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Algebraic Properties of Interval -Valued Quadri Partitioned Neutrosophic Fuzzy Matrices and their Application in Multi-Criteria Decision- Making Problem

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Abstract – In this study, we introduce two novel matrix concepts in the neutrosophic fuzzy domain: range-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices and kernel-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices. These matrices are defined analogously to EP-matrices within the complex domain. Initially, we establish fundamental characterizations of range-symmetric matrices and then derive the necessary and sufficient conditions under which an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices becomes kernel-symmetric . A detailed analysis follows to explore the relationship between range-symmetric and kernel-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices. Additionally, we introduce the concepts of Kernel and k-Kernel Symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, providing illustrative examples to demonstrate their application. Basic results for kernel-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices are derived, highlighting that while k-symmetric implies k- kernel-symmetric in Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, the converse does not necessarily hold. We further discuss the connections between kernel-symmetric, k- kernel-symmetric and the Moore-Penrose inverse of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, supported by numerical examples. The study culminates in an algorithm tailored for solving multi-criteria decision-making problems using Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, validated through an illustrative example that demonstrates its practical utility.

Keywords: Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, range-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, kernel-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, Moore-Penrose inverse, Decision-making.

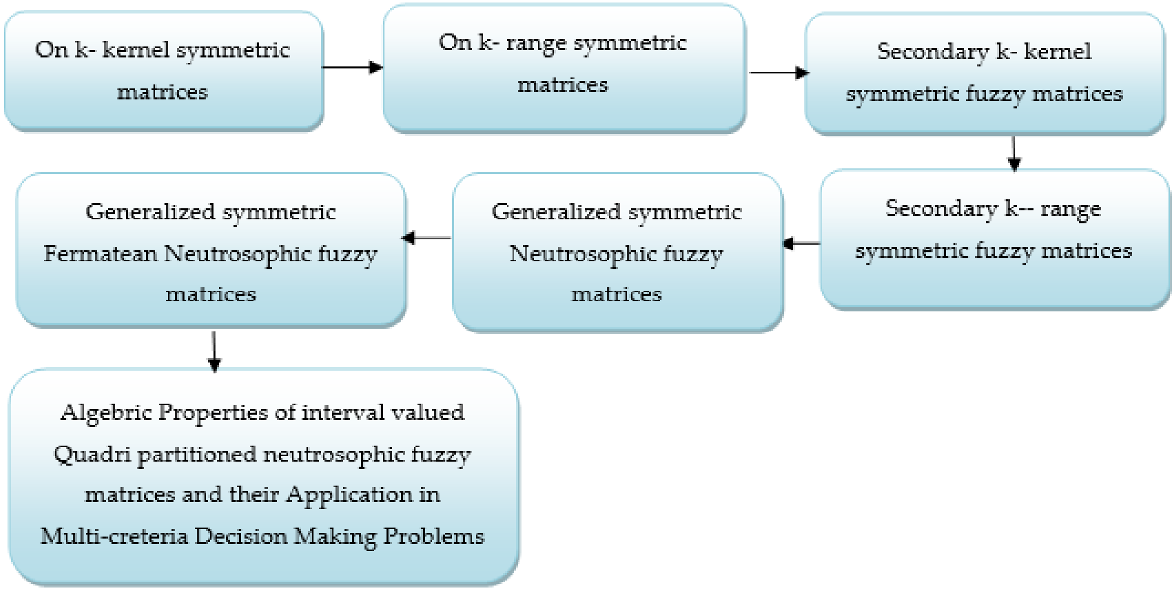
1.Introduction

In recent years, decision-making under uncertainty has gained prominence across various fields due to the growing need to manage vague, imprecise, and indeterminate information effectively. Neutrosophic sets and their extensions provide a flexible framework for modeling such uncertain environments, enabling nuanced representations of truth, indeterminacy, falsity, and hesitation values independently. Among these extensions, Quadri Partitioned Neutrosophic Fuzzy Matrices offer a structured approach to representing multi-dimensional data, making them particularly suitable for applications where handling nuanced and parametric uncertainty systematically is critical. This paper seeks to address the limitations of current fuzzy and neutrosophic matrix approaches in representing and analyzing complex decision-making scenarios. Specifically, it introduces and explores Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, focusing on their structural properties and applications in predictive and multi-criteria decision-making frameworks.

The theoretical foundations of fuzzy sets, introduced by Zadeh [1] in 1965, revolutionized the mathematical treatment of vagueness in real-world problems by allowing for degrees of membership. Subsequent advancements, such as Atanassov's [2] Intuitionistic Fuzzy Sets and Smarandache's [3] Neutrosophic Sets, further enriched uncertainty modeling by incorporating non-membership and indeterminacy. Neutrosophic sets have spurred the development of neutrosophic matrices, enabling the representation of multi-dimensional uncertainty in systems involving conflicting and ambiguous information. Recent studies have extended fuzzy and neutrosophic matrix theory to specialized forms, such as interval-valued matrices and quadri-partitioned structures, which have demonstrated utility in domains like health risk assessment, supplier evaluation, and financial decision-making.

However, existing research often lacks a thorough exploration of advanced properties, such as secondary symmetry, pseudo-similarity, and k-kernel characteristics, which are crucial for improving the robustness and adaptability of decision models. While Anandhkumar [23],[34] et al. have laid a foundation for understanding range and kernel symmetry in neutrosophic fuzzy matrices, there remains a significant gap in applying these principles to Interval-Valued Quadri Partitioned

Neutrosophic Fuzzy Matrices. Addressing this gap can enhance decision-making accuracy and reliability in high-stakes and uncertain environments.



This study introduces the concept of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, extending the principles of range and kernel symmetry to these structures. The research develops necessary and sufficient conditions for range-symmetric, k-range symmetric, kernel symmetric, and k-kernel symmetric matrices, providing a deeper understanding of their algebraic properties. Furthermore, it explores the implications of k-symmetric matrices, demonstrating that k-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices are also k-kernel symmetric and k-range symmetric, with specific scenarios where the converse does not hold.

To bridge theory and application, the study constructs an algorithm-based model utilizing these matrices for decision-making problems. The model's applicability is illustrated through a detailed example, showcasing its potential for real-world applications in areas such as financial risk assessment, healthcare, and supply chain management. This work not only fills critical gaps in the literature but also establishes a robust framework for decision-making under uncertainty, paving the way for future research in this evolving field.

The objectives of this study are threefold, focusing on the introduction of new concepts, the development of theoretical foundations, and the creation of application-driven algorithms. The primary aim is to define Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices and explore their underlying mathematical structures, introducing range-symmetric and kernel-symmetric configurations alongside their k-extensions while establishing their algebraic properties. A secondary focus lies in deriving necessary and sufficient conditions for range-symmetric, k-range symmetric, kernel symmetric, and k-kernel symmetric matrices, demonstrating that k-symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices imply k-kernel symmetry and k-range symmetry, while investigating cases where the converse may not hold. Finally, the study seeks to develop an algorithm-based model specifically tailored for decision-making problems by leveraging these matrices' unique properties. This model's practical relevance is demonstrated through its application in predictive and multi-criteria decision-making scenarios, thereby bridging the gap between theoretical advancements and real-world utility. Through these objectives, the research aims to establish a comprehensive framework for the effective application of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, enhancing both their theoretical depth and practical relevance.

The present study aims to address key limitations in existing fuzzy and neutrosophic matrix theories by developing a comprehensive framework based on Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices. Previous research has successfully applied concepts like Fuzzy Soft Sets, Interval-Valued Fuzzy Soft Sets, and Quadri Partitioned Neutrosophic Soft Sets to solve decision-making problems involving uncertainty and hesitation. However, these models often fail to account for parametric indeterminacy in a neutrosophic environment with dependent components. This work introduces a novel methodology to handle indeterminacy parametrically, offering a more flexible framework for decision-makers. By developing range-symmetric, k-range symmetric, kernel symmetric, and k-kernel symmetric properties for Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, this study addresses the critical need for refined uncertainty modeling. A comparative analysis of the proposed model with existing soft models highlights its advantages, showcasing its potential to revolutionize decision-making in uncertain environments. This integrated section lays a strong foundation for understanding the study's motivation and objectives, ensuring a seamless transition into the methods and results.

Review of Literature

Decision-making under uncertainty has been a critical area of research across various domains, driven by the increasing complexity of modern systems. The advent of fuzzy set theory by Zadeh in 1965 [1] laid the foundation for handling vagueness and imprecision mathematically. This pioneering work introduced the concept of degrees of membership, revolutionizing the approach to uncertainty. Following this, Intuitionistic Fuzzy Sets by Atanassov [2,22] expanded the framework by incorporating membership, non-membership, and hesitation degrees, enabling more nuanced representations of uncertainty.

