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TOPSIS approach for MADM based on Quadripartitioned single valued neutrosophic refined Hamacher aggregation operations

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Abstract. Hamacher operators are extensively utilized in multicriteria attribute group decision-making (MAGDM) problems due to their remarkable adaptability provided by an adjustable parameter. Here, the Hamacher T-norm and T-conorm operations for two $\mathcal{QSVNRN}$s are formulated.Using these Hamacher operations, we present the quadripartitioned single-valued neutrosophic refined Hamacher weighted averaging ($\mathcal{QSVNRHWA}$) operators within the $\mathcal{QSVNR}$ framework and analyze their properties.Finally, we explore a TOPSIS-based approach for multi-attribute decision-making problems that employs the $\mathcal{QSVNRHWA}$ operators, demonstrating its application in evaluating practical scenarios related to converting solid waste into energy.

Keywords: Quadripartitioned Single Valued Neutrosophic Refined Set,Quadripartitioned Single Valued Neutrosophic Refined Number,Multi attribute decision making, Aggregation Operator, TOPSIS.

1. **Introduction**

In real life, we often encounter issues involving inconsistent, indeterminate, and incomplete information that cannot be accurately represented by precise numbers. In response to this, Lotfi Aliasker Zadeh[34], the pioneer of fuzzy logic, introduced fuzzy sets in 1965 as a way to effectively address these challenges. Later, in 1986, Krassimir Atanassov[1] expanded upon Zadeh's concept by introducing intuitionistic fuzzy sets, which incorporate varying degrees of both membership and non-membership.

Florentin Smarandache[23,24] became familiar with the neutrosophic set, which underpins contemporary mathematical theories. Neutrosophic sets can generate intuitionistic fuzzy sets(IF), fuzzy sets(FS), and classical sets. The term "neutrosophic" originates from Neutrosophy, a philosophy that encourages the exploration of neutral thought. The main distinction among fuzzy, intuitionistic fuzzy, and neutrosophic logic or sets is this concept of neutrality. This idea is crucial for differentiating the three associated functions in the neutrosophic framework: the truth membership function (T), the indeterminate membership function (I), and the false membership function (F). Additionally, inspired by practical needs, Haibin Wang and his team created single-valued neutrosophic sets by streamlining the neutrosophic set for real-world applications.

Yager[31] pioneered the theory of bags, which extends classical set theory by permitting the repetition of elements, thus introducing the foundation of the multiset. This concept, allowing multiple occurrences of the same element within a set, was later rigorously defined by Blizard [4] and Calude et al.[8].To synthesize the principles of fuzzy multisets with IFS, Shinoj et al. [25] developed the notion of intuitionistic fuzzy multisets(IFMS). Nevertheless, both fuzzy and intuitionistic fuzzy multisets exhibit limitations in addressing higher-order uncertainties, often required in complex data analyses.To overcome these constraints, Chatterjee et al.[21] introduced the single-valued neutrosophic multiset, which refined the representation of uncertainty by incorporating distinct truth, indeterminacy, and falsity values. Building on this framework, Smarandache [26] advanced neutrosophic logic in 2013 by proposing n-valued refined neutrosophic logic. This sophisticated model decomposes the neutrosophic components T_1,T_2,...,T_m and I_1,I_2,...,I_p and F_1,F_2,...,F_r allowing for a more granular depiction of uncertainty across diverse applications.Deli et al.[12]expanded on these ideas by investigating the fundamental properties of neutrosophic refined sets, which generalize both fuzzy and intuitionistic fuzzy multisets, thereby offering enhanced flexibility in representing complex information. Additionally, Ye et al.[33] developed sophisticated distance and similarity measures tailored for single-valued neutrosophic multisets. These metrics were subsequently applied to fields such as medical diagnostics, where data often present ambiguities, contradictions, and incompleteness, illustrating the model's practical utility in handling intricate, uncertain datasets.

In four-valued logic, Belnap [6] created a new concept in which any information is represented by four parameters: T, F, none, and both. These parameters stand for true, false, neither true nor false, and both true and false, respectively. Smarandache [24] presented four numerical valued neutrosophic logic, which separates indeterminacy into two terms: Contradiction (C) and Unknown (U). Belnap's four-valued logic provides the foundation of this reasoning.Thus, the Quadripartitioned Single Valued Neutrosophic Set (QSVNS), which has four components T, C, U, and F in the real unit interval [0,1], was established by Rajashi Chatterjee et al.[22].

The TOPSIS method integrates a range of valuable and effective models, including the ideal positive and negative solutions, discrimination measures, and closeness measures, to determine rankings based on proximity to ideal solutions. Initially introduced by researchers in 1981, the foundation of the TOPSIS concept was laid by Hwang and Yoon [15]. Yoon [32] then presented the formal theory in 1987, and Hwang et al. provided further evaluation in 1993 [16].For example, imagine a person shopping for a new mobile phone, assessing options based on features like RAM, storage, display size, battery life, and price. Overwhelmed by the variety, the customer may find it difficult to choose the most suitable option. TOPSIS provides an effective solution by ranking the options based on the weight and significance of each feature.Later,Chu and Lin [9] introduced the fuzzy TOPSIS technique, and Wang and Elhag [30] applied a TOPSIS method with FS theory for bridge risk assessments. Several other scholars, including Chen and Tsao [10], Sun and Lin [28], Dymova et al. [12], Ashtiani et al. [3], and Memari et al. [20], applied TOPSIS within FS theory contexts. Furthermore, Shen et al. [28] proposed a TOPSIS method for A-IF set theory, and Joshi and Kumar [18] incorporated entropy measures in TOPSIS for A-IF information. Additional applications include Liu [19] developing a TOPSIS approach for physical education issues, while Zulqarnain and Dayan [35] applied TOPSIS to address challenges in the automotive industry.

Many important norms have been identified by researchers, but the Hamacher t-norms, introduced by Hamacher in 1970, have had a significant and well-regarded influence on the field of aggregation operators in computing. Additionally, Bellman and Zadeh [5] contributed to the development of Hamacher aggregation theory, particularly in decision-making with fuzzy information. Huang [14] explored the use of Hamacher aggregation operators for A-IF sets and their role in decision-making, while Garg [13] proposed Hamacher aggregation operators and entropy measures specifically for A-IF data. More recently, Cakir and Ulukan [7] made advancements in the theory of A-IF Hamacher aggregation operators. These aggregation operators, based on Hamacher norms, are particularly powerful due to their adjustable parameter range of $0 \leq \alpha \leq \infty$.

