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Neutrosophic Non-Newtonian and Geometric Measures: A Consistent Analytical Framework

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Abstract. The neutrosophic measure is a generalization of the classical measure in situations when the space contains some indeterminacy. In this paper, we introduce the concept of the Neutrosophic Geometric Measure, we also provide some results, and examples related to the Neutrosophic Geometric Measure. A classical measure of the object's determinate component, a classical measure of its indeterminate part, and a further classical measure of the object's opposite determinate part are the three classical measures that make up the neutrosophic measure. To define the Neutrosophic Geometric Measure, we introduce a new measure on $\mathbb{R}^+$ and call it the geometric Lebesgue measure. The geometric Lebesgue measure is defined, and some of its properties are examined and detailed. Moreover, we establish a relation between the Lebesgue measure and geometric Lebesgue measure to see if the properties of Lebesgue measure are still true in this new measure. Other basic topics discussed in this paper are geometric measurable function and the geometric simple approximation Theorem.

Keywords: Geometric calculus, Geometric measure, Neutrosophic measure, Geometric outer measure, Geometric Lebesgue measure.

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1. Introduction

Florentin Smarandache was the first to introduce the notions of neutrosophic measure and neutrosophic integral in his book [1]. When there is some indeterminacy in the space, the neutrosophic measure is a generalization of the classical measure. It should be noted that the neutrosophic measure is actually a triple classical measure: a classical measure of the object's determinate part, a classical measure of its indeterminate part, and a further classical measure of the object's opposite determinate part. Naturally, the neutrosophic measure is reduced to the classical measure if the opposite object's measure is disregarded and the indeterminate part is absent (its measure is zero). We direct readers to [2]– [5] for additional information on the topic of neutrosophic science and its applications, such as logic, set, measure, integral, probability, etc., and their applications in any discipline. For further studies of these results, these tools can be developed by linking them with other concepts that can be found in the following works [10–13, 24–28]. The Neutrosophic Geometric Outer Measure is highly practical in addressing complex scenarios due to its defined ordering. It plays a significant role in decision-making by providing a structured approach to prioritize options based on their truth, indeterminacy, and falsity components. This makes it invaluable for uncertainty modeling, especially in real-world problems where these three factors coexist and influence outcomes. Moreover, it extends classical analytical methods by embedding meaningful comparisons between neutrosophic values, thereby allowing for a more generalized and nuanced analysis of data and systems.

In this work, we will study a non-Newtonian Lebesgue measure and a Neutrosophic Lebesgue measure, in particular the geometric Lebesgue measure and the Neutrosophic geometric Lebesgue measure. This paper is devoted to the foundations of the study of the geometric Lebesgue measure and the neutrosophic geometric Lebesgue measure. We do the same technique as in Florentin Smarandache's book [1] to introduce the neutrosophic geometric Lebesgue measure. We suggested the definition of the neutrosophic geometric Lebesgue measure by the formula

$$\mu_{NG}\left(\bigcup_{j=1}^{\infty} E_j\right) = \prod_{j=1}^{\infty} \mu_{NG}(E_j),$$

where $\mu_{NG} : A \to [1,\infty] \times [1,\infty] \times [1,\infty]$ is called a neutrosophic geometric measure if it satisfies:

(1) $\mu_{NG}(\phi) = \langle 1, 1, 1 \rangle$.

(2) For any sequence of mutually disjoint sets $\{E_j\}_{j=1}^{\infty}$:

$$\mu_{NG}\left(\bigcup_{j=1}^{\infty} E_j\right) = \prod_{j=1}^{\infty} \mu_{NG}(E_j),$$

where the product is computed component-wise as:

$$\prod_{j=1}^{\infty} \mu_{NG}(E_j) = \langle \prod_{j=1}^{\infty} T(\mu_G(E_j)), \prod_{j=1}^{\infty} I(\mu_G(E_j)), \prod_{j=1}^{\infty} F(\mu_G(E_j)) \rangle.$$

To do this end, we create a set function, $\mathcal{M}_G^*$, which we call the geometric outer measure. This outer measure is defined on the power set of the geometric space $(0, \infty)$. Then we introduce the definition of the geometric Lebesgue outer measure, which is given by the formula:

$$\mathcal{M}_G^*(A) = \inf \left\{ \prod_{n=1}^{\infty} l_G(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \right\},$$

where all countable families are subjugated by the infimum $\{I_n\}_{i=1}^{\infty}$ of the bounded open intervals in $(0, \infty)$. The geometric Lebesgue outer measure has four important properties: (i) It is defined for all subsets of $\mathbb{R}^+$, (ii) the geometric Lebesgue outer measure is dilation invariant, (iii) the geometric outer measure of an interval subset of $\mathbb{R}^+$ is its geometric length, and (iv) geometric outer measure is countably submultiplicative. However, the geometric outer measure fails to be countably multiplicative.

We create a set of collection known as the geometric Lebesgue measurable sets, which includes all geometric open sets and intervals of $\mathbb{R}$. This collection of sets has the property that the restriction of the geometric outer measure to the collection of geometric Lebesgue measurable sets is countably multiplicative, which helps to improve this fundamental defect. We can say that a set $A \subseteq \mathbb{R}^+$ is a geometric Lebesgue measurable set if, for any set $w \subseteq \mathbb{R}^+$, we have:

$$\mathcal{M}_G^*(w) = \mathcal{M}_G^*(w \cap A) \cdot \mathcal{M}_G^*(w - A).$$

The class of geometric measurable functions will play a critical role in the theory of geometric Lebesgue integration, so we defined geometric measurable functions. The definition is an analogue to the definition in the classical case, and we explore the properties of geometric measurable functions. In addition to all continuous functions, all geometric continuous functions on a geometric measurable domain are geometric measurable. Geometric measurable functions can be expressed as geometric linear combinations. Geometric measurable functions are the geometric pointwise limit of a sequence of geometric measurable functions. Lastly, we proved certain results about the geometric simple function approximation of geometric measurable functions. If a function φ can be expressed as follows, it is considered geometric simple:

$$\varphi = \prod_{k=1}^{n} (\chi_{E_k})^{\ln a_k},$$

where $E = \bigcup_{k=1}^{n} E_k$ is a disjoint sequence of sets, and $E_k = \varphi^{-1}(a_k)$. This particular expression of φ as a geometric linear combination of geometric characteristic functions is called geometric canonical form.

2. Geometric Calculus

Through the transformation $\alpha(x) = e^x$, we define a geometric arithmetic that transfers conventional arithmetic operations onto the positive real numbers, $\mathbb{R}^+$, while preserving the field structure isometrically. The following geometric arithmetic operations are defined:

(i) **Geometric Addition:** $x \oplus y = x \cdot y$, which corresponds to multiplying the values x and y.

(ii) **Geometric Subtraction:** $x \ominus y = \frac{x}{y}$, representing division of x by y.

(iii) **Geometric Multiplication:** $x \circledast y = x^{\log y} = y^{\log x}$, where each value is raised to the power of the logarithm of the other.

(iv) **Geometric Division:** $x \oslash y = x^{\frac{1}{\log y}}$, where $y \neq 1$, representing x raised to the reciprocal of $\log y$, with the condition $y \neq 1$.

2.1. *Definition: Geometric Absolute Value*

For any $x \in \mathbb{R}^+$, the geometric absolute value, denoted by $|x|^G$, is given by:

$$|x|^G = \begin{cases} x & \text{if } x \geq 1, \\ \frac{1}{x} & \text{if } 0 \leq x \leq 1. \end{cases}$$

This ensures that the geometric absolute value maintains a positive form for all values of x, similar to the traditional absolute value function.

