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Neutrosophic Formulation of the Quadratic Transmuted Generalized Exponential Distribution: Properties and Applications

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Abstract. The Quadratic Transmuted Generalized Exponential Distribution (QTGED) enhances the generalized exponential distribution, making it a significant development for handling complex decision-making contexts. Traditional statistical distributions often focus on representing degrees of truth or membership in fuzzy sets, yet they struggle to capture situations involving incomplete, vague, or contradictory data accurately. This study introduces the Neutrosophic Quadratic Transmuted Generalized Exponential Distribution (NQTGED), specifically designed to address indeterminacy and transmuted data. Neutrosophic theory is essential here, as it overcomes the limitations of classical and fuzzy set theories by effectively managing uncertainty, indeterminacy, and inconsistency in data. By simultaneously representing truth, indeterminacy, and falsity, neutrosophic sets offer a comprehensive framework for modeling uncertainty. Traditional distributions lack the adaptability needed for evolving data complexities, often falling short when faced with non-standard data distributions or outliers. Addressing these challenges requires innovative approaches that incorporate advanced mathematical models for uncertainty. This is especially valuable in real-world situations, where data is frequently incomplete, imprecise, or contradictory, and sometimes transmuted. The study derives various mathematical properties of the model, assesses parameter estimation using maximum likelihood and simulation, and demonstrates practical applications with cancer remission data. Simulation results reveal that Neutrosophic Average Biases (NABs) and Neutrosophic Mean Square Errors (NMSEs) decrease as sample sizes increase, indicating strong and accurate parameter estimation. NQTGED provides superior fit and performance, offering significant insights for applications in reliability engineering and biomedical sciences.

Keywords: Neutrosophic; indeterminacy; transmutation; generalized exponential distribution; maximum likelihood estimator; Monte Carlo simulation.

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1. Introduction

Neutrosophic theory emerges as a significant advancement over classical and fuzzy set theories by offering a robust framework to model uncertainty, indeterminacy, and inconsistency in data. Florentin Smarandache's [1] introduces Neutrosophic Sets represents a significant advancement in modeling uncertainty by integrating three core elements: truth, falsity, and indeterminacy. Classical sets operate on binary logic, representing data as either true or false, while fuzzy sets extend this by allowing degrees of truth. However, both approaches fall short in scenarios involving indeterminate or contradictory information.This triadic framework allows Neutrosophic Sets to more effectively address the complexities and subtleties found in real-world systems compared to classical and fuzzy sets. Neutrosophic sets address this gap by enabling the simultaneous representation of truth, indeterminacy, and falsity, thus providing a more nuanced and flexible approach to data analysis. This capability is particularly crucial in fields such as engineering, medical sciences, and social sciences, where data is frequently incomplete or imprecise, and decision-making processes require a comprehensive understanding of all aspects of uncertainty. The introduction of neutrosophic theory into these domains enhances the ability to interpret complex systems accurately, making it an indispensable tool for modern data analysis and informed decision-making.

In real-world scenarios, ambiguity and uncertainty often hinder the precise assignment of statistical values, making classical probability inadequate for accurate results [2, 3]. Neutrosophic statistics, with its framework for managing ambiguous and inconsistent data, addresses these challenges effectively. This approach has been advanced through applications of fuzzy logic and neutrosophy [4–8], with significant contributions from Smarandache's pioneering work [9]. On top of this, extensions of fuzzy sets, such as Picture Fuzzy Graphs (PFGs) and Picture Fuzzy Soft Graphs (PFSGs), enhance the modeling of vagueness by introducing parameters like cardinality and domination [10]. In software engineering, hybrid models combining Interval Type-2 Fuzzy Logic Systems with Artificial Neural Networks improve reliability predictions [11]. Moreover, recent studies on fuzzy functions reveal that somewhat fuzzy continuous functions are equivalent to somewhat fuzzy semicontinuous functions, while being weaker than fuzzy semicontinuous functions [12]. Recent studies in urban sustainability emphasize structured frameworks for addressing challenges like water scarcity and waste management using advanced fuzzy methodologies [13]. Neutrosophic philosophy acknowledges three key factors: truth membership, indeterminacy membership, and falsity membership, each representing the degree of truth, ambiguity, or untruth associated with an observation or hypothesis [6]. This makes them highly useful in fields such as decision-making, artificial intelligence, and risk assessment. Furthermore, the foundational principles of revolutionary topologies highlight how Neutrosophic Sets excel at representing contradictions and ambiguities in diverse scenarios [14].

While the application of neutrosophic probability theory holds practical relevance, it remains an area that has not received significant attention. However, recent years have witnessed an increasing focus on the neutrosophic statistics approach and its diverse applications across various fields. Neutrosophic probability distributions, such as the Neutrosophic Weibull distribution proposed by Alhasan and Smarandache [15], have been developed to address indeterminacy in real-life scenarios, often surpassing classical statistics in problem-solving effectiveness. Additionally, distributions such as Neutrosophic Uniform, Neutrosophic Exponential, and Neutrosophic Poisson have been numerically solved [7–10], [16]. Furthermore, the neutrosophic interpretation of normal and binomial distributions has been extensively explored through various examples [17]. Aslam and Ahtisham [3] introduced the neutrosophic variant of the Rayleigh distribution, while Nayana et al. [18] developed a novel neutrosophic model using the DUS-Weibull transformation. Rao et al. [19] provided insights into the neutrosophic generalized exponential distribution, elucidating its properties and applications. Recent studies [20–25] showcase advancements in neutrosophical probability distributions.

Neutrosophic statistics finds applications across diverse fields, including decision-making, pattern identification, data mining, and image processing in various industries [26–29]. The Box and Muller Technique for generating neutrosophic random variables with a normal distribution exemplifies the effective use of neutrosophic methods in operations research by enhancing decision-making and minimizing risk. This technique integrates goal functions, constraints, and optimal solutions within a neutrosophic framework, enabling simulations to tackle complex problems that are not easily represented mathematically [30]. The neutrosophic theory offers numerous applications, including the treatment of static models, integration of renewable energy sources such as photovoltaic panels, wind turbines, and addressing challenges related to COVID-19 and its Omicron mutation. The neutrosophic approach has recently been applied to tackle the complex challenges posed by the COVID-19 pandemic, utilizing multi-criteria decision-making (MCDM) methods like the Analytic Hierarchy Process (AHP) within a neutrosophic framework to manage uncertain data [31]. Recently, neutrosophic methods have expanded into Industry 5.0, introducing technology-driven inventory models where demand and cost parameters adapt to technological advancements, thereby supporting production optimization in dynamic manufacturing settings [32]. In parallel, plithogenic forest hypersoft sets (PFHS) have been developed to enhance multi-criteria decision-making, allowing complex attribute handling for applications like manufacturing plant site selection [33]. Moreover, generating gamma-distributed neutrosophic variables broadens the applicability of neutrosophic simulations, converting random numbers to improve precision in engineering and other fields [34]. Neutrosophic methods are employed to evaluate sustainable road transport systems, focusing on reducing carbon emissions and improving energy efficiency, while using MCDM models to balance various sustainability criteria. The composition method facilitates the generation of Poisson-distributed neutrosophic variables, enhancing fidelity in representing real-world environments [35]. The integration of neutrosophic sets with Decision-Making Trial and Evaluation Laboratory (DEMATEL) techniques in this research provides a structured approach to assess and enhance the sustainability of road transport systems [36].In cryptography, neutrosophic integers contribute to innovative encryption and decryption schemes, showcasing their potential in secure information processing [37]. Furthermore, neutrosophic stratified sampling applied to climate data yields reliable estimators with lower bias and improved mean square error (MSE), demonstrating the superior performance of neutrosophic methods over traditional approaches [38]. In addition, advancements in group decision-making models have highlighted the limitations of traditional neutrosophic AHP approaches. Recent developments have introduced enhanced models using neutrosophic trapezoidal numbers to address these limitations, providing more robust solutions for complex decision-making scenarios [39]. Furthermore, neutrosophic logic aids in supply chain risk management by modeling uncertainties and ranking risks using a hybrid AHP and Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) approach, providing valuable insights for risk mitigation [40]. While traditional mathematics prioritizes precision, real-world problems often involve ambiguous data, necessitating the use of mathematical concepts grounded in uncertainty. Despite its practical utility, neutrosophic probability theory has not received widespread attention, although it has been the subject of some studies. Recent research efforts have focused on various aspects of neutrosophic statistics, including correlation, regression analysis, test procedures, and probability distributions [24].

An influential extension of the exponential distribution, known as the Generalized Exponential Distribution (GED), was introduced by Gupta and Kundu [41]- [42]. Widely acknowledged in the literature, this two-parameter model has gained popularity for analyzing lifetime data. Serving as a versatile alternative to traditional distributions like Weibull or Gamma, the generalized exponential distribution finds applications in various fields including reliability analysis, hydrology, quality control, and medicine [41]- [48].