The development of neutrosophic sets by Smarandache [3] further extended the landscape of uncertainty modeling by introducing three independent components: truth, indeterminacy, and falsity. This flexibility has made neutrosophic sets particularly useful in applications requiring simultaneous handling of contradictory and incomplete information. Neutrosophic fuzzy matrices, as an extension of this theory, have proven effective in capturing complex relationships and multi-dimensional uncertainties [35–39].

In the realm of fuzzy matrices, foundational studies by Kim and Roush [4] established generalized fuzzy matrix theory, while Meenakshi [5] contributed significantly to their algebraic properties. Special forms of fuzzy matrices, such as κ-Hermitian, κ-real, and EPr matrices [6,7], have been instrumental in addressing structured data analysis in diverse fields. Interval-valued fuzzy matrices introduced by Shyamal and Pal [12,20] provided an additional layer of robustness by allowing interval representation for matrix elements, making them particularly valuable in areas such as risk analysis and decision support systems [40,43].

More recently, neutrosophic fuzzy matrices have gained attention due to their ability to model multi-layered uncertainties. Research by Anandhkumar et al. [23–29] has delved into properties like pseudo-similarity, secondary symmetry, and partial ordering, providing a refined understanding of these structures. The introduction of quadri-partitioned and interval-valued neutrosophic fuzzy matrices has further expanded their applicability, with studies highlighting their effectiveness in high-stakes domains such as health risk assessment, supplier evaluation, and financial decision-making [36,39,41,42].

Parallel to these developments, soft set theories have emerged as a complementary approach to decision-making. Roy and Maji [46] emphasized the computational utility of fuzzy soft sets, while Yang et al. [47] combined interval-valued fuzzy sets with soft set theory, enriching the theoretical foundation. Neutrosophic fuzzy soft sets, introduced by Said and Smarandache [48,51], and quadripartitioned neutrosophic fuzzy soft sets proposed by Mary [52] have further demonstrated their relevance in addressing indeterminacy and hesitation in decision-making frameworks. J, J., & S, R. [55] have studied Some Operations on Neutrosophic Hypersoft Matrices and Their Applications. Ranulfo Paiva Barbosa and Smarandache [56] have discussed. Pura Vida Neutrosophic Algebra. Anandhkumar et al [57] have studied Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems. Radhika et al [58,59] have focused on Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making and On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices, Prathab et al [60] have present Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses.

The literature also includes innovative approaches to symmetry in neutrosophic matrices. Studies on k-symmetric and k-kernel matrices by Anandhkumar et al. [27,28] have provided deeper insights into matrix behavior under various algebraic operations. These contributions underscore the potential of advanced neutrosophic matrix properties in enhancing decision-making accuracy and robustness in uncertain environments. Despite these advancements, several research gaps remain. Existing literature often lacks a comprehensive exploration of the interaction between advanced matrix properties and real-world decision frameworks. Moreover, the behavior of interval-valued quadri-partitioned neutrosophic fuzzy matrices under complex conditions, such as multi-attribute or multi-objective criteria, remains underexplored. Addressing these gaps through extended algebraic characterization and empirical validation is essential for advancing the practical utility of these models in high-uncertainty scenarios.

1.1 Notations:

For Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices of $S = \langle [S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U \rangle \in \left(IVQPNFM\right)_n$ where

$[S^T, S^C, S^I, S^F]_L{}^{\mathrm{T}}$: Transpose of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

R($[S^T, S^C, S^I, S^F]_L$) : Row space of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

C($[S^T, S^C, S^I, S^F]_L$) : Column space of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

N($[S^T, S^C, S^I, S^F]_L$) : Null Space of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

$[S^T, S^C, S^I, S^F]_L{}^{+}$: Moore-Penrose inverse of Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $[S^T, S^C, S^I, S^F]_L$.

2. Preliminaries

Definition: 2.1 [5] (page no 118) (EP- Matrices)} An EP- matrix (or range-symmetric matrix) is a square Fuzzy matrix A whose range is equal to the range of its transpose A^T and it's denoted by $R(A) = R(A^T)$.

Definition 2.2 [53] Let X is an initial universe set and E is a set of parameters. Consider a non-empty set A where A ⊆ E. Let P(X) denote the set of all Quadri Partitioned Neutrosophic soft sets of X. The collection (F, A) is termed the Quadri Partitioned Neutrosophic soft sets over X, where F is a mapping given by F : A⟶ P(X). Here,

$A = \{< x, T_A(x), C_A(x), U_A(x), FA(x) >: x \in U\}$ with T_A , F_A, C_A ,$U_A : X \longrightarrow [0,1]$ and $0 \le T_A(x) + C_A(x) + U_A(x) + F_A(x) \le 4$. In this context

- $T_A(x)$ is the TM,
- $C_A(x)$ is CM,
- $U_A(x)$ is IM,
- $F_A(x)$ is the FM,

3. Interval -Valued Quadri Partitioned Neutrosophic Fuzzy Matrices.

The following properties are essential for proving the theorems and examples presented in this work.

(P_1) $K = K^T$, $K^2 = I$ and $\kappa(y) = Ky$ for every

$$S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U> \in \left(IVSQPNFM\right)_n$$

(P_2) $N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L K) = N(K[S^T, S^C, S^I, S^F]_L)$

$N([S^T, S^C, S^I, S^F]_U) = N([S^T, S^C, S^I, S^F]_U K) = N(K[S^T, S^C, S^I, S^F]_U)$

(P_3) $([S^T, S^C, S^I, S^F]_L K)^+ = K[S^T, S^C, S^I, S^F]_L{}^+$ and

$(K[S^T, S^C, S^I, S^F]_L)^+ = [S^T, S^C, S^I, S^F]_L{}^+ K$ exists, if $[S^T, S^C, S^I, S^F]_L{}^+$ exists.

$([S^T, S^C, S^I, S^F]_U K)^+ = K[S^T, S^C, S^I, S^F]_U{}^+$ and $(K[S^T, S^C, S^I, S^F]_U)^+ = [S^T, S^C, S^I, S^F]_U{}^+ K$

exists, if $[S^T, S^C, S^I, S^F]_L{}^+$ exists.

(P_4) $[S^T, S^C, S^I, S^F]_L{}^T$ is a g-inverse of $[S^T, S^C, S^I, S^F]_L$ iff $[S^T, S^C, S^I, S^F]_L{}^+$ exist.

$[S^T, S^C, S^I, S^F]_U{}^T$ is a g-inverse of $[S^T, S^C, S^I, S^F]_U$ iff $[S^T, S^C, S^I, S^F]_U{}^+$ exist.

Definition: 3.1: An Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices S of order m×n is well-defined as $S = [x_{ij}, < T_{ij\mu}, C_{ij\lambda}\ U_{iju}, F_{ij\nu} >]_{m\times n}$ where $T_{ij\mu}$, $C_{ij\lambda}\ U_{iju}$, and $F_{ij\nu}$ are the subsets of [0,1] which are denoted by $T_{ij\mu} = [T_{ij\mu L}, T_{ij\mu U}]$, $C_{ij\lambda} = [C_{ij\lambda L}, C_{ij\lambda U}]$ and $U_{iju} = [U_{ijuL}, U_{ijuU}]$ and $F_{ij\nu} = [F_{ij\nu L}, F_{ij\nu U}]$ which maintaining the condition $0 \le T_{ij\mu L} + C_{ij\lambda L} + U_{ijuL} + F_{ij\nu L} \le 4$, $0 \le T_{ij\mu U} + C_{ij\lambda U} + U_{ijuU} + F_{ij\nu U} \le 4$, and $0 \le T_{ij\mu L} \le T_{ij\mu U} \le 1$, $0 \le C_{ij\lambda L} \le C_{ij\lambda U} \le 1$, $0 \le U_{ijuL} \le U_{ijuU} \le 1$, $0 \le F_{ij\nu L} \le F_{ij\nu U} \le 1$.

Example 3.1 Let us consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices.

$$S = \begin{bmatrix} <[1,1],[1,1],[0,0],[0,0]> & <[0.1,0.3],[0.2,0.4],[0.2,0.5],[0.3,0.5]> \\ <[0.1,0.3],[0.2,0.4],[0.2,0.5],[0.3,0.5]> & <[1,1],[1,1],[0,0],[0,0]> \end{bmatrix}$$

Lower Limit Quadri Partitioned Neutrosophic Fuzzy Matrices,

$$[S_T, S_C, S_I, S_F]_L = \begin{bmatrix} <1,1,0,0> & <0.1,0.2,0,2,0.3> \\ <0.1,0.2,0.2,0.3> & <1,1,0,0> \end{bmatrix},$$

Upper Limit Quadri Partitioned Neutrosophic Fuzzy Matrices,

$$[S_T, S_C, S_I, S_F]_U = \begin{bmatrix} <1,1,0,0> & <0.3,0.4,0.5,0.5> \\ <0.3,0.4,0.5,0.5> & <1,1,0,0> \end{bmatrix}$$

Definition: 3.2 Let $S=<[S^T,S^C,S^I,S^F]_L,[S^T,S^C,S^I,S^F]_U>\in\left(IVSQPNFM\right)_n$ be a Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, if R [$[S^T,S^C,S^I,S^F]_L$] = R [$[S^T,S^C,S^I,S^F]_L$ ^{T}], then S is called as Lower range symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices if R[$[S^T,S^C,S^I,S^F]_U$] = R [$[S^T,S^C,S^I,S^F]_U$ ^{T}] then S is called as Upper range symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices.