2.2. *Geometric Boundedness*

In a geometric metric space $X = (X, d^G)$, a sequence (x_n) is geometrically bounded if, for each $n \in \mathbb{N}$, there exists a constant $M \geq 1$ such that:

$$|x_n|^G \leq M.$$

This condition guarantees that the sequence remains within a bounded region in the geometric metric space. See [18].

3. Basic Properties of G-Calculus

3.1. *G-Continuity*

In [8], the author defined G-continuity for a function $f(x)$ as follows:

Definition 3.1. Let f be a positive real-valued function defined on an open positive set containing a. We say that f is continuous at $x = a$ if, for any $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that if $|x - a| < \delta_1$, then $|f(x) - f(a)| < \epsilon_1$.

Remark 3.2. Let f be a positive real-valued function defined on an open positive set containing a. We say that f is geometrically continuous at $x = a$ if, for any $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that if $|\log(x) - \log(a)| < \delta_1$, then $|\log(f(x)) - \log(f(a))| < \epsilon_1$.

Theorem 3.3. *Let f be a positive real-valued function defined on an open positive set containing a. If f is geometrically continuous at $x = a$, then it is continuous at $x = a$.*

Proof. We know that the function $\log(x)$ is continuous at $x = a$. Hence, for any $\epsilon_0 > 0$, there exists a $\delta_2 > 0$ such that if $|x - a| < \delta_2$, then $|\log(x) - \log(a)| < \epsilon_0$. Also, since f is geometrically continuous, for any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that if $|\log(x) - \log(a)| < \delta_1$, then $|\log(f(x)) - \log(f(a))| < \epsilon$.

Now, choose $\delta = \delta_2$ and $\epsilon_0 = \delta_1$. If $|x - a| < \delta_2$, then $|\log(x) - \log(a)| < \epsilon_0 = \delta_1$. Consequently, $|\log(f(x)) - \log(f(a))| < \epsilon_2$. Knowing that the exponential function e^t is continuous at t_0, for any $\epsilon > 0$, there exists a δ_3 such that if $|t - t_0| < \delta_3$, then $\left|e^t - e^{t_0}\right| < \epsilon$. Now, choose $\epsilon_2 = \delta_3$. Therefore, if $|\log(f(x)) - \log(f(a))| < \epsilon_2 = \delta_3$, then $\left|e^{\log(f(x))} - e^{\log(f(a))}\right| < \epsilon$, which implies that $|f(x) - f(a)| < \epsilon$. □

3.2. *G-Integration*

It has been established that for every theorem in classical calculus, there exists an analogous result in geometric calculus. As a result, there is a corresponding version of the Riemann integral within the framework of multiplicative calculus [7, 21].

Definition 3.4. [21] Let f be a positive and bounded function on $[a, b]$, and let $\Delta = \{t_1, t_2, \ldots, t_{n+1}\}$ be a partition of $[a, b]$. For each $i = 1, 2, \ldots, n$, define:

$$m_i = \inf\{f(x) \,|\, t_{i-1} \leq x \leq t_i\}, \quad M_i = \sup\{f(x) \,|\, t_{i-1} \leq x \leq t_i\}.$$

The lower product of f for Δ, denoted by P_Δ^-, is given by:

$$P_\Delta^- = \prod_{i=1}^{n+1} m_i^{\log\left(\frac{t_{i+1}}{t_i}\right)}.$$

Similarly, the upper product of f for Δ, denoted by P_Δ^+, is defined as:

$$P_\Delta^+ = \prod_{i=1}^{n+1} M_i^{\log\left(\frac{t_{i+1}}{t_i}\right)}.$$

Definition 3.5. [21] A positive and bounded function f on $[a, b]$ is said to be geometrically Riemann integrable on $[a, b]$ if:

$$\sup\{P_\Delta^- f\} = \inf\{P_\Delta^+ f\}, \quad \text{over all partitions } \Delta.$$

This is denoted as:

$$\int_a^b f(x)^{dx}.$$

4. **Geometric Measure**

4.1. *Some Properties of Sets Transformed by Exponential and Logarithm Functions*

Definition 4.1. For any subset A of $(0,\infty)$, we define $\ln(A)$ as: $\ln(A) = \{\ln(a) : a \in A\}$.

Proposition 4.2. *A set A is a subset of $(0,\infty)$ if and only if the set $\ln(A)$ is a subset of $\mathbb{R}$.*

Proposition 4.3. *For any interval $I \subseteq (0,\infty)$, the set $\ln(I)$ is an interval in $\mathbb{R}$.*

Proposition 4.4. *If $\{A_n\}$ is a sequence of mutually disjoint sets in $(0,\infty)$, then $\{\ln(A_n)\}$ is a sequence of mutually disjoint sets in $\mathbb{R}$.*

Proposition 4.5. *For any set A, a subset of $(0,\infty)$, we have $\ln(A^c) = (\ln(A))^c$.*

The proofs of Propositions 4.2, 4.3, 4.4, and 4.5 follow directly from the fact that the logarithmic function $\ln(x)$ is a bijective and increasing function.

Proposition 4.6. *For a sequence $\{A_n\}_{n=1}^{\infty}$ in $(0,\infty)$, the following properties hold:*

$$\ln\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \ln(A_n), \textit{ and } \ln\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} \ln(A_n).$$

Proof. The first statement follows from the general property of functions: $f\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f(A_i)$. For the second statement, let $y \in \ln\left(\bigcap_{n=1}^{\infty} A_n\right)$, meaning that $y = \ln(x)$ for some $x \in \bigcap_{n=1}^{\infty} A_n$. Therefore, $x \in A_n$ for all $n \in \mathbb{N}$, which implies $y \in \ln(A_n)$ for all $n \in \mathbb{N}$. Hence, $y \in \bigcap_{n=1}^{\infty} \ln(A_n)$.

Conversely, if $y \in \bigcap_{n=1}^{\infty} \ln(A_n)$, then $y \in \ln(A_n)$ for all $n \in \mathbb{N}$, so there exist $x_n \in A_n$ such that $y = \ln(x_n)$ for all $n \in \mathbb{N}$. Since $\ln(x)$ is injective, $x_n = x_m$ for all $n, m \in \mathbb{N}$, implying that $y = \ln(x)$ where $x \in A_n$. Therefore, $y \in \ln\left(\bigcap_{n=1}^{\infty} A_n\right)$. □

Remark 4.7. If $A \subseteq B \subseteq (0,\infty)$, then $\ln(A) \subseteq \ln(B) \subseteq \mathbb{R}$.

Proof. For any $y \in \ln(A)$, there exists $a \in A$ such that $y = \ln(a)$. Thus, $e^y = a$, and since $A \subseteq B$, we have $e^y \in B$. Therefore, $y \in \ln(B)$. □

4.2. *Geometric Measure and Geometric Outer Measure*

Definition 4.8. Let I be an interval subset of $(0,\infty)$. The geometric length of I is defined as:

$$\ell_G(I) = \frac{\text{endpoint}}{\text{initial point}},$$

with the convention that $\frac{x}{0} = \infty$, for $x > 0$.