The groundbreaking research conducted by Shaw and Buckley [49] introduced the quadratic transmuted family of distributions, opening a new avenue for enhancing existing probability models to better capture the quadratic patterns inherent in data.

Khan et al. [50] introduced the Transmuted Generalized Exponential Distributions (TGED), stemming partly from advancements made in the transmuted family of lifetime distributions. They utilized the Quadratic Rank Transmutation Map (QRTM) technique and designated G(x) as the Cumulative Distribution Function (CDF) of the GED, serving as the baseline model. Additionally, they provided a comprehensive exploration of the mathematical properties associated with the three-parameter TGED, with the anticipation of its potential applications in reliability engineering and the biomedical sciences.

The study of transmuted neutrosophic distributions has garnered significant attention in recent years. Jabarali et al. [51] proposed Generalized p-Transmuted Neutrosophic Distributions (GTND), which delve into the theory and applications of various distributions, including exponential, Weibull, and Rayleigh, in both quadratic and cubic transmuted cases. This research emphasizes the theoretical foundations and practical implications of GTND, contributing to a deeper understanding of its potential applications across various fields. Additionally, the reliability and hazard functions of these distributions are analyzed, shedding light on their performance and reliability. This investigation provides valuable insights for researchers and practitioners.

The introduction of the Neutrosophic Quadratic Transmuted Generalized Exponential Distribution (NQTGED) extends the existing Quadratic Transmuted Generalized Exponential Distribution by incorporating neutrosophic elements, addressing uncertainties, vagueness, and indeterminacy in the data. This new distribution offers enhanced flexibility and versatility in modeling complex real-world scenarios, potentially leading to more accurate and robust statistical analyses. Motivated by the need to handle ambiguity and uncertainty, advance neutrosophic statistical theory, and provide practical applications for data characterized by indeterminacy, the researchers have developed this new distribution. The studies referenced and the reviews of the existing literature have inspired the researchers to embark on the development of a neutrosophic quadratic transmuted generalized exponential distribution, along with an exploration of its associated properties.

In the realm of statistical distribution theory, the accurate modeling of uncertainty, indeterminacy, and inconsistency in data remains a persistent challenge. Traditional statistical distributions, such as the GED and its extensions like the QTGED, primarily focus on capturing degrees of truth or membership in fuzzy sets. However, they often struggle to effectively represent scenarios where data is incomplete, vague, or contradictory.

The limitations of existing frameworks become particularly pronounced in fields such as engineering, medical sciences, and social sciences, where datasets frequently exhibit complex and nuanced patterns that defy simple categorization into true or false states. For instance, in biomedical research, the dynamics of disease progression or treatment outcomes may vary significantly among individuals, leading to data that is inherently uncertain or indeterminate.

Moreover, traditional distributions lack the flexibility to adapt to these diverse and evolving data complexities. They may fail to provide robust estimations or accurate predictions when faced with non-standard data distributions or outliers. This gap underscores the need for novel approaches that can integrate advanced mathematical frameworks capable of handling the full spectrum of uncertainty.

The proposed work is motivated by the need to better handle uncertainty, indeterminacy, and inconsistency in data analysis, areas where traditional statistical distributions and classical or fuzzy set theories fall short. Real-world data often exhibits complexity, vagueness, and contradictions that existing models cannot adequately capture. Specifically, the limitations of the QTGED in addressing indeterminate data highlight the necessity for a more comprehensive approach. This research addresses gaps such as the inadequate handling of indeterminacy by classical and fuzzy set theories, limited flexibility in existing models to manage imprecise and inconsistent data, and the need for enhanced robustness in real-world applications. The objectives of the proposed work include developing the NQTGED to integrate neutrosophic logic, deriving its mathematical properties, establishing parameter estimation methods using the maximum likelihood approach, and validating its practical applicability with real-world data. This includes comparing its performance against existing distributions using goodness-of-fit tests. The justification for introducing NQTGED lies in its potential to provide more accurate and reliable models for data analysis, particularly in contexts where data is uncertain or indeterminate. By incorporating neutrosophic logic, the proposed distribution offers significant advancements over existing models, contributing to more informed and accurate decision-making processes in various fields.

This paper is structured into seven sections. Section 2 outlines the preliminaries and research framework for the proposed distribution. Subsequent sections delve into properties and discuss various mathematical characteristics (Section 3). Section 4 presents a simulation study assessing the flexibility of the estimates. The penultimate section 5 includes a real-life data analysis, followed by a discussion of the study's implications in the section 6. Finally, the paper concludes with a summary in the last section.

2. **Preliminaries and Research Framework**

This section discusses the preliminaries and research framework of the study.

2.1. *Preliminaries*

In classical data analysis, information is typically represented by precise values. However, in neutrosophic statistics, data can manifest in various forms due to the potential for indeterminacy across different problem contexts. Neutrosophic statistics extends the principles of classical statistics. Neutrosophic numbers adhere to a standard format derived from classical statistics:

$$X_N = E + I$$

Here, the data is divided into two components: 'E' represents the exact or determined data, while 'I' signifies the uncertain, imprecise, or indeterminate part. Both 'E' and 'I' can take on any real number value. This format is equivalent to $X_N \in [X_L, X_U]$. To distinguish the neutrosophic random variable, it is denoted with the subscript N, as seen in the notation X_N.

Definition 1. Generalized Exponential distribution

[39] Let X be a continuous random variable, the Probability Density Function (PDF) of GED is given as,

$$g(x) = \begin{cases} \frac{\alpha}{\theta} exp\left(-\frac{x}{\theta}\right)\left(1 - exp\left(-\frac{x}{\theta}\right)\right)^{\alpha-1} & ; x > 0, \alpha > 0, \ and\ \theta > 0 \\ 0 & otherwise \end{cases} \tag{1}$$

where, α is the shape parameter and θ is the scale parameter.

The Cumulative Density Function (CDF) of GED is given as,

$$G(x) = \begin{cases} \left[1 - exp\left(-\frac{x}{\theta}\right)\right]^{\alpha} & ; x > 0, \alpha > 0, \ and\ \theta > 0 \\ 0 & otherwise \end{cases} \tag{2}$$

Definition 2. Quadratic Transmuted Distribution

[47] According to the QRTM approach, the CDF and probability density function (PDF) satisfy the following relationship:

$$F(x) = (1 + \lambda)G(x) - \lambda[G(x)]^2 \tag{3}$$

and

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)];\ x > 0; |\lambda| \le 1 \tag{4}$$

where, λ is the transmuted parameter.

G(x) and g(x) are the CDF and PDF of the baseline distribution respectively.

Definition 3. Transmuted Generalized Exponential Distribution

[48] If a random variable X follows the TGED with parameters α , θ and λ, then its PDF and CDF is as follows:

$$f(x) = \begin{cases} \frac{\alpha}{\theta} exp\left(-\frac{x}{\theta}\right)\left(1 - exp\left(-\frac{x}{\theta}\right)\right)^{\alpha-1}\left[1 + \lambda - 2\lambda\left(1 - exp\left(-\frac{x}{\theta}\right)\right)^{\alpha}\right] & ; x > 0, \alpha > 0, \theta > 0\ and\ |\lambda| \le 1 \\ 0 & otherwise \end{cases} \tag{5}$$

and

$$F(x) = \begin{cases} \left[1 - exp\left(-\frac{x}{\theta}\right)\right]^{\alpha}\left[1 + \lambda - \lambda\left(1 - exp\left(-\frac{x}{\theta}\right)\right)^{\alpha}\right] & ; x > 0, \alpha > 0, \theta > 0\ and\ |\lambda| \le 1 \\ 0 & otherwise \end{cases} \tag{6}$$

If $\alpha = 1$, TGED reduces to Transmuted Exponential Distribution.

2.2. *Conceptual Framework*

The conceptual framework, where the core concept is the NQTGED. This central arrow represents the innovative contribution of the paper. Surrounding NQTGED are elements such as traditional statistical distributions or QTGED, illustrating how NQTGED builds upon and modifies existing frameworks. Connected to NQTGED is another labeled neutrosophic logic, indicating how neutrosophic logic is integrated into the distribution. Neutrosophic logic allows NQTGED to handle uncertainty, indeterminacy, and inconsistency in data, capabilities that traditional distributions may not fully address. Arrows extend from various data sources, such as real-world data (e.g., cancer patient remission times), pointing towards NQTGED, demonstrating its application and validation with empirical data. Comparative arrows between NQTGED and traditional frameworks highlight NQTGED's advancements: it offers greater flexibility, improved accuracy in modeling real-world phenomena, and employs advanced methodological techniques like Bayesian estimation and robust methods. This diagram visually captures how NQTGED expands upon traditional frameworks through neutrosophic integration, making it a powerful tool for analyzing complex datasets with enhanced precision and applicability (Figure 1).

