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Innovative Efficiency Measurement Using Stackelberg Games and Neutrosophic Numbers

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Abstract: This paper addresses the challenge of measuring efficiency in systems with two components using Stackelberg game theory and neutrosophic numbers to handle data uncertainty. Unlike traditional Data Envelopment Analysis (DEA) models that treat Decision Making Units (DMUs) as black boxes, we propose a new neutrosophic Network DEA (NDEA) model to compute efficiency scores for each division. By applying Stackelberg game theory, we break down the system's efficiency scores into sub-system efficiencies, providing a more detailed assessment of performance. Our proposed model represents a significant contribution to the literature by developing a two-stage Network Slack Based Measure (NSBM) model. This model addresses conflicts arising from the dual role of intermediate measures and assumes a Stackelberg-game relationship between the two stages while ensuring continuity in the flow of links between them. Additionally, the model imposes a penalty on the follower's objective to discourage deviation from the leader's objectives, thereby enhancing feasibility robustness. To handle uncertainty within the model, we further refine it into a two-stage framework that leverages Pareto efficiency concepts to establish lower and upper bounds for DMU efficiencies. To validate our approach, we utilized data from the Iranian Airline Industry. This empirical study evaluates the efficiencies of 13 Iranian airlines, illustrating how our methodology effectively captures and quantifies operational efficiencies within uncertainty.

Keywords: Stackelberg game, neutrosophic numbers, data envelopment analysis, SBM

1. Introduction

Data Envelopment Analysis (DEA), developed by the revolutionary work of Charnes et al. [1], is a powerful tool for efficiency evaluation of peer decision-making units, commonly referred to as DMUs in the literature. These DMUs consume multiple inputs to produce multiple outputs. Traditional DEA models treat DMUs as black boxes and do not account for the internal structure of the DMUs. In simpler terms, traditional DEA models do not offer sufficient detail for the Decision Maker (DM) to identify the sources of inefficiency within the DMU that may arise from interactions among different stages of production [2]. To address this issue, scholars and researchers have developed Network DEA models to evaluate efficiency scores by considering the internal structure of DMUs. This has led to the formulation of intermediate measures that define the interaction between different stages of production. Intermediate products are produced from one stage and consumed by other stages during the production process. Therefore, this variable plays a dual role as both an output from one stage and an input to another. The earlier stage focuses on maximizing its output efficiency, while the later stage emphasizes minimizing input. This dual role of intermediate measures introduces a challenge in evaluating efficiency. To address the conflicting nature of these measures, numerous scholars and researchers have proposed various approaches. For instance, Kao and Hwang [3] utilized a geometric multiplicative model, whereas Chen [4] opted for a weighted additive model. Broadly, the differences between models arise from the diverse assumptions regarding the incorporation of intermediate measures into constraints. In Kao's study [5] three categories were defined: independent, relational, and cooperative. The cooperative approach is further divided into two subtypes: one where the goal is set based on the current level and another aiming to optimize the goal. Kao [6] also described the relational scenario as non-cooperative. Moreover, Kao [6] discussed the cooperative scenario. Within academic literature, this approach is commonly recognized as the continuity condition, an idea innovatively introduced by Tone and Tsutsui [7]. Several researchers have expanded on the continuity concept, such as Tone and Tsutsui [7], to develop SBM network models for evaluating efficiency, including works by Gerami and Mozaffari [8]. Hassanzadeh and Mostafaee [9] proposed six scenarios for assessing intermediate measures in NDEA, focusing on controlling links at different stages. These scenarios include link control by earlier or later stages, both stages using non-cooperative methods, both stages adopting cooperative methods, or mixed approaches with cooperative and non-cooperative elements. Tone and Tsutsui [7] referred to link control methods with cooperative and free links at various stages.

Rasinojehdehi et al. [10] proposed a Network SBM model in which the role of intermediate measures is determined endogenously by the model to enhance the objective function.

According to the literature review conducted by Alves and Meza [11], game theory-based DEA approaches have been less explored in the DEA literature.

Game theory is rooted in the pioneering work of Von Neumann and Morgenstern [12]. It was further developed by John Forbes Nash who formulated the strategic interactions among rational DMs mathematically. Its applications span across various domains in social sciences, and it has been extensively utilized in fields associated with decision-making. Myerson [13] defined Game theory as the study of mathematical models for strategic interaction between rational DMs. In the following we review some studies in the literature based on the integration of game theory and DEA.

Liang et al. [14] proposed their two stage DEA model utilizing the concepts from noncooperative and cooperative games, illustrating with an example of a manufacturer and retailer scenario. Yu & Rakshit [15] integrated DEA and game theory in the airline industry to evaluate input and output targets. Their proposed model provides an impartial, rational negotiation between inputs and outputs, aiming to attain reasonable and optimal solutions for both inputs and outputs. Chang et al. [16] in their applied the first DEA-based Nash bargaining models to assist acquirer companies in acquiring their most desired target companies. Du et al. [17] applied Nash bargaining game model for measuring the efficiency of DMUs with two-stage structure. In their proposed model, the stages are treated as players negotiating for a better pay off, represented by the DEA ratio efficiency score. The approach is formulated as a cooperative game model. In their study they conclude that, in the presence of only one intermediate measure between the two stages, their proposed bargaining method produces same results as applying the standard DEA models to each stage separately; see also [18].

All the studies mentioned above are based on accurate data. In real-world scenarios, uncertainty is often an inseparable part of the available data. To address this issue, many scholars in the DEA literature applied fuzzy sets and established various fuzzy DEA approaches [19], [20]. Khalili-Damghani and Tavana [21] introduced a two-stage Stackelberg fuzzy DEA model to evaluate the efficiency of 52 chemistry departments in UK universities. Their framework integrates fuzzy data into the production process of a two-stage network, aiming to evaluate both system-level and process-level efficiencies for parallel systems functioning in a fuzzy environment using the Stackelberg game approach.

As highlighted by Voskoglou [22], no single universally optimal model exists for addressing uncertainty. The efficacy of a model is influenced by factors such as its structure, the data available, and the knowledge already established about the issue being addressed. Fundamentally, uncertainty encompasses the complete set of information available in a situation. While fuzzy set theory has been instrumental in addressing vagueness or partial inclusion of elements within a set, there are still certain situations that cannot be effectively managed using fuzzy sets. To address these limitations, Zadeh [23] introduced concepts such as interval-valued fuzzy sets and interval-valued intuitionistic fuzzy sets, designed to handle imprecise, incomplete, and indeterminate data, which are prevalent in real-world scenarios.

Florentin Smarandache [24], a mathematician and Romanian-American writer, conceptualized neutrosophic sets in 1995 as a broad extension of fuzzy sets and intuitionistic fuzzy sets. Wang et al. [25] developed Single-Valued Neutrosophic Sets (SVNSs) as a distinct subset of neutrosophic sets, which has spurred substantial research efforts in recent years focusing on scientific problems rooted in SVNSs and their extensions. These sets align closely with human reasoning and address the gaps in incomplete knowledge that arise from the environment [26].

A notable illustration of the flexibility of these sets is their application in performance analysis, which has substantially improved the precision in modeling uncertainty. In 2018, Edalatpanah [27] pioneered the integration of neutrosophic input and output data into the DEA model, marking the first application of neutrosophic sets in this context. This represented a significant breakthrough, offering a novel approach to handle both uncertainty and indeterminacy in DEA analysis. Since then, the innovative model has gained widespread attention among researchers, who have applied it to address a broad range of uncertainty-related challenges across diverse fields, further enhancing its relevance and applicability in complex decision-making environments. Kahraman et al. [28] applied an integrated approach combining a neutrosophic version of AHP and DEA models to assess the performance of private universities in Turkey. Edalatpanah and Smarandache [29] further advanced the Neu-DEA model by transforming the input-oriented Neu-DEA framework into a corresponding crisp DEA model through the use of natural logarithms, enhancing the model's computational feasibility and practical application. Edalatpanah [30] further expanded the Neu-DEA model by integrating Triangular Neutrosophic Numbers (TNNs), enhancing the model's ability to address uncertainty and indeterminacy in input-output data. Rasinojehdehi and Valami [31] proposed a neutrosophic network DEA model to assess the performance in Airline industry.

El-Demerdash et al. [32] proposed a Neutrosophic Input-Oriented DEA (NIODEA) model to address the uncertainty and imprecision present in real-world data for performance evaluation. The model incorporates both neutrosophic and deterministic input/output variables using a scoring function, allowing for a more accurate assessment of Decision-Making Units under uncertain conditions. This approach overcomes the limitations of traditional DEA models, which assume deterministic inputs and outputs, and provides decision-makers with a tool to optimize resource allocation. The model’s applicability is demonstrated through a practical example.

Mohanta et al. [33] propose a modified version of Khatter’s methodology for tackling Neutrosophic Data Envelopment Analysis (Neu-DEA) by incorporating TNNs to handle imprecise and ambiguous data. The study redefines the possibility mean for TNNs and simplifies the Neu-DEA model into a crisp Linear Programming (LP) problem with a risk parameter representing the decision-maker’s attitude towards risk. The approach ranks Decision-Making Units (DMUs) by determining their efficiency levels under various risk parameters, and its effectiveness is demonstrated through a numerical example.