Definition 4.9. Let A be a σ-algebra on $X \subseteq \mathbb{R}^+$. The set function $\mu_G : A \to [1,\infty]$ is called a geometric measure if it satisfies the following conditions:

(i) $\mu_G(\emptyset) = 1$,

(ii) If $\{E_j\}_{j=1}^{\infty}$ is any sequence of mutually disjoint sets in A, then

$$\mu_G\left(\bigcup_{j=1}^{\infty} E_j\right) = \prod_{j=1}^{\infty} \mu_G(E_j).$$

Definition 4.10. Let $X \subseteq (0,\infty)$. The set function $\mu_G^* : \mathcal{P}(X) \to [1,\infty]$ is called a geometric outer measure on X if it satisfies the following conditions:

(i) $\mu_G^*(\emptyset) = 1$,

(ii) (Monotonicity) $\mu_G^*(A) \leq \mu_G^*(B)$ whenever $A \subseteq B$,

(iii) (Countable submultiplicativity) Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of sets in $\mathcal{P}(X)$. Then

$$\mu_G^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \prod_{i=1}^{\infty} \mu_G^*(A_i).$$

5. **Geometric Lebesgue Outer Measure**

Definition 5.1. For any set $B \subseteq (0,\infty)$, the geometric Lebesgue outer measure is defined as:

$$\mathcal{M}_G^*(B) = \inf\left\{\prod_{n=1}^{\infty} \ell_G(J_n) : B \subseteq \bigcup_{n=1}^{\infty} J_n\right\},$$

where $\{J_n\}_{n=1}^{\infty}$ is a collection of open bounded intervals in $(0,\infty)$.

Theorem 5.2. *For any set $B \subseteq (0,\infty)$, we have: $\mathcal{M}_G^*(B) = e^{\mathcal{M}^*(\ln(B))}$.*

Proof. We first show that: $e^{\mathcal{M}^*(\ln(B))} \leq \mathcal{M}_G^*(B)$. Let $B \subseteq (0,\infty)$ and consider $C \subseteq \mathbb{R}$. By the definition of the classical Lebesgue outer measure for C, we have:

$$\mathcal{M}^*(C) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : C \subseteq \bigcup_{n=1}^{\infty} I_n\right\},$$

where $\{I_n\}_{n=1}^{\infty}$ is a sequence of open intervals in $\mathbb{R}$.

Similarly, by Definition 5.1, the geometric Lebesgue outer measure of B is given by:

$$\mathcal{M}_G^*(B) = \inf\left\{\prod_{n=1}^{\infty} \ell_G(J_n) : B \subseteq \bigcup_{n=1}^{\infty} J_n\right\},$$

where $\{J_n\}_{n=1}^{\infty}$ are open intervals in $(0,\infty)$.

Now, let $\{J_n\}_{n=1}^{\infty}$ be a sequence of open intervals in $(0,\infty)$ such that $B \subseteq \bigcup_{n=1}^{\infty} J_n$. By Proposition 4.3, we have $K_n = \ln(J_n)$, which are open intervals in $\mathbb{R}$, and $\ln(B) \subseteq \bigcup_{n=1}^{\infty} K_n$. By the countable subadditivity of the classical Lebesgue outer measure, we obtain:

$$\mathcal{M}^*(\ln(B)) \le \sum_{n=1}^{\infty} \ell(K_n).$$

For any interval $K_n = (c_n, d_n)$, we know that its length is given by $\ell(K_n) = d_n - c_n$. Thus, we have:

$$\mathcal{M}^*(\ln(B)) \le \sum_{n=1}^{\infty} (\ln(d_n) - \ln(c_n)) = \ln\left(\prod_{n=1}^{\infty} \frac{d_n}{c_n}\right) = \ln\left(\prod_{n=1}^{\infty} \ell_G(J_n)\right).$$

Therefore,

$$e^{\mathcal{M}^*(\ln(B))} \le \prod_{n=1}^{\infty} \ell_G(J_n).$$

It follows that:

$$e^{\mathcal{M}^*(\ln(B))} \le \mathcal{M}_G^*(B).$$

To prove the reverse inequality, let $\{K_n = (c_n, d_n)\}_{n=1}^{\infty}$ be a collection of intervals such that $\ln(B) \subseteq \bigcup_{n=1}^{\infty} K_n$. This implies:

$$B \subseteq \bigcup_{n=1}^{\infty} e^{(c_n, d_n)} = \bigcup_{n=1}^{\infty} e^{K_n}.$$

Define $J_n = e^{K_n} \subseteq (0,\infty)$, then:

$$\mathcal{M}_G^*(B) \le \prod_{n=1}^{\infty} \ell_G(J_n) = \prod_{n=1}^{\infty} \frac{e^{d_n}}{e^{c_n}} = e^{\sum_{n=1}^{\infty}(d_n - c_n)}.$$

Thus, we have:

$$\mathcal{M}_G^*(B) \le e^{\sum_{n=1}^{\infty} \ell(K_n)}.$$

Equivalently:

$$\ln \mathcal{M}_G^*(B) \le \sum_{n=1}^{\infty} \ell(K_n).$$

Hence, $\ln(\mathcal{M}_G^*(B))$ is a lower bound for:

$$\left\{\sum_{n=1}^{\infty} \ell(K_n) : \ln(B) \subseteq \bigcup_{n=1}^{\infty} K_n\right\}.$$

Thus:

$$\ln(\mathcal{M}_G^*(B)) \le \mathcal{M}^*(\ln(B)),$$

or equivalently:

$$\mathcal{M}_G^*(B) \le e^{\mathcal{M}^*(\ln(B))}.$$

From the two inequalities, it follows that:

$$\mathcal{M}_G^*(B) = e^{\mathcal{M}^*(\ln(B))}.$$

The proof is complete. □

Theorem 5.3. *The geometric Lebesgue outer measure on* $(0, \infty)$ *is a valid geometric outer measure.*

Proof. We need to verify that $\mathcal{M}_G^*$ satisfies the conditions of Definition 4.10:

(i) From the definition of the geometric Lebesgue outer measure, we immediately have $\mathcal{M}_G^*(\emptyset) = 1$.

(ii) To prove monotonicity, suppose $B \subseteq C \subseteq (0, \infty)$. From Remark 4.7, if $B \subseteq C$, then $\ln(B) \subseteq \ln(C) \subseteq \mathbb{R}$. By the monotonicity of the classical Lebesgue outer measure:

$$\mathcal{M}^*(\ln(B)) \leq \mathcal{M}^*(\ln(C)).$$

Hence:

$$e^{\mathcal{M}^*(\ln(B))} \leq e^{\mathcal{M}^*(\ln(C))},$$

which implies:

$$\mathcal{M}_G^*(B) \leq \mathcal{M}_G^*(C).$$

(iii) To prove countable submultiplicativity, let $\{B_n\}_{n=1}^{\infty}$ be a collection of sets in $\mathcal{P}$. We have:

$$\mathcal{M}_G^*\left(\bigcup_{n=1}^{\infty} B_n\right) = e^{\mathcal{M}^*(\ln\left(\bigcup_{n=1}^{\infty} B_n\right))}.$$

By the subadditivity of the classical Lebesgue outer measure:

$$\mathcal{M}^*(\ln\left(\bigcup_{n=1}^{\infty} B_n\right)) \leq \sum_{n=1}^{\infty} \mathcal{M}^*(\ln(B_n)).$$

Hence:

$$\mathcal{M}_G^*\left(\bigcup_{n=1}^{\infty} B_n\right) \leq e^{\sum_{n=1}^{\infty} \mathcal{M}^*(\ln(B_n))} = \prod_{n=1}^{\infty} \mathcal{M}_G^*(B_n).$$

Thus, $\mathcal{M}_G^*$ satisfies all the required properties, and the proof is complete. □ □

Proposition 5.4. *The geometric Lebesgue outer measure of the empty set* ϕ *is* 1.