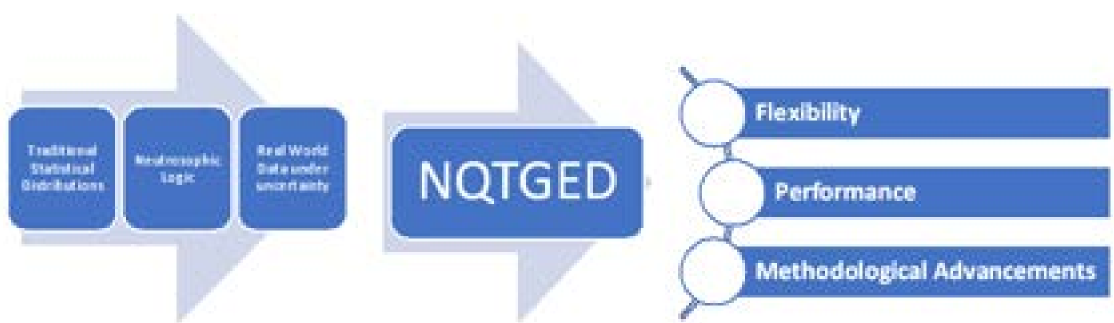


FIGURE 1. Conceptual Framework of the Proposed Work

The primary contributions of this paper include:

- **Development of NQTGED:** Introducing a novel distribution that integrates neutrosophic logic to handle uncertainty and indeterminacy.
- **Mathematical Properties:** Deriving key mathematical properties of NQTGED, such as moments, reliability measures, and order statistics.

- **Methodological Advancements:** Proposing and validating parameter estimation methods using advanced techniques like maximum likelihood estimation under neutrosophic uncertainty.
- **Application in Real-World Scenarios:** Demonstrating the practical applicability of NQTGED through case studies, such as analyzing remission times in cancer patients, to showcase its superiority over traditional distributions.

2.3. *Theoretical Framework*

Definition 4. *A neutrosophic random variable X follows a NQTGED, characterized by neutrosophic shape parameter α_N, neutrosophic scale parameter θ_N, and a neutrosophic transmuted parameter λ_N. The Neutrosophic Probability Density Function (NPDF) of the proposed distribution is defined as:*

$$f_N(x) = \begin{cases} \frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N - 1} & \\ \quad \times \left[1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] & if\ x > 0,\ \alpha_N > 0,\ \theta_N > 0,\ and\ \ |\lambda_N| \leq 1 \\ 0 & otherwise \end{cases} \tag{7}$$

The Neutrosophic Cumulative Distribution Function (NCDF) of NQTGED is,

$$F_N(x) = \begin{cases} \left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N} & \\ \quad \times \left[1 + \lambda_N - \lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] & if\ x > 0,\ \alpha_N > 0,\ \theta_N > 0,\ and\ \ |\lambda_N| \leq 1 \\ 0 & otherwise \end{cases} \tag{8}$$

Also, the neutrosophic baseline distribution is the Neutrosophic Generalized Exponential Distribution (NGED).

Figure 2 illustrates the probability density curves for the NQTGED, maintaining a constant transmuted parameter while varying the shape parameter and scale parameter within specified intervals. In the neutrosophic framework, the curve is depicted as a thick layer rather than a single curve. The thick layers indicate the range of possible densities, reflecting the uncertainty and variability captured by the neutrosophic parameters. The curves show a noticeable distortion towards the right side. They reveal an exponential decline, commencing from an infinite point when α_N<1. On the other hand, if $\alpha_N \geq 1$, the exponential decline persists, but it initiates from a specific position on the y-axis and also shows a unimodal behavior.

The NCDF curves of the proposed model are presented in Figure 3. In each panel depicted in Figure 3, the NCDF curve exhibits a non-decreasing pattern, covering the range from 0 to

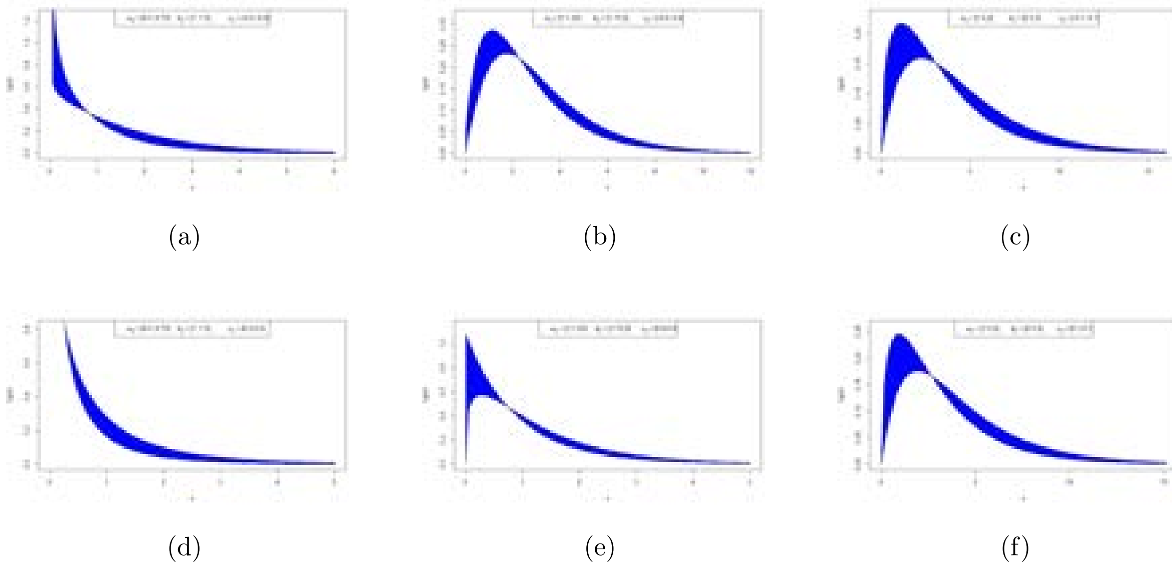


Figure 2. NPDF Plot of NQTGED

1. The inclusion of neutrosophic parameters ensures that the distribution can model a variety of real-world scenarios where data uncertainty is significant.

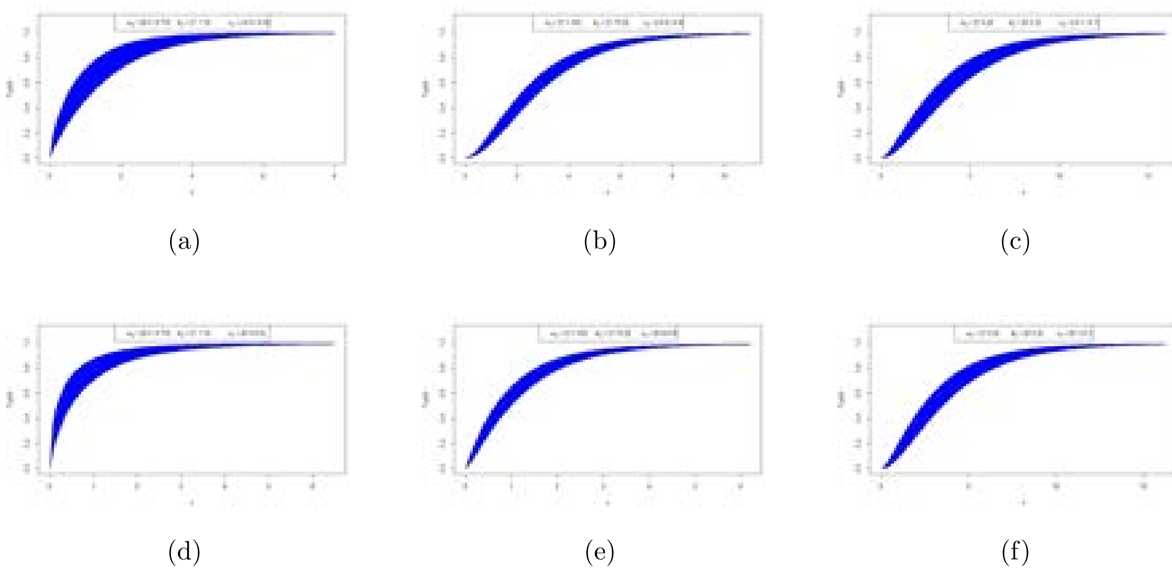


Figure 3. NCDF Plot of NQTGED

When analyzing the panels (a), (b) and (c), we observe that lower values of α_N lead to a gradual increase in the NCDF, suggesting a slower accumulation of probability and a more spread-out distribution. Higher values of α_N result in a steeper NCDF, indicating rapid probability accumulation and a more peaked distribution. The scale parameter θ_N stretches or compresses the NCDF horizontally, reflecting wider or tighter clustering of values, respectively.