Example: 3.2 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

	$<[0.4,0.5],[0,0.3],[0.7,0.8],[1,1]>$	$<[1,1],[0,0.5],[1,1],[1,1]>$	$<[0.2,0.3],[0.3,0.4],[0.4,0.5],[0.6,0.7]>$
S=	$<[1,1],[0,0.5],[1,1],[1,1]>$	$<[1,1],[0,1],[1,1],[1,1]>$	$<[1,1],[0,0.9],[1,1],[1,1]>$
	$<[0.2,0.3],[0.3,0.4],[0.4,0.5],[0.6,0.7]>$	$<[1,1],[0,0.9],[1,1],[1,1]>$	$<[1,1],[0,1],[1,1],[1,1]>$

$$[S^T,S^C,S^I,S^F]_L=\begin{bmatrix}<0.4,0,1,0.7,1> & <1,0,1,1> & <0.2,0.3,0.4,0.6> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.6> & <1,0,1,1> & <1,0,1,1>\end{bmatrix}$$

Here, R [$[S^T,S^C,S^I,S^F]_L$] = R [$[S^T,S^C,S^I,S^F]_L$ ^{T}]

$$[S^T,S^C,S^I,S^F]_U=\begin{bmatrix}<0.5,0,3,0.8,1> & <1,0.5,1,1> & <0.3,0.4,0.5,0.7> \\ <1,0.5,1,1> & <1,0,1,1> & <1,0.9,1,1> \\ <0.3,0.4,0.5,0.7> & <1,0.9,1,1> & <1,0,1,1>\end{bmatrix}$$

Here, R [$[S^T,S^C,S^I,S^F]_U$] = R [$[S^T,S^C,S^I,S^F]_U$ ^{T}]

The following Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices does not satisfies the range symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices condition.

$$[S^T,S^C,S^I,S^F]=\begin{bmatrix}<0.4,0,1,0.9> & <1,0,1,1> & <0.2,0.3,0.4,0.6> \\ <1,0,1,0.9> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.4> & <1,0,1,1> & <1,0,1,1>\end{bmatrix}$$

$$[S^T,S^C,S^I,S^F]^T = \begin{bmatrix} <0.4,0,1,0.9> & <1,0,1,0.9> & <0.2,0.3,0.4,0.4> \\ <1,0,1,1> & <1,0,1,1> & <1,0,1,1> \\ <0.2,0.3,0.4,0.6> & <1,0,1,1> & <1,0,1,1> \end{bmatrix}$$

$\left[(0.4,0,1,0.9)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)\right] \in R([S^T,S^C,S^I,S^F])$,

$\left[(0.4,0,1,0.9)\ (1,0,1,1)\ (0.2,0.3,0.4,0.6)\right] \notin R([S^T,S^C,S^I,S^F]^T)$

$\left[(1,0,1,0.9)\ (1,0,1,1)\ (1,0,1,1)\right] \in R([S^T,S^C,S^I,S^F])$,

$\left[(1,0,1,0.9)\ (1,0,1,1)\ (1,0,1,1)\right] \in R([S^T,S^C,S^I,S^F]^T)$

$\left[(0.2,0.3,0.4,0.4)\ (1,0,1,1)\ (1,0,1,1)\right] \in R([S^T,S^C,S^I,S^F])$,

$\left[(0.2,0.3,0.4,0.4)\ (1,0,1,1)\ (1,0,1,1)\right] \notin R([S^T,S^C,S^I,S^F]^T)$

$R([S^T,S^C,S^I,S^F]) \notin R([S^T,S^C,S^I,S^F]^T)$

Note: 3.1 For Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices S with det $[S^T,S^C,S^I,S^F]_L$ > <0,0,1,1> has non- zero rows and non-columns, hereafter N($[S^T,S^C,S^I,S^F]_L$) = <0,0,1,1> = N($[S^T,S^C,S^I,S^F]_L$ T). Furthermore, a symmetric matrix $[S^T,S^C,S^I,S^F]_L$ = $[S^T,S^C,S^I,S^F]_L$ T that is N($[S^T,S^C,S^I,S^F]_L$)= N($[S^T,S^C,S^I,S^F]_L$ T). Similarly, for Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices S with det $[S^T,S^C,S^I,S^F]_U$ ><0,0,1,1> has non- zero rows and columns, in future N($[S^T,S^C,S^I,S^F]_U$) = <0,0,1,1> = N($[S^T,S^C,S^I,S^F]_U$ T). Furthermore, a symmetric matrix $[S^T,S^C,S^I,S^F]_U$ = $[S^T,S^C,S^I,S^F]_U$ T that is N($[S^T,S^C,S^I,S^F]_U$)= N($[S^T,S^C,S^I,S^F]_U$ T).

Definition: 3.3 Let $S=<[S^T,S^C,S^I,S^F]_L,[S^T,S^C,S^I,S^F]_U> \in \left(IVSQPNFM\right)_n$ be a Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices, if R [$[S^T,S^C,S^I,S^F]_L$] = R [k $[S^T,S^C,S^I,S^F]_L$ T k], then S is called as Lower k- range symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices if R[$[S^T,S^C,S^I,S^F]_U$]

= R [k$[S^T, S^C, S^I, S^F]_U$ Tk] then S is called as Upper k-range symmetric Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices.

Definition : 3.4 Let $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U> \in (IVSQPNFM)_n$ if N($[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$ T) and S is called Lower kernel symmetric - Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices where N($[S^T, S^C, S^I, S^F]_L$) = {x/x $[S^T, S^C, S^I, S^F]_L$ = (0,0,1,1) and x ∈ $F_{1\times n}$} if N($[S^T, S^C, S^I, S^F]_U$) = N($[S^T, S^C, S^I, S^F]_U$ T) and S is called Upper kernel symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices where N($[S^T, S^C, S^I, S^F]_U$) = {x/x$[S^T, S^C, S^I, S^F]_U$ = (0,0,1,1) and x ∈ $F_{1\times n}$} .

Example: 3.3 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

S=	<[0.4, 0.5],[0.4, 0.5],[0.6, 0.8], [0.9,1]>	<[0.6,1],[0.7, 0.8],[0.7,1], [0.3,1]>	<[0.5, 0.6],[0.6, 0.6],[0.7, 0.8], [0.4, 0.7]>
	<[0.6,1],[0.8, 0.8],[0.7,1], [0.5,1]>	<[0.7,1],[0.9,1],[0.2,1], [0.3,1]>	<[0.3,1],[0.7, 0.9],[0.2,1], [0.4,1]>
	<[0.7, 0.7],[0.6, 0.7],[0.7, 0.8], [0.3, 0.7]>	<[0.5,1],[0.6, 0.9],[0.6,1], [0.8,1]>	<[0.5,1],[0.5,1],[0.6,1], [0.4,1]>

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.4, 0.4, 0.6, 0.9> & <1, 0.8, 1, 1> & <0.6, 0.6, 0.8, 0.7> \\ <0.6, 0.8, 0.7, 0.5> & <0.7, 0.9, 0.2, 0.3> & <0.3, 0.7, 0.2, 0.5> \\ <0.7, 0.6, 0.7, 0.3> & <0.5, 0.6, 0.6, 0.8> & <0.5, 0.5, 0.6, 0.4> \end{bmatrix}$$

$$N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L{}^T) = (0, 0, 1, 1).$$

$$[S^T, S^C, S^I, S^F]_U = \begin{bmatrix} <0.5, 0.5, 0.8, 0.1> & <0.6, 0.7, 0.7, 0.3> & <0.5, 0.6, 0.7, 0.4> \\ <1, 0.8, 1, 1> & <1, 1, 1, 1> & <1, 0.9, 1, 1> \\ <0.7, 0.7, 0.8, 0.7> & <1, 0.9, 1, 1> & <1, 1, 1, 1> \end{bmatrix}$$

$$N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L{}^T) = (0, 0, 1, 1).$$

Definition 3.5. Permutation Quadri Partitioned Neutrosophic Fuzzy Matrices: If each row and each column contains correctly one <1,1,0,0> and all other entries are <0,0,1,1> in a square Quadri Partitioned Neutrosophic Fuzzy Matrices, it is known as Quadri Partitioned Neutrosophic Fuzzy Matrices.