In the literature on game theory-based network DEA, there's a noticeable absence of studies conducted with uncertain data, particularly within the neutrosophic environment. Also, the models that effectively combine game theory with network DEA while maintaining the continuity of linking activities among stages in an uncertain environment hasn't been developed yet. In this study, we propose a novel game theory-based NDEA approach within the SBM framework, following the continuity concept introduced by Tone and Tsutsui [7]. The proposed model contributes significantly to the literature by developing a two-stage network SBM model to address conflicts arising from the dual role of intermediate measures, assuming a Stackelberg-game relationship between the two stages while maintaining the continuity flow of links between stages. Additionally, the proposed model imposes a penalty on the follower's objective to discourage deviation from the leader's objective. Another contribution of this paper is the incorporation of inefficiencies related to link flows into the objective function, as highlighted by Alves and Meza [11] as a significant area for future research in NSBM model development. Moreover, the incorporation of uncertain data and formulation of the model in a neutrosophic environment are further contributions of this paper. This allows the DM to handle the ambiguity and uncertainty inherent in real-world data, enhancing the robustness and applicability of the proposed approach.

The remainder of this paper is organized as follows. Section 2 presents dome backgrounds and fundamental concepts required for understanding this study. Section 3 presents the proposed method. Subsequently, in Section 4, a case study is presented to illustrate the application and validate the proposed model. Finally, Section 5 concludes the paper.

2. Preliminaries

In this section we review some fundamental concepts id neutrosophic logic and Stackelberg model.

2.1. Neutrosophic logic

A SVNS A on U is of the form presented in equation (1) [24]:

$$A = \{(x, T(x), I(x), F(x)) : x \in U, T(x), I(x), F(x) \in [0,1], 0 \leq T(x) + I(x) + F(x) \leq 3\}, \tag{1}$$

In equation (4), T(x), I(x), and F(x) represent the degrees of truth, indeterminacy, and falsity membership of x in A, respectively, which are referred to as the neutrosophic components of x. The term "neutrosophy" originates from the combination of the adjective "neutral" and the Greek word "sophia" (wisdom). According to its introducer, Smarandache, it denotes "the knowledge of neutral thought". In other words, Indeterminacy is generally understood to encompass everything that lies between the opposites of truth and falsity [34]. For instance, let the set U represents the students of a class, where A denotes the SVNS of the good students in U. Each student x in U can be characterized by a neutrosophic triplet (t, i, f) with respect to A, where t, i, and f lie within the range [0, 1]. if (0.7, 0.1, 0.4) ∈ A, then it implies that there is a 70% probability for student x to be classified as a good student, a 10% probability for x's status to be indeterminate regarding A, and a 40% probability for x not being considered a good student.

Edalatpanah & Smarandache [29] introduced the Triangular Neutrosophic Number (TNNs) $\widetilde{Y} = \prec (r_1, r_2, r_3), (\omega, \theta, \chi) \succ$ with the following truth, indeterminacy, and falsehood membership functions of x as shown respectively in equations (2-a) to (2-c):

$$T_{\widetilde{Y}}(x) = \begin{cases} \frac{(x - r_1)}{(r_2 - r_1)}\omega, & r_1 \leq x < r_2, \\ \omega, & x = r_2, \\ \frac{(r_3 - x)}{(r_3 - r_2)}\omega, & r_2 \leq x < r_3, \\ 0, & \text{otherwise.} \end{cases} \tag{2-a}$$

$$I_{\widetilde{Y}}(x) = \begin{cases} \frac{(r_2 - x)}{(r_2 - r_1)}\theta, & r_1 \leq x < r_2, \\ \theta, & x = r_2, \\ \frac{(x - r_3)}{(r_3 - r_2)}\theta, & r_2 \leq x < r_3, \\ 1, & \text{otherwise.} \end{cases} \tag{2-b}$$

$$F_{\tilde{\Upsilon}}(x) = \begin{cases} \frac{(r_2 - x)}{(r_2 - r_1)}\chi, & r_1 \le x < r_2, \\ \chi, & x = r_2, \\ \frac{(x - r_3)}{(r_3 - r_2)}\chi, & r_2 \le x < r_3, \\ 1, & \text{otherwise.} \end{cases} \quad (2\text{-c})$$

Where, $0 \le T_{\tilde{\Upsilon}}(x) + I_{\tilde{\Upsilon}}(x) + F_{\tilde{\Upsilon}}(x) \le 3, x \in \tilde{\Upsilon}$.

Akram et al. (2019) introduced the aggregate coefficient $\varphi_{(\alpha,\beta,\gamma)}$ of $\tilde{\Upsilon} = \prec (r_1, r_2, r_3), (\omega, \theta, \chi) \succ$ which is a TNN as presented in equation (3).

$$\varphi_{(\alpha,\beta,\gamma)} = \left[\tilde{\Upsilon}^{L}_{(\alpha,\beta,\gamma)}, \tilde{\Upsilon}^{U}_{(\alpha,\beta,\gamma)}\right] = [r_1 + (r_2 - r_3)\hbar, r_3 - (r_3 - r_2)\hbar]. \quad (2\text{-c})$$

Where $\hbar$, is a parameter that reflects the decision-maker's attitude toward uncertainty

2.2. Stackelberg game

The Stackelberg model, introduced by Heinrich Freiherr von Stackelberg in 1934, is a strategic economic game involving a hierarchical setup where a leading firm and a following firm compete for market quantity, acting sequentially. Extensions to the original static two-person non-cooperative nonzero-sum Stackelberg game were made by Chen and Cruz [35] and Simaan and Cruz [36], who developed a dynamic version incorporating asymmetric information criteria. Aiyoshi and Shimizu [37] further proposed a generalized bi-level Stackelberg game structure, employing an interior penalty, or barrier method, to address non-linear min-max optimization challenges in such games.

Game theory serves as a widely recognized analytical tool for examining interactions among multiple participants. Its application spans numerous disciplines. Games can be categorized by features such as cooperation levels, simultaneity of actions, information availability, symmetry, and sum properties. This study centers on the Stackelberg game, a sequential non-cooperative game model characterized by perfect information.

In Stackelberg games, participants take actions in a predefined sequence. Typically, the first player to act is the leader, while the second player is referred to as the follower. The leader's decisions are often made with the follower's potential responses in mind. The follower, in turn, reacts to the leader's choices.

3. The Proposed Model

Without loss of generality, let us assume that the first stage takes on the role of the leader, thereby holding greater importance. In turn, the second stage acts as the follower, making decisions influenced by the leader's actions. Fig. 1 demonstrates the methodology adopted in this study, which is rooted in the Stackelberg game framework, to evaluate the efficiency scores of both the leader and the followers.

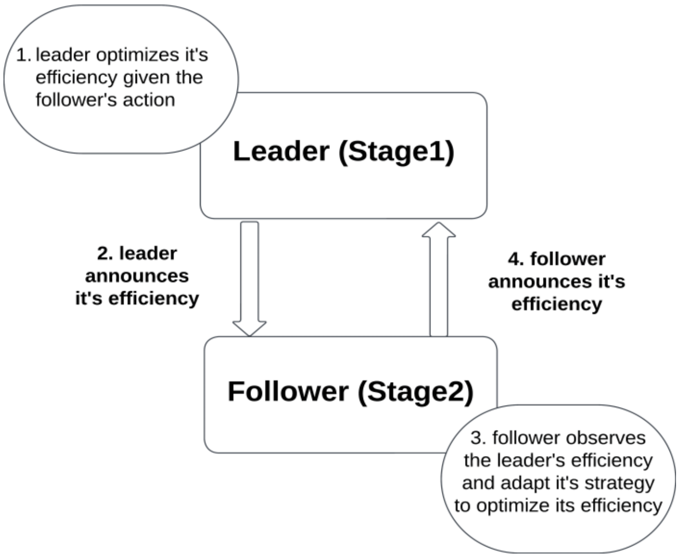

Figure 1. Methodology for measuring efficiency scores using the Stackelberg game.