In panels (d), (e) and (f), variations in the transmuted parameter λ_N show that positive values steepen the NCDF, concentrating probability in a smaller range, while negative values result in a flatter NCDF, spreading the probability mass more broadly.

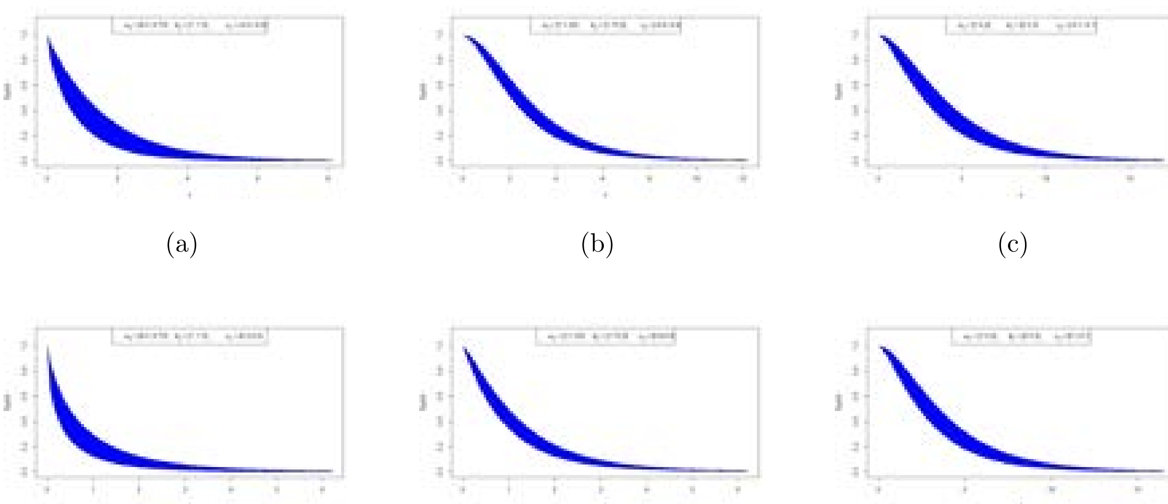
Definition 5. *Surival and Hazard Function*
The Neutrosophic Survival Function (NSF) and Neutrosophic Hazard Function (NHF) of the NQTGED are respectively formulated as follows:

$$S_N(x) = 1 - \left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N} \left[1 + \lambda_N - \lambda_N \left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] \tag{9}$$

and

$$h_N(x) = \frac{\frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N - 1}\left[1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]}{1 - \left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N}\left[1 + \lambda_N - \lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]} \tag{10}$$

Figure 4 showcases the NSF Plot of NQTGED, illustrating survival functions under different parameter settings to highlight the distribution's robustness. The panels (a), (b) and (c), with negative λ_N values, shows that increasing α_N leads to a steeper decline in survival probabilities, while increasing θ_N results in a more gradual decline. Negative λ_N values generally slow the decline, extending survival times. The panels (d), (e) and (f), with positive λ_N values, reveals similar trends where higher α_N steepens the decline, and higher θ_N makes it more gradual. Positive λ_N values tend to quicken the decline, shortening survival times.

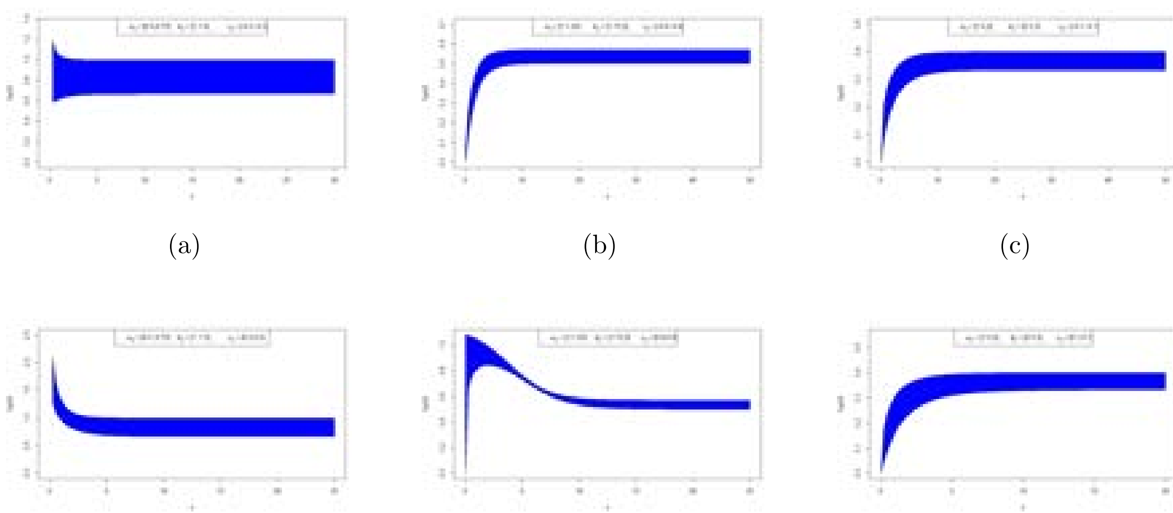


(a) (b) (c)

(d) (e) (f)

Figure 4. NSF Plot of NQTGED

Notably, Figure 5 highlights the distinctive bathtub-shaped and increasing behavior of the hazard rate in the NQTGED. Panels (a), (b) and (c) illustrates that negative λ_N values show that as α_N increases, the initial hazard rate rises sharply, while higher θ_N results in a slower increase in the hazard rate over time. Negative λ_N values generally lower the hazard rate, delaying its significant rise. In panels (d), (e), and (f), positive λ_N values, shows that increasing α_N leads to a more pronounced initial increase in the hazard rate, while higher θ_N causes a slower rise. Positive λ_N values quickly elevate the hazard rate, reducing the time before it peaks. These insights illustrate the substantial effects of α_N, θ_N, and λ_N on hazard rate functions, emphasizing the adaptability of the NQTGED distribution in modeling different hazard scenarios. The bathtub-shaped hazard rate benefits practical applications by identifying early failures for quick fixes, offering a stable period for routine maintenance, and predicting end-of-life replacements. This approach improves resource allocation and reliability. It accommodates uncertainty and indeterminacy, making the model versatile and realistic.



(a) (b) (c)

(d) (e) (f)

FIGURE 5. NHRF Plot of NQTGED

3. **Properties**

Theorem 1. *If $X \sim NQTGED(x; \alpha_N, \theta_N, \lambda_N)$, then the neutrosophic r^{th} moment of X is:*

$$\mu'_{Nr} = \sum_{j=0}^{\infty} (-1)^j \alpha_N \frac{\Gamma(r+1)}{(j+1)^{r+1}} \theta_N^r \left((1+\lambda_N) \binom{\alpha_N - 1}{j} - 2\lambda_N \binom{2\alpha_N - 1}{j} \right)$$

Proof. The r^{th} moment associated with the NQTGED are detailed as follows,

$$\mu'_{Nr} = \int_0^\infty x^r \frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right) \left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N - 1} \left[1 + \lambda_N - 2\lambda_N \left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] dx \quad (11)$$

Employing the binomial expansion equation (11) simplifies to

$$\mu'_{Nr} = (1+\lambda_N)\sum_{j=0}\infty\frac{\alpha_N}{\theta_N}(-1)^j\binom{\alpha_N-1}{j}exp-\frac{xj}{\theta_N}\int_0^\infty x^r exp-\frac{xj}{\theta_N}dx$$

$$-2\lambda_N\sum_{j=0}\infty\frac{\alpha_N}{\theta_N}(-1)^j\binom{2\alpha_N-1}{j}exp-\frac{xj}{\theta_N}\int_0^\infty x^r exp-\frac{xj}{\theta_N}dx$$

$$-2\lambda_N\sum_{j=0}\infty\frac{\alpha_N}{\theta_N}(-1)^j\binom{2\alpha_N-1}{j}exp-\frac{xj}{\theta_N}\int_0^\infty x^r exp-\frac{xj}{\theta_N}dx$$

Ultimately, we acquire

$$\mu'_{Nr} = \sum_{j=0}^{\infty}(-1)^j\alpha_N\frac{\Gamma(r+1)}{(j+1)^{r+1}}\theta_N^r\left((1+\lambda_N)\binom{\alpha_N-1}{j}-2\lambda_N\binom{2\alpha_N-1}{j}\right) \tag{12}$$