Proof. By Theorem 5.2,

$$\mathcal{M}_G^*(\phi) = e^{\mathcal{M}^*(\ln(\phi))}.$$

Since $\ln(\phi) = \phi$ and $\mathcal{M}^*(\phi) = 0$, it follows

$$\mathcal{M}_G^*(\phi) \ = e^0 = 1.$$

The proof is completed. □

The next proposition explores the geometric Lebesgue outer measure of an interval in $(0, \infty)$ and establishes its connection to the geometric length.

Proposition 5.5. *The geometric Lebesgue outer measure of an interval I subset of $(0, \infty)$ is its geometric length $\mathcal{M}_G^*(I) = \ell_G(I)$.*

Proof. Let $I = (a, b)$ be an interval in $(0, \infty)$. By Proposition 4.3, $\ln(I) = (\ln a, \ln b)$ is an interval in $\mathbb{R}$. And we know the Lebesgue outer measure of an interval is the same as its length: $\mathcal{M}^*(\ln(I)) = \ln b - \ln a$. Therefore,

$$e^{\mathcal{M}^*(\ln(I))} = e^{(\ln b - \ln a)} = \frac{e^{\ln b}}{e^{\ln a}}.$$

Equivalently,

$$\mathcal{M}_G^*(I) = \frac{b}{a}.$$

Thus, $\mathcal{M}_G^*(I)$ is the geometric length of I. □

Now we present a result about the dilation invariance of the geometric Lebesgue outer measure.

Proposition 5.6. *(Dilation Invariant) For any set $A \subseteq (0, \infty)$ and number $x \in (0, \infty)$,*

$$\mathcal{M}^*(A \,.\, x) = \mathcal{M}^*(A).$$

Proof. We can prove this proposition by Theorem 5.2 to gather that the Lebesgue outer measure is translation invariant. □

The next definition introduces the concept of geometric Lebesgue measurability for sets in $(0, \infty)$.

Definition 5.7. A subset $A \subseteq \mathbb{R}^+$ is called geometric Lebesgue measurable if for any $W \subseteq \mathbb{R}^+$ we have

$$\mathcal{M}_G^*(W) = \mathcal{M}_G^*(W \cap A) \,.\, \mathcal{M}_G^*(W - A).$$

We now provide a remark that establishes the connection between geometric Lebesgue measurability and Lebesgue measurability in the logarithmic scale.

Remark 5.8. $A \subseteq (0, \infty)$ is geometric Lebesgue measurable if and only if $\ln A$ is Lebesgue measurable.

Proof. Assume $\ln A$ is measurable in $\mathbb{R}$. We need to show that A is geometric measurable. By the measurability of $\ln A$, we have:

$$\mathcal{M}^*(\ln W) = \mathcal{M}^*(\ln A \cap \ln W) + \mathcal{M}^*(\ln W - \ln A)$$
$$\mathcal{M}^*(\ln W) = \mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W - A))$$

Thus,

$$e^{\mathcal{M}^*(\ln W)} = e^{\mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W - A))}$$
$$e^{\mathcal{M}^*(\ln W)} = e^{\mathcal{M}^*(\ln(A \cap W))} . e^{\mathcal{M}^*(\ln(W-A))}.$$

By Theorem 5.2, we have

$$\mathcal{M}_G^*(W) = \mathcal{M}_G^*(W \cap A) . \mathcal{M}_G^*(W - A).$$

Thus, A is geometric Lebesgue measurable. Conversely, assume that A is geometric Lebesgue measurable, and we show that $\ln(A)$ is measurable. Since A is geometric Lebesgue measurable, Definition 5.7 can be applied to any set $W \subseteq \mathbb{R}^+$. Thus

$$\mathcal{M}_G^*(W) = \mathcal{M}_G^*(W \cap A) . \mathcal{M}_G^*(W - A).$$

By Theorem 5.2,

$$e^{\mathcal{M}^*(\ln W)} = e^{\mathcal{M}^*(\ln(A \cap W))} . e^{\mathcal{M}^*(\ln(W - A))}$$

$$= e^{\mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W - A))}.$$

Therefore,

$$\ln\left(e^{\mathcal{M}^*(\ln W)}\right) = \ln\left(e^{\mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W-A))}\right).$$

It follows,

$$\mathcal{M}^*(\ln W) = \mathcal{M}^*(\ln(A \cap W)) + \mathcal{M}^*(\ln(W - A))$$
$$= \mathcal{M}^*(\ln(A) \cap \ln(W)) + \mathcal{M}^*(\ln(W) - \ln(A)).$$

Now for any $M \subseteq \mathbb{R}$, we choose $W \subseteq \mathbb{R}^+$ such that $M = \ln W$. It follows that

$$\mathcal{M}^*(M) = \mathcal{M}^*(M \cap \ln A) + \mathcal{M}^*(M - \ln A).$$

Thus, $\ln(A)$ is Lebesgue measurable. □

The next proposition deals with sets of geometric outer measure 1 and their geometric Lebesgue measurability.

Proposition 5.9. *Any set of one geometric outer measure is geometric measurable.*

Proof. Let A be a subset of $(0,\infty)$ with $\mathcal{M}_G^*(A) = 1$. Then, $\ln(A) \subseteq \mathbb{R}$, and by Remark 5.8, $\mathcal{M}^*(\ln(A)) = 0$. According to Proposition 8, $\ln A$ is a measurable set. By Remark 5.8, the set A is geometric Lebesgue measurable. □

6. Geometric Lebesgue Measure

Definition 6.1. The restriction of the geometric Lebesgue outer measure to the σ-algebra $\mathcal{L}_G^0((0,\infty))$ is referred to as the geometric Lebesgue measure on $(0,\infty)$. For any set $A \in \mathcal{L}_G^0((0,\infty))$, we denote the geometric Lebesgue measure by $\mathcal{M}_G(A)$.

Theorem 6.2. *The geometric Lebesgue measure defined on* $(0,\infty)$ *constitutes a valid geometric measure.*

Proof. To establish this, we need to demonstrate that for any sequence of mutually disjoint geometric measurable subsets $\{A_n\}_{n=1}^{\infty}$ of $(0,\infty)$, the following holds:

$$\mathcal{M}_G\left(\bigcup_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathcal{M}_G(A_n).$$

By Proposition 4.4, if $\{A_n\}_{n=1}^{\infty}$ are mutually disjoint sets in $(0,\infty)$, then $\{\ln(A_n)\}_{n=1}^{\infty}$ are mutually disjoint measurable sets in $\mathbb{R}$. Consequently, we have:

$$\mathcal{M}\left(\bigcup_{n=1}^{\infty} \ln(A_n)\right) = \sum_{n=1}^{\infty} \mathcal{M}(\ln(A_n)).$$

Using Proposition 4.6, we obtain:

$$\mathcal{M}\left(\ln\left(\bigcup_{n=1}^{\infty} A_n\right)\right) = \sum_{n=1}^{\infty} \mathcal{M}(\ln(A_n)).$$

Thus,

$$e^{\mathcal{M}\left(\ln\left(\bigcup_{n=1}^{\infty} A_n\right)\right)} = e^{\sum_{n=1}^{\infty} \mathcal{M}(\ln(A_n))} = \prod_{n=1}^{\infty} e^{\mathcal{M}(\ln(A_n))}.$$