In the first step, we optimize the leader's score without considering the followers, using the SBM model presented in equation (4). It is important to note that in this paper, we utilize a non-oriented model under the Constant Return to Scale (CRS) assumption for both the leader and follower.

$$\tilde{\psi}_p^{*1} = \min \frac{1-\frac{1}{m_1}(\sum_{i=1}^{m_1}\frac{\tilde{s}_{ip}^{1-}}{\tilde{x}_{ip}^{1}})}{1+\frac{1}{D}(\sum_{d=1}^{D}\frac{\tilde{s}_{dp}^{(1,2)+}}{\tilde{z}_{dp}^{(1,2)}})},$$

$$\sum_{j=1}^{n}\lambda_j^1 x_{ij}^1 + \tilde{s}_{ip}^{1-} = \tilde{x}_{ip}^1, \forall i \in I_k,\ \forall k, \tag{4}$$

$$\sum_{j=1}^{n}\lambda_j^1 \tilde{z}_{dp}^{(1,2)} - \tilde{s}_{dp}^{(1,2)+} = \tilde{z}_{dp}^{(1,2)} \text{as output from leader},\ \forall d \in l_{(k,h)}, \forall (k,h),$$

$$\lambda_j^1 \geq 0\ , s_{ip}^{1-} \geq 0\ \geq 0, s_{dp}^{(1,2)+} \geq 0, \quad (\forall j, k),$$

The model presented in equation (4) acts as a primary SBM model used for evaluating the efficiency of the leader, assumed to be the first stage, where $\tilde{s}_{ip}^{1-}$and $\tilde{s}_{dp}^{(1,2)+}$ represent the optimal solutions. Given the interconnected nature of the two subprocesses through intermediate measures, and with the follower's decisions expected to mirror those of the leader, it is imperative to include $\tilde{s}_{ip}^{1-}$and $\tilde{s}_{dp}^{(1,2)+}$ in measuring the efficiency of the follower. Now, we optimize the score of stage 2 while keeping the score of stage 1 at its optimal level ($\tilde{\psi}_p^1$=$\tilde{\psi}_p^{*1}$). Given the nature of the minimization problem, this constraint can be expressed as $\tilde{\psi}_p^1 \geq \tilde{\psi}_p^{*1}$. Model presented in equation (5) to equation (12) computes the score of stage 2.

$$\tilde{\psi}_p^{*2} = \min\left\{\frac{1-\frac{1}{m_{2+D}}(\sum_{i=1}^{m_1}\frac{\tilde{s}_{ip}^{2-}}{\tilde{x}_{ip}^{1}}+\sum_{d=1}^{D}\frac{\tilde{s}_{dp}^{(1,2)-}}{\tilde{z}_{dp}^{(1,2)}}))}{1+\frac{1}{s}(\sum_{d=1}^{D}\frac{\tilde{s}_{rp}^{2+}}{\tilde{y}_{rp}^{2}})}\right\} \tag{5}$$

$$\sum_{j=1}^{n}\lambda_j^1 \tilde{x}_{ij}^1 + \tilde{s}_{ip}^{1-} = \tilde{x}_{ip}^1, \forall i \in I_k,\ \forall k, \tag{6}$$

$$\sum_{j=1}^{n}\lambda_j^2 \tilde{y}_{rj}^2 - \tilde{s}_{rp}^{2+} = \tilde{y}_{rp}^2, \tag{7}$$

$$\sum_{j=1}^{n} \lambda_j^2 \tilde{z}_{jp}^{(1,2)} + g\tilde{s}_{dp}^{(1,2)-} = \tilde{z}_{dp}^{(1,2)}, \tag{8}$$

$$\sum_{j=1}^{n} \lambda_j^1 \tilde{z}_{jp}^{(1,2)} - (1-g)\tilde{s}_{dp}^{(1,2)+} = \tilde{z}_{dp}^{(1,2)}, \tag{9}$$

$$\sum_{j=1}^{n} \lambda_j^1 \tilde{z}_{dp}^{(1,2)} = \sum_{j=1}^{n} \lambda_j^2 \tilde{z}_{dp}^{(1,2)}, \tag{10}$$

$$\frac{1-\frac{1}{m_1}(\sum_{i=1}^{m_1}\frac{\tilde{s}_{ip}^{1-}}{\tilde{x}_{ij}^{1}})}{1+\frac{1}{D}(\sum_{d=1}^{D}\frac{\tilde{s}_{dp}^{(1,2)+}}{\tilde{z}_{dp}^{(1,2)}})} \geq \widetilde{\Psi}_p^{*1}, \quad \forall d \in l_{(k,h)}, \forall (k,h), \tag{11}$$

$$\lambda_j^1 \geq 0, \lambda_j^2 \geq 0 \ , \tilde{s}_{ip}^{1-} \geq 0 \ \geq 0, \tilde{s}_{dp}^{(1,2)+} \geq 0, S \geq 0, (\forall j, k), \tag{12}$$

Where g is a binary variable that determines the role of intermediate measure $\tilde{z}_d^{(1,2)}$ as either an input to the follower or an output from the leader. This enables the follower to improve its objective function independently of the leader's decisions and seek its optimum efficiency by just keeping the optimum efficiency of leader. With this assumption, the leader can anticipate the follower's best move before making its own

As mentioned earlier, the presented models operate under the Constant Returns-to-Scale (CRS) assumption. However, by incorporating relations $\sum_{j=1}^{n} \lambda_j^1 = 1$ into Model (4) and $\sum_{j=1}^{n} \lambda_j^1 = \sum_{j=1}^{n} \lambda_j^2 = 1$ into Model presented in equations (5-12), calculations can be changed to the Variable Returns-to-Scale (VRS) assumption.

To ensure the feasibility and robustness of the proposed model, we allow the follower a slight deviation from the leader's optimal strategy by imposing a penalty term on the follower's objective function. This penalty ensures that the leader's score closely matches the observed value. Any slight deviation from this, denoted by $S^{penalty} where\ S^{penalty} \geq 0$, incurs a penalty for the follower, with the severity determined by the factor π. This underscores the importance of closely aligning with the leader's performance. To implement this, we modify equations (11) to (13) and the objective function (5) to (14).

$$\frac{1-\frac{1}{m_1}(\sum_{i=1}^{m_1}\frac{\tilde{s}_{ip}^{1-}}{\tilde{x}_{ij}^{1}})}{1+\frac{1}{D}(\sum_{d=1}^{D}\frac{\tilde{s}_{dp}^{(1,2)+}}{\tilde{z}_{dp}^{(1,2)}})} + S^{penalty} \geq \widetilde{\Psi}_p^{*1}, \tag{13}$$

$$\widetilde{\Psi}_p^{*2} = \min \left\{ \frac{1-\frac{1}{m_{2+D}}(\sum_{i=1}^{m_1}\frac{\tilde{s}_{ip}^{2-}}{\tilde{x}_{ip}^{1}} + \sum_{d=1}^{D}\frac{\tilde{s}_{dp}^{(1,2)-}}{\tilde{z}_{dp}^{(1,2)}}))}{1+\frac{1}{s}(\sum_{d=1}^{D}\frac{\tilde{s}_{rp}^{2+}}{\tilde{y}_{rp}^{2}})} + \pi S^{penalty} \right\}, \tag{14}$$

Essentially, π serves as a factor determining the strength of the penalty applied to the follower's performance relative to the leader's efficiency. Increasing the value of π would result in a stronger penalty for the follower if they deviate from the leader's efficiency. On the other hand, decreasing the value of π would lead to a weaker penalty for such deviations.

Since all the data presented in the models for efficiency evaluation of both the leader and follower are of the neutrosophic data type, the resulting efficiency is also expressed as a neutrosophic number. Therefore, in the next section, we aim to convert the model into an interval-based model.

3.1. Converting the proposed model to interval-based model

As previously mentioned, the efficiency derived from the proposed model is expressed as a neutrosophic number. The neutrosophic values in the model are represented by convex neutrosophic numbers $\tilde{x}_{ij}^{tk}$, $\tilde{y}_{rj}^{tk}$ and $\tilde{z}_{dj}^{t(k,h)\beta}$, each characterized by truth, indeterminacy, and falsity membership functions as presented in equations (15) to (17):

$$\tilde{X}_{ij}^{q} = \left\{ \tilde{x}_{ij}^{q}, T_{\tilde{x}_{ij}^{tq}}, I_{\tilde{x}_{ij}^{tq}}, F_{\tilde{x}_{ij}^{tq}} \mid \tilde{x}_{ij}^{tq} \in S(\tilde{x}_{ij}^{tq}) \right\}, i \in \{1,2\}, \tag{15}$$

$$\tilde{Y}_{rj}^{q} = \left\{ \tilde{y}_{rj}^{q}, T_{\tilde{y}_{rj}^{q}}, I_{\tilde{y}_{rj}^{q}}, F_{\tilde{y}_{rj}^{q}} \mid \tilde{y}_{rj}^{q} \in S(\tilde{y}_{rj}^{tk}) \right\}, \tag{16}$$

$$\tilde{Z}_{dj}^{t(1,2)} = \left\{ \tilde{z}_{dj}^{t(1,2)}, T_{\tilde{z}_{dj}^{t(1,2)}}, I_{\tilde{z}_{dj}^{t(1,2)}}, F_{\tilde{z}_{dj}^{t(1,2)}} \mid \tilde{z}_{dj}^{t(1,2)} \in S(\tilde{z}_{dj}^{t(1,2)}) \right\}. \tag{17}$$

Where S(.) denotes the support set of the neutrosophic number, it is sufficient to determine the boundaries of the aggregate coefficient introduced in Definition 4. Therefore, for all the data, the upper and lower boundaries of the neutrosophic variable in a φ variation degree can be measured as presented in equations (18)-(20):

$$(X_{ij}^{q})_{\varphi}^{L} \le x_{ij}^{q} \le (X_{ij}^{q})_{\varphi}^{U}, \tag{18}$$

$$(Y_{ij}^{q})_{\varphi}^{L} \le y_{ij}^{q} \le (Y_{ij}^{q})_{\varphi}^{U}, \tag{19}$$

$$(Z_{dj}^{(1,2)})_{\varphi}^{L} \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_{\varphi}^{U}, \tag{20}$$

By utilizing the φ variation degree of neutrosophic numbers, the neutrosophic model of the leader can be simply transformed into the two-stage programming models represented in equations (21) and (22).