□

Theorem 2. *If $X \sim NQTGED(x;\alpha_N,\theta_N,\lambda_N)$, then the neutrosophic Moment Generating Function of X is:*

$$M_{NX}(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\frac{t^r}{r!}\alpha_N\theta_N^r\frac{\Gamma(r+1)}{(j+1)^{r+1}}\left((1+\lambda_N)\binom{\alpha_N-1}{j}-2\lambda_N\binom{2\alpha_N-1}{j}\right)$$

Proof.

$$\begin{aligned} M_{NX}(t) &= E\left[e^{tx}\right] \\ &= \int_0^\infty e^{tx} f_N(x)dx \\ &= \int_0^\infty\left(\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r\right)f_N(x)dx \\ &= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'_{Nr} \end{aligned}$$

Using the equation (13), it yields,

$$M_{NX}(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\frac{t^r}{r!}\alpha_N\theta_N^r\frac{\Gamma(r+1)}{(j+1)^{r+1}}\left((1+\lambda_N)\binom{\alpha_N-1}{j}-2\lambda_N\binom{2\alpha_N-1}{j}\right) \tag{13}$$

□

Theorem 3. *The neutrosophic MGF of NQTGED can be expressed in terms of gamma function as:*

$$(1+\lambda_N)\alpha_N\frac{\Gamma(1-t\theta_N)\Gamma\alpha_N}{\Gamma(\alpha_N-t\theta_N+1)}-2\lambda_N\alpha_N\frac{\Gamma(1-t\theta_N)\Gamma2\alpha_N}{\Gamma(2\alpha_N-t\theta_N+1)}$$

Proof.

$$\begin{aligned} M_{NX}(t) &= E\left[e^{tx}\right] \\ &= \int_0^\infty e^{tx} f_N(x)dx \\ &= \int_0^\infty e^{tx}\frac{\alpha_N}{\theta_N}exp\left(-\frac{x}{\theta_N}\right)\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N-1} \\ &\left[1+\lambda_N-2\lambda_N\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]dx \end{aligned} \tag{14}$$

Setting $y = exp\left(-\frac{x}{\theta_N}\right)$, $x = -\theta_N lny$ and $dx = \frac{-\theta_N}{y}dy$. Making substitution in equation (14),

$$\begin{aligned} M_{NX}(t) &= \alpha_N(1+\lambda_N)\int_0^1 y^{-t\theta-N}[1-y]^{\alpha_N-1}dy - 2\alpha_N\lambda_N\int_0^1 y^{-t\theta-N}[1-y]^{2\alpha_N-1}dy \\ &= \alpha_N(1+\lambda_N)\frac{\Gamma\alpha_N\Gamma(1-t\theta_N)}{\Gamma(\alpha_N-t\theta_N+1)} - 2\lambda_N\alpha_N\frac{\Gamma 2\alpha_N\Gamma(1-t\theta_N)}{\Gamma(2\alpha_N-t\theta_N+1)} \end{aligned} \tag{15}$$

□

Upon taking the derivative of $ln(M_{NX}(t))$ and assessing it at t = 0, we derive the mean and variance of X as

$$E_N(x) = \alpha_N\theta_N(1+\lambda_N)\left[\psi(\alpha_N+1)-\psi(1)\right] - 2\alpha_N\theta_N\lambda_N\left[\psi(2\alpha_N+1)-\psi(1)\right] \tag{16}$$

and

$$V_N(x) = \alpha_N\theta_N^2(1+\lambda_N)\left[\psi^{'}(1)-\psi^{'}(\alpha_N+1)\right] - 2\alpha_N\theta_N^2\lambda_N\left[\psi^{'}(1)-\psi^{'}(2\alpha_N+1)\right] \tag{17}$$

The qth quantile x_q of the NQTGED can be determined from equation (8) as follows;

$$x_q = -\theta_N ln\left[1-\left[\frac{(\lambda_N+1)-\sqrt{(\lambda_N+1)^2-4\lambda_N q}}{2\lambda_N}\right]^{\frac{1}{\alpha_N}}\right] \tag{18}$$

Specifically, the median of the distribution is;

$$x_{0.5} = -\theta_N ln\left[1-\left[\frac{(\lambda_N+1)-\sqrt{(\lambda_N+1)^2-8\lambda_N}}{2\lambda_N}\right]^{\frac{1}{\alpha_N}}\right]$$

3.1. *Random Number Generation and Estimation of Parameters of NQTGED*

By employing the inversion method, random numbers can be generated from the NQTGED according to the equation as

$$\left[1-exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N}\left[1+\lambda_N-\lambda_N\left(1-exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right] = u \tag{19}$$

where, $u \sim U(0,1)$. By solving equation (19) for x becomes,

$$x = -\theta_N ln \left[1 - \left[\frac{(\lambda_N + 1) - \sqrt{(\lambda_N + 1)^2 - 4\lambda_N u}}{2\lambda_N}\right]^{\frac{1}{\alpha_N}}\right] \tag{20}$$

Equation (20) can be utilized to generate random numbers provided the parameters α_N, θ_N and λ_N are known.

3.2. *Order Statistics*

Consider a set of n independent and identically distributed (i.i.d) random variables denoted as $X_{N1}, X_{N2}, ..., X_{Nn}$, each associated with their respective order statistics $X_{N(1)}, X_{N(2)}, ..., X_{N(n)}$, derived from NQTGED with density function $f_N(x)$ and distribution function $F_N(x)$. Consequently, the j^{th} order statistics for the PDF and the CDF are expressed as:

$$f_{x_j}(x) = \frac{n!}{(j-1)!(n-j)!} f_N(x)(F_N(x))^{j-1}(1 - F_N(x))^{n-j} \ \forall \ j = 1, 2, ..., n$$

The PDF of largest order statistic of NQTGED is given by,

$$f_{x_{(n)}}(x) = n\left(\frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N - 1}\left[1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]\right)$$
$$\left(\left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N}\left[1 + \lambda_N - \lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]\right)^{n-1}$$

The PDF of the smallest order statistic of NQTGED is given by,

$$f_{x_{(1)}}(x) = n\left(\frac{\alpha_N}{\theta_N} exp\left(-\frac{x}{\theta_N}\right)\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N - 1}\left[1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]\right)$$
$$\left(1 - \left[1 - exp\left(-\frac{x}{\theta_N}\right)\right]^{\alpha_N}\left[1 + \lambda_N - \lambda_N\left(1 - exp\left(-\frac{x}{\theta_N}\right)\right)^{\alpha_N}\right]\right)^{n-1}$$

3.3. *Maximum Likelihood Estimates of NQTGED*

The maximum likelihood estimates (MLEs) of the parameters inherent in the neutrosophic quadratic transmuted exponential distribution function are as follows: Let X_{N1}, X_{N2},...,X_{Nn} be a random sample from NQTGED. Then, the log-likelihood function is provided as:

$$logL = nlog\alpha_N - nlog\theta_N - \sum_{i=1}^{n} \frac{x_i}{\theta_N} + (\alpha_N - 1)\sum_{i=1}^{n} log\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)$$
$$+ \sum_{i=1}^{n} log\left(1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha}{}_{N}\right) \tag{21}$$

The MLEs of α_N, θ_N and λ_N are indicated as $\hat{\alpha}_N$, $\hat{\theta}_N$ and $\hat{\lambda}_N$ respectively, and are derived by optimizing equation (21). And these are the solutions of the following equations:

$$\frac{\partial LogL}{\partial \alpha_N} = \frac{n}{\alpha_N} + \sum_{i=1}^{n} log\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right) + \sum_{i=1}^{n} \frac{-2\lambda_N\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N} log\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)}{1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N}} \tag{22}$$

$$\frac{\partial LogL}{\partial \theta_N} = \frac{-n}{\theta_N} + \sum_{i=1}^{n} \frac{x_i}{\theta_N^2} + (\alpha_N - 1)\sum_{i=1}^{n} \frac{x_i exp\left(\frac{x_i}{\theta_N}\right)}{1 - exp\left(\frac{x_i}{\theta_N}\right)} - \sum_{i=1}^{n} \frac{2\lambda_N\alpha_N\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N - 1} x_i exp\left(-\frac{x_i}{\theta_N}\right)}{1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N}} \tag{23}$$

and

$$\frac{\partial LogL}{\partial \lambda_N} = \sum_{i=1}^{n} \frac{1 - 2\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N}}{1 + \lambda_N - 2\lambda_N\left(1 - exp\left(-\frac{x_i}{\theta_N}\right)\right)^{\alpha_N}} \tag{24}$$

Due to the complexity in computing the nonlinear equations, solving equations (22), (23) and (24) can be accomplished using the inversion method through statistical software R. This results in obtaining the maximum likelihood estimates of α_N, θ_N and λ_N, starting from a selected initial value.