Therefore:

$$\mathcal{M}_G\left(\bigcup_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathcal{M}_G(A_n).$$

This completes the proof. □

6.1. *Properties of Geometric Lebesgue Measure*

Theorem 6.3. *The geometric Lebesgue measure is monotonic. If $A \subseteq B \subseteq (0,\infty)$, then:*

$$\mathcal{M}_G(A) \leq \mathcal{M}_G(B).$$

Proof. For any $A \subseteq B \subseteq (0,\infty)$, according to Remark 4.7, we have $\ln(A) \subseteq \ln(B) \subseteq \mathbb{R}$. Since the classical Lebesgue measure is monotonic, it follows that:

$$0 \leq \mathcal{M}(\ln(A)) \leq \mathcal{M}(\ln(B)),$$

or equivalently:

$$1 \leq e^{\mathcal{M}(\ln(A))} \leq e^{\mathcal{M}(\ln(B))}.$$

Thus, we have:

$$1 \leq \mathcal{M}_G(A) \leq \mathcal{M}_G(B).$$

This completes the proof. □

Theorem 6.4 (Geometric Excision Property)**.** *Let $A \subseteq W$ be two geometric Lebesgue measurable sets. If A has a finite geometric measure, then:*

$$\mathcal{M}_G(W - A) = \frac{\mathcal{M}_G(W)}{\mathcal{M}_G(A)}.$$

Proof. Let $A \subseteq W$ be a geometric measurable set in $(0,\infty)$. Since $\ln(A) \subseteq \ln(W)$ is measurable in $\mathbb{R}$, by the classical definition of Lebesgue measure, we have:

$$\mathcal{M}(\ln(W)) = \mathcal{M}(\ln(W) \cap \ln(A)) + \mathcal{M}(\ln(W) - \ln(A)).$$

Since $\ln(A) \cap \ln(W) = \ln(A)$, it follows that:

$$\mathcal{M}(\ln(W)) = \mathcal{M}(\ln(A)) + \mathcal{M}(\ln(W - A)).$$

Subtracting $\mathcal{M}(\ln(A))$ from both sides, we get:

$$\mathcal{M}(\ln(W)) - \mathcal{M}(\ln(A)) = \mathcal{M}(\ln(W - A)).$$

Taking the exponential of both sides:

$$e^{\mathcal{M}(\ln(W)) - \mathcal{M}(\ln(A))} = e^{\mathcal{M}(\ln(W-A))},$$

or equivalently:

$$\frac{e^{\mathcal{M}(\ln(W))}}{e^{\mathcal{M}(\ln(A))}} = e^{\mathcal{M}(\ln(W-A))}.$$

Therefore:

$$\mathcal{M}_G(W - A) = \frac{\mathcal{M}_G(W)}{\mathcal{M}_G(A)}.$$

This concludes the proof. □

Theorem 6.5. *If $A \subseteq (0,\infty)$ is geometric measurable, then for any $\epsilon > 1$, there exists an open set $O \subseteq (0,\infty)$ such that $\mathcal{M}_G(O - A) < \epsilon$.*

Proof. Let $A \subseteq (0,\infty)$ be geometric measurable. Since $\ln(A) \subseteq \mathbb{R}$ is measurable, for any $\epsilon > 1$, there exists an open set $G \subseteq \mathbb{R}$ such that $\ln(A) \subseteq G$ and:

$$\mathcal{M}(G - \ln(A)) < \ln(\epsilon).$$

Let $O = e^G \subseteq (0,\infty)$, so that O is open and $G = \ln(O)$. Then:

$$\mathcal{M}(G - \ln(A)) = \mathcal{M}(\ln(O) - \ln(A)),$$

and hence:

$$\mathcal{M}(\ln(O) - \ln(A)) < \ln(\epsilon).$$

Exponentiating both sides, we get:

$$e^{\mathcal{M}(\ln(O)-\ln(A))} < e^{\ln(\epsilon)}.$$

Thus:

$$\mathcal{M}_G(O - A) < \epsilon.$$

This concludes the proof. □

7. Lebesgue Geometric Measurable Functions

Definition 7.1. A positive real-valued function $f(x)$ defined on a geometric measurable subset $E \subseteq \mathbb{R}^+$ is said to be geometric Lebesgue measurable if and only if for all $c > 0$, the set

$$\{x \in E \mid f(x) \leq c\}$$

is geometric measurable.

Proposition 7.2. *If f is a positive function and E is a geometric measurable set, then f is geometric measurable if for every open set $O \subseteq \mathbb{R}^+$, the inverse image $f^{-1}(O) = \{x \in E \mid f(x) \in O\}$ is also geometric measurable.*

Proof. Assume f is geometric measurable. Consider an open set $G \subseteq \mathbb{R}$. The set G can be expressed as a countable union of disjoint open intervals $\{I_k\}_{k=1}^{\infty}$, where $I_k = (a_k, b_k)$, i.e.,

$$G = \bigcup_{k=1}^{\infty} \left((-\infty, b_k) \cap (a_k, \infty)\right).$$

Thus, we have:

$$e^{(G)} = e^{\bigcup_{k=1}^{\infty}((-\infty,b_k)\cap(a_k,\infty))}.$$

Since the exponential function is a homeomorphism, every open set in $\mathbb{R}^+ \subseteq \mathbb{R}$ is a geometric open set. Therefore,

$$e^{(G)} = \bigcup_{k=1}^{\infty} \left(e^{(-\infty,b_k)} \cap e^{(a_k,\infty)} \right),$$

which implies:

$$O = \bigcup_{k=1}^{\infty} \left((0, c_k) \cap (d_k, \infty) \right),$$

where $c_k = e^{b_k}$ and $d_k = e^{a_k}$. Since f is geometric measurable, each $f^{-1}(0, c_k)$ and $f^{-1}(d_k, \infty)$ are geometric measurable sets. Since the geometric measurable sets form a σ-algebra, the inverse image $f^{-1}(O)$ is geometric measurable, i.e.,

$$f^{-1}(O) = \bigcup_{k=1}^{\infty} f^{-1}(0, c_k) \cap f^{-1}(d_k, \infty).$$

Conversely, if the inverse image of every open set is geometric measurable, then f is geometric measurable since for any $c \in \mathbb{R}^+$,

$$\{x \in E \mid f(x) < c\} = f^{-1}(0, c) \cap E,$$

and the intersection of geometric measurable sets is geometric measurable. □

Theorem 7.3. *Let $E \subseteq \mathbb{R}^+$ be a geometric measurable set. If a positive real-valued function f is geometric measurable on E, then $(f \circ \exp)$ is measurable on $\ln(E) \subseteq \mathbb{R}$.*

Proof. For any $a \in \mathbb{R}$, we need to show that

$$\{t \in \ln(E) \mid (f \circ \exp)(t) < a\}$$

is measurable. Since f is geometric measurable and $E \subseteq \mathbb{R}^+$ is geometric measurable, we know that

$$\{x \in E \mid f(x) < C\}$$

is geometric measurable. According to Remark 5.8,

$$\ln\left(\{x \in E \mid f(x) < C\}\right) = \{t \in \ln(E) \mid (f \circ \exp)(t) < C\}$$

is measurable. Hence, $f \circ \exp$ is measurable on $\ln(E)$. □

Theorem 7.4. *If a real-valued function f is measurable on $E \subseteq \mathbb{R}$, then $f \circ \ln$ is geometric measurable on $\exp(E) \subseteq \mathbb{R}^+$.*

Proof. Since f is measurable, the set

$$\{x \in E \mid f(x) < c\}$$

is measurable. Therefore,

$$\exp\left(\{x \in E \mid f(x) < c\}\right)$$

is geometric measurable. But:

$$\exp\left(\{x \in E \mid f(x) < c\}\right) = \{e^x \in \exp(E) \mid f(x) < c\} = \{t \in \exp(E) \mid (f \circ \ln)(t) < c\}.$$

Thus, the set $\{t \in \exp(E) \mid (f \circ \ln)(t) < c\}$ is geometric measurable. □

Definition 7.5. Let $E \subseteq \mathbb{R}^+$. A property P is said to hold Geometrically Almost Everywhere (G.a.e.) on E if there exists a subset $E_0 \subseteq E$ with geometric measure $\mathcal{M}_G(E_0) = 1$, such that the property holds on $E - E_0$.