The paper is structured as follows: Section 2 introduces fundamental concepts related to neutrosophic sets, neutrosophic refined sets, quadripartitioned single-valued neutrosophic sets, and Hamacher operations. Section 3 focuses on enhancing the Hamacher operations for quadripartitioned single-valued neutrosophic refined numbers. In Section 4, quadripartitioned single-valued neutrosophic refined Hamacher weighted aggregation operators are introduced, along with their essential properties. Section 5 explores the TOPSIS method using QSVNRHWA information and demonstrates a MADM procedure to assess uncertain and unreliable data. This method utilizes the TOPSIS technique and derived operators are developed to enhance the accuracy and significance of the information.Section 7 concludes the paper and discusses future research opportunities.

2. **Preliminaries**

Definition 2.1. [22] Let $\mathcal{A}^*_{\mathcal{QR}}$ be a universe. A quadripartitioned single valued neutrosophic set $\mathcal{QSVNS}$ $\mathcal{K}_{\mathcal{QNR}}$ on $\mathcal{A}^*_{\mathcal{QR}}$ is defined by by a truth-membership function $\breve{\mathfrak{T}}_{\mathcal{K}}(\mathfrak{a}_1)$, a contradiction membership function $\breve{\mathfrak{D}}_{\mathcal{K}}(\mathfrak{a}_1)$,an unknown membership function $\breve{\mathfrak{Y}}_{\mathcal{K}}(\mathfrak{a}_1)$ and a falsity membership function $\breve{\mathfrak{F}}_{\mathcal{K}}(\mathfrak{a}_1)$ such that for each $\mathfrak{a}_1 \in \mathcal{A}^*_{\mathcal{QR}}$, $\breve{\mathfrak{T}}_{\mathcal{K}}$,$\breve{\mathfrak{D}}_{\mathcal{K}}$,$\breve{\mathfrak{Y}}_{\mathcal{K}}$,$\breve{\mathfrak{F}}_{\mathcal{K}} \in$ [0,1] and $0 \leq \breve{\mathfrak{T}}_{\mathcal{K}}(\mathfrak{a}_1)+\breve{\mathfrak{T}}_{\mathcal{K}}(\mathfrak{a}_1)+\breve{\mathfrak{T}}_{\mathcal{K}}(\mathfrak{a}_1)+\breve{\mathfrak{T}}_{\mathcal{K}}(\mathfrak{a}_1) \leq 4$.

Definition 2.2. [2] Let $\mathcal{A}^*_{\mathcal{QR}}$ be a universe. A Quadripartitioned Single Valued Neutrosophic Refined Set(briefly, $\mathcal{QSVNRS}$) $\mathcal{K}_{\mathcal{QNR}}$ on $\mathcal{A}^*_{\mathcal{QR}}$ is defined by

$$\mathcal{K}_{\mathcal{QNR}} = \{\langle\ \mathfrak{a}_1,(\breve{\mathfrak{T}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{T}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{T}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)),(\breve{\mathfrak{D}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{D}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{D}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)),$$

$$(\breve{\mathfrak{Y}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{Y}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{Y}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)),(\breve{\mathfrak{F}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{F}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{F}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1))\rangle{:}\mathfrak{a}_1{\in}\mathcal{A}^*_{\mathcal{QR}}\}$$

where $\breve{\mathfrak{T}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{T}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{T}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1){:}\mathcal{A}^*_{\mathcal{QR}}\to[0,1]$,
$\breve{\mathfrak{D}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\ \breve{\mathfrak{D}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{D}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1){:}\mathcal{A}^*_{\mathcal{QR}}\to[0,1]$,
$\breve{\mathfrak{Y}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{Y}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{Y}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1){:}\mathcal{A}^*_{\mathcal{QR}}\to[0,1]$ and
$\breve{\mathfrak{F}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{F}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{F}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1){:}\mathcal{A}^*_{\mathcal{QR}}\to[0,1]$ such that
$0 \leq \breve{\mathfrak{T}}^{\mathfrak{j}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)+\ \breve{\mathfrak{D}}^{\mathfrak{j}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)+\ \breve{\mathfrak{Y}}^{\mathfrak{j}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)+\ \breve{\mathfrak{F}}^{\mathfrak{j}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1) \leq 4$ ($\mathfrak{j}$=1,2,...$\mathfrak{p}$) for any $\mathfrak{a}_1 \in \mathcal{A}^*_{\mathcal{QR}}$.
Here $\breve{\mathfrak{T}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{T}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{T}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{D}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{D}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{D}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)$,
$\breve{\mathfrak{Y}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{Y}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{Y}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{F}}^1_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),\breve{\mathfrak{F}}^2_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1),....,\breve{\mathfrak{F}}^{\mathfrak{p}}_{\mathcal{K}_{\mathcal{QNR}}}(\mathfrak{a}_1)$ is the truth ,a contradiction, an unknown and a falsity membership sequence of the element $\mathfrak{a}_1$, respectively.And $\mathfrak{p}$ is often referred to as $\mathcal{QSVNRS}(\mathcal{K}_{\mathcal{QNR}})$ dimension.

Definition 2.3. [14] Let $\mathcal{K}_{\mathcal{QNR}} = (\breve{\mathfrak{T}}_{\mathcal{K}},\breve{\mathfrak{F}}_{\mathcal{K}})$ be a collection of $\mathcal{IFN}$s and Let $\mathcal{IFHWA}{:}\mathcal{Q}^{\setminus} \to \mathcal{Q}$,if

$$\mathcal{IFHWA}_w = (\mathfrak{a}_1,\mathfrak{a}_2,....,\mathfrak{a}_n) = \bigoplus_{j=1}^{n} (w_j\mathfrak{a}_j)$$

where w = $(w_1,w_2,....,w_n)^T$ be the weight vector of $\mathfrak{a}_n$($\mathfrak{j}$=1,2...$\mathfrak{n}$) and $w_j > 0, \sum_{j=1}^{n} w_j = 1$,then $\mathcal{IFHWA}$ is called the $\mathcal{IFHWA}$ operator.