Example: 3.4 Consider a Quadri Partitioned Neutrosophic Fuzzy Matrices,

$$K = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

4. Theorems and Results

Theorem:4.1 For an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U>$, $T = <[T^T, T^C, T^I, T^F]_L, [T^T, T^C, T^I, T^F]_U>$ and K be a Quadri Partitioned Neutrosophic Fuzzy permutation Matrices if $N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L) \Leftrightarrow N(K[S^T, S^C, S^I, S^F]_L K^T) = N(K[T^T, T^C, T^I, T^F]_L K^T)$ and $N([S^T, S^C, S^I, S^F]_U) = N([T^T, T^C, T^I, T^F]_U) \Leftrightarrow N(K[S^T, S^C, S^I, S^F]_U K^T) = N(K[T^T, T^C, T^I, T^F]_U K^T)$.

Proof: Let $w \in N(K[S^T, S^C, S^I, S^F]_L K^T)$

$\Rightarrow w(K[S^T, S^C, S^I, S^F]_L K^T) = (0,0,1,1)$

$\Rightarrow yK^T = (0,0,1,1)$ where $y = wK([S^T, S^C, S^I, S^F]_L)$

$\Rightarrow y \in N(K^T)$

$\det K = \det K^T > (0,0,1,1)$

Then, $N(K^T) = (0,0,1,1)$

Hereafter, $y = (0,0,1,1) \Rightarrow wK([S^T, S^C, S^I, S^F]_L) = (0,0,1,1)$

$\Rightarrow wK \in N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$

$\Rightarrow wK([T^T, T^C, T^I, T^F]_L)K^T = (0,0,1,1) \Rightarrow w \in N(K([T^T, T^C, T^I, T^F]_L)K^T)$

$N(K[S^T, S^C, S^I, S^F]_L K^T) \subseteq N(K[T^T, T^C, T^I, T^F]_L K^T)$

Similarly, $N(K[T^T, T^C, T^I, T^F]_L K^T) \subseteq N(K[S^T, S^C, S^I, S^F]_L K^T)$

Therefore, $N([S^T,S^C,S^I,S^F]_L)=N([T^T,T^C,T^I,T^F]_L)$

$\Leftrightarrow N(K[S^T,S^C,S^I,S^F]_L K^T)=N(K[T^T,T^C,T^I,T^F]_L K^T)$

Conversely, if $N(K[S^T,S^C,S^I,S^F]_L K^T)=N(K[T^T,T^C,T^I,T^F]_L K^T)$,

N $[S^T,S^C,S^I,S^F]_L$ = N (K^T (K$[S^T,S^C,S^I,S^F]_L$ K^T) K) = N (K^T (K$([T^T,T^C,T^I,T^F]_L)$ K^T) K)

$N([S^T,S^C,S^I,S^F]_L)=N([T^T,T^C,T^I,T^F]_L)$

Similarly, $N([S^T,S^C,S^I,S^F]_U)=N([T^T,T^C,T^I,T^F]_U)$

$\Leftrightarrow N(K[S^T,S^C,S^I,S^F]_U\ K^T)=N(K[T^T,T^C,T^I,T^F]_U K^T)$.

Example: 4.1 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T,S^C,S^I,S^F]_L=\begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix}$$

$$K=\begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

$$[T^T,T^C,T^I,T^F]_L=\begin{bmatrix} <0.5,0.3,0.7,0.9> & <0.4,0.3,0.6,0.4> & <0.3,0.4,0.4,0.7> \\ <0.5,0.7,0.5,0.7> & <0.4,0.8,0.2,0.8> & <0.2,0.5,0.2,0.3> \\ <0.2,0.4,0.4,0.5> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.8> \end{bmatrix}$$

$$K[S^T,S^C,S^I,S^F]_L K^T=\begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

$$\begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix}$$

$$\begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

$$=\begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix}$$

Let $w \in N(K[S^T, S^C, S^I, S^F]_L K^T)$

$$w = [(0,0,1,1) \quad (0,0,1,1) \quad (0,0,1,1)]$$

$$\Rightarrow w(K[S^T, S^C, S^I, S^F]_L K^T) = [(0,0,1,1) \quad (0,0,1,1) \quad (0,0,1,1)]$$

$$\begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix} = (0,0,1,1)$$

$\Rightarrow yK^T = (0,0,1,1)$ where $y = wK([S^T, S^C, S^I, S^F]_L)$. Hence, $y = (0,0,1,1)$

$$K[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.3,0.3,0.4,0.9> & <0.5,0.3,0.6,0.8> & <0.2,0.3,0.4,0.4> \\ <0.5,0.4,0.7,0.6> & <0.6,0.4,0.8,0.1> & <0.8,0.2,0.5,0.1> \\ <0.2,0.4,0.4,0.4> & <0.4,0.4,0.5,0.9> & <0.4,0.5,0.6,0.5> \end{bmatrix}$$

$$\Rightarrow wK([S^T, S^C, S^I, S^F]_L) = (0,0,1,1)$$

$$\Rightarrow wK \in N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$$

$$\Rightarrow wK([T^T, T^C, T^I, T^F]_L)K^T = (0,0,1,1)$$

$$N([S^T, S^C, S^I, S^F]_L) = N([T^T, T^C, T^I, T^F]_L)$$

$$\Leftrightarrow N(K[S^T, S^C, S^I, S^F]_L K^T) = N(K[T^T, T^C, T^I, T^F]_L K^T).$$

Similarly, $N([S^T, S^C, S^I, S^F]_U) = N([T^T, T^C, T^I, T^F]_U)$

$$\Leftrightarrow N(K[S^T, S^C, S^I, S^F]_U K^T) = N(K[T^T, T^C, T^I, T^F]_U K^T).$$

Theorem:4.2 For an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U> \in (IVSQPNFM)_n$ and K be Quadri Partitioned Neutrosophic Fuzzy Permutation Matrices if $N([S^T, S^C, S^I, S^F]_L) = N([S^T, S^C, S^I, S^F]_L{}^T)$

⇔N(K$[S^T,S^C,S^I,S^F]_L$ K^T) = N(K $[S^T,S^C,S^I,S^F]_L$ ^T K^T) and N(K$[S^T,S^C,S^I,S^F]_U$ K^T) = N(K $[S^T,S^C,S^I,S^F]_U$ ^T K^T) ⇔

$$N([S^T,S^C,S^I,S^F]_U) = N([S^T,S^C,S^I,S^F]_U{}^T)$$

Proof: The proof follows a structure like that of Theorem 9.1

Example: 4.2 Consider an Quadri Partitioned Neutrosophic Fuzzy permutation Matrices

$$K = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

$$[S^T,S^C,S^I,S^F]_L = \begin{bmatrix} <0.7,0.3,0.5,0.9> & <0.5,0.3,0.6,0.7> & <0.2,0.3,0.4,0.8> \\ <0.4,0.7,0.6,0.6> & <0.4,0.8,0.1,0.7> & <0.2,0.5,0.1,0.3> \\ <0.2,0.4,0.4,0.5> & <0.4,0.4,0.5,0.8> & <0.4,0.5,0.6,0.4> \end{bmatrix}$$

$$[S^T,S^C,S^I,S^F]_U = \begin{bmatrix} <0.8,0.4,0.6,1> & <0.6,0.4,0.7,0.8> & <0.3,0.5,0.5,0.9> \\ <0.5,0.8,0.7,0.7> & <0.5,0.9,0.2,0.8> & <0.3,0.6,0.2,0.4> \\ <0.3,0.5,0.5,0.6> & <0.5,0.5,0.6,0.9> & <0.5,0.6,0.7,0.5> \end{bmatrix}$$

Theorem: 4.3 For $S = <[S^T,S^C,S^I,S^F]_L,[S^T,S^C,S^I,S^F]_U>$ is kernel symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, then N($[S^T,S^C,S^I,S^F]_L$ $[S^T,S^C,S^I,S^F]_L$ ^T) = N($[S^T,S^C,S^I,S^F]_L$) = N($[S^T,S^C,S^I,S^F]_L$ ^T $[S^T,S^C,S^I,S^F]_L$) and N($[S^T,S^C,S^I,S^F]_U$ $[S^T,S^C,S^I,S^F]_U$ ^T) = N($[S^T,S^C,S^I,S^F]_U$) = N($[S^T,S^C,S^I,S^F]_U$ ^T $[S^T,S^C,S^I,S^F]_U$).