$$\begin{matrix} (\psi_p^{1*})_{\varphi}^{U} = \max \\ (X_{ij}^{1})_{\varphi}^{L} \le x_{ij}^{1} \le (X_{ij}^{1})_{\varphi}^{U} \\ (Z_{dj}^{(1,2)})_{\varphi}^{L} \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_{\varphi}^{U} \\ \forall d, \forall i \end{matrix} = \begin{cases} \min \dfrac{1 - \frac{1}{m_1}(\sum_{i=1}^{m_1} \frac{s_{ip}^{1-}}{x_{ip}^{1}})}{1 + \frac{1}{D}(\sum_{d=1}^{D} \frac{s_{dp}^{(1,2)+}}{z_{dp}^{(1,2)}})} \\ \sum_{j=1}^{n} \lambda_j^1 x_{ij}^1 + s_{ip}^{1-} = x_{ip}^1 \\ \sum_{j=1}^{n} \lambda_j^1 z_{dp}^{(1,2)} - s_{dp}^{(1,2)+} = z_{dp}^{(1,2)} \\ \lambda_j^1 \ge 0, s_{ip}^{1-} \ge 0 \ \ge 0, s_{dp}^{(1,2)+} \ge 0 \end{cases}, \tag{21}$$

$$\begin{matrix} (\psi_p^{1*})_{\varphi}^{U} = \min \\ (X_{ij}^{1})_{\varphi}^{L} \le x_{ij}^{1} \le (X_{ij}^{1})_{\varphi}^{U} \\ (Z_{dj}^{(1,2)})_{\varphi}^{L} \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_{\varphi}^{U} \\ \forall d, \forall i \end{matrix} = \begin{cases} \min \dfrac{1 - \frac{1}{m_1}(\sum_{i=1}^{m_1} \frac{s_{ip}^{1-}}{x_{ip}^{1}})}{1 + \frac{1}{D}(\sum_{d=1}^{D} \frac{s_{dp}^{(1,2)+}}{z_{dp}^{(1,2)}})} \\ \sum_{j=1}^{n} \lambda_j^1 x_{ij}^1 + s_{ip}^{1-} = x_{ip}^1 \\ \sum_{j=1}^{n} \lambda_j^1 z_{dp}^{(1,2)} - s_{dp}^{(1,2)+} = z_{dp}^{(1,2)} \\ \lambda_j^1 \ge 0, s_{ip}^{1-} \ge 0 \ \ge 0, s_{dp}^{(1,2)+} \ge 0 \end{cases}, \tag{22}$$

Similarly, the neutrosophic model of the follower can be transformed into the two-stage programming models represented in equations (23) and (24).

$$
\begin{aligned}
&(\psi_p^{2*})_\varphi^U = \underset{\substack{(X_{ij}^1)_\varphi^L \le x_{ij}^1 \le (X_{ij}^1)_\varphi^U \\ (Z_{dj}^{(1,2)})_\varphi^L \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_\varphi^U \\ \forall d,\forall i}}{\mathrm{Max}} =\\
&\begin{cases}
\min\left\{\dfrac{1-\frac{1}{m_2+D}\left(\sum_{i=1}^{m_1}\frac{s_{ip}^{2-}}{x_{ip}^1}+\sum_{d=1}^{D}\frac{s_{dp}^{(1,2)-}}{z_{dp}^{(1,2)}}\right)}{1+\frac{1}{s}\left(\sum_{d=1}^{D}\frac{s_{rp}^{2+}}{y_{rp}^2}\right)}\right\}, \\
\sum_{j=1}^{n}\lambda_j^1 x_{ij}^1 + s_{ip}^{1-} = x_{ip}^1, \\
\sum_{j=1}^{n}\lambda_j^2 y_{rj}^2 - s_{rp}^{2+} = y_{rp}^2, \\
\sum_{j=1}^{n}\lambda_j^1 z_{dp}^{(1,2)} - (1-g)s_{dp}^{(1,2)+} = z_{dp}^{(1,2)}, \\
\sum_{j=1}^{n}\lambda_j^2 z_{dp}^{(1,2)} + g s_{dp}^{(1,2)-} = z_{dp}^{(1,2)}, \\
\sum_{j=1}^{n}\lambda_j^1 z_{dp}^{(1,2)} = \sum_{j=1}^{n}\lambda_j^2 z_{dp}^{(1,2)}, \\
\dfrac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{s_{ip}^{1-}}{x_{ip}^1}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{s_{dp}^{(1,2)+}}{z_{dp}^{(1,2)}}\right)} \ge (\psi_p^{1*})_\varphi^U, \\
\lambda_j^1,\lambda_j^1 \ge 0\,;\, s_{ip}^{1-} \ge 0\,;\, s_{rp}^{2+} \ge 0, s_{dp}^{(1,2)+} \ge 0;\ s_{dp}^{(1,2)-};\, g\epsilon\{0,1\};\, S \ge 0,
\end{cases}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&(\psi_p^{2*})_\varphi^L = \underset{\substack{(X_{ij}^1)_\varphi^L \le x_{ij}^1 \le (X_{ij}^1)_\varphi^U \\ (Z_{dj}^{(1,2)})_\varphi^L \le z_{dj}^{(1,2)} \le (Z_{dj}^{(1,2)})_\varphi^U \\ \forall d,\forall i}}{\mathrm{Min}} =\\
&\begin{cases}
\min\left\{\dfrac{1-\frac{1}{m_2+D}\left(\sum_{i=1}^{m_1}\frac{s_{ip}^{2-}}{x_{ip}^1}+\sum_{d=1}^{D}\frac{s_{dp}^{(1,2)-}}{z_{dp}^{(1,2)}}\right)}{1+\frac{1}{s}\left(\sum_{d=1}^{D}\frac{s_{rp}^{2+}}{y_{rp}^2}\right)}\right\}, \\
\sum_{j=1}^{n}\lambda_j^1 x_{ij}^1 + s_{ip}^{1-} = x_{ip}^1, \\
\sum_{j=1}^{n}\lambda_j^2 y_{rj}^2 - s_{rp}^{2+} = y_{rp}^2, \\
\sum_{j=1}^{n}\lambda_j^1 z_{dp}^{(1,2)} - (1-g)s_{dp}^{(1,2)+} = z_{dp}^{(1,2)}, \\
\sum_{j=1}^{n}\lambda_j^2 z_{dp}^{(1,2)} + g s_{dp}^{(1,2)-} = z_{dp}^{(1,2)}, \\
\sum_{j=1}^{n}\lambda_j^1 z_{dp}^{(1,2)} = \sum_{j=1}^{n}\lambda_j^2 z_{dp}^{(1,2)}, \\
\dfrac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{s_{ip}^{1-}}{x_{ip}^1}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{s_{dp}^{(1,2)+}}{z_{dp}^{(1,2)}}\right)} = (\psi_p^{1*})_\varphi^L, \\
\lambda_j^1,\lambda_j^1 \ge 0\,;\, s_{ip}^{1-} \ge 0\,;\, s_{rp}^{2+} \ge 0, s_{dp}^{(1,2)+} \ge 0;\ s_{dp}^{(1,2)-};\, g\epsilon\{0,1\};\, S \ge 0,
\end{cases}
\end{aligned}
\tag{24}
$$

By comparing the two-stage models for both leader and follower, it becomes clear that $(\psi_p^{1*})_\varphi^L \le (\psi_p^{1*})_\varphi^U$ and $(\psi_p^{2*})_\varphi^L \le (\psi_p^{2*})_\varphi^U$. Therefore, the intervals $\left[(\psi_p^{1*})_\varphi^U, (\psi_p^{1*})_\varphi^U\right]$ and $\left[(\psi_p^{2*})_\varphi^U, (\psi_p^{2*})_\varphi^U\right]$, represent the efficiency score boundaries for the leader and follower, respectively, at the degree of variation for φ. Following the concept of Pareto efficiency, for a $DMUp$ with a specific aggregate coefficient $\varphi_{(\alpha,\beta,\gamma)} = \varphi$, the upper limit of efficiency occurs when the $DMUp$ under evaluation minimizes its input and maximizes its output, while other DMUs maximize their input and minimize their output. Conversely, the lower limit occurs when the $DMUp$ maximizes its input and minimizes its output, while other DMUs minimize their input and maximize their output. Models (25) and (26) measure the upper and lower bounds of the leader's efficiency at a given aggregate coefficient of φ, respectively, while Models (27) and (28) determine the lower and upper scores for the follower, respectively.