3. **Hamacher Weighted Aggregation Operators of $\mathcal{QSVNRN}$s**

Definition 3.1. Let $k_1^* = \langle\ \breve{\mathfrak{T}}_1^j, \breve{\mathfrak{D}}_1^j, \breve{\mathfrak{Y}}_1^j, \breve{\mathfrak{F}}_1^j$: j = 1,2,...$\mathfrak{p}\rangle$ and $k_2^* = \langle\ \breve{\mathfrak{T}}_2^j, \breve{\mathfrak{D}}_2^j, \breve{\mathfrak{Y}}_2^j, \breve{\mathfrak{F}}_2^j$: j = 1,2,...$\mathfrak{p}$ $\rangle$ be two $\mathcal{QSVNRN}$s and $\breve{\eta} > 0$ for any real number, then we define Hamacher $\mathbb{T}$-norm and Hamacher $\mathbb{T}$-conorm with $\breve{\beta} > 0$

i) $$k_1^* \oplus k_2^* = \left\langle \frac{\breve{\mathfrak{T}}_1^j + \breve{\mathfrak{T}}_2^j - \breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j - (1-\breve{\beta})\breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j}{1-(1-\breve{\beta})\breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j}, \frac{\breve{\mathfrak{D}}_1^j + \breve{\mathfrak{D}}_2^j - \breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j - (1-\breve{\beta})\breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j}{1-(1-\breve{\beta})\breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j},\right.$$
$$\left. \frac{\breve{\mathfrak{Y}}_1^j\breve{\mathfrak{Y}}_2^j}{\breve{\beta} + (1-\breve{\beta})(\breve{\mathfrak{Y}}_1^j + \breve{\mathfrak{Y}}_2^j - \breve{\mathfrak{Y}}_1^j\breve{\mathfrak{Y}}_2^j}, \frac{\breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j}{\breve{\beta} + (1-\breve{\beta})(\breve{\mathfrak{F}}_1^j + \breve{\mathfrak{F}}_2^j - \breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j)} \right\rangle.$$

ii) $$k_1^* \otimes k_2^* = \left\langle \frac{\breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j}{\breve{\beta} + (1-\breve{\beta})(\breve{\mathfrak{T}}_1^j + \breve{\mathfrak{T}}_2^j - \breve{\mathfrak{T}}_1^j\breve{\mathfrak{T}}_2^j}, \frac{\breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j}{\breve{\beta} + (1-\breve{\beta})(\breve{\mathfrak{D}}_1^j + \breve{\mathfrak{D}}_2^j - \breve{\mathfrak{D}}_1^j\breve{\mathfrak{D}}_2^j},\right.$$
$$\left. \frac{\breve{\mathfrak{Y}}_1^j + \breve{\mathfrak{Y}}_2^j - \breve{\mathfrak{Y}}_1^j\breve{\mathfrak{Y}}_2^j - (1-\breve{\beta})\breve{\mathfrak{Y}}_1^j\breve{\mathfrak{Y}}_2^j}{1-(1-\breve{\beta})\breve{\mathfrak{Y}}_1^j\breve{\mathfrak{Y}}_2^j}, \frac{\breve{\mathfrak{F}}_1^j + \breve{\mathfrak{F}}_2^j - \breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j - (1-\breve{\beta})\breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j}{1-(1-\breve{\beta})\breve{\mathfrak{F}}_1^j\breve{\mathfrak{F}}_2^j} \right\rangle.$$

iii) $$\breve{\eta}k_1^* = \left\langle \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}) - (1-\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_1^j)^{\breve{\eta}} + (\breve{\beta}-1)(1-\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}}, \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}) - (1-\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_1^j)^{\breve{\eta}} + (\breve{\beta}-1)(1-\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}},\right.$$
$$\left. \frac{\breve{\beta}(\breve{\mathfrak{Y}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_1^j))^{\breve{\eta}} + (\breve{\beta}-1)(\breve{\mathfrak{Y}}_1^j)^{\breve{\eta}}}, \frac{\breve{\beta}(\breve{\mathfrak{F}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_1^j))^{\breve{\eta}} + (\breve{\beta}-1)(\breve{\mathfrak{F}}_1^j)^{\breve{\eta}}} \right\rangle.$$

iv) $$(k_1^*)^{\breve{\eta}} = \left\langle \frac{(\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{T}}_1^j))^{\breve{\eta}} + (\breve{\beta}-1)(\breve{\mathfrak{T}}_1^j)^{\breve{\eta}}}, \frac{(\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{D}}_1^j))^{\breve{\eta}} + (\breve{\beta}-1)(\breve{\mathfrak{D}}_1^j)^{\breve{\eta}}},\right.$$
$$\left. \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{Y}}_1^j)^{\breve{\eta}} - (1-\breve{\mathfrak{Y}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)\breve{\mathfrak{Y}}_1^j)^{\breve{\eta}} + (\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_1^j)^{\breve{\eta}}}, \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{F}}_1^j)^{\breve{\eta}} - (1-\breve{\mathfrak{F}}_1^j)^{\breve{\eta}}}{(1+(\breve{\beta}-1)\breve{\mathfrak{F}}_1^j)^{\breve{\eta}} + (\breve{\beta}-1)(1-\breve{\mathfrak{F}}_1^j)^{\breve{\eta}}} \right\rangle.$$

Example 3.2. Let two $\mathcal{QSVNRN}$s are $k_1^* = (\langle 0.5,0.3,0.2,0.2\rangle,\langle 0.5,0.6,0.3,0.3\rangle,\langle 0.4,0.7,0.2,0.3\rangle)$ and $k_2^* = (\langle 0.3,0.6,0.3,0.1\rangle,\langle 0.5,0.5,0.2,0.3\rangle,\langle 0.6,0.9,0.3,0.3\rangle$ and $\breve{\eta} = 2$.Then for $\breve{\beta} = 3$.