Example: 4.3 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T,S^C,S^I,S^F]_L = \begin{bmatrix} <0.1,0.3,0.5,0.8> & <0.4,0.3,0.6,0.7> & <0.2,0.3,0.4,0.3> \\ <0.4,0.7,0.6,0.4> & <0.4,0.7,0.1,0.5> & <0.2,0.5,0.1,0.2> \\ <0.2,0.3,0.5,0.6> & <0.4,0.3,0.5,0.9> & <0.4,0.2,0.1,0.6> \end{bmatrix}$$

$$[S^T,S^C,S^I,S^F]_U = \begin{bmatrix} <0.2,0.5,0.5,0.9> & <0.5,0.4,0.7,0.9> & <0.4,0.5,0.7,0.4> \\ <0.5,0.8,0.7,0.5> & <0.5,0.7,0.2,0.6> & <0.2,0.7,0.2,0.4> \\ <0.2,0.5,0.7,0.8> & <0.5,0.4,0.5,0.9> & <0.5,0.2,0.1,0.6> \end{bmatrix}$$

$$K=\begin{bmatrix}(1,1,0,0) & (0,0,1,1) & (0,0,1,1)\\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1)\\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0)\end{bmatrix}$$

Theorem:4.4 For a Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices $S=<[S^T,S^C,S^I,S^F]_L,[S^T,S^C,S^I,S^F]_U>$, $T=<[T^T,T^C,T^I,T^F]_L,[T^T,T^C,T^I,T^F]_U>$ and K be a Quadri Partitioned Neutrosophic Fuzzy permutation Matrices, R ($[S^T,S^C,S^I,S^F]_L$) = R ($[T^T,T^C,T^I,T^F]_L$) $\Leftrightarrow$ R (K $[S^T,S^C,S^I,S^F]_L$ K^T) = R (K $[T^T,T^C,T^I,T^F]_L$ K^T) and R ($[S^T,S^C,S^I,S^F]_U$) = R ($[T^T,T^C,T^I,T^F]_U$) $\Leftrightarrow$ R (K $[S^T,S^C,S^I,S^F]_U$ K^T) = R (K $[T^T,T^C,T^I,T^F]_U$ K^T).

Proof: Let R ($[S^T,S^C,S^I,S^F]_L$) = R ($[T^T,T^C,T^I,T^F]_L$)

Then, R ($[S^T,S^C,S^I,S^F]_L$ K^T) = R ($[S^T,S^C,S^I,S^F]_L$) K^T

= R ($[S^T,S^C,S^I,S^F]_L$) K^T

= R ($[S^T,S^C,S^I,S^F]_L$ K^T)

Let $z\in\left\{R(K[S^T,S^C,S^I,S^F]_L K^T)\right\}$

$z=w(K[S^T,S^C,S^I,S^F]_L K^T)$ for some $w\in V^n$

$z=r[S^T,S^C,S^I,S^F]_L K^T,\ r=wK$

$z\in R\left([S^T,S^C,S^I,S^F]_L K^T\right)=R\left([T^T,T^C,T^I,T^F]_L\left(K^T\right)\right)$

$z=u[T^T,T^C,T^I,T^F]_L K^T$ for some $u\in V^n$

$z=\left(uK^T\right)K[T^T,T^C,T^I,T^F]_L K^T$

$z=vK[T^T,T^C,T^I,T^F]_L K^T$ for some $v\in V^n$

$z\in R(K[T^T,T^C,T^I,T^F]_L K^T)$

Therefore, $R(K[S^T,S^C,S^I,S^F]_L K^T)\subseteq R(K[T^T,T^C,T^I,T^F]_L K^T)$

Similarly, $R(K[T^T,T^C,T^I,T^F]_L K^T) \subseteq R(K[S^T,S^C,S^I,S^F]_L K^T)$

Therefore, R(K$[S^T,S^C,S^I,S^F]_L$ K^T) = R(K$[T^T,T^C,T^I,T^F]_L$ K^T)

Similarly, ($[S^T,S^C,S^I,S^F]_U$) = R ($[T^T,T^C,T^I,T^F]_U$) $\Leftrightarrow$ R (K $[S^T,S^C,S^I,S^F]_U$ K^T) = R (K $[T^T,T^C,T^I,T^F]_U$ K^T).

Conversely, Let R(K$[S^T,S^C,S^I,S^F]_L$ K^T) = R(K$[T^T,T^C,T^I,T^F]_L$ K^T)

$$R([S^T,S^C,S^I,S^F]_L) = R[K^T(K[S^T,S^C,S^I,S^F]_L K^T)K]$$

$$= R[K^T(K[T^T,T^C,T^I,T^F]_L K^T)K]$$

$$= R([T^T,T^C,T^I,T^F]_L)$$

$$R([S^T,S^C,S^I,S^F]_L) = R([T^T,T^C,T^I,T^F]_L)$$

Similarly,

$$R([S^T,S^C,S^I,S^F]_U) = R([T^T,T^C,T^I,T^F]_U)$$

Example: 4.4 Consider an Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T,S^C,S^I,S^F]_L = \begin{bmatrix} <0.2,0.4,0.7,0.9> & <0.9,0.9,0.9,0.9> & <0.2,0.3,0.4,0.8> \\ <0.4,0.3,0.6,0.6> & <0.4,0.7,0.1,0.6> & <0.3,0.4,0.5,0.9> \\ <0.1,0.2,0.3,0.4> & <0.7,0.8,0.2,0.1> & <0.8,0.5,0.7,0.9> \end{bmatrix}$$

$$[T^T,T^C,T^I,T^F]_L = \begin{bmatrix} <0.1,0.2,0.3,0.4> & <0.7,0.8,0.2,0.1> & <0.8,0.5,0.7,0.9> \\ <0.4,0.3,0.6,0.5> & <0.4,0.7,0.1,0.6> & <0.3,0.4,0.5,0.9> \\ <0.2,0.4,0.7,0.9> & <0.9,0.9,0.9,0.9> & <0.2,0.3,0.4,0.8> \end{bmatrix}$$

$$K = \begin{bmatrix} (1,1,0,0) & (0,0,1,1) & (0,0,1,1) \\ (0,0,1,1) & (1,1,0,0) & (0,0,1,1) \\ (0,0,1,1) & (0,0,1,1) & (1,1,0,0) \end{bmatrix}$$

Theorem:4.5 For $S = <[S^T,S^C,S^I,S^F]_L,[S^T,S^C,S^I,S^F]_U>$ be the Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices and K be a Quadri Partitioned Neutrosophic Fuzzy permutation Matrices, R($[S^T,S^C,S^I,S^F]_L$) = R($[S^T,S^C,S^I,S^F]_L$) $\Leftrightarrow$ R(K$[S^T,S^C,S^I,S^F]_L$ K^T) =

R(K $[S^T, S^C, S^I, S^F]_L$ T KT) and R($[S^T, S^C, S^I, S^F]_U$) = R($[S^T, S^C, S^I, S^F]_U$) ⇔ R(K $[S^T, S^C, S^I, S^F]_U$ KT) = R(K $[S^T, S^C, S^I, S^F]_U$ T KT).

Proof: The proof follows a structure like that of Theorem 4.4

Example: 4.5 Consider an Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} <0.2, 0.1, 0.3, 0.9> & <0.3, 0.3, 0.2, 0.8> & <0.4, 0.5, 0.6, 0.7> \\ <0.3, 0.3, 0.2, 0.8> & <0.4, 0.7, 0.1, 0.3> & <0.4, 0.5, 0.2, 0.7> \\ <0.4, 0.5, 0.6, 0.7> & <0.4, 0.5, 0.2, 0.7> & <0.3, 1, 1, 0.6> \end{bmatrix}$$

Theorem:4.6 For a Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U>$, $T = <[T^T, T^C, T^I, T^F]_L, [T^T, T^C, T^I, T^F]_U>$ and K be a Quadri Partitioned Neutrosophic Fuzzy permutation Matrices C($[S^T, S^C, S^I, S^F]_L$) = C($[T^T, T^C, T^I, T^F]_L$) ⇔ C(K $[S^T, S^C, S^I, S^F]_L$ KT) = C(K $[T^T, T^C, T^I, T^F]_L$ KT) and C($[S^T, S^C, S^I, S^F]_U$) = C($[T^T, T^C, T^I, T^F]_U$) ⇔ C(K $[S^T, S^C, S^I, S^F]_U$ KT) = C(K $[T^T, T^C, T^I, T^F]_U$ KT).

Proof: The proof follows a structure like that of Theorem 4.4

5. k-Kernel Symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

Definition: 5.1 Let $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U>$ is called Lower k-kernel symmetric Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices if N($[S^T, S^C, S^I, S^F]_L$) = N(K $[S^T, S^C, S^I, S^F]_L$ TK) and Upper k-Kernel symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices if N($[S^T, S^C, S^I, S^F]_U$) = N(K $[S^T, S^C, S^I, S^F]_U$ TK).

Note:5.1 Let $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U>$ is k-Symmetric Interval-Valued symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices implies it is k-kernel - Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices, i.e $[S^T, S^C, S^I, S^F]_L$ = K($[S^T, S^C, S^I, S^F]_L$ T)K, spontaneously implies N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T} K) but converse need not be true.