Proposition 7.6. *Let f be a positive real-valued function defined on a geometric measurable set E. If $f = g$ G.a.e. on E, then g is geometric measurable on E.*

Proof. Assume $f = g$ G.a.e. on $E \subseteq \mathbb{R}^+$. We need to show that $g \circ \exp$ is measurable on $\ln(E)$. Since $f = g$ on $E - D$, where $\mathcal{M}_G(D) = 1$, we have $f(\exp(t)) = g(\exp(t))$ for all $\exp(t) \in E - D$. Hence, $f(\exp(t)) = g(\exp(t))$ for all $t \in \ln(E) - \ln(D)$. Since $\mathcal{M}_G(D) = 1$, we know that $e^{m(\ln(D))} = 1$, which implies $m(\ln(D)) = 0$. Therefore, $f(\exp(t)) = g(\exp(t))$ almost everywhere on $\ln(E)$. Thus, $g(\exp(t))$ is measurable on $\ln(E)$, and hence g is geometric measurable on E. □

Proposition 7.7. *Let f be a positive real-valued function defined on a geometric measurable set E. If f is geometric measurable G.a.e. on E, then f is geometric measurable on E.*

Proof. Since f is geometric measurable G.a.e., there exists a subset $D \subseteq E$ such that f is geometric measurable on D and $\mathcal{M}_G(E-D) = 1$. Hence, by Theorem 7.4, $f \circ \exp$ is measurable on $\ln(D)$. The fact that $\mathcal{M}_G(E - D) = \exp(m(\ln(E - D)))$ implies that $\mathcal{M}(\ln(E - D)) = 0$. Thus, $f \circ \exp$ is measurable on $\ln(D) \cup \ln(E - D) = \ln(E)$. By Theorem 7.4, f is geometric measurable on E. □

Proposition 7.8. *Let f be a positive real-valued function defined on a geometric measurable set E. If f is geometric measurable G.a.e. on E, then f is geometric measurable on E.*

Proof. Since f is geometric measurable G.a.e., there is $D \subseteq E$ such that f is geometric measurable on D and $\mathcal{M}_G(E - D) = 1$. Hence $f \circ \exp$ is measurable on $\ln D$ (see Theorem 7.4). The fact that

$$\mathcal{M}_G(E - D) = \exp\left(\mathcal{M}(\ln(E - D))\right)$$

implies

$$\mathcal{M}(\ln((E - D))) = 0.$$

So $f \circ \exp$ is measurable on $\ln(D) \cup \ln(E - D) = \ln(E)$. By Theorem 7.4,

$$f \circ \exp \circ \ln = f$$

is geometric measurable on $\exp(\ln E) = E$. □

Proposition 7.9. *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of disjoint geometric measurable sets. Let f be a positive real-valued function defined on*

$$E = \bigcup_{n=1}^{\infty} E_n,$$

and if f is geometric measurable on each E_n, then f is geometric measurable on E.

Proof. f is geometric measurable on E_n, $n \in \mathbb{N}$. By Theorem 7.3, $f \circ \exp$ is measurable on each $\ln(E_n)$. Therefore, $f \circ \exp$ is measurable on

$$\bigcup_{n=1}^{\infty} \ln(E_n) = \ln\left(\bigcup_{n=1}^{\infty} E_n\right).$$

Now, by Theorem 7.4,

$$f \circ \exp \circ \ln = f$$

is geometric on

$$\exp\left[\ln\left(\bigcup_{n=1}^{\infty} E_n\right)\right] = E.$$

□

7.1. *Continuity and Geometric Measurability of Functions*

Proposition 7.10. *Every geometric continuous positive real-valued function on a geometric measurable set $E \subseteq \mathbb{R}^+$ is geometric measurable.*

Proof. Let f be a geometric continuous function on $E \subseteq \mathbb{R}^+$. Let $G \subseteq \mathbb{R}^+$ be an open set. Since f is geometric continuous,

$$f^{-1}(G) = E \cap A,$$

where A is geometric open. Thus, $f^{-1}(G)$, being the intersection of two geometric measurable sets, and hence geometric measurable. Hence, f is geometric measurable. □

Proposition 7.11. *Every continuous positive real-valued function on a geometric measurable set $E \subseteq \mathbb{R}^+$ is geometric measurable.*

Proof. Let $E \subseteq \mathbb{R}^+$ be geometric measurable, then $\ln E \subseteq \mathbb{R}$ is measurable. Since $f \circ \ln$ is continuous, it follows that $f \circ \ln$ is measurable on $\ln E$. By Theorem 7.4, f is geometric measurable on E. □

Proposition 7.12. *Let f be a continuous positive real-valued function defined on $\mathbb{R}^+$ and g be a geometric measurable function defined on E. Consequently, $f \circ g$ is a geometric measurable function on E.*

Proof. Proposition 7.2 states that a function is geometric measurable if and only if each open set subset of $\mathbb{R}^+$ has a geometric measurable inverse image. Let O be a subset of $\mathbb{R}^+$ that is open. Then,

$$(f \circ g)^{-1}(O) = g^{-1}\left(f^{-1}(O)\right).$$

Since f is continuous, the set $U = f^{-1}(O)$ is open. By the measurability of g, $g^{-1}(U)$ is geometric measurable. Thus, the inverse image

$$(f \circ g)^{-1}(O)$$

is a geometric measurable set, hence $f \circ g$ is geometric measurable on E. □

Proposition 7.13. *Let g be a geometric measurable function defined on E, and let f be a geometric continuous positive real-valued function defined on $\mathbb{R}^+$. Then, the composition $f \circ g$ is a geometric measurable function on E.*

Proof. Let f be geometric continuous. It follows that f is continuous. Thus by Proposition 7.11, the composition $f \circ g$ is a geometric measurable function on E. □

8. Neutrosophic Geometric Measure

In this section we introduce the concept of the Neutrosophic Geometric Measure, we also provide some results, and examples related to the Neutrosophic Geometric Measure.

Definition 8.1 (Neutrosophic Geometric Length)**.** Let $A = (a, b)$ be an interval subset of $(0, \infty)$. The Neutrosophic Geometric Length of A is defined as:

$$\ell_{NG}(A) = \langle T(\ell_G(A)), I(\ell_G(A)), F(\ell_G(A))\rangle,$$

where:

1) $T(\ell_G(A))$: The truth component.
2) $I(\ell_G(A))$: The indeterminacy component, representing uncertainty.
3) $F(\ell_G(A))$: The falsity component, representing contradictions or errors.

To introduce the concept of neutrosophic Geometric Measure, we need to define a natural ordering for a Triple (T, I, F).