$k_1^* \oplus k_2^* = \langle 0.949,0.919,0.500,0.455\rangle$

$k_1^* \otimes k_2^* = \langle 0.371,0.246,0.972,0.969\rangle$

$\breve{\eta} k_1^* = \langle 0.941, 0.821, 0.381, 0.280 \rangle$

$(k_1^*)^{\breve{\eta}} = \langle 0.496, 0.381, 0.986, 0.931 \rangle$

Proposition 3.3. Let k_1^* and k_2^* be two $\mathcal{QSVNRN}$s and $\breve{\beta}, \breve{\beta}' \geq 0$.

a $k_1^* \oplus k_2^* = k_2^* \oplus k_1^*$.
b $k_1^* \otimes k_2^* = k_2^* \otimes k_1^*$.
c $\breve{\beta}(k_1^* \oplus k_2^*) = \breve{\beta}\ k_1^* \oplus \breve{\beta}\ k_2^*$.
d $\breve{\beta}\ k_1^* \oplus \breve{\beta}'\ k_1^* = (\breve{\beta}+\breve{\beta}')k_1^*$.
e $(k_1^* \otimes k_2^*)^{\breve{\beta}} = (k_1^*)^{\breve{\beta}} \otimes (k_2^*)^{\breve{\beta}}$.
f $(k_1^*)^{\breve{\beta}} \otimes (k_1^*)^{\breve{\beta}'} = (k_1^*)^{\breve{\beta}+\breve{\beta}'}$

Proof. They are easily seen from the formulas in Definition 3.1, hence omitted □

Definition 3.4. Let $k_r^* = \langle\ \breve{\mathfrak{T}}_r^j,\ \breve{\mathfrak{D}}_r^j,\ \breve{\mathfrak{Y}}_r^j, \breve{\mathfrak{F}}_r^j\ :\ \mathrm{j} = 1,2,...,\mathfrak{p} \rangle$ (r = 1,2,...,$\mathfrak{n}$) is a collection of $\mathcal{QSVNRN}$s.Then the Quadripartitioned single valued neutrosophic refined hamacher weighted average operator($\mathcal{QSVNRHWA}$) can be defined as follows:

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*, k_2^*, ..., k_{\mathfrak{n}}^*) = \bigoplus_{r=1}^{\mathfrak{n}} (\ddot{\varpi}_r k_r^*).$$

where $\ddot{\varpi} = (\ddot{\varpi}_1,\ \ddot{\varpi}_2, ..., \ddot{\varpi}_{\mathfrak{n}})$be a weighted vector of k_r^* such that $\ddot{\varpi}_r \in [0,1]$ and $\sum\limits_{r=1}^{\mathfrak{n}} (\ddot{\varpi}_r)$ = 1.

Theorem 3.5. Let $k_r^* = \langle\ \breve{\mathfrak{T}}_r^j,\ \breve{\mathfrak{D}}_r^j,\ \breve{\mathfrak{Y}}_r^j, \breve{\mathfrak{F}}_r^j\ :\ \mathrm{j} = 1,2,...,\mathfrak{p} \rangle$ (r = 1,2,...,$\mathfrak{n}$) is a collection of $\mathcal{QSVNRN}$s then their aggregated value of $\mathcal{QSVNRHWA}$ operator is again a $\mathcal{QSVNRN}$s can be defined as follows

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*, k_2^*, ..., k_{\mathfrak{n}}^*) = \bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}} (\ddot{\varpi}_r k_r^*)$$

$$= \left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r} - \prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod\limits_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-{}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r} - \prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \right.$$

$$\left. \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_r^j))^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

......(5.2.1)

where $\ddot{\varpi} = (\ddot{\varpi}_1,\ \ddot{\varpi}_2, ..., \ddot{\varpi}_{\mathfrak{n}})$ be a weighted vector of k_r^* such that $\ddot{\varpi}_r \in [0,1]$ and $\sum_{r=1}^{\mathfrak{n}} (\ddot{\varpi}_r) =$ 1, $\breve{\beta} > 0$.

Proof. we can prove this theorem by mathematical induction.

For $\mathfrak{n}$=1,by Eq(5.2.1),we get

$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*) = (\ddot{\varpi}_1 k_1^*) = k_1^*$.

$$= \left\langle \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_1^j)-(1-\breve{\mathfrak{T}}_1^j)}{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_1^j)+(\breve{\beta}-1)(1-\breve{\mathfrak{T}}_1^j)}, \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_1^j)-(1-\breve{\mathfrak{D}}_1^j)}{(1+(\breve{\eta}-1)\breve{\mathfrak{D}}_1^j)+(\breve{\beta}-1)(1-\breve{\mathfrak{D}}_1^j)}, \frac{\breve{\beta}(\breve{\mathfrak{Y}}_1^j)}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_1^j))+(\breve{\beta}-1)(\breve{\mathfrak{Y}}_1^j)}, \frac{\breve{\beta}(\breve{\mathfrak{F}}_1^j)}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_1^j))+(\breve{\beta}-1)(\breve{\mathfrak{F}}_1^j)} \right\rangle$$

Thus,satisfied for $\mathfrak{n}$=1.

put $\mathfrak{n} = \upsilon$ for Eq(5.2.1),then

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_\upsilon^*) = \bigoplus_{r=1}^{\upsilon}{}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*)$$

$$= \left\langle \frac{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(1-{}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

If Eq(5.2.1) is true for $\mathfrak{n} = \upsilon$.

For $\mathfrak{n} = \upsilon$+1,then

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_{\upsilon+1}^*) = \bigoplus_{r=1}^{\upsilon+1}{}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*) \oplus_{\mathcal{H}} (\ddot{\varpi}_{\upsilon+1} k_{\upsilon+1}^*)$$

$$= \left\langle \frac{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(1-{}_1^j)^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

$$\oplus_{\mathcal{H}} \left\langle \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_{\upsilon+1}^j)-(1-\breve{\mathfrak{T}}_{\upsilon+1}^j)}{(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_{\upsilon+1}^j)+(\breve{\beta}-1)(1-\breve{\mathfrak{T}}_{\upsilon+1}^j)}, \frac{(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_{\upsilon+1}^j)-(1-\breve{\mathfrak{D}}_{\upsilon+1}^j)}{(1+(\breve{\eta}-1)\breve{\mathfrak{D}}_{\upsilon+1}^j)+(\breve{\beta}-1)(1-\breve{\mathfrak{D}}_{\upsilon+1}^j)}, \frac{\breve{\beta}(\breve{\mathfrak{Y}}_{\upsilon+1}^j)}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_{\upsilon+1}^j))+(\breve{\beta}-1)(\breve{\mathfrak{Y}}_{\upsilon+1}^j)}, \frac{\breve{\beta}(\breve{\mathfrak{F}}_{\upsilon+1}^j)}{(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_{\upsilon+1}^j))+(\breve{\beta}-1)(\breve{\mathfrak{F}}_{\upsilon+1}^j)} \right\rangle$$

$$= \left\langle \frac{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon+1}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon+1}(1-{}^{j}_{1})^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\upsilon+1}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod\limits_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon+1}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon+1}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon+1}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\upsilon+1}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\upsilon+1}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\upsilon+1}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

Thus,Eq(5.2.1) is true for $\mathfrak{n} = \upsilon+1$.Therefore,Eq(5.2.1) is satisfied for every $\mathfrak{n}$. □

Theorem 3.6. (Idempotency)
Let k_r^* (r=1,2,...,$\mathfrak{n}$) is a collection of $\mathcal{QSVNRN}$s.If $k_r^* = k^*$ for all (r=1,2,...,$\mathfrak{n}$) then $\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,....,k_{\mathfrak{n}}^*) = k^*$.