4. Computational Study of NQTGED

In this section, to assess the performance of the proposed NGE distribution model, a simulation study is conducted. A Monte Carlo simulation is performed using R software across various sample sizes: specifically, n = 20, 50, 100, and 150, along with different neutrosophic parameter settings: (1) with $\alpha \in$ [0.5, 0.75], $\theta \in$ [1.5, 1.5], and $\lambda \in$ [-0.5, -0.5], (2) with $\alpha \in$ [0.5, 0.75], $\theta \in$ [1.5, 1.5], and $\lambda \in$ [0.5, 0.5], and (3) with $\alpha \in$ [1.5, 1.5], $\theta \in$ [0.5, 0.75], and $\lambda \in$ [-0.5, -0.5]. The simulation is repeated 1000 times. The estimated parameters of the NQTGED and their performance are evaluated in terms of Neutrosophic Average Estimates (NAEs), Neutrosophic Average Biases (NABs), and Neutrosophic Mean Square Errors (NMSEs) through simulation analysis. The simulation results for NAEs, NABs, and NMSEs are presented in Tables 1-3.

TABLE 1. **α_N=[0.5,0.75],θ_N=[1.5,1.5] and λ_N=[-0.5,-0.5]**

	NAE			NAB			NMSE		
n	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$
20	[0.5967,0.7898]	1.5964	-0.4484	[0.0967,0.0398]	0.0964	0.0516	[0.0619,0.1027]	0.2992	0.2181
50	[0.5496,0.7734]	1.5785	-0.4669	[0.0496,0.0233]	0.0785	0.0331	[0.0365,0.0495]	0.1202	0.1873
100	[0.5198,0.7610]	1.5661	-0.5073	[0.0198,0.0110]	0.0662	-0.0073	[0.0212,0.0358]	0.0647	0.1565
150	[0.5064,0.7572]	1.5618	-0.5065	[0.0064,0.0072]	0.0618	-0.0065	[0.0169,0.0323]	0.0571	0.1364

Table 2. α_N=**[0.5,0.75]**,θ_N=**[1.5,1.5] and** λ_N=**[0.5,0.5]**

	NAE			NAB			NMSE		
n	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$
20	[0.5299,0.7951]	1.5881	0.5299	[0.0299,0.0451]	0.0881	0.0299	[0.0223,0.0675]	0.4570	0.2067
50	[0.5112,0.7565]	1.5754	0.5175	[0.0112,0.0065]	0.0754	0.0175	[0.0081,0.0194]	0.1801	0.1971
100	[0.5097,0.7534]	1.5177	0.5155	[0.0097,0.0034]	0.0177	0.0155	[0.0029,0.0067]	0.1610	0.1613
150	[0.5001,0.7529]	1.5047	0.5095	[0.001,0.0029]	0.0047	0.0095	[0.0026,0.0056]	0.1299	0.1327

Table 3. α_N=**[1.5,1.5]**,θ_N=**[0.5,0.75] and** λ_N=**[-0.5,-0.5]**

	NAE			NAB			NMSE		
n	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$	$\hat{\alpha}_N$	$\hat{\theta}_N$	$\hat{\lambda}_N$
20	1.5227	[0.5218,0.7725]	-0.5280	0.0227	[0.0218,0.0225]	-0.0280	0.4255	[0.0236,0.0426]	0.2147
50	1.5163	[0.5213,0.7694]	-0.5191	0.0163	[0.0213,0.0194]	-0.0191	0.1554	[0.0106,0.0181]	0.1917
100	1.5112	[0.5181,0.7688]	-0.5090	0.0112	[0.0181,0.0188]	-0.0090	1.0923	[0.0065,0.0101]	0.1702
150	1.5022	[0.5147,0.7674]	-0.5079	0.0022	[0.0470,0.0174]	-0.0058	0.1036	[0.0042,0.0063]	0.1435

5. Application

In this section, to assess the interest in the NQTGED, a practical application utilized a real-world dataset comprising remission times of 128 cancer patients, measured in months, in Table 4. These remission times, sourced from subsets of bladder cancer studies referenced in [52], serve descriptive purposes. In a neutrosophic context, [19, 22] utilized the remission periods dataset to formulate the neutrosophic generalized exponential distribution. Furthermore, [51] conducted a comparison between the Neutrosophic Rayleigh Distribution and the Quadratic Transmuted Neutrosophic Rayleigh Distribution (QTNRD), adding depth to the study of neutrosophic statistical models. According to their research, the remission periods of cancer patients exhibit a good fit with the NGED model. The dataset's historical application in developing neutrosophic statistical models positions it as an excellent benchmark for showcasing the efficacy of the NQTGED. The analysis focused on evaluating how well the NQTGED represents the remission times by employing a range of goodness-of-fit measures, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Criterion (HQC), and the Kolmogorov-Smirnov (KS) test, which are furnished for comparison across QTNRD, Neutrosophic Exponential Distribution (NED) and NGED, as presented in Table 5. This thorough evaluation underscores the potential of the NQTGED for modeling complex datasets within the realm of medical statistics and beyond.

TABLE 4. **Remission periods of 128 cancer patients**

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.2	2.23	3.52	4.98	6.97
9.02	13.29	0.4	2.26	3.57	5.06	7.09	9.22	13.8	25.74	0.5	2.46	3.64
5.09	**[7.26, 8.2]**	9.47	14.24	25.82	0.51	2.54	3.7	5.17	7.28	9.74	14.76	**[5.3, 7.1]**
0.81	2.62	3.82	5.32	7.32	10.06	**[12, 14.77]**	32.15	2.64	3.88	5.32	7.39	10.34
14.83	34.26	0.9	2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23
5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	**[15, 17.2]**	46.12	1.26
2.83	4.33	5.49	7.66	11.25	17.14	**[75.02, 81]**	1.35	2.87	5.62	7.87	11.64	17.36
1.4	3.02	4.34	5.71	7.93	11.79	18.1	1.46	4.4	5.85	8.26	11.98	19.13
1.76	3.25	4.5	6.25	8.37	12.02	**[1.5, 3.2]**	3.31	4.51	6.54	**[7.5, 8.2]**	12.03	
20.28	2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69	

The indeterminant/ imprecise values are represented as bold values.

TABLE 5. **Estimation of Parameters and model adequacy measures of remission periods of cancer patients**

	QTNRD	NED	NGED	NQTGED
Parameter	$\hat{\theta}_N$=[11.0574,11.3365] $\hat{\lambda}_N$=[0.7949,0.8052]	$\hat{\theta}_N$=[0.1096,0.1081]	$\hat{\theta}_N$=[7.9506,8.0568] $\hat{\alpha}_N$=[1.2390,1.2397]	$\hat{\theta}_N$=[10.9445,11.2182] $\hat{\alpha}_N$=[1.3256,1.3236] $\hat{\lambda}_N$=[0.7006,0.7169]
Log Liklihood	[-463.1498,-466.6273]	[-410.9358,-412.6880]	[-409.4565,-411.2037]	[-407.2067,-408.8213]
AIC	[930.2996,937.2546]	[825.8715,829.3760]	[822.9129,826.4074]	[820.4134,823.6425]
BIC	[936.0037,942.9587]	[841.2796,844.7842]	[838.3210,841.8155]	[828.9695,832.1987]
HQC	[932.6172,939.5722]	[825.0304,828.5348]	[825.2306,828.7250]	[823.8898,827.1190]
KS-Value	[0.2805,0.2807]	[0.0815,0.0869]	[0.0752,0.0759]	[0.0551,0.055114]
P-value	[$3x10^{-}9$,$3x10^{-}9$]	[0.3631,0.2886]	[0.4642, 0.3736]	[0.8323,0.8317]

6. Result Comparison

The computational study yielded significant insights into the performance of the NQTGED. As anticipated, the results demonstrated a clear relationship between sample size and estimation accuracy. Specifically, from Table 1-3, as the sample size increased from n=20 to n=150, the NABs and NMSEs consistently decreased across all parameter settings. This trend indicates that the NQTGED's parameter estimates become more reliable with larger samples, showcasing its robustness in handling varying neutrosophic parameters. The results demonstrate that the NQTGED can effectively provide accurate estimates, making it a valuable tool for statistical analysis where uncertainty is inherent.

The goodness-of-fit tests, as shown in Table 5, revealed that the NQTGED fitted the remission times data well, surpassing the fits of alternative distributions. From Table 5, the NQTGED exhibited the lowest AIC, BIC and HQC values, indicating a more parsimonious model while still accurately capturing the underlying patterns in the data. Likewise, KS test statistic further corroborated the superior fit of the NQTGED.

An algorithm for constructing the NQTGED as well as the flow chart of this study continues as follows:

Step 1: Define GED as the baseline distribution as given in **Definition 1**.

