates.

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