Proof. Assume $k_r^* = k^*$ for all (r=1,2,...,$\mathfrak{n}$).
By Theorem (4.2) we obtain that
$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_{\mathfrak{n}}^*) = \bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*) = \bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k^*) =$

$$\left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}^j)^{\ddot{\varpi}_r}}{\prod\limits_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-{}^{j}_{1})^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}^j)^{\ddot{\varpi}_r}} \right\rangle = k^* \ \square$$

Theorem 3.7. (Monotonicity)
Let k_r^* and $k_r^{*'}$ (r=1,2,...,$\mathfrak{n}$) be two collections of $\mathcal{QSVNRN}$s.If $k_r^* \leq k_r^{*'}$ for all r=1,2,...,$\mathfrak{n}$ then $\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,....,k_{\mathfrak{n}}^*) \leq \mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^{*'},k_2^{*'},....,k_{\mathfrak{n}}^{*'})$.

Proof. If $k_r^* \leq k_r^{*'}$ then we have
$\breve{\mathfrak{T}}_r^j \leq \breve{\mathfrak{T}}_r^{j'}$, $\breve{\mathfrak{D}}_r^j \leq \breve{\mathfrak{D}}_r^{j'}$,$\breve{\mathfrak{Y}}_r^j \leq \breve{\mathfrak{Y}}_r^{j'}$,$\breve{\mathfrak{F}}_r^j \leq \breve{\mathfrak{F}}_r^{j'}$ for all r=1,2,...,$\mathfrak{n}$.
With these assumptions,we find that

$$\left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod\limits_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-{}^{j}_{1})^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

$$\leq \left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^{j'})^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^{j'})^{\ddot{\varpi}_r}}{\prod\limits_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^{j'})^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-{}^{j}_{1})^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^{j'})^{\ddot{\varpi}_r}-\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^{j'})^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^{j'})^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^{j'})^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}_r^{j'})^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_r^{j'}))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}_r^{j'})^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^{j'})^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^{j'}))^{\ddot{\varpi}_r}+(\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^{j'})^{\ddot{\varpi}_r}} \right\rangle$$

Then $\bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k_r^*) = \bigoplus_{r=1}^{\mathfrak{n}}{}_{\mathcal{H}}(\ddot{\varpi}_r k^{*'})$,so $\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,....,k_{\mathfrak{n}}^*) \leq \mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^{*'},k_2^{*'},....,k_{\mathfrak{n}}^{*'})$. □

Theorem 3.8. (Boundedness)
Let k_r^* (r=1,2,...,$\mathfrak{n}$) be a collection of $\mathcal{QSVNRN}$s.Then $k_{min}^* \leq \mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,....,k_{\mathfrak{n}}^*) \leq k_{max}^*$.

Proof. $k_{min}^* = \langle\ \min(\breve{\mathfrak{T}}_r^j),\ \min(\breve{\mathfrak{D}}_r^j),\ \max(\breve{\mathfrak{Y}}_r^j),\ \max(\breve{\mathfrak{F}}_r^j)\text{: r=1,2,...,}\mathfrak{n}\ \rangle$
and $k_{max}^* = \langle\ \max(\breve{\mathfrak{T}}_r^j),\ \max(\breve{\mathfrak{D}}_r^j),\ \min(\breve{\mathfrak{Y}}_r^j),\ \min(\breve{\mathfrak{F}}_r^j)\text{: r=1,2,...,}\mathfrak{n}\ \rangle$
They can be proved using similar techniques, therefore omitted. □

4. TOPSIS method for MADM problems using $\mathcal{QSVNRHWA}$

One of the conventional methods for handling MADM problems is TOPSIS. It is used to rank the practical options in order of priority and select the best option based on the available data. Here, we will demonstrate how to solve the MADM problem using the TOPSIS technique, which is based on the $\mathcal{QSVNRHWA}$ operator.
Let us consider a collection of evaluation attributes $\hat{\Omega} = \{\hat{\Omega}_1,\hat{\Omega}_2,...,\hat{\Omega}_{\mathfrak{n}}\}$, $\hat{\zeta} = \{\hat{\zeta}_1,\hat{\zeta}_2,...,\hat{\zeta}_{\mathfrak{m}}\}$ be the set of feasible Preferences(Alternatives) and $\ddot{\kappa}_{\alpha\beta}$, $\alpha = 1,2,...,\mathfrak{m}$; $\beta = 1,2,...,\mathfrak{n}$ is the score of preferencei $\hat{\zeta}_\alpha$ with respect to attributes $\hat{\Omega}_\beta$.Let $\ddot{\varpi} = (\ddot{\varpi}_1,\ \ddot{\varpi}_2,....,\ddot{\varpi}_{\mathfrak{q}})$ be a weighted vector satisfying $0 \leq \ddot{\varpi}_\beta \leq 1$ and $\sum_{\beta=1}^{\mathfrak{n}}(\ddot{\varpi}_\beta) = 1$.
TOPSIS approach is summarized as follows:
Step 1: Computation of normalized decision matrix.
Formulate the normalized value $\breve{\omega}_{\alpha\beta}$ is a follows:

$$\text{For the profit matrix, } \breve{\omega}_{\alpha\beta}^N = \frac{\breve{\omega}_{\alpha\beta} - \breve{\omega}_\beta^-}{\breve{\omega}_\beta^+ - \breve{\omega}_\beta^-}$$

where $\breve{\omega}_\beta^+ = \max\limits_\beta(\breve{\omega}_{\alpha\beta})$ and $\breve{\omega}_\beta^- = \min\limits_\beta(\breve{\omega}_{\alpha\beta})$or setting $\breve{\omega}_\beta^+$ is the best position and $\breve{\omega}_\beta^-$ is the worst position.