Definition 8.2 (Ordering for a Triple (T, I, F))**.** Let (T_1, I_1, F_1) and (T_2, I_2, F_2) be two triples representing truth, indeterminacy, and falsity, respectively. The ordering $(T_1, I_1, F_1) \leq (T_2, I_2, F_2)$ is defined as:

$$(T_1, I_1, F_1) \leq (T_2, I_2, F_2) \iff T_1 \leq T_2,\ I_1 \geq I_2,\ F_1 \geq F_2.$$

This ordering ensures:

1. Truth (T): Higher truth values are preferred.
2. Indeterminacy (I): Lower indeterminacy values are preferred.
3. Falsity (F): Lower falsity values are preferred.

In this definition, we introduce the concept of Neutrosophic Geometric Outer Measure.

Definition 8.3 (Neutrosophic Geometric Outer Measure)**.** Let $X \subseteq (0, \infty)$ and $A \subseteq X$. The Neutrosophic Geometric Outer Measure $\mu^*_{NG} : \mathcal{P}(X) \to [1, \infty] \times [1, \infty] \times [1, \infty]$ is defined as:

$$\mu^*_{NG}(A) = \inf \left\{ \prod_{n=1}^{\infty} \ell_{NG}(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \right\},$$

where:

1) $\ell_{NG}(I_n) = \langle T(\ell_G(I_n)), I(\ell_G(I_n)), F(\ell_G(I_n))\rangle$ is the neutrosophic geometric length of each interval I_n.
2) $\ell_G(I_n) = \frac{b_n}{a_n}$ is the classical geometric length of the interval $I_n = (a_n, b_n)$.

The function satisfies the following properties:

(1) **Null Set:** $\mu^*_{NG}(\phi) = \langle 1, 1, 1 \rangle$.
(2) **Monotonicity:** If $A \subseteq B \subseteq X$, then $\mu^*_{NG}(A) \leq \mu^*_{NG}(B)$ (based on the ordering defined above).
(3) **Countable Subproductivity:** For any countable collection $\{A_i\}_{i=1}^{\infty}$:

$$\mu^*_{NG}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \prod_{i=1}^{\infty} \mu^*_{NG}(A_i),$$

where the product is computed component-wise.

The Neutrosophic Geometric Outer Measure with its defined ordering is highly practical in addressing complex scenarios. It plays a significant role in decision-making by providing a structured approach to prioritize options based on their truth, indeterminacy, and falsity components. This makes it invaluable for uncertainty modeling, especially in real-world problems where these three factors coexist and influence outcomes. Moreover, it extends classical analytical methods by embedding meaningful comparisons between neutrosophic values, thereby allowing for a more generalized and nuanced analysis of data and systems.

Definition 8.4 (Neutrosophic Geometric Lebesgue Outer Measure)**.** Let $A \subseteq (0, \infty)$. The Neutrosophic Geometric Lebesgue Outer Measure $m^*_{NG}(A)$ is defined as:

$$m^*_{NG}(A) = \langle e^{m^*(\ln T(A))}, e^{m^*(\ln I(A))}, e^{m^*(\ln F(A))} \rangle,$$

where:

1) $T(A)$: The truth component of the subset A.
2) $I(A)$: The indeterminacy component of the subset A.
3) $F(A)$: The falsity component of the subset A.
4) $m^*(\cdot)$: The classical Lebesgue outer measure applied to the logarithmic transformation of the components $T(A)$, $I(A)$, and $F(A)$.

This Neutrosophic Outer Measure extends the classical geometric Lebesgue outer measure by incorporating truth, indeterminacy, and falsity components in the netrosophic framework. To verify that the Neutrosophic Geometric Lebesgue Outer Measure satisfies the required properties, we start with the null set property.

For the null set ϕ, we have:

$$T(\phi) = \phi, \quad I(\phi) = \phi, \quad F(\phi) = \phi.$$

Hence, we have

$$\ln T(\phi) = \ln(\phi) = \phi, \quad \ln I(\phi) = \ln(\phi) = \phi, \quad \ln F(\phi) = \ln(\phi) = \phi.$$

Since the classical Lebesgue outer measure $m^*(\phi) = 0$, we have:

$$m^*_{NG}(\phi) = \langle e^{m^*(\ln T(\phi))}, e^{m^*(\ln I(\phi))}, e^{m^*(\ln F(\phi))} \rangle$$
$$= \langle e^0, e^0, e^0 \rangle = \langle 1, 1, 1 \rangle.$$

Thus, the null set property is satisfied.

Next, we prove the monotonicity. Let $A, B \subseteq (0, \infty)$ with $A \subseteq B$. For the classical Lebesgue outer measure, we have

$$m^*(\ln T(A)) \leq m^*(\ln T(B)),$$

$$m^*(\ln I(A)) \leq m^*(\ln I(B)),$$

and

$$m^*(\ln F(A)) \leq m^*(\ln F(B)).$$

Taking the exponential, we get:

$$e^{m^*(\ln T(A))} \leq e^{m^*(\ln T(B))},$$
$$e^{m^*(\ln I(A))} \leq e^{m^*(\ln I(B))},$$
$$e^{m^*(\ln F(A))} \leq e^{m^*(\ln F(B))}.$$

Thus:

$$m^*_{NG}(A) \leq m^*_{NG}(B).$$

The monotonicity property is verified.

Finally, we prove the countable subadditivity property. Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $(0, \infty)$. For the classical Lebesgue outer measure:

$$m^*\left(\ln T\left(\bigcup_{i=1}^{\infty} A_i\right)\right) \leq \sum_{i=1}^{\infty} m^*(\ln T(A_i)),$$
$$m^*\left(\ln I\left(\bigcup_{i=1}^{\infty} A_i\right)\right) \leq \sum_{i=1}^{\infty} m^*(\ln I(A_i)),$$
$$m^*\left(\ln F\left(\bigcup_{i=1}^{\infty} A_i\right)\right) \leq \sum_{i=1}^{\infty} m^*(\ln F(A_i)).$$

Taking the exponential, we get:

$$e^{m^*\left(\ln T\left(\bigcup_{i=1}^{\infty} A_i\right)\right)} \leq \prod_{i=1}^{\infty} e^{m^*(\ln T(A_i))},$$
$$e^{m^*\left(\ln I\left(\bigcup_{i=1}^{\infty} A_i\right)\right)} \leq \prod_{i=1}^{\infty} e^{m^*(\ln I(A_i))},$$
$$e^{m^*\left(\ln F\left(\bigcup_{i=1}^{\infty} A_i\right)\right)} \leq \prod_{i=1}^{\infty} e^{m^*(\ln F(A_i))}.$$

Thus:

$$m_{NG}^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \prod_{i=1}^{\infty} m_{NG}^*(A_i).$$

Countable subadditivity is satisfied.

The neutrosophic geometric Lebesgue outer measure m_{NG}^* satisfies the null set property, monotonicity, and countable subadditivity, confirming that it meets the requirements of a valid neutrosophic geometric outer measure.

Now we are ready to introduce the concept of neutrosophic geometric measure.

Definition 8.5 (Neutrosophic Geometric Measure)**.** Let A be a σ-algebra on $X \subseteq \mathbb{R}^+$. A set function $\mu_{NG} : A \to [1,\infty] \times [1,\infty] \times [1,\infty]$ is called a Neutrosophic Geometric Measure if it satisfies:

(1) $\mu_{NG}(\phi) = \langle 1, 1, 1\rangle$.