Step 2: Apply the Quadratic Transmutation to transform GED into TGED, following **Definition 3**.

Step 3: Introduce neutrosophic parameters to incorporate indeterminacy, forming the NQTGED, as specified in **Definition 4**.

Step 4: Plot the NPDF, NCDF, NSF and NHRF of NQTGED.

Step 5: Derive key mathematical properties for NQTGED.

Step 6: Run Monte Carlo simulations across different sample sizes and neutrosophic parameter settings to test model reliability.

Step 7: Evaluate model fit using real data and compare criteria like AIC, BIC, HQC, and the KS test with other models.

Step 8: Summarize the results to highlight NTGED's effectiveness in fitting data ith indeterminacy.

7. Discussion and Implication of the Study

In the realm of statistical distribution theory, effectively modeling uncertainty, indeterminacy, and inconsistency in data remains a formidable challenge. Traditional statistical distributions, such as the GED and its extensions like the QTGED, primarily focus on capturing degrees of truth or membership in fuzzy sets. However, these frameworks often struggle to accurately represent scenarios where data is incomplete, vague, or contradictory.

The limitations of existing methodologies become particularly evident in fields such as engineering, medical sciences, and social sciences, where datasets frequently exhibit complex and nuanced patterns that defy simple categorization into true or false states. For instance, in biomedical research, understanding disease progression or treatment outcomes requires models that can adeptly handle uncertainties inherent in clinical data.

Moreover, traditional distributions lack the flexibility to adapt to diverse and evolving data complexities. They may falter in providing robust estimations or accurate predictions when faced with non-standard data distributions or outliers. Addressing these challenges necessitates innovative approaches capable of integrating advanced mathematical frameworks to comprehensively model uncertainty.

To bridge these gaps, this paper introduces the NQTGED. This distribution extends the QTGED by incorporating neutrosophic logic, which allows for the simultaneous representation of truth, indeterminacy, and falsity. Neutrosophic logic provides a more nuanced and flexible approach compared to traditional methods, as it can quantitatively capture uncertainties that classical and fuzzy set theories struggle to accommodate.

Firstly, the introduction of the NQTGED represents a significant advancement in probability modeling, particularly in handling situations characterized by ambiguity and indeterminacy. By incorporating neutrosophic principles into the quadratic transmuted generalized exponential distribution, the NQTGED offers a more flexible framework for modeling real-world data that may exhibit uncertain or imprecise characteristics.

The simulation studies conducted to assess the performance of the NQTGED model under different sample sizes and neutrosophic parameter settings provide valuable insights into its robustness and accuracy. The findings indicate that the neutrosophic maximum likelihood estimation (MLE) for the NQTGED yields accurate parameter estimation, particularly with larger sample sizes in the uncertainty scenario. This suggests the potential practical applicability of the NQTGED model in various fields, including reliability engineering and biomedical sciences during uncertain transmuted cases.

Moreover, the practical application of the NQTGED model to a real-world dataset of cancer patient remission times highlights its relevance and effectiveness in capturing complex data patterns. The Log-likelihood and the goodness-of-fit tests, including the AIC, BIC, HQC and Kolmogorov–Smirnov (KS) test, suggest that the NQTGED emerges as a plausible fit for the remission times, outperforming other potential distributions such as the QTNRD, NED and NGED.

Overall, the study's findings underscore the importance and potential impact of neutrosophic probability modeling in addressing the challenges posed by uncertain and imprecise data. The development and exploration of the NQTGED model offer researchers and practitioners a powerful tool for analyzing and interpreting complex datasets, with implications for various fields ranging from reliability engineering to biomedical sciences. Further research and applications of neutrosophic probability theory are warranted to continue advancing our understanding and utilization of probabilistic models in the face of ambiguity and uncertainty.

8. **Conclusion**

In conclusion, the NQTGED introduced in this study offers a flexible and robust framework for modeling uncertain and imprecise data. Through thorough exploration and practical application, the NQTGED has shown promising results in accurately capturing complex data patterns. Its development signifies a significant advancement in probabilistic modeling, with implications for various fields including reliability engineering and biomedical sciences. Further research and application of the NQTGED model are warranted to continue advancing our understanding and utilization of neutrosophic probability theory in real-world scenarios. Future plans include deepening theoretical foundations by exploring additional mathematical properties and extending methodological advancements beyond maximum likelihood estimation to include Bayesian approaches and handling censored data. The application scope will expand beyond cancer patient remission times to encompass diverse fields like finance and environmental sciences, validating NQTGED's robustness across different domains.

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References

1. Smarandache, F. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*, Neutrosophic Systems With Applications, 2023, 12, 68-76. https://doi.org/10.61356/j.nswa.2023.115
2. Cruzaty, L.E.V.; Tomalá, M.R. and Gallo,C.M.C.*A Neutrosophic Statistic Method to Predict Tax Time Series in Ecuador*. Neutrosophic Sets and Systems, 2020, 34, 33-39. https://doi.org/10.5281/ZENODO.3843289
3. Aslam, M.A.*Neutrosophic Rayleigh distribution with some basic properties and application*. Neutrosophic Sets in Decision Analysis and Operations Research, 2020, IGI Global,119-128. https://doi.org/10.4018/978-1-7998-2555-5.ch006
4. Ashbacher, C. *Introduction to Neutrosophic logic*. Infinite Study, 2014. https://doi.org/10.5281/zenodo.8819
5. Atanassov, K.T. *Intuitionistic fuzzy sets*. Fuzzy Sets Syst, 1986, 20(1):87–96. http://dx.doi.org/10.1016/S0165-0114(86)80034-3
6. Smarandache, F. *Introduction to Neutrosophic Statistics*, Sitech and Education Publisher, Craiova.Romania-Educational Publisher, Columbus, Ohio, USA, 2014. http://dx.doi.org/10.13140/2.1.2780.1289
7. Zeina, M.B.; Hatip,A. *Neutrosophic Random Variables*. Neutrosophic Sets and Systems, 2021, 39(1), 4. http://dx.doi.org/10.5281/zenodo.4444987.
8. Bělohlávek, R.; Dauben, J.W. and Klir,G.J. Fuzzy logic and mathematics: a historical perspective, Oxford University Press, 2017. http://dx.doi.org/10.1093/oso/9780190200015.001.0001
9. Smarandache, F. Neutrosophy: Neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis.1998.
10. Jaikumar, R. V., Sundareswaran, R., Shanmugapriya, M., Broumi, S., & Al-Hawary, T. A. *Vulnerability parameters in picture fuzzy soft graphs and their applications to locate a diagnosis center in cities*. Journal of Fuzzy Extension and Applications, 2024, 5(1), 86-99. https://doi.org/10.22105/jfea.2024.428227.1336
11. Umoeka, I. J.; Akwukwuma, V. N. *Interval type-2 fuzzy logic system for early software reliability prediction*. Journal of fuzzy extension and applications, 2023, 4(3), 188-206. https://doi.org/10.22105/jfea.2023.379363.1241
12. Ameen, Z. A. A note on somewhat fuzzy continuity. arXiv preprint arXiv:2305.05376. http://dx.doi.org/10.22105/jfea.2022.356442.1230