$$\text{For the cost matrix, } \breve{\omega}_{\alpha\beta}^N = \frac{\breve{\omega}_\beta^- - \breve{\omega}_{\alpha\beta}}{\breve{\omega}_\beta^+ - \breve{\omega}_\beta^-}.$$

Step 2: Compute the $\mathcal{QSVNRN}$ for each alternative $\hat{\zeta}_\alpha$ ($\alpha = 1,2,...,\mathfrak{m}$) by using the operator $\mathcal{QSVNRHWA}$.

$$\mathcal{QSVNRHWA}_{\ddot{\varpi}}(k_1^*,k_2^*,...,k_{\mathfrak{n}}^*) = \bigoplus_{r=1}^{\mathfrak{n}}(\ddot{\varpi}_r k_r^*)$$

$$= \left\langle \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r} - \prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{T}}_r^j)^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-{}^{j}_{1})^{\ddot{\varpi}_r}}, \frac{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r} - \prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(1-\breve{\mathfrak{D}}_r^j)^{\ddot{\varpi}_r}}, \right.$$
$$\left. \frac{\breve{\beta}\prod\limits_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{Y}}_r^j))^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{Y}}_r^j)^{\ddot{\varpi}_r}}, \frac{\breve{\beta}\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}}{\prod_{r=1}^{\mathfrak{n}}(1+(\breve{\beta}-1)(1-\breve{\mathfrak{F}}_r^j))^{\ddot{\varpi}_r} + (\breve{\beta}-1)\prod_{r=1}^{\mathfrak{n}}(\breve{\mathfrak{F}}_r^j)^{\ddot{\varpi}_r}} \right\rangle$$

where $\ddot{\varpi} = (\ddot{\varpi}_1, \ddot{\varpi}_2,....,\ddot{\varpi}_{\mathfrak{n}})$ be a weighted vector of k_r^* such that $\ddot{\varpi}_r \in [0,1]$ and $\sum_{r=1}^{\mathfrak{n}}(\ddot{\varpi}_r) = 1$, $\breve{\beta} > 0$.

Step 3: Determine the quadripartitioned single valued neutrosophic refined positive ideal solution ($\mathcal{QSVNRPIS}$) and quadripartitioned single valued neutrosophic refined negative ideal solutions ($\mathcal{QSVNRNIS}$).

$$\mathcal{QSVNRPIS} = \ddot{\mathfrak{D}}_\beta^+ = (\ddot{\varepsilon}_1^+, \ddot{\varepsilon}_2^+,....,\ddot{\varepsilon}_{\mathfrak{n}}^+)$$

$$= \left\langle \max_\beta(\breve{\mathfrak{T}}_{\alpha\beta}), \max_\beta(\breve{\mathfrak{D}}_{\alpha\beta}), \min_\beta(\breve{\mathfrak{Y}}_{\alpha\beta}), \min_\beta(\breve{\mathfrak{F}}_{\alpha\beta}) \right\rangle, \ \alpha = 1,2,...,\mathfrak{m} \ ; \ \beta = 1,2,...,\mathfrak{n}.$$

$$\mathcal{QSVNRNIS} = \ddot{\mathfrak{D}}_\beta^- = (\ddot{\varepsilon}_1^-, \ddot{\varepsilon}_2^-,....,\ddot{\varepsilon}_{\mathfrak{n}}^-)$$

$$= \left\langle \min_\beta(\breve{\mathfrak{T}}_{\alpha\beta}), \min_\beta(\breve{\mathfrak{D}}_{\alpha\beta}), \max_\beta(\breve{\mathfrak{Y}}_{\alpha\beta}), \max_\beta(\breve{\mathfrak{F}}_{\alpha\beta}) \right\rangle, \ \alpha = 1,2,...,\mathfrak{m} \ ; \ \beta = 1,2,...,\mathfrak{n}.$$

Step 4: Calculate the distance measures of a perferencei.

For the seperation measures, $\hat{\xi}^+$ and $\hat{\xi}^-$ can be calculated by using Normalized Hamming distance as follows:

$$\hat{\xi}_\alpha^+ = \sum_{\beta=1}^{\mathfrak{q}} \ddot{\varpi}_\beta d(\hat{\zeta}_{\alpha\beta}, \ddot{\mathfrak{D}}_\beta^+) \text{ and } \hat{\xi}_\alpha^- = \sum_{\beta=1}^{\mathfrak{q}} \ddot{\varpi}_\beta d(\hat{\zeta}_{\alpha\beta}, \ddot{\mathfrak{D}}_\beta^-), \ \alpha = 1,2,...,\mathfrak{m}.$$

Step 5: Formulate the relative closeness coefficient($\mathcal{RCC}$) for the alternative $\hat{\zeta}_\alpha$,which is defined by

$$\breve{\Gamma}_\alpha = \frac{\hat{\xi}_\alpha^+}{(\hat{\xi}_\alpha^+ + \hat{\xi}_\alpha^-)}, \ \alpha = 1,2,...,\mathfrak{m} \ ; \ 0 \leq \breve{\Gamma}_\alpha \leq 1.$$

Step 6: Ranking the alternatives in decreasing or ascending order of their $\mathcal{RCC}$.The greater value $\breve{\Gamma}_\alpha$ indicates desirable alternative $\hat{\zeta}_\alpha$.

4.1. *Application*

In today's world, the amount of solid waste produced rises daily as a result of both population increase and the development of several technologies. While the government is employing various disposal techniques to cut down on garbage, they are still searching for the best way to solve this issue without negatively affecting society or the environment. Currently, solid waste management refers to the undesired or worthless solid materials produced by human activity in commercial, industrial, or domestic settings. Therefore, it is essential to ethically reduce solid waste and convert it into energy.