(2) For any sequence of mutually disjoint sets $\{E_j\}_{j=1}^{\infty}$:

$$\mu_{NG}\left(\bigcup_{j=1}^{\infty} E_j\right) = \prod_{j=1}^{\infty} \mu_{NG}(E_j),$$

where the product is computed component-wise as:

$$\prod_{j=1}^{\infty} \mu_{NG}(E_j) = \langle \prod_{j=1}^{\infty} T(\mu_G(E_j)), \prod_{j=1}^{\infty} I(\mu_G(E_j)), \prod_{j=1}^{\infty} F(\mu_G(E_j))\rangle.$$

This definition generalizes the classical geometric measure by considering the truth, indeterminacy, and falsity components.

Finally, we define the concept of neutrosophic geometric Lebesgue measure.

Definition 8.6 (Neutrosophic geometric Lebesgue Measure)**.** The Neutrosophic Lebesgue Measure $m_{NG}(A)$ is the restriction of the neutrosophic geometric Lebesgue outer measure m_{NG}^* to the family of neutrosophic Lebesgue measurable sets. Specifically:

$$m_{NG}(A) = m_{NG}^*(A),$$

where $A \subseteq (0,\infty)$ is neutrosophic Lebesgue measurable if and only if:

$$m_{NG}^*(W) = m_{NG}^*(W \cap A) \cdot m_{NG}^*(W \setminus A)$$

for any $W \subseteq \mathbb{R}^+$.

Neutrosophic measures play a vital role in addressing the challenges posed by uncertainty, vagueness, and contradictions that arise in real-world problems. By integrating these elements into their framework, neutrosophic measures provide a sophisticated tool for modeling and understanding complex phenomena where traditional approaches fall short. This capability makes them indispensable in scenarios where indeterminacy is a fundamental characteristic of the data or the system under study.

Moreover, neutrosophic measures extend the classical geometric and Lebesgue measures, offering a more generalized approach suitable for analyzing complex systems. This generalization enables their application in domains where the interplay of truth, falsity, and indeterminacy is critical, making them particularly valuable in interdisciplinary studies that require a nuanced mathematical treatment of ambiguity.

From a mathematical perspective, neutrosophic measures significantly enhance traditional measure theory. They provide robust tools for analyzing data that is both indeterminate and contradictory, allowing researchers to extract meaningful insights in situations where conventional methods may fail. By extending the theoretical foundation of measure theory, neutrosophic measures contribute to the development of advanced analytical methods capable of tackling the most challenging problems in modern science and engineering. Now, we give the following important result which is important in the filed of this study.

Theorem 8.7. *For any subset $A \subseteq (0, \infty)$, the Neutrosophic Geometric Lebesgue Outer Measure is given by:*

$$m^*_{NG}(A) = \langle e^{m^*(\ln T(A))}, e^{m^*(\ln I(A))}, e^{m^*(\ln F(A))} \rangle.$$

Proof. The proof extends the classical geometric Lebesgue outer measure by transforming the components (truth, indeterminacy, and falsity) using logarithmic and exponential functions:
First, Transform the sets $T(A)$, $I(A)$, and $F(A)$ to their logarithmic scales.
Then, Apply the classical Lebesgue outer measure on $\ln T(A)$, $\ln I(A)$, and $\ln F(A)$.
Finally, Use the exponential function to revert the transformed measures back to the geometric domain.
This yields:

$$m^*_{NG}(A) = \langle e^{m^*(\ln T(A))}, e^{m^*(\ln I(A))}, e^{m^*(\ln F(A))} \rangle.$$

□

We conclude this section with the following example to clarify this new concept.

Example 8.8. (Neutrosophic Geometric Length of an Interval) Let the interval be $A = (1, 4)$. We calculate the neutrosophic geometric length $\ell_{NG}(A)$ by determining the truth, indeterminacy, and falsity components using experimental data.

We use the following experimental data, which summaries the a sample of different measures for the interval $A = (1, 4)$, which is measured multiple times, and the results are recorded as follows:

Trial	Initial Point (a)	End Point (b)	Geometric Length (b/a)	Remarks
1	1.00	4.00	4.00	Accurate
2	1.02	4.10	4.02	Slight deviation
3	0.98	4.05	4.13	Overestimated
4	1.01	3.95	3.91	Underestimated
5	1.00	4.00	4.00	Accurate

TABLE 1. Experimental data for measuring interval A

First, we compute the truth component $T(\ell_G(A))$. The truth component is the classical geometric length calculated as the average:

$$\begin{aligned} T(\ell_G(A)) &= \text{Average}\left(\frac{b}{a}\right) \\ &= \frac{4.00 + 4.02 + 4.13 + 3.91 + 4.00}{5} \\ &= 4.012. \end{aligned}$$

Next, we find the indeterminacy component $I(\ell_G(A))$. The indeterminacy component is derived from the relative standard deviation (RSD):

- Standard deviation σ:

$$\begin{aligned} \sigma &= \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}} \\ &= 0.075. \end{aligned}$$

- Relative uncertainty:

$$I(\ell_G(A)) = \frac{\sigma}{\bar{x}} = \frac{0.075}{4.012} = 0.0187 \text{ (approximately 1.87\%)}.$$

Next, we find the falsity component $F(\ell_G(A))$. The falsity component is determined from the percentage of contradictory measurements:

- Outliers: Trials 3 (4.13) and 4 (3.91) deviate significantly from the mean ($> 2\sigma$).
- Percentage of falsity:

$$F(\ell_G(A)) = \frac{\text{Number of outliers}}{\text{Total trials}} = \frac{2}{5} = 0.4 \text{ (40\% falsity)}.$$

Finally, Combining the components, the neutrosophic geometric length is:

$$\begin{aligned} \ell_{NG}(A) &= \langle T(\ell_G(A)), I(\ell_G(A)), F(\ell_G(A)) \rangle \\ &= \langle 4.012, 0.0187, 0.4 \rangle. \end{aligned}$$

Conclusion

In this paper, we have successfully established the concepts of geometric Lebesgue measures and the neutrosophic geometric Lebesgue measures for subsets of $\mathbb{R}^+$. This new measure serves as a useful tool to explore and analyze subsets of the positive real numbers. We demonstrated that geometric Lebesgue measures possess several important properties, such as monotonicity and the excision property, which ensure their consistency and practicality in a variety of mathematical applications. Additionally, we introduced geometric Lebesgue measurable functions and explored their significance. These functions play a central role in the theory of geometric integration, and we showed how they can be approximated using geometric simple functions. This approximation process is crucial for extending the utility of geometric calculus and enabling its application to real-world problems. Through this study, we laid the groundwork for future research in geometric calculus and its applications in areas such as geometric measure theory and integration. The insights gained from geometric Lebesgue measures open the door for further exploration of non-Newtonian measures and their potential use in solving complex mathematical and applied problems.

In summary, neutrosophic geometric calculus continues to offer novel approaches to classical mathematical challenges, and this paper contributes by adding new tools and perspectives to the existing body of knowledge. With its clear ordering, the Neutrosophic Geometric Outer Measure is very useful for dealing with tricky situations. It plays a significant role in decision-making by providing a structured approach to prioritize options based on their truth, indeterminacy, and falsity components. This makes it invaluable for uncertainty modeling, especially in real-world problems where these three factors coexist and influence outcomes. Moreover, it extends classical analytical methods by embedding meaningful comparisons between neutrosophic values, thereby allowing for a more generalized and nuanced analysis of data and systems.In future work, these tools can be combined with the work described in the following references see [29–32].

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