Similarly, $[S^T, S^C, S^I, S^F]_U$ = K($[S^T, S^C, S^I, S^F]_U$ ^{T})K, spontaneously implies N($[S^T, S^C, S^I, S^F]_U$) = N(K$[S^T, S^C, S^I, S^F]_U$ ^{T} K) however, the converse does not necessarily hold.

Example 5.1 We show that kernel symmetric implies k-kernel symmetric; however, the converse does not necessarily apply

Example: 5.1 Consider an Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices

$$[S^T, S^C, S^I, S^F]_L = \begin{bmatrix} (0,0,0.5) & (0,0,0.4) & (0.3,0.4,0.5) \\ (0.5,0.4,0.6) & (0.1,0.4,0.6) & (0,0,0.4) \\ (0.4,0.5,0.3) & (0.3,0.4,0.5) & (0,0,0.3) \end{bmatrix}$$

$$K = \begin{bmatrix} (0,0,1) & (0,0,1) & (1,1,0) \\ (0,0,1) & (1,1,0) & (0,0,1) \\ (1,1,0) & (0,0,1) & (0,0,1) \end{bmatrix}$$

Therefore, $[S^T, S^C, S^I, S^F]_L \neq$ K$[S^T, S^C, S^I, S^F]_L$ ^{T} K

But, N ($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T} K) = $(0,0,1,1)$

Similarly, $[S^T, S^C, S^I, S^F]_U \neq$ K$[S^T, S^C, S^I, S^F]_U$ ^{T} K

But, N ($[S^T, S^C, S^I, S^F]_U$) = N(K$[S^T, S^C, S^I, S^F]_U$ ^{T} K) = $(0,0,1,1)$

Theorem: 5.1 The following conditions are equivalent for $S = <[S^T, S^C, S^I, S^F]_L, [S^T, S^C, S^I, S^F]_U> \in (IVSQPNFM)_n$

(i) N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T}K)

N($[S^T, S^C, S^I, S^F]_U$) = N(K$[S^T, S^C, S^I, S^F]_U$ ^{T}K)

(ii) N(K$[S^T, S^C, S^I, S^F]_L$) = N((K$[S^T, S^C, S^I, S^F]_L$)^T)

N(K$[S^T, S^C, S^I, S^F]_U$) = N((K$[S^T, S^C, S^I, S^F]_U$)^T)

(iii) N($[S^T, S^C, S^I, S^F]_L$ K) = N(($[S^T, S^C, S^I, S^F]_L$ K)^T)

N($[S^T, S^C, S^I, S^F]_U$ K) = N(($[S^T, S^C, S^I, S^F]_U$ K)^T)

(iv) N($[S^T, S^C, S^I, S^F]_L{}^T$) = N(K$[S^T, S^C, S^I, S^F]_L$)

N($[S^T, S^C, S^I, S^F]_U{}^T$) = N(K$[S^T, S^C, S^I, S^F]_U$)

(v) N($[S^T, S^C, S^I, S^F]_L$) = N(($[S^T, S^C, S^I, S^F]_L$ K)^T)

N($[S^T, S^C, S^I, S^F]_U$) = N(($[S^T, S^C, S^I, S^F]_U$ K)^T)

(vi) $[S^T, S^C, S^I, S^F]_L{}^+$ is k- kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices and $[S^T, S^C, S^I, S^F]_U{}^+$ is k- kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

(vii) N($[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L{}^+$ K)

N($[S^T, S^C, S^I, S^F]_U$) = N($[S^T, S^C, S^I, S^F]_U$ K)

(viii) K $[S^T, S^C, S^I, S^F]_L{}^+ [S^T, S^C, S^I, S^F]_L = [S^T, S^C, S^I, S^F]_L [S^T, S^C, S^I, S^F]_L{}^+$ K

K $[S^T, S^C, S^I, S^F]_U{}^+ [S^T, S^C, S^I, S^F]_U = [S^T, S^C, S^I, S^F]_U [S^T, S^C, S^I, S^F]_U{}^+$ K

(ix) $[S^T, S^C, S^I, S^F]_L{}^+ [S^T, S^C, S^I, S^F]_L$ K = K$[S^T, S^C, S^I, S^F]_L [S^T, S^C, S^I, S^F]_L{}^+$

$[S^T, S^C, S^I, S^F]_U{}^+ [S^T, S^C, S^I, S^F]_U$ K = K$[S^T, S^C, S^I, S^F]_U [S^T, S^C, S^I, S^F]_U{}^+$

Proof: (i) $\Leftrightarrow$ (ii)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L{}^T$K)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L{}^T$K) (By P_2) (K^2 =I)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N((K$[S^T, S^C, S^I, S^F]_L$)^T) (Because , $(KP)^T = P^T K^T = P^T K$)

$\Leftrightarrow$ K$[S^T, S^C, S^I, S^F]_L$ is kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

Similarly, N($[S^T, S^C, S^I, S^F]_U$) = N(K$[S^T, S^C, S^I, S^F]_U{}^T$K).

Condition (ii) is true

Condition (i) $\Leftrightarrow$ (iii)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T}K)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$ K) = N(K$[S^T, S^C, S^I, S^F]_L$ ^{T}) (By P2) (K^2 = I)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$ K) = N(($[S^T, S^C, S^I, S^F]_L$ K) ^{T}) (Because , $(PK)^T = K^TP^T = KP^T$)

$[S^T, S^C, S^I, S^F]_L$ K is kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

Similarly, N($[S^T, S^C, S^I, S^F]_U$ K) = N(($[S^T, S^C, S^I, S^F]_U$ K)^T).

Therefore, (iii) holds

(ii) $\Leftrightarrow$ (iv)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$)^T=N($[S^T, S^C, S^I, S^F]_L$ ^{T}K)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$ ^{T})

Similarly, N($[S^T, S^C, S^I, S^F]_U$ ^{T}) = N(K$[S^T, S^C, S^I, S^F]_U$), (By P2)

Therefore, (iv) holds
(iii) $\Leftrightarrow$ (v)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$ K) = N(($[S^T, S^C, S^I, S^F]_L$ K)^T)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(($[S^T, S^C, S^I, S^F]_L$ K)^T) (By P2)

Similarly, N($[S^T, S^C, S^I, S^F]_U$) = N(($[S^T, S^C, S^I, S^F]_U$ K)^T)

(ii) $\Leftrightarrow$ (vi)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$)^T

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$ ^{T}K) (By P2)

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$ $^+$K)

Since N(K$[S^T, S^C, S^I, S^F]_L$ $^+$K) = N($^+$K) $[S^T, S^C, S^I, S^F]_L$

$\Leftrightarrow$ N(K$[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$$^+$)

$\Leftrightarrow$ $[S^T, S^C, S^I, S^F]_L$$^+$ is k-kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

Similarly, $[S^T, S^C, S^I, S^F]_U{}^+$ is k- kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices. Condition (i) $\Leftrightarrow$ (vii)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$^{T}K)

$\Leftrightarrow$ N() = N(K$[S^T, S^C, S^I, S^F]_L$^{T}K) = N($[S^T, S^C, S^I, S^F]_L$^{T}K)

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N(K$[S^T, S^C, S^I, S^F]_L$)^T

$\Leftrightarrow$ N($[S^T, S^C, S^I, S^F]_L$) = N($[S^T, S^C, S^I, S^F]_L$$^+$K) (By P_2)

Similarly, N($[S^T, S^C, S^I, S^F]_U$) = N($[S^T, S^C, S^I, S^F]_U$K),

(i) $\Leftrightarrow$ (viii)

$[S^T, S^C, S^I, S^F]_L$K is kernel symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices

$\Leftrightarrow$ ($[S^T, S^C, S^I, S^F]_L$K)($[S^T, S^C, S^I, S^F]_L$K)$^+$ = ($[S^T, S^C, S^I, S^F]_L$K)$^+$($[S^T, S^C, S^I, S^F]_L$K)

$\Leftrightarrow$ ($[S^T, S^C, S^I, S^F]_L$K)(K$[S^T, S^C, S^I, S^F]_L$$^+$) = (K$[S^T, S^C, S^I, S^F]_L$$^+$)($[S^T, S^C, S^I, S^F]_L$K)

$\Leftrightarrow$ $[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L$$^+$ = K$[S^T, S^C, S^I, S^F]_L$$^+$$[S^T, S^C, S^I, S^F]_L$K

$\Leftrightarrow$ $[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L$$^+$K = K$[S^T, S^C, S^I, S^F]_L$$^+$$[S^T, S^C, S^I, S^F]_L$

(viii) $\Leftrightarrow$ (ix)

$\Leftrightarrow$ K$[S^T, S^C, S^I, S^F]_L$$^+$$[S^T, S^C, S^I, S^F]_L$ = $[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L$$^+$ K

$\Leftrightarrow$ $[S^T, S^C, S^I, S^F]_L$$^+$ $[S^T, S^C, S^I, S^F]_L$K = K$[S^T, S^C, S^I, S^F]_L$ $[S^T, S^C, S^I, S^F]_L$$^+$.

