NSS

Neutrosophic Sets and Systems, Vol. 80, 2025

University of New Mexico

Intuitionistic Quadri-Partitioned Neutrosophic Sets with Extended Operators

Sai Anu P[1], Arulselvam A[2] and Bobin A[3,*]

[1]Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai, Tamilnadu; saianupriya@gmail.com

[2]Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai, Tamilnadu; arulselvam.a91@gmail.com.

[3]Department of Mathematics, IFET College of Engineering (An Autonomous Institution), Villupuram, Tamilnadu.

[*]Correspondence: bobinalbert@gmail.com

Abstract. We present the concept of intuitionistic quadri-partitioned neutrosophic sets (IQ-PNSs) and extend it to intuitionistic quadri-partitioned neutrosophic soft sets (IQ-PNSSs). These sets introduce a constraint where the combined membership values for truth and falsity must not exceed one, while the indeterminacy and contradiction membership values are restricted to the interval $[0, 1]$, ensuring the total sum of all membership values remains below three. In addition, we define necessity and possibility operators, along with two novel operators, ($\oplus$ and $\ominus$), on IQ-PNSSs and explore their properties. Furthermore, we introduce and analyze the operators Δ_μ, $\Delta_{\mu,\nu}$, and $\delta_{\mu,\nu}$, as well as the concentration ($\mathcal{CO}$) and dilation ($\mathcal{DO}$) operators on IQ-PNSSs. This framework effectively addresses complex uncertainties, particularly in situations where conventional approaches, such as those used for modeling ambivalent psychological behaviors, are insufficient.

Keywords:soft set; intuitionistic set; neutrosophic set; quadri-partitioned neutrosophic set.

---

1. **Introduction**

The concept of fuzzy sets (FS) was first introduced by Zadeh [31], where the membership of each element is represented by a value within the closed interval $[0, 1]$. Later, Atanassov [3] expanded this concept by proposing intuitionistic fuzzy sets (IFS), incorporating both membership and non-membership values, with the condition that their combined total does not exceed one. Building on this foundation, Atanassov [4] explored various properties of IFS. Subsequent advancements were made by Smarandache [26], who introduced the neutrosophic set (NS), characterized by three membership functions: truth, indeterminacy, and falsity. Neutrosophic theory has been applied extensively across numerous fields, particularly in decision-making. To make neutrosophic theory more practical, Wang et al. [30] introduced the single-valued neutrosophic set (SVNS), which provides a more adaptable framework for real-world applications. Researchers like Smarandache [27] have applied neutrosophic theory to psychological concepts such as the soul, and Christianto and Smarandache [17] reviewed the use of neutrosophic logic in cultural psychology. Similarly, Chicaiza et al. [12] investigated emotional intelligence among students using neutrosophic psychology, while Smarandache [28] introduced the concepts of dependence and independence in FS and NS. Smarandache [29] also discussed hypersoft sets as an extension of soft set theory. The growing use of NS and SVNS in psychology demonstrates their significance.

In the field of medical diagnosis, Shahzadi et al. [25] employed SVNSs, while Broumi et al. [9] developed single-valued neutrosophic graphs as a generalization of fuzzy and intuitionistic fuzzy graphs. Other significant advancements include Hamidi et al.'s [18] application of neutrosophic graphs in sensor network analysis, and Beg and Tabasam's [5] exploration of comparative linguistic expressions using hesitant IFSs. Researchers such as Anita et al. [2] utilized interval-valued IFS for multi-criteria decision-making (MCDM), while Jianming et al. [20] proposed weighted aggregation operators for neutrosophic cubic sets (NCSs) in similar applications. The development of algebraic and Einstein operators on NCSs by Majid Khan et al. [22] has further advanced MCDM methodologies. Other contributions, such as Chinnadurai et al.'s [13] creation of a ranking system for neutrosophic environments, and Chinnadurai and Bobin's [14] application of prospect theory to MCDM, emphasize the practical applications of these models. Additionally, Chatterjee et al. [10] and Abdel Basset et al. [1] have contributed to the development of similarity measures (SMs) within the neutrosophic framework, including bipolar SVNSs for diagnosing bipolar disorder behaviors. Programming tools for comparing NSs, as developed by Saranya et al. [24], and similarity measures based on Hausdorff and Euclidean distances, introduced by Broumi and Smarandache [7] and Lix et al. [21], have further strengthened these approaches. In a specialized context, Hashim et al. [19] applied similarity measures in neutrosophic bipolar fuzzy sets to assist the HOPE foundation in establishing a children's hospital.