$$(\psi_p^{1*})_\varphi^U = \min \frac{1 - \frac{1}{m_1}\left(\sum_{i=1}^{m_1} \frac{(s_{ip}^{1-})_\varphi^L}{(X_{ip}^1)_\varphi^L}\right)}{1 + \frac{1}{D}\left(\sum_{d=1}^{D} \frac{(s_{dP}^{(1,2)+})_\varphi^U}{(Z_{dP}^{(1,2)})_\varphi^U}\right)},$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^1 (X_{ij}^1)_\varphi^U + \lambda_p^1 (X_{ip}^1)_\varphi^L + (s_{ip}^{1-})_\varphi^L = (X_{ip}^1)_\varphi^L, \tag{25}$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + \lambda_p^1 (Z_{dp}^{(1,2)})_\varphi^U - (s_{dp}^{(1,2)+})_\varphi^U = (Z_{dp}^{(1,2)})_\varphi^U,$$
$$\lambda_j^1 \geq 0,\ (s_{ij}^{1-})_\varphi^L \geq 0,\ (s_{dP}^{(1,2)+})_\varphi^U \geq 0,$$

$$(\psi_p^{1*})_\varphi^L = \min \frac{1 - \frac{1}{m_1}\left(\sum_{i=1}^{m_1} \frac{(s_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right)}{1 + \frac{1}{D}\left(\sum_{d=1}^{D} \frac{(s_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)},$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^1 (X_{ij}^1)_\varphi^L + \lambda_p^1 (X_{ip}^1)_\varphi^U + (s_{ip}^{1-})_\varphi^U = (X_{ip}^1)_\varphi^U, \tag{26}$$
$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^U + \lambda_p^1 (Z_{dp}^{(1,2)})_\varphi^L - (s_{dp}^{(1,2)+})_\varphi^L = (Z_{dp}^{(1,2)})_\varphi^L,$$
$$\lambda_j^1 \geq 0,\ (s_{ij}^{1-})_\varphi^U \geq 0,\ (s_{dP}^{(1,2)+})_\varphi^L \geq 0,$$

The models developed to calculate the efficiency boundaries for the leader and follower are nonlinear, particularly the models for the follower, which involve mixed integer nonlinear programming that is challenging to solve. The models for the leader can be easily transformed to the linear form by applying Charnes-Cooper transformation [38], but doing the same for the follower's models is challenging. In the following, we apply Charnes-Cooper transformation along with other techniques to change variables and linearize the models for evaluating the efficiency boundaries of the follower.

$$
\begin{gathered}
(\psi_p^{2*})_\varphi^U = \min\left\{\frac{1-\frac{1}{m_2+D}\left(\sum_{i=1}^{m_1}\frac{(s_{ij}^{1-})_\varphi^L}{(X_{ip}^1)_\varphi^L}+\sum_{d=1}^{D}\frac{(s_{dp}^{(1,2)-})_\varphi^L}{(Z_{dj}^{(1,2)})_\varphi^L}\right)}{1+\frac{1}{s}\left(\sum_{d=1}^{D}\frac{(s_{rp}^{2+})_\varphi^U}{(Y_{ip}^2)_\varphi^U}\right)}\right\}, \\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(X_{ij}^1)_\varphi^U+\lambda_p^1(X_{ip}^1)_\varphi^L+(s_{ip}^{1-})_\varphi^L=(X_{ip}^1)_\varphi^L, \\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Y_{rj}^2)_\varphi^L+\lambda_p^2(Y_{rp}^1)_\varphi^U-(s_{rp}^{2+})_\varphi^U=(Y_{rp}^2)_\varphi^U, \\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^1(Z_{dj}^{(1,2)})_\varphi^L+g(s_{dp}^{(1,2)-})_\varphi^L=(Z_{dj}^{(1,2)})_\varphi^L, \\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^2(Z_{dj}^{(1,2)})_\varphi^U-(1-g)(s_{dp}^{(1,2)+})_\varphi^U=(Z_{dj}^{(1,2)})_\varphi^U, \\
\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^1(Z_{dj}^{(1,2)})_\varphi^L=\sum_{\substack{j=1\\ j\neq p}}^{n}\lambda_j^2(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^2(Z_{dj}^{(1,2)})_\varphi^U, \\
\frac{1-\frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{(s_{ip}^{1-})_\varphi^L}{(X_{ip}^1)_\varphi^L}\right)}{1+\frac{1}{D}\left(\sum_{d=1}^{D}\frac{(s_{dP}^{(1,2)+})_\varphi^U}{(Z_{dP}^{(1,2)})_\varphi^U}\right)}=(\psi_p^{1*})_\varphi^U, \\
\lambda_j^1,\lambda_j^2\geq 0\,;\,(s_{ip}^{1-})_\varphi^U\geq 0\,;\,(s_{ip}^{2+})_\varphi^L,(s_{dp}^{(1,2)-})_\varphi^U\geq 0; \\
(s_{dp}^{(1,2)+})_\varphi^L;\, g\epsilon\{0,1\};\, S^U\geq 0.
\end{gathered}
\tag{27}
$$

$$(\psi_p^{2*})_\varphi^L = \min\left\{\frac{1-\frac{1}{m_2+D}(\sum_{i=1}^{m_1}\frac{(s_{ij}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}+\sum_{d=1}^{D}\frac{(s_{dp}^{(1,2)-})_\varphi^U}{(Z_{dj}^{(1,2)})_\varphi^U}))}{1+\frac{1}{s}(\sum_{d=1}^{D}\frac{(s_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L})}\right\},$$

$$\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^1(X_{ij}^1)_\varphi^L+\lambda_p^1(X_{ip}^1)_\varphi^U+(s_{ip}^{1-})_\varphi^U=(X_{ip}^1)_\varphi^U,$$

$$\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^2(Y_{rj}^2)_\varphi^U+\lambda_p^2(Y_{rp}^1)_\varphi^L-(s_{rp}^{2+})_\varphi^L=(Y_{rp}^2)_\varphi^L,$$

$$\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^1(Z_{dj}^{(1,2)})_\varphi^U+g(s_{dp}^{(1,2)-})_\varphi^U=(Z_{dj}^{(1,2)})_\varphi^U, \tag{28}$$

$$\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^2(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^2(Z_{dj}^{(1,2)})_\varphi^L-(1-g)(s_{dp}^{(1,2)+})_\varphi^L=(Z_{dj}^{(1,2)})_\varphi^L,$$

$$\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L+\lambda_p^1(Z_{dj}^{(1,2)})_\varphi^U=\sum_{\substack{j=1\\j\neq p}}^{n}\lambda_j^2(Z_{dj}^{(1,2)})_\varphi^U+\lambda_p^2(Z_{dj}^{(1,2)})_\varphi^L,$$

$$\frac{1-\frac{1}{m_1}(\sum_{i=1}^{m_1}\frac{(s_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U})}{1+\frac{1}{D}(\sum_{d=1}^{D}\frac{(s_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L})}=(\psi_p^{1*})_\varphi^L,$$

$$\lambda_j^1,\lambda_j^2\geq 0\,;(s_{ip}^{1-})_\varphi^U\geq 0\,;(s_{ip}^{2+})_\varphi^L,(s_{dp}^{(1,2)-})_\varphi^U\geq 0;\ (s_{dp}^{(1,2)+})_\varphi^L;$$
$$g\epsilon\{0,1\}.$$

3.2. Linearization of the proposed models

As mentioned earlier, linearizing the follower's models is challenging. In this subsection, we utilize the Charnes-Cooper transformation and some other techniques to change variables and linearize the models for evaluating the follower's efficiency boundaries. We linearize the model for measuring the lower bound score of the follower. The same approach can be applied to the leader. To do this, let's define the variable t as in equation (29):

$$\frac{1}{1+\frac{1}{s}(\sum_{d=1}^{D}\frac{t(s_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L})}=t, \tag{29}$$

Then the model outlined in equation (28) can be rewritten as presented in equation (30).

$$(\psi_p^{2*})_\varphi^L = \min\left\{t - \frac{1}{m_2 + D}\left(\sum_{i=1}^{m_1}\frac{t(s_{ij}^{2-})_\varphi^U}{(X_{ip}^2)_\varphi^U} + \sum_{d=1}^{D}\frac{t(s_{dp}^{(1,2)-})_\varphi^U}{(Z_{dj}^{(1,2)})_\varphi^U}\right)\right\},$$

$$t + \frac{1}{s}\left(\sum_{d=1}^{D}\frac{t(s_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L}\right) = 1,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} t\lambda_j^1 (X_{ij}^1)_\varphi^L + t\lambda_p^1 (X_{ip}^1)_\varphi^U + t(s_{ip}^{1-})_\varphi^U = t(X_{ip}^1)_\varphi^U,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} t\lambda_j^2 (Y_{rj}^2)_\varphi^U + t\lambda_p^2 (Y_{rp}^1)_\varphi^L - t(s_{rp}^{2+})_\varphi^L = t(Y_{rp}^2)_\varphi^L,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} t\lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + t\lambda_p^1 (Z_{dj}^{(1,2)})_\varphi^U + gt(s_{dp}^{(1,2)-})_\varphi^U = t(Z_{dj}^{(1,2)})_\varphi^U, \tag{30}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} t\lambda_j^2 (Z_{dj}^{(1,2)})_\varphi^U + t\lambda_p^2 (Z_{dj}^{(1,2)})_\varphi^L - (1-g)t(s_{dp}^{(1,2)+})_\varphi^L = t(Z_{dj}^{(1,2)})_\varphi^L,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} t\lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + t\lambda_p^1 (Z_{dj}^{(1,2)})_\varphi^U = \sum_{\substack{j=1\\ j\neq p}}^{n} t\lambda_j^2 (Z_{dj}^{(1,2)})_\varphi^U + t\lambda_p^2 (Z_{dj}^{(1,2)})_\varphi^L,$$

$$t - \frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{t(s_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right) - (\psi_p^{1*})_\varphi^L\left(t + \frac{1}{D}\left(\sum_{d=1}^{D}\frac{t(s_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)\right) \geq 0,$$

$$\lambda_j^1, \lambda_j^2 \geq 0\ ;\ (s_{ip}^{1-})_\varphi^U \geq 0\ ;\ (s_{ip}^{2+})_\varphi^L, (s_{dp}^{(1,2)-})_\varphi^U \geq 0;\ (s_{dp}^{(1,2)+})_\varphi^L,$$

$$g \in \{0,1\},$$

In the next step, we replace the nonlinear terms $g(S_{dp}^{(1,2)-})_\varphi^U$ and $(1-g)(S_{dp}^{(1,2)+})_\varphi^L$ respectively by $(h_{dp}^{(1,2)-})_\varphi^U$ and $(h_{dp}^{(1,2)+})_\varphi^U$ .Then we add the constraints (31)-(36) to the problem.