Similarly, $[S^T, S^C, S^I, S^F]_U{}^+$ $[S^T, S^C, S^I, S^F]_U$K = K$[S^T, S^C, S^I, S^F]_U$ $[S^T, S^C, S^I, S^F]_U{}^+$

Therefore, (ix) is holds.

6. An Algorithm Based on IVQPNSS in a Decision-Making Problem

Definition 6.1. Let Y = {y1,y2,.....,yn} be an initial universe and E = {e_1,e_2,.....,e_m} be a set of parameters. Then, for an **IVQPNSS** (η,E) over Y the degree of **TM** and the degree of **CM** of an element y_i to η (e_j) denoted by $T_{\eta(e_j)}(y_i)=\left[\underline{T}_{\eta(e_j)}(y_i),\bar{T}_{\eta(e_j)}(y_i)\right]$ and $C_{\eta(e_j)}(y_i)=\left[\underline{C}_{\eta(e_j)}(y_i),\bar{C}_{\eta(e_j)}(y_i)\right]$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{T_{\eta(e_j)}}(y_i)=\sum_{K=1}^{N}\left[\left(\underline{T}_{\eta(e_j)}(y_i)+\bar{T}_{\eta(e_j)}(y_i)\right)-\left(\underline{T}_{\eta(e_j)}(y_k)+\bar{T}_{\eta(e_j)}(y_k)\right)\right]$$

$$S_{C_{\eta(e_j)}}(y_i)=\sum_{K=1}^{N}\left[\left(\underline{C}_{\eta(e_j)}(y_i)+\bar{C}_{\eta(e_j)}(y_i)\right)-\left(\underline{C}_{\eta(e_j)}(y_k)+\bar{C}_{\eta(e_j)}(y_k)\right)\right]$$

Definition 6.2. Let Y = {y1,y2,.....,yn} be an initial universe and E = {e_1,e_2,.....,e_m} be a set of parameters. Then, for an **IVQPNSS** (η,E) over X the degree of **UM** and the degree of − **FM** of an element xi to η (e_j) denoted by $U_{\eta(e_j)}(y_i)=\left[\underline{U}_{\eta(e_j)}(y_i),\bar{U}_{\eta(e_j)}(y_i)\right]$ and $F_{\eta(e_j)}(y_i)=\left[\underline{F}_{\eta(e_j)}(y_i),\bar{F}_{\eta(e_j)}(y_i)\right]$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{U_{\eta(e_j)}}(y_i)=-\sum_{K=1}^{N}\left[\left(\underline{U}_{\eta(e_j)}(y_i)+\bar{U}_{\eta(e_j)}(y_i)\right)-\left(\underline{U}_{\eta(e_j)}(y_k)+\bar{U}_{\eta(e_j)}(y_k)\right)\right]$$

$$S_{F_{\eta(e_j)}}(y_i)=\sum_{K=1}^{N}\left[\left(\underline{F}_{\eta(e_j)}(y_i)+\bar{F}_{\eta(e_j)}(y_i)\right)-\left(\underline{F}_{\eta(e_j)}(y_k)+\bar{F}_{\eta(e_j)}(y_k)\right)\right]$$

Definition 6.3. Let X = {y1,y2,.....,yn} be an initial universe and E = {e_1,e_2,.....,e_m} be a set of parameters. For an **IVQPNSS** (η,E) over Y, the scores of the **TM**, CM, **UM**, and **FM** of y_i for each e_j be denoted by $S_{T_{\eta(e_j)}}(y_i)$, $S_{C_{\eta(e_j)}}(y_i)$, $S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$ respectively. Then, the total score of x_i for each e_j is denoted by $\square_{T_{\eta(e_j)}}(y_i)=S_{T_{\eta(e_j)}}(y_i)+S_{C_{\eta(e_j)}}(y_i)+S_{U_{\eta(e_j)}}(y_i)+S_{F_{\eta(e_j)}}(y_i)$

Based on the above definitions, we give the steps of the proposed algorithm as follows:

Algorithm: Step 1: Input Matrix Representation

In this initial step, the decision-maker provides the data in the form of an interval-valued Quadri Partitioned Neutrosophic soft set. The universal set Y = {y1,y2,.....,yn} represents the elements under consideration, and the parameter set E = {e_1,e_2,.....,e_m} denotes the criteria or attributes. The data is structured into a tabular matrix, where each element represents the interval-valued Quadri Partitioned Neutrosophic soft value corresponding to a specific element y_i and criterion e_j. This step establishes the foundation for further computation by organizing the data systematically.

Step 2: Compute Individual Components: Using the interval-valued Quadri Partitioned Neutrosophic matrix from Step 1, Definitions 11.1 and 11.2 are applied to calculate the components $S_{T_{\eta(e_j)}}(y_i)$, $S_{C_{\eta(e_j)}}(y_i)$, $S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$. These represent truth, indeterminacy, falsity, and hesitation degrees, respectively, for each element x_i under the criterion e_j. This computation ensures that the nuanced aspects of each element's evaluation are captured.

Step 3: Score Calculation for Individual Criteria

With the components computed in Step 2, Definition 11.3 is utilized to determine the score $\square_{T_{\eta(e_j)}}(x_i)$ for each element x_i under each criterion e_j. The score reflects a composite evaluation of the truth, indeterminacy, falsity, and hesitation degrees. This step condenses the multi-dimensional information into a single score for each element-criterion pair, facilitating easier comparison.

Step 4: Overall Score Computation

The overall score u_i for each element x_i is calculated by aggregating the scores $v_i = \square_{T_{\eta(e_1)}}(y_i) + \square_{T_{\eta(e_2)}}(y_i) + \square_{T_{\eta(e_3)}}(y_i) + \ldots + \square_{T_{\eta(e_m)}}(y_i)$ across all criteria e_j. This provides a holistic evaluation of each element concerning all the given parameters. The aggregation formula ensures that the influence of all criteria is considered, offering a balanced view of each element's performance.

Step 5: Optimal Choice Identification

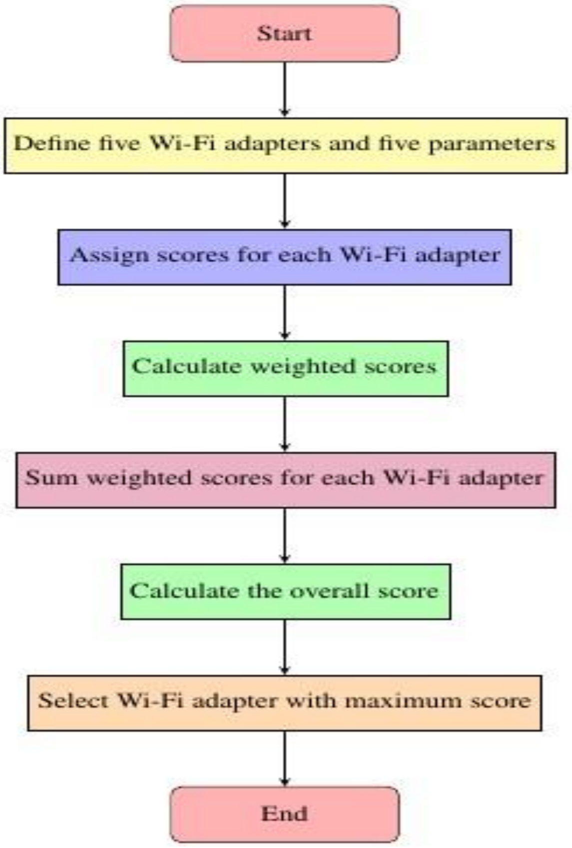
After computing the overall scores for all elements, the element x_k with the maximum score $v_k = \max_{x_i \in X}\{v_i\}$. is identified as the optimal choice. This step concludes the decision-making process by pinpointing the best candidate based on the computed scores.

Step 6: Handling Ties and Reassessment

In cases where a tie occurs, indicating that two or more elements share the highest score, either all tied elements are considered as optimal choices, or the decision-making process is revisited. Reassessment involves consulting experts and potentially redefining the parameters or input values, followed by repeating the previous steps. This ensures that the final decision is both robust and aligned with expert judgment.

7. **To demonstrate the practical application of the algorithm, we present the following example.**

We consider the following example. Illustrative Example to implement the proposed algorithm successfully in a real life context, we consider the following problem: Suppose Mr. Z wants to purchase a Wi-Fi adapters to carry out his official work. But he has limited knowledge to select a good quality WiFi adapters. The selection process is complicated due to various conflicting factors involved during decision making. To solve the purpose, Mr. Z consulted with some experts cum decision makers having IT backgrounds. The decision makers evaluated the five Wi-Fi adapters according to the fixed criteria. To select the best alternative, the evaluation procedure is executed as follows:



Step1. Consider a set of five Wi-Fi adapters be Y={y_1,y_2,y_3,y_4,y_5}and a set of parameters be E={e_i:i=1,2,3,4,5},where

e_1= Speed (Bandwidth),

e_2 = Compatibility,

e_3 = Range and Coverage,

e_4 = Interface Type, and

e_5 = Portability and Design.