Let us consider a set of solid waste-to-energy technologies: 1)Biological conversion($\hat{\zeta}_1$); 2)Bioelectrochemical conversion($\hat{\zeta}_2$);3)Thermal conversion($\hat{\zeta}_3$);4) Mechanical conversion ($\hat{\zeta}_4$).These four kinds of solid waste-to-energy technologies are evaluated based on five attributes: :1) Economy($\hat{\Omega}_1$);2) Technical aspects($\hat{\Omega}_2$) 3)Environment($\hat{\Omega}_3$) 4) Ecosystem ($\hat{\Omega}_4$) 5) Society ($\hat{\Omega}_5$).The weight vector assigned for the above five attributes is as follows $\ddot{\varpi}$ = (0.2,0.1,0.3,0.2,0.2).Hence the proposed operator $\mathcal{QSVNRHWA}$ and here to solve MADM problem under $\mathcal{QSVNRN}$ information.

Step 1: Compute the normalized decision matrix.

The decision matrix for the profit type can be expressed as

	$\hat{\Omega}_1$	$\hat{\Omega}_2$	$\hat{\Omega}_3$	$\hat{\Omega}_4$	$\hat{\Omega}_5$
$\hat{\zeta}_1$	$\langle$0.6,0.3,0.5,0.4$\rangle$	$\langle$0.6,0.6,0.4,0.3$\rangle$	$\langle$0.5,0.6,0.5,0.4$\rangle$	$\langle$0.4,0.5,0.4,0.2$\rangle$	$\langle$0.8,0.6,0.4,0.3$\rangle$
	$\langle$0.5,0.6,0.2,0.3$\rangle$	$\langle$0.9,0.7,0.3,0.1$\rangle$	$\langle$0.6,0.4,0.3,0.6$\rangle$	$\langle$0.6,0.7,0.5,0.6$\rangle$	$\langle$0.6,0.7,0.2,0.5$\rangle$
	$\langle$0.6,0.7,0.5,0.2$\rangle$	$\langle$0.8,0.8,0.3,0.4$\rangle$	$\langle$0.7,0.5,0.6,0.3$\rangle$	$\langle$0.7,0.4,0.6,0.5$\rangle$	$\langle$0.7,0.5,0.6,0.2$\rangle$
$\hat{\zeta}_2$	$\langle$0.5,0.7,0.3,0.5$\rangle$	$\langle$0.5,0.5,0.4,0.2$\rangle$	$\langle$0.4,0.4,0.7,0.6$\rangle$	$\langle$0.5,0.3,0.6,0.4$\rangle$	$\langle$0.8,0.5,0.4,0.4$\rangle$
	$\langle$0.6,0.4,0.3,0.4$\rangle$	$\langle$0.8,0.7,0.3,0.4$\rangle$	$\langle$0.7,0.7,0.3,0.2$\rangle$	$\langle$0.9,0.6,0.4,0.5$\rangle$	$\langle$0.4,0.4,0.5,0.6$\rangle$
	$\langle$0.8,0.5,0.7,0.3$\rangle$	$\langle$0.7,0.6,0.6,0.3$\rangle$	$\langle$0.6,0.5,0.3,0.4$\rangle$	$\langle$0.7,0.8,0.4,0.2$\rangle$	$\langle$0.5,0.7,0.6,0.2$\rangle$
$\hat{\zeta}_3$	$\langle$0.7,0.5,0.6,0.4$\rangle$	$\langle$0.7,0.6,0.4,0.3$\rangle$	$\langle$0.6,0.5,0.7,0.4$\rangle$	$\langle$0.5,0.7,0.6,0.4$\rangle$	$\langle$0.8,0.5,0.6,0.5$\rangle$
	$\langle$0.5,0.8,0.4,0.6$\rangle$	$\langle$0.8,0.8,0.2,0.2$\rangle$	$\langle$0.5,0.6,0.4,0.2$\rangle$	$\langle$0.7,0.6,0.5,0.3$\rangle$	$\langle$0.7,0.6,0.3,0.3$\rangle$
	$\langle$0.6,0.3,0.5,0.2$\rangle$	$\langle$0.9,0.7,0.3,0.2$\rangle$	$\langle$0.4,0.4,0.3,0.6$\rangle$	$\langle$0.5,0.3,0.5,0.7$\rangle$	$\langle$0.5,0.7,0.5,0.3$\rangle$
$\hat{\zeta}_4$	$\langle$0.6,0.4,0.3,0.2$\rangle$	$\langle$0.7,0.6,0.5,0.3$\rangle$	$\langle$0.4,0.6,0.3,0.5$\rangle$	$\langle$0.8,0.5,0.6,0.4$\rangle$	$\langle$0.4,0.6,0.7,0.3$\rangle$
	$\langle$0.7,0.5,0.6,0.8$\rangle$	$\langle$0.5,0.5,0.6,0.7$\rangle$	$\langle$0.8,0.5,0.6,0.3$\rangle$	$\langle$0.5,0.4,0.3,0.5$\rangle$	$\langle$0.6,0.3,0.4,0.7$\rangle$
	$\langle$0.3,0.6,0.5,0.4$\rangle$	$\langle$0.9,0.9,0.4,0.4$\rangle$	$\langle$0.5,0.6,0.4,0.2$\rangle$	$\langle$0.6,0.8,0.4,0.2$\rangle$	$\langle$0.7,0.6,0.4,0.3$\rangle$

Step 2: Computed $\mathcal{QSVNRHWA}$ operator with $\breve{\beta} = 2$,we obtain the collective $\mathcal{QSVNR}$ decision matrix.

	$\hat{\Omega}_1$	$\hat{\Omega}_2$	$\hat{\Omega}_3$
$\hat{\zeta}_1$	$\langle 0.281,0.243,0.667,0.560\rangle$	$\langle 0.294,0.232,0.794,0.730\rangle$	$\langle 0.462,0.302,0.639,0.587\rangle$
$\hat{\zeta}_2$	$\langle 0.369,0.247,0.715,0.679\rangle$	$\langle 0.212,0.165,0.855,0.768\rangle$	$\langle 0.413,0.361,0.587,0.522\rangle$
$\hat{\zeta}_3$	$\langle 0.321,0.256,0.795,0.667\rangle$	$\langle 0.314,0.232,0.768,0.723\rangle$	$\langle 0.302,0.241,0.644,0.522\rangle$
$\hat{\zeta}_4$	$\langle 0.243,0.205,0.753,0.757\rangle$	$\langle 0.251,0.230,0.896,0.876\rangle$	$\langle 0.429,0.408,0.587,0.430\rangle$