$$(h_{dp}^{(1,2)-})_\varphi^U = g(S_{dp}^{(1,2)-})_\varphi^U, \tag{31}$$

$$(h_{dp}^{(1,2)+})_\varphi^U = (1-g)(S_{dp}^{(1,2)+})_\varphi^L, \tag{32}$$

$$(S_{dp}^{(1,2)-})_\varphi^U - M(1-g) \leq (h_{dp}^{(1,2)-})_\varphi^U \leq (S_{dp}^{(1,2)-})_\varphi^U + M(1-g), \tag{33}$$

$$(h_{dp}^{(1,2)-})_\varphi^U \leq Mg, \tag{34}$$

$$(S_{dp}^{(1,2)+})_\varphi^U - Mg \leq (h_{dp}^{(1,2)+})_\varphi^U \leq (S_{dp}^{(1,2)+})_\varphi^U + Mg, \tag{35}$$

$$(h_{dp}^{(1,2)+})_\varphi^U \leq M(1-g), \tag{36}$$

Now, in the final stage of linearization, we define $t(s_{rp}^{2+})_\alpha^U = (S_{rp}^{2+})_\alpha^U$, $t(s_{dp}^{(1,2)-})_\alpha^L = (S_{dp}^{(1,2)-})_\alpha^L$, $t(s_{dp}^{(1,2)+})_\alpha^U = (S_{dp}^{(1,2)+})_\alpha^U$, $t(s_{ip}^{1-})_\alpha^L = (S_{ip}^{1-})_\alpha^L$، $t\lambda_j^1 = \Lambda_j^1$, $t\lambda_j^2 = \Lambda_j^2$ and substitute them into the model. Consequently, the equivalent linear representation of the model is obtained, as presented in Model (37).

$$(\psi_p^{2*})_\varphi^L = \min\left\{t - \frac{1}{m_2 + D}\left(\sum_{i=1}^{m_1} \frac{(S_{ij}^{2-})_\varphi^U}{(X_{ip}^2)_\varphi^U} + \sum_{d=1}^{D} \frac{(S_{dp}^{(1,2)-})_\varphi^U}{(Z_{dj}^{(1,2)})_\varphi^U}\right)\right\}$$

$$t + \frac{1}{s}\left(\sum_{d=1}^{D} \frac{(S_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L}\right) = 1,$$

$$\text{s.t.} \sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^1 (X_{ij}^1)_\varphi^L + \Lambda_p^1 (X_{ip}^1)_\varphi^U + (S_{ip}^{1-})_\varphi^U = t(X_{ip}^1)_\varphi^U,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^2 (Y_{rj}^2)_\varphi^U + \Lambda_p^2 (Y_{rp}^1)_\varphi^L - (S_{rp}^{2+})_\varphi^L = t(Y_{rp}^2)_\varphi^L,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + \Lambda_p^1 (Z_{dj}^{(1,2)})_\varphi^U + (h_{dp}^{(1,2)-})_\varphi^U = t(Z_{dj}^{(1,2)})_\varphi^U,$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^2 (Z_{dj}^{(1,2)})_\varphi^U + \Lambda_p^2 (Z_{dj}^{(1,2)})_\varphi^L - (h_{dp}^{(1,2)+})_\varphi^L = t(Z_{dj}^{(1,2)})_\varphi^L \tag{37}$$

$$(S_{dp}^{(1,2)-})_\varphi^U - M(1-g) \le (h_{dp}^{(1,2)-})_\varphi^U \le (S_{dp}^{(1,2)-})_\varphi^U + M(1-g),$$

$$(h_{dp}^{(1,2)-})_\varphi^U \le Mg,$$

$$(S_{dp}^{(1,2)+})_\varphi^U - Mg \le (h_{dp}^{(1,2)+})_\varphi^U \le (S_{dp}^{(1,2)+})_\varphi^U + Mg,$$

$$(h_{dp}^{(1,2)+})_\varphi^U \le M(1-g),$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^1 (Z_{dj}^{(1,2)})_\varphi^L + \Lambda_p^1 (Z_{dj}^{(1,2)})_\varphi^U = \sum_{\substack{j=1 \\ j\neq p}}^{n} \Lambda_j^2 (Z_{dj}^{(1,2)})_\varphi^U + \Lambda_p^2 (Z_{dj}^{(1,2)})_\varphi^L,$$

$$t - \frac{1}{m_1}\left(\sum_{i=1}^{m_1} \frac{(S_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right) - (\psi_p^{1*})_\varphi^L \left(t + \frac{1}{D}\left(\sum_{d=1}^{D} \frac{(S_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)\right) = 0,$$

$$\Lambda_j^1, \Lambda_j^2 \ge 0\, ;\, (S_{ip}^{1-})_\varphi^U \ge 0\, ;\, (S_{ip}^{2+})_\varphi^L, (S_{dp}^{(1,2)-})_\varphi^U \ge 0;\ (S_{dp}^{(1,2)+})_\varphi^L,$$

$$g \epsilon \{0,1\},$$

As discussed earlier, to ensure the feasibility robustness of the proposed model, we can impose a penalty on the follower for any deviation from the leader's efficiency, which is presented in Model (38).

$$(\psi_p^{2*})_\varphi^L = \min\left\{t - \frac{1}{m_2 + D}\left(\sum_{i=1}^{m_1}\frac{(S_{ij}^{2-})_\varphi^U}{(X_{ip}^2)_\varphi^U} + \sum_{d=1}^{D}\frac{(S_{dp}^{(1,2)-})_\varphi^U}{(Z_{dj}^{(1,2)})_\varphi^U}\right) + \pi S^{penalty}\right\},$$

$$t + \frac{1}{s}\left(\sum_{d=1}^{D}\frac{(S_{rp}^{2+})_\varphi^L}{(Y_{ip}^2)_\varphi^L}\right) = 1,$$

$$\text{s.t.} \sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^1(X_{ij}^1)_\varphi^L + \Lambda_p^1(X_{ip}^1)_\varphi^U + (S_{ip}^{1-})_\varphi^U = t(X_{ip}^1)_\varphi^U,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^2(Y_{rj}^2)_\varphi^U + \Lambda_p^2(Y_{rp}^1)_\varphi^L - (S_{rp}^{2+})_\varphi^L = t(Y_{rp}^2)_\varphi^L,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L + \Lambda_p^1(Z_{dj}^{(1,2)})_\varphi^U + (h_{dp}^{(1,2)-})_\varphi^U = t(Z_{dj}^{(1,2)})_\varphi^U,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^2(Z_{dj}^{(1,2)})_\varphi^U + \Lambda_p^2(Z_{dj}^{(1,2)})_\varphi^L - (h_{dp}^{(1,2)+})_\varphi^L = t(Z_{dj}^{(1,2)})_\varphi^L, \quad (37)$$

$$(S_{dp}^{(1,2)-})_\varphi^U - M(1-g) \leq (h_{dp}^{(1,2)-})_\varphi^U \leq (S_{dp}^{(1,2)-})_\varphi^U + M(1-g),$$

$$(h_{dp}^{(1,2)-})_\varphi^U \leq Mg,$$

$$(S_{dp}^{(1,2)+})_\varphi^U - Mg \leq (h_{dp}^{(1,2)+})_\varphi^U \leq (S_{dp}^{(1,2)+})_\varphi^U + Mg,$$

$$(h_{dp}^{(1,2)+})_\varphi^U \leq M(1-g),$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^1(Z_{dj}^{(1,2)})_\varphi^L + \Lambda_p^1(Z_{dj}^{(1,2)})_\varphi^U = \sum_{\substack{j=1\\ j\neq p}}^{n}\Lambda_j^2(Z_{dj}^{(1,2)})_\varphi^U + \Lambda_p^2(Z_{dj}^{(1,2)})_\varphi^L,$$

$$t - \frac{1}{m_1}\left(\sum_{i=1}^{m_1}\frac{(S_{ip}^{1-})_\varphi^U}{(X_{ip}^1)_\varphi^U}\right) - (\psi_p^{1*})_\varphi^L\left(t + \frac{1}{D}\left(\sum_{d=1}^{D}\frac{(S_{dP}^{(1,2)+})_\varphi^L}{(Z_{dP}^{(1,2)})_\varphi^L}\right)\right) - S^{penalty} = 0,$$

$$\Lambda_j^1, \Lambda_j^2 \geq 0\ ;\ (S_{ip}^{1-})_\varphi^U \geq 0\ ;\ (S_{ip}^{2+})_\varphi^L, (S_{dp}^{(1,2)-})_\varphi^U \geq 0;\ (S_{dp}^{(1,2)+})_\varphi^L,$$

$$g \in \{0,1\}; S^{penalty} \geq 0.$$

Increasing the value of π would result in a stronger penalty for the follower if they deviate from the leader's strategy. On the other hand, decreasing the value of π would lead to a weaker penalty for such deviations. Essentially, π serves as a factor determining the strength of the penalty applied to the follower's performance relative to the leader's efficiency.