13. Kiptum, C. K.; Bouraima, M. B.; Badi, I., Zonon, B. I. P.; Ndiema, K. M., & Qiu, Y. (2023). *Assessment of the challenges to urban sustainable development using an interval-valued fermatean fuzzy approach.* Systemic analytics, 1(1), 11-26. http://dx.doi.org/10.22105/SA.2021.281500.1061
14. Smarandache, F. (2023). *Foundation of Revolutionary Topologies: An Overview, Examples, Trend Analysis, Research Issues, Challenges, and Future Directions.* Neutrosophic Systems With Applications, 13, 45-66. https://doi.org/10.61356/j.nswa.2024.125
15. Alhasan, K. F. H.; Smarandache, F. *Neutrosophic Weibull distribution and Neutrosophic Family Weibull Distribution.* Infinite Study, 2019. http://dx.doi.org/10.5281/zenodo.3382540
16. Alhabib, R.; Ranna, M. M.; Farah, H; & Salama, A. A. *Some neutrosophic probability distributions.*Neutrosophic Sets Systems,2018, 22, 30–38. http://dx.doi.org/10.5281/zenodo.2160478
17. Patro, S. K.; Smarandache, F. T*he Neutrosophic Statistical Distribution, More Problems, More Solutions.* Infinite Study, 2016.
18. Nayana,B. M.; Anakha,K. K.; Chacko,V. M.; Aslam,M. ; Albassam,M. *A new neutrosophic model using DUS-Weibull transformation with application.* Complex & Intelligent Systems, 2022,8(5), 4079-4088. http://dx.doi.org/10.1007/s40747-022-00698-6
19. Rao,G. S.; Norouzirad, M. and Mazarei,D. *Neutrosophic Generalized Exponential Distribution with Application.* Neutrosophic Sets and Systems, 2023, 55(1), 28. http://dx.doi.org/10.5281/zenodo.7832786
20. Khan,Z.; Amin,A.; Khan, S. A.; Gulistan, M. *Statistical Development of the Neutrosophic Lognormal Model with Application to Environmental Data.* Neutrosophic Sets and Systems,2021, 47(1), 1-11. http://dx.doi.org/10.5281/zenodo.5775076
21. Khan,Z.; Al-Bossly, A.; Almazah, M. M. A.; Alduais,F.S. *On Statistical Development of Neutrosophic Gamma Distribution with Applications to Complex Data Analysis.* Complexity,2021, 2021, 1-8. http://dx.doi.org/10.1155/2021/3701236
22. Duan, W. Q.; Khan,Z.; Gulistan,M.; Khurshid, A. *Neutrosophic Exponential Distribution: Modeling and Applications for Complex Data Analysis.* Complexity, 2021, 1-8. http://dx.doi.org/10.1155/2021/5970613
23. Khan,Z.; Almazah,M.M.A.; Odhah,O. H.; Alshanbari, H.M. *Generalized pareto model: properties and applications in neutrosophic data modeling.* Mathematical Problems in Engineering, 2022.http://dx.doi.org/10.1155/2022/3686968
24. M. Y. Mustafa and Z. Algamal,*Neutrosophic inverse power Lindley distribution: A modeling and application for bladder cancer patients*, International Journal of Neutrosophic Science **21**(2), 216-223, 2023. http://dx.doi.org/10.54216/IJNS.210218
25. Haq, M. A. *Neutrosophic Kumaraswamy Distribution with Engineering Application.* Neutrosophic Sets and Systems,2022, 49, 269-276. http://dx.doi.org/10.5281/zenodo.6426401
26. Guan,H.; Dai, Z.; Guan,S.; Zhao,A. *A neutrosophic forecasting model for time series based on first-order state and information entropy of high-order fluctuation.*2019, Entropy, 21(5),455. http://dx.doi.org/10.3390/e21050455
27. Mao,X.; Guoxi,Z.; Fallah,M.; Edalatpanah, S. A.*A Neutrosophic-Based Approach in Data Envelopment Analysis with Undesirable Outputs.* Mathematical Problems in Engineering, 2020, 1-8. http://dx.doi.org/10.1155/2020/7626102
28. Aslam, M. *A New Failure-Censored Reliability Test Using Neutrosophic Statistical Interval Method.* International Journal of Fuzzy Systems,2019, 21, 1214-1220. http://dx.doi.org/10.1007/s40815-018-0588-y
29. Taş,F.; Topal,S.; Smarandache,F.*Clustering Neutrosophic Data Sets and Neutrosophic Valued Metric Spaces.* Symmetry, 2018, 10(10), 430. http://dx.doi.org/10.3390/sym10100430
30. M. A. Jdid, F. Smarandache. *The Box and Muller Technique for Generating Neutrosophic Random Variables Following a Normal Distribution.* International Journal of Neutrosophic Science, 2024, 23(4), 83–87. http://dx.doi.org/10.54216/IJNS.230406

31. A. El-Douh, A. *Assessment the Health Sustainability using Neutrosophic MCDM Methodology: Case Study COVID-19*. Sustainable Machine Intelligence Journal, 2023, 3(1), 1–10. http://dx.doi.org/10.61185/SMIJ.2023.33101
32. Theeba, A., Martin, N., & Smarandache, F. *Neutrosophic Industry 5.0 Inventory Model with Technology Driven Demand and Costs Parameters*. Neutrosophic Sets and Systems, 2024, 73, 103–119.
33. Sathya, P., Martin, N., & Smarandache, F. *Plithogenic Forest Hypersoft Sets in Plithogenic Contradiction Based Multi-Criteria Decision Making*. Neutrosophic Sets and Systems, 2024, 73, 668–693.
34. M. Jdid, F. Smarandache, K. Al Shaqsi. *Generating Neutrosophic Random Variables Based on Gamma Distribution*. Infinite Study, 2024. http://dx.doi.org/10.61356/j.plc.2024.18760
35. M. Jdid, F. Smarandache. *Generating Neutrosophic Random Variables Following the Poisson Distribution Using the Composition Method (The Mixed Method of Inverse Transformation Method and Rejection Method)*. Infinite Study, 2024.
36. Nabeeh, N. *Assessment and Contrast the Sustainable Growth of Various Road Transport Systems using Intelligent Neutrosophic Multi-Criteria Decision-Making Model*. Sustainable Machine Intelligence Journal, 2023, 2(2), 1–12. http://dx.doi.org/10.61185/SMIJ.2023.22102
37. Abobala, M.; Sankari, H., & Zeina, M. B. *On novel security systems based on the 2-cyclic refined integers and the foundations of 2-cyclic refined number theory*. Journal of Fuzzy Extension and Applications, 2024, 5(1), 69-85. http://dx.doi.org/10.22105/jfea.2024.423818.1321
38. A. Singh, H. Kulkarni, F. Smarandache, G. K. Vishwakarma. *Computation of Separate Ratio and Regression Estimator Under Neutrosophic Stratified Sampling: An Application to Climate Data*. Journal of Fuzzy Extension and Applications, 2024. https://doi.org/10.22105/jfea.2024.422211.1313
39. Bhat, S. A. *An enhanced AHP group decision-making model employing neutrosophic trapezoidal numbers*. 2023, J. Oper. Strateg Anal, 1(2), 81-89. http://dx.doi.org/10.56578/josa010205
40. Razdan, M. R.; Aghasi, S.; & Davoodi, S. M. R. *Ranking factors affecting supply chain risk with a combined approach of neutrosophic analytical hierarchy process and TOPSIS*. 2024, Journal of applied research on industrial engineering. https://doi.org/10.22105/jarie.2021.303869.1379
41. Gupta,R. D.; Kundu,D. *Generalized exponential distributions*. Australian & New Zealand Journal of Statistics,1999, 41(2),173-188.
42. Gupta,R. D.; Kundu, D.*Generalized exponential distributions: different methods of estimation*. Journal of Statistical Computation and Simulation,2001, 69(4), 315-338. http://dx.doi.org/10.1080/00949650108812098
43. Gupta,R. D.; Kundu,D. *Generalized exponential distributions: statistical inferences*. Journal of Statistical theory and Applications,2002, 1(1), 101-118.
44. Gupta,R. D.; Kundu, D. *Closeness of gamma and generalized exponential distribution*. Communications in statistics-theory and methods,2003,32(4), 705-721. https://doi.org/10.1081/STA-120018824
45. Gupta, R. D.; Kundu,D. *Discriminating between gamma and generalized exponential distributions*. Journal of Statistical Computation and Simulation,2004, 74(2), 107-121. https://doi.org/10.1080/0094965031000114359
46. Gupta,R. D.; Kundu,D. *Generalized exponential distribution: Existing results and some recent developments*. Journal of Statistical Planning and Inference,2007, 137(11), 3537-3547. http://dx.doi.org/10.1016/j.jspi.2007.03.030
47. Gupta, R. D.; Kundu,D. *Generalized exponential distribution: Bayesian estimations*. Computational Statistics & Data Analysis, 2008, 52(4), 1873-1883. http://dx.doi.org/10.1016/j.csda.2007.06.004
48. Rao,G.S. *Estimation of reliability in multicomponent stress-strength based on generalized exponential distribution*. Revista Colombiana de Estadística,2012, 35(1), 67-76.

49. Shaw,W. T.; Buckley,I. R. The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew kurtotic-normal distribution from a rank transmutation map, Research report, King's College, London, U.K. arXiv preprint arXiv: 0901.0434, 2009. https://doi.org/10.48550/arXiv.0901.0434
50. Khan,M. S.; King,R. & Hudson,I. L. *Transmuted generalized exponential distribution: A generalization of the exponential distribution with applications to survival data*. Communications in Statistics-Simulation and Computation,2017, 46(6), 4377-4398. http://dx.doi.org/10.1080/03610918.2015.1118503
51. Jabarali, A.; Mohana, K.; Devanayan, D.W.P; Devanayan,P; Krishnaprasad; Sandhya,D.; SV,P.K. *Generalized p-Transmuted Neutrosophic Distributions: Theory and its Applications*. Journal of International Journal of Neutrosophic Science, 24(3), 201-219, 2024. https://doi.org/10.54216/IJNS.240318
52. Lee,E.T.; Wang,J. *Statistical Methods for Survival Data Analysis* . John Wiley & Sons, Hoboken, NJ, USA **476**, 2003. https://doi.org/10.1002/0471458546

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