Based on the opinions of the decision makers, the decision matrix of the set of five alternatives and five evaluation criteria under the interval valued Neutrosophic soft set environment is shown in Table1.

Here's an example of assigning weights to the five parameters based on their importance in the decision making process for selecting a Wi-Fi adapter.

Step1: Begin by considering a set of five Wi-Fi adapters represented as Y ={y_1,y_2,y_3,y_4,y_5} , along with a set of assessment criteria denoted by E = {e_i :i=1,2,3,4,5}

Speed Bandwidth:40 This is often themost important parameter as it affects data transfer rates and performance.

Compatibility:20

Ensures the adapter works with your system and network standards, crucial for usability.

Range and Coverage: 15 Determines the signal strength and connection stability across distances.

Interface Type: 15 USB or PCIe; affects ease of installation, portability, and data transfer rates.
Portability and Design: 10 Impacts how convenient the adapter is for travel or integration into a setup.
Step 2: Assign Scores for Each Course Rate each course for each parameter on a scale from 0 to 1 (where 0 is the lowest and 1 is the highest). Table: 1. Tabular representation of interval valued Neutrosophic Fuzzy Matrices to describe the f ive Wi-Fi adapters.

Step 3. The score of the truth-membership degrees f shown in Table: 2. The score of the indeterminacy-membership degrees is shown in Table: 3, Score of the false-membership degrees Table: 4 and Total score function Table: 5.

Table: 1 Scores for Each Wi-Fi adapters

X/E	e_1	e_2	e_3	e_4	e_5
y_1	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>	< [0.4,0.5], [0.2,0.4], [0.3,0.8], [0.6,0.7]>
y_2	< [0.3,0.4], [0.5,0.6], [0.6,0.9], [0.8,0.9]>	< [0.1,0.2], [0.2,0.4], [0.5,0.6], [0.8,0.9]>	< [0.4,0.5], [0.3,0.4], [0.6,0.8], [0.7,0.7]>	< [0.3,0.5], [0.2,0.6], [0.3,0.7], [0.5,0.7]>	< [0.7,0.9], [0.3,0.4], [0.5,0.9], [0.7,0.9]>
y_3	< [0.3,0.6], [0.5,0.6], [0.7,0.8], [0.8,0.9]>	< [0.6,0.7], [0.2,0.4], [0.4,0.8], [0.7,0.8]>	< [0.1,0.5], [0.3,0.4], [0.5,0.8], [0.7,0.7]>	< [0.6,0.6], [0.3,0.4], [0.7,0.8], [0.8,0.9]>	< [0.1,0.2], [0.4,0.8], [0.5,0.8], [0.1,0.6]>
y_4	< [0.1,0.7], [0.5,0.9], [0.6,0.9], [0.7,0.9]>	< [0.2,0.5], [0.6,0.7], [0.3,0.8], [0.7,0.8]>	< [0.3,0.5], [0.2,0.4], [0.5,0.8], [0.8,0.9]>	< [0.2,0.6], [0.2,0.4], [0.6,0.8], [0.1,0.7]>	< [0.2,0.5], [0.2,0.7], [0.6,0.8], [0.6,0.9]>
y_5	< [0.4,0.7], [0.1,0.4], [0.3,0.9], [0.6,0.8]>	< [0.1,0.5], [0.2,0.3], [0.4,0.8], [0.6,0.9]>	< [0.4,0.4], [0.3,0.4], [0.5,0.8], [0.6,0.7]>	< [0.4,0.5], [0.3,0.4], [0.3,0.5], [0.6,0.6]>	< [0.6,0.7], [0.2,0.4], [0.4,0.8], [0.5,0.7]>

Step 2. The score of the truth-membership degrees $S_{T_{\eta(e_j)}}(y_i)$ for (η ,E) is exposed in Table 2.

X/E	e_1	e_2	e_3	e_4	e_5
y_1	0.1	0.7	0.5	-0.1	-.0.3
y_2	-0.9	-2.3	0.5	0.6	3.2
y_3	0.1	2.7	-1	1.4	-3.3
y_4	-0.4	1.1	0	-0.6	-.0.8

y5	1.1	-0.8	-1.6	-0.1	1.7

The score of the contradiction-membership degrees $S_{C_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 3.

Table 4. Tabular illustration of the score of contradiction-membership degree

X/E	e1	e2	e3	e4	e5
y1	-1.4	-0.8	-1	-1	-1.8
y2	1.1	-0.8	-0.5	-0.6	-1.3
y3	1.1	-0.8	-0.5	-1.1	1.2
y4	2.6	2.7	-1	-2.2	-0.3
y5	1.1	-1.3	-0.5	-1.1	-1.8

The score of the unknown-membership degrees $S_{U_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 4.

Table 5. Tabular illustration of the score of unknown-membership degree

X/E	e1	e2	e3	e4	e5
y1	1.1	1.7	1.5	0.9	0.7
y2	3.1	1.7	3	0.4	2.2
y3	3.1	2.2	2.5	2.9	1.7
y4	3.1	1.7	2.5	2.4	2.2
y5	1.6	2.2	2.5	-0.6	1.2

The score of the false-membership degrees $S_{F_{\eta(e_j)}}(y_i)$ for (η,E) is exposed in Table 5.

Table 6. Tabular illustration of the score of false-membership degree

X/E	e1	e2	e3	e4	e5
x1	2.1	2.7	2.5	1.9	1.7
x2	4.1	4.7	3	1.4	3.2
x3	3.1	3.7	3	3.9	-1.3
x4	3.1	1.7	3.5	-0.6	2.7
x5	2.6	3.7	2.5	1.4	1.2

Step 3. By using Table 3 to Table 6, the score □ $\eta_{(ej)}$ (x_i) for (η,E) is exhibited in Table 6.

Table 7. Tabular illustration of the score □ $\eta_{(ej)}$ (x_i).

X/E	e1	e2	e3	e4	e5
x1	1.9	4.3	3.5	1.7	0.4
x2	7.4	3.3	6	1.8	7.3
x3	7.4	7.8	4	6	-1.7
x4	8.4	7.2	5	-1	3.8
x5	6.4	3.8	2.9	-0.4	2.3

Step 4. Now, we calculate the overall score given as:

$v_1 = 11.8$, $v_2 = 25.8$, $v_3 = 23.5$, $v_4 = 23.4$, $v_5 = 15$.

Step 5. Thus, $v_k = \max_{yi\in Y} \{ v_1, v_2, v_3, v_4, v_5\} = v_2$. Therefore, v_2 is the optimal choice object for the decision maker. If v_2 is not available in the market, then he/she will choose v_3 or v_4 .

Step 6. As there is no tie in the optimal choice so there is no need to reassess data in the given problem.

13. Conclusion and Future work

In conclusion, this study makes a significant contribution to the field of neutrosophic fuzzy matrices by introducing two innovative concepts: range symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices and kernel symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices. Through rigorous characterization and analysis, we establish essential conditions for the kernel-symmetry of Interval-Valued Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices and elucidate the intricate relationships between range-symmetry and kernel-symmetry. The development of the concepts of Kernel and k-Kernel Symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices enhances the theoretical understanding of these matrix types, while the derivation of basic results underscores the distinctions between k-symmetry and k-Kernel symmetry. Additionally, the study sheds light on the interplay between kernel symmetry, k-kernel symmetry, and the Moore-Penrose inverse, providing valuable insights into their applications in practical scenarios. By proposing an algorithm for multi-criteria decision-making using Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices, validated with a real-world example, we demonstrate the practical implications of our findings. Overall, this research lays a robust foundation for further exploration in the neutrosophic fuzzy domain and highlights the potential of these matrix concepts in addressing complex decision-making challenges. Future work can explore several avenues to enhance the findings of this study. First, researchers could extend the concepts of Range-Symmetric and Kernel-Symmetric Interval-Valued Quadri Partitioned Neutrosophic Fuzzy Matrices to encompass higher-order matrices or other fuzzy structures, thereby deepening our understanding of their properties. Additionally, practical applications of the developed algorithms in diverse real-world multi-criteria decision-making scenarios, such as finance and healthcare, could validate their effectiveness. There is also potential to create numerical methods and computational tools to facilitate efficient calculations and manipulations of these matrices, making them more accessible for practical use. Furthermore, theoretical extensions investigating relationships between various classes of neutrosophic fuzzy matrices could provide insights into their interconnectedness. Lastly, combining range symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices and kernel symmetric Interval valued Quadri Partitioned Neutrosophic Fuzzy matrices with other mathematical models, like interval-valued fuzzy sets, may yield hybrid approaches that better capture uncertainty and imprecision in decision-making processes.

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