In this section, we applied the leader-follower Stackelberg game to propose a novel network SBM model in a neutrosophic environment. To address the inherent uncertainty within the model, we convert it into a two-stage programming framework. By employing the concept of Pareto efficiency, we develop the upper and lower levels of efficiency within the φ variation degree of neutrosophic numbers. Subsequently, the models are transformed into linear programming to reduce complexity. In the next section, we present a case study to validate the proposed model.

4. An Application in the Airline Industry

In this section, we utilize the proposed model to evaluate the efficiency of airlines in Iran. Initially, we explore the interconnected divisions of these airlines, structured as a network. Subsequently, we introduce and implement the model under investigation. Finally, we present and analyze the obtained results. In the following we use the example of Rasinojehdehi and Bagherzadeh Valami [10] to illustrate our methodology.

In Rasinojehdehi and Bagherzadeh Valami [10] the production and consumption efficiencies of 13 Iranian airlines are investigated. The production division (Division 1) utilizes inputs such as the number of employees and number of aircraft to produce Tons-kilometers supplied per year and seat-kilometers supplied per year. These outputs serve as intermediate inputs for the consumption division. In the consumption division (Division 2), airlines utilize external inputs such as Airport coverage on domestic flights and Airport coverage on foreign flights, along with the intermediate inputs from the production division. From these inputs, they produce Passenger share of foreign flights, Passenger share of domestic flights, Ton-kilometers performed, and Passenger-kilometers performed. The results are summarized in Table 1, which displays the lower and upper bounds of efficiency scores for the leader and follower at various levels of truth, indeterminacy, and falsity. In this context, the production division is designated as the leader, with the consumption division serving as the follower. The results are obtained under constant return to scale assumption. It should be emphasized that, the penalty for deviation from the leader's efficiency is set to 10 ($\pi = 10$). This parameter underscores the significant consequence imposed on deviations from the optimal efficiency established by the leader.

Table 1. Scores of the leader and follower at different levels of uncertainty.

Level of Truth, indeterminacy and falsity	$\varphi_{(0,0.1,0.1)}$				$\varphi_{(0.2,0.7,0.9)}$				$\varphi_{(0.5,0.8,0.9)}$			
DMU	E_L^-	E_L^+	E_F^-	E_F^+	E_L^-	E_L^+	E_F^-	E_F^+	E_L^-	E_L^+	E_F^-	E_F^+
DMU1	0.69	1.00	0.79	1.00	0.74	1.00	0.79	1.00	0.83	1.00	0.88	1.00
DMU2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU3	0.66	0.83	0.74	0.81	0.76	0.83	0.74	0.81	0.85	0.93	0.94	1.00
DMU4	0.89	1.00	1.00	1.00	0.91	1.00	1.00	1.00	0.91	1.00	1.00	1.00
DMU5	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU6	0.24	0.39	0.31	0.42	0.36	0.49	0.51	0.68	0.44	0.59	0.58	0.79
DMU7	0.55	0.69	0.63	0.75	0.65	0.79	0.68	0.85	0.75	0.83	0.79	0.95
DMU8	0.52	0.90	0.68	1.00	0.56	1.00	0.79	1.00	0.65	1.00	0.81	1.00
DMU9	0.66	0.84	0.74	1.00	0.74	0.94	0.83	1.00	0.79	1.00	0.88	1.00
DMU10	0.70	0.85	0.83	0.95	0.81	0.88	0.91	0.95	1.00	1.00	1.00	1.00
DMU11	0.89	1.00	1.00	1.00	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU12	0.76	1.00	1.00	1.00	0.83	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DMU13	0.88	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Average	0.722	0.885	0.825	0.918	0.792	0.918	0.865	0.945	0.863	0.950	0.914	0.980

In Table 1, for each DMU, efficiency is measured through both lower and upper bounds for the Leader (E_L^- and E_L^+) and Follower ($E_F^-$and E_F^+) divisions, providing a nuanced view of performance across various operational scenarios. According to the results, DMU2 consistently achieves perfect efficiency scores of 1.00 across all φ values, demonstrating exceptionally stable performance where both lower and upper bounds align perfectly. In contrast, DMU6 shows more variability in efficiency scores, especially at higher levels of indeterminacy and falsity ($\varphi_{(0.5,0.8,0.9)}$). This variability suggests opportunities for operational adjustments to enhance performance reliability. DMU6 notably has the lowest scores for both the Leader and Follower divisions. Specifically, at $\varphi_{(0,0.1,0.1)}$, DMU6 scored 0.24 for the Leader division and 0.31 for the Follower division at the lower bound. These scores indicate significant challenges for DMU6 in achieving efficiency compared to other units in the study. These findings underscore the importance for DMU6 to address operational issues, improve resource management, and enhance overall performance. By doing so, DMU6 can better align with efficiency targets and improve its competitiveness in the studied environment. Fig. 2 to Fig. 5 compare the scores of the leader and follower at different levels of uncertainty.

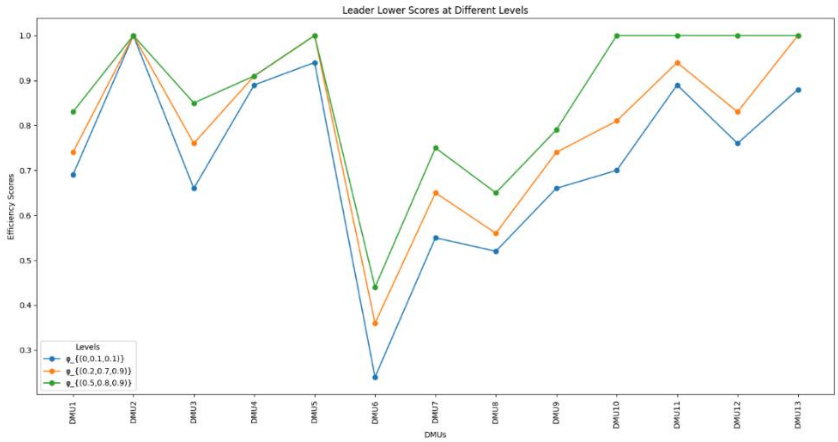

Figure 2. Leader's lower score at different levels of uncertainty.

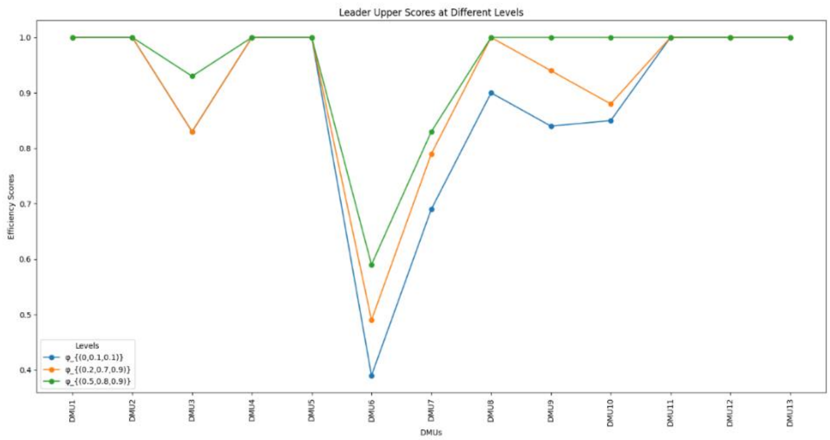

Figure 3. Leader's upper score at different levels of uncertainty.

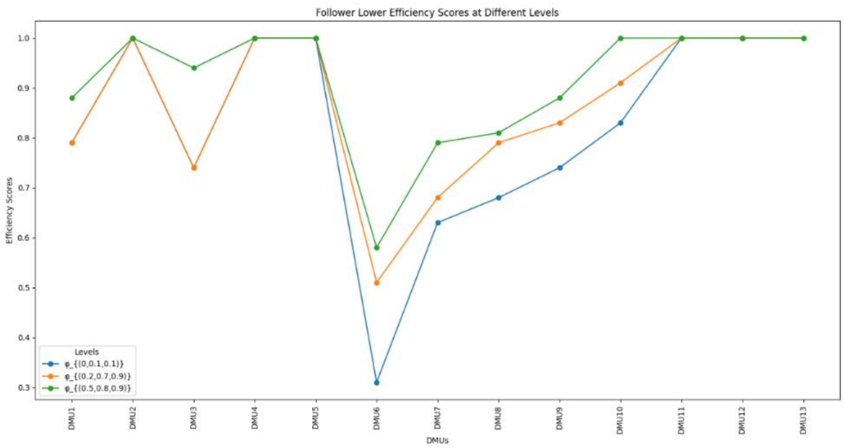

Figure 4. Follower's lower score at different levels of uncertainty.