	$\hat{\Omega}_4$	$\hat{\Omega}_5$
$\hat{\zeta}_1$	$\langle 0.312,0.258,0.795,0.703\rangle$	$\langle 0.440,0.321,0.667,0.679\rangle$
$\hat{\zeta}_2$	$\langle 0.540,0.306,0.757,0.631\rangle$	$\langle 0.297,0.247,0.795,0.667\rangle$
$\hat{\zeta}_3$	$\langle 0.315,0.244,0.834,0.757\rangle$	$\langle 0.406,0.321,0.753,0.640\rangle$
$\hat{\zeta}_4$	$\langle 0.424,0.319,0.715,0.631\rangle$	$\langle 0.285,0.202,0.798,0.715\rangle$

Step 3: Calculate the $\mathcal{QSVNRPIS}$ and $\mathcal{QSVNRNIS}$ are defined as

${\ddot{\mathfrak{D}}_\beta}^+$	$\mathcal{QSVNRPIS}$
${\ddot{\mathfrak{D}}_1}^+$	$\langle 0.369,0.256,0.667,0.560\rangle$
${\ddot{\mathfrak{D}}_2}^+$	$\langle 0.294,0.232,0.794,0.730\rangle$
${\ddot{\mathfrak{D}}_3}^+$	$\langle 0.462,0.408,0.587,0.430\rangle$
${\ddot{\mathfrak{D}}_4}^+$	$\langle 0.540,0.319,0.715,0.631\rangle$
${\ddot{\mathfrak{D}}_5}^+$	$\langle 0.440,0.321,0.667,0.640\rangle$

${\ddot{\mathfrak{D}}_\beta}^-$	$\mathcal{QSVNRNIS}$
${\ddot{\mathfrak{D}}_1}^-$	$\langle 0.243,0.205,0.795,0.757\rangle$
${\ddot{\mathfrak{D}}_2}^-$	$\langle 0.212,0.165,0.896,0.876\rangle$
${\ddot{\mathfrak{D}}_3}^-$	$\langle 0.302,0.241,0.639,0.587\rangle$
${\ddot{\mathfrak{D}}_4}^-$	$\langle 0.312,0.244,0.834,0.757\rangle$
${\ddot{\mathfrak{D}}_5}^-$	$\langle 0.285,0.202,0.798,0.715\rangle$

Step 4: Calculate the distance measures of a perferencei.

For the seperation measures, $\hat{\xi}^+$ and $\hat{\xi}^-$ can be calculated by using Normalized Hamming distance of each perference.

$$\hat{\xi}^+_\alpha = \sum_{\beta=1}^5 \ddot{\varpi}_\beta d(\ddot{\mathfrak{P}}_{\alpha\beta}, \ddot{\mathfrak{D}}^+_\beta) \text{ and } \hat{\xi}^-_\alpha = \sum_{\beta=1}^4 \ddot{\varpi}_\beta d(\ddot{\mathfrak{P}}_{\alpha\beta}, \ddot{\mathfrak{D}}^-_\beta), \alpha = 1,2,3,4.$$

$$\hat{\xi}^+_1 = 0.0106,\ \hat{\xi}^+_2 = 0.0101,\ \hat{\xi}^+_3 = 0.0169,\ \hat{\xi}^+_4 = 0.0126.$$

$$\hat{\xi}^-_1 = 0.0148,\ \hat{\xi}^-_2 = 0.0152,\ \hat{\xi}^-_3 = 0.0091,\ \hat{\xi}^-_4 = 0.0128.$$

Step 5: Calculate($\mathcal{RCC}$) for each alternative $\hat{\zeta}_\alpha$,which is defined by

$$\breve{\Gamma}_\alpha = \frac{\hat{\xi}^+_\alpha}{(\hat{\xi}^+_\alpha + \hat{\xi}^-_\alpha)}, \alpha = 1,2,3,4.$$

$$\Rightarrow \breve{\Gamma}_1 = 0.5825,\ \breve{\Gamma}_2 = 0.6013,\ \breve{\Gamma}_3 = 0.3505,\ \breve{\Gamma}_4 = 0.5040.$$

Step 5: Select the desirable alterative $\hat{\zeta}_\alpha$.

The four alternatives are ranked based on the $\mathcal{RCC}$ values for each preference.

$$\hat{\zeta}_2 > \hat{\zeta}_1 > \hat{\zeta}_4 > \hat{\zeta}_3.$$

Therefore $\hat{\zeta}_2$ is the desirable.

Results:

The highest $\mathfrak{RCC}$ indicates that Bioelectrochemical conversion is the optimal approach for solid waste-to-energy technology due to its environmentally friendly nature. This process generates more green energy, contributing to environmental cleanliness and offering significant employment opportunities to our society.

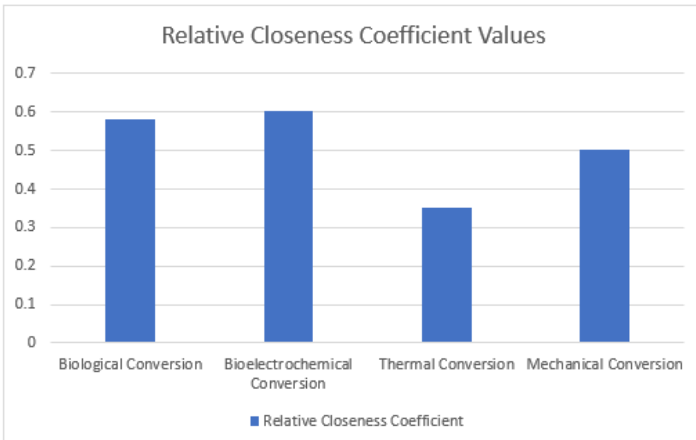

Figure 1. Rating values assigned to the alternatives

5. Conclusion

In this study, we introduced a comparative approach for two quadripartitioned single-valued neutrosophic refined numbers.We proposed new aggregation operators for these quadripartitioned single-valued neutrosophic refined sets, based on Hamacher t-norm and Hamacher t-conorm, and examined their fundamental properties. These operators were incorporated into the TOPSIS method, and the approach was extended to solve MADM problems involving quadripartitioned single-valued neutrosophic refined numbers. A practical application was provided, demonstrating the conversion of solid waste into energy, a solution relevant to current global issues. For future research, we aim to expand the proposed approach and applying it to a range of decision-making challenges, including information management, project selection, and other decision-making areas.

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