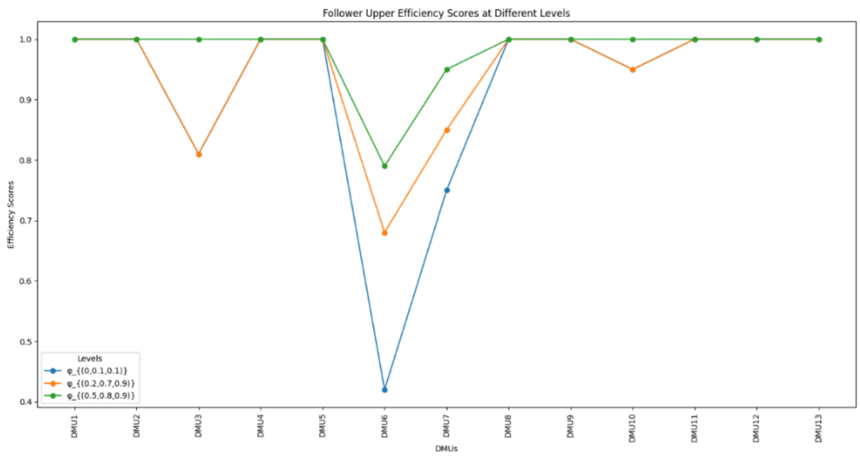

Figure 5. Follower's upper score at different levels of uncertainty.

4.1. Policy recommendations

Based on the findings of our model, several policy recommendations can be drawn to enhance operational efficiency in the Iranian airline industry:

1. Improve Resource Allocation for Underperforming Units:

Airlines such as DMU6, which consistently show lower efficiency scores across various levels of uncertainty, should focus on improving resource management in both their production and consumption divisions. This could involve optimizing workforce deployment and better utilization of aircraft to align closer with the performance of more efficient airlines like DMU2.

2. Targeted Interventions for Efficiency Improvement:

The variability in efficiency scores at different levels of uncertainty, particularly for DMU6, suggests that management should invest in operational adjustments that specifically address areas prone to inefficiencies. This could include investing in better route planning, aircraft maintenance, or optimizing flight schedules to reduce inefficiency in both divisions.

3. Implement a Penalty-Based Performance Framework:

The penalty for deviation from the leader's efficiency score, as shown in our model, could be used by airlines to incentivize higher performance. Policymakers should consider adopting a framework where airlines are rewarded for staying within optimal efficiency ranges and penalized for significant deviations, especially in the follower division, to promote consistency across all operational stages.

4.2.Comparison of results with an existing approach

In this subsection, we compare the results of the proposed model with those of Rasinojehdehi et al. [10]. Fig. 6 and Fig. 7 illustrate comparison of the average efficiency scores for the generated by our model and the corresponding results from Rasinojehdehi and Bagherzadeh Valami [10]. This comparison highlights the improvements and differences in performance evaluation, showcasing the strengths of the proposed dynamic network DEA model in handling the dual role of intermediate measures.

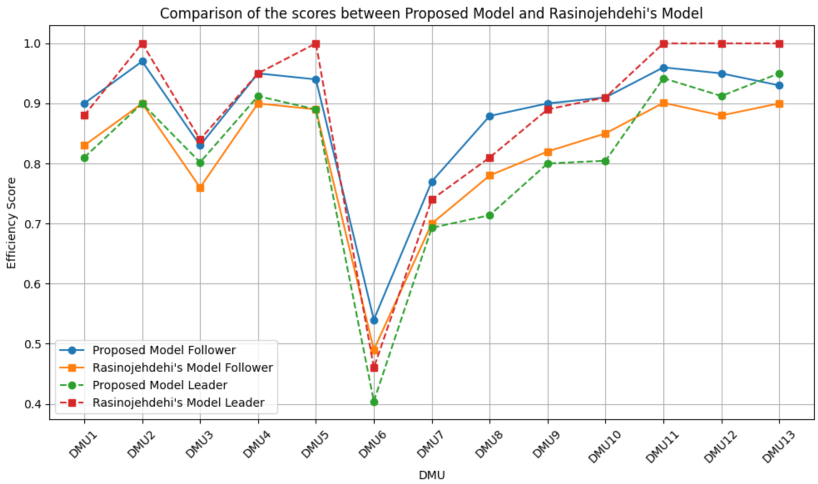

Figure 6. Comparison of average stage scores between our model and the model by Rasinojehdehi and Bagherzadeh Valami [10].

The comparison reveals that the average scores for the leader's stage in the proposed model are lower than those of Rasinojehdehi et al. [10], reflecting a more stringent evaluation of the leader's performance. This stricter assessment highlights the proposed model's ability to differentiate between varying levels of efficiency, especially in the leader's stage. Furthermore, the proposed model exhibits superior discriminating power, enabling a clearer distinction between efficient and inefficient DMUs. This enhancement stems from the model's focus on optimizing the leader's performance, which drives the system by compelling followers to adopt the strategy set by the leader, thereby improving overall system cohesion and performance stability.

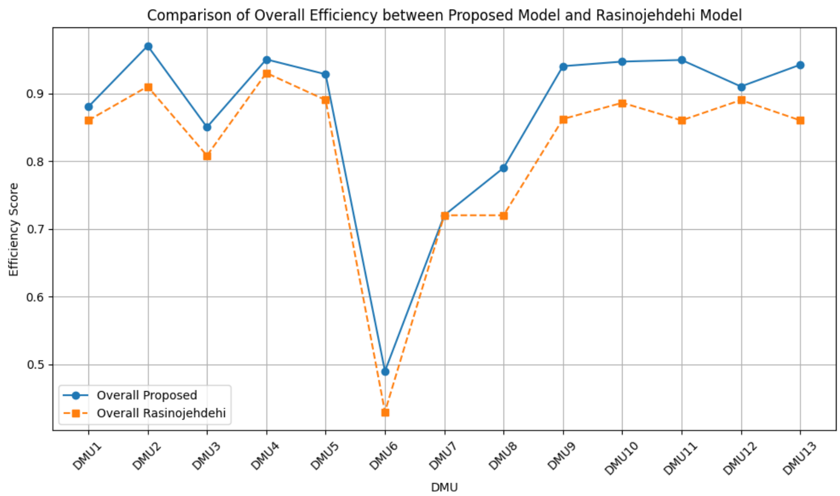

Figure 7. Comparison of the average overall score between our model and the model by Rasinojehdehi and Bagherzadeh Valami [10].

In the proposed model, the leader is tasked with determining the allocation of intermediate measures, ensuring that resources are allocated in a manner that optimizes the leader's efficiency. In contrast, Rasinojehdehi et al.'s [10] model focuses on distributing intermediate measures to minimize the overall system score, emphasizing a more collective approach. This fundamental difference explains why the average leader's score in the proposed model is lower, as it places greater emphasis on the leader's strategic role. However, despite the stricter assessment of the leader's efficiency, the overall system score remains comparable between the two models. This is because the followers in the proposed model are guided by the leader's strategy, leading to a more balanced, aligned, and cohesive system performance.

5. Conclusion

In this paper, a novel approach for the performance evaluation of two-stage network systems, addressing the dual role of intermediate measures, is presented. To handle uncertainty within real-world data, SVNSs, a special case of neutrosophic sets, are employed. The Stackelberg game theory is utilized to explain the relationship between the stages and address the issue of the dual role of intermediate measures. The proposed model is developed within the SBM framework, which not only maintains the continuity of linking activities among the stages but also incorporates their inefficiency into the efficiency evaluation. Additionally, the model incorporates a penalty on the follower's objective for deviations from the leader's efficiency. This penalty enhances the model's feasibility robustness by underscoring the significant consequences imposed on deviations from the optimal efficiency established by the leader. To address uncertainty, the model is refined into a two-stage framework using Pareto efficiency concepts to establish lower and upper bounds for DMU efficiencies. Data from the Iranian Airline Industry, was used to validate our approach. This empirical study evaluates the production and consumption efficiencies of 13 Iranian airlines, demonstrating how our methodology effectively captures and quantifies operational efficiencies amidst uncertainty.

5.1. Limitations and future directions

While the proposed model has proven effective in capturing operational efficiencies under uncertainty, there are several limitations that should be considered. First, the model assumes a constant return to scale, which may not reflect real-world variability in airline operations. Future studies could explore the use of variable returns to scale to account for this. Additionally, the current model focuses on a two-stage process; however, many real-world systems are dynamic and involve more complex multi-stage interactions. Extending the model to dynamic NDEA or multi-stage systems could provide deeper insights into efficiency changes over time.

Furthermore, while Stackelberg game theory is used to explain the relationship between stages, other game theory models, such as cooperative games, could be integrated to explore different interaction dynamics between stages. Comparative studies could enhance the robustness of efficiency measurements by analyzing how various game theory models affect the outcomes.

Lastly, the model's applicability has been demonstrated in the airline industry, but future research should test its adaptability in other industries or sectors where uncertainty and multi-stage processes are prevalent.

Statements & Declarations

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All authors contributed to the study conception and design.

Compliance with Ethical Standards

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of Interest

The authors declare that they have no conflict of interest.

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