In set theory, Bhowmik and Pal [6] introduced the intuitionistic neutrosophic set (INS) and studied its properties, while Broumi and Smarandache [8] extended this work by defining the intuitionistic neutrosophic soft set (INSS) and examining its characteristics. Chinnadurai and Bobin [15] made a significant contribution to decision-making methodologies by introducing the Simplified-Intuitionistic Neutrosophic Soft Set (INSS) framework. Additionally, they [16] proposed a novel approach to addressing decision-making challenges in uncertain environments through the development of the Interval-Valued INSS. This framework is particularly focused on its application in diagnosing psychiatric disorders, a domain where ambiguity and incomplete information often hinder effective decision-making. However, no existing set addresses the scenario where truth and falsity membership values are interdependent, while indeterminacy and contradiction are independent. To fill this gap, we propose intuitionistic quadri-partitioned neutrosophic soft sets (IQ-PNSS), which offer more flexible decision-making conditions. This study highlights the relevance of IQ-PNSS, particularly in situations where experts assign membership values for truth, indeterminacy, contradiction, and falsity under specific constraints.

2. **Preliminaries**

Let $\mathcal{U}$ be the universal set, with $\triangle \in \mathcal{U}$, and let $\mathcal{M}$ represent a set of parameters, where Λ is a subset of $\mathcal{M}$. Moreover, let $\mathcal{I}^Q$ stand for the set containing all IQ-PNS over $\mathcal{U}$. The following notations will be used consistently throughout the manuscript, unless explicitly stated otherwise.

Definition 2.1. [31] A FS is represented as $\mathcal{F} = \{(\triangle, \mathcal{P}_{\mathcal{F}}(\triangle)) : \triangle \in \mathcal{U}\}$, where the function $\mathcal{P}_{\mathcal{F}} : \mathcal{U} \to [0, 1]$ assigns a membership degree to each element $\triangle \in \mathcal{U}$.

Definition 2.2. [3] An IFS is given as $\mathcal{I} = \{(\triangle, \mathcal{P}_{\mathcal{I}}(\triangle), \mathcal{S}_{\mathcal{I}}(\triangle)) : \triangle \in \mathcal{U}\}$, where $\mathcal{P}_{\mathcal{I}}(\triangle) : \mathcal{U} \to [0, 1]$ and $\mathcal{S}_{\mathcal{I}}(\triangle) : \mathcal{U} \to [0, 1]$ denote the membership and non-membership degrees of each element $\triangle \in \mathcal{U}$, respectively. For all $\triangle \in \mathcal{U}$, $0 \le \mathcal{P}_{\mathcal{I}}(\triangle) + \mathcal{S}_{\mathcal{I}}(\triangle) \le 1$, and the degree of hesitation is given by $\pi_{\mathcal{I}}(\triangle) = 1 - \mathcal{P}_{\mathcal{I}}(\triangle) - \mathcal{S}_{\mathcal{I}}(\triangle)$.

Definition 2.3. [30] A SVNS is defined as $\mathcal{N} = \{\langle \triangle, \mathcal{P}_{\mathcal{N}(\triangle)}, \mathcal{Q}_{\mathcal{N}}(\triangle), \mathcal{S}_{\mathcal{N}}(\triangle)\rangle : \triangle \in \mathcal{U}\}$, where $\mathcal{P}_{\mathcal{N}}(\triangle) : \mathcal{U} \to [0, 1]$, $\mathcal{Q}_{\mathcal{N}}(\triangle) : \mathcal{U} \to [0, 1]$ and $\mathcal{S}_{\mathcal{N}}(\triangle) : \mathcal{U} \to [0, 1]$ represent the degrees of truth, indeterminacy, and falsity membership, respectively. For each $\triangle \in \mathcal{U}$, $0 \le \mathcal{P}_{\mathcal{N}}(\triangle) + \mathcal{Q}_{\mathcal{N}}(\triangle) + \mathcal{S}_{\mathcal{N}}(\triangle) \le 3$.

Definition 2.4. [23] A soft set (SS) over a universe $\mathcal{U}$ is a pair (δ, Λ), where $\delta : \Lambda \to \mathcal{P}(\mathcal{U})$. For each $e \in \Lambda$, $\delta(e) = 1$ implies $\triangle \in \delta(e)$, and $\delta(e) = 0$ implies $\triangle \notin \delta(e)$. Therefore, a soft set is not an ordinary set but rather a parameterized collection of subsets of $\mathcal{U}$.

Definition 2.5. [11] A quadri-partitioned neutrosophic set (Q-PNS) is given as $\mathfrak{N} = \{\langle \triangle, \mathcal{P}_{\mathfrak{N}}(\triangle), \mathcal{Q}_{\mathfrak{N}}(\triangle), \mathcal{R}_{\mathfrak{N}}(\triangle), \mathcal{S}_{\mathfrak{N}}(\triangle)\rangle : \triangle \in \mathcal{U}\}$, where $\mathcal{P}_{\mathfrak{N}}(\triangle) : \mathcal{U} \to [0, 1]$, $\mathcal{Q}_{\mathfrak{N}}(\triangle) : \mathcal{U} \to [0, 1]$, $\mathcal{R}_{\mathfrak{N}}(\triangle) : \mathcal{U} \to [0, 1]$ and $\mathcal{S}_{\mathfrak{N}}(\triangle) : \mathcal{U} \to [0, 1]$ represent the membership degrees for truth, indeterminacy, contradiction, and falsity, respectively. For every $\triangle \in \mathcal{U}$, it holds that $0 \le \mathcal{P}_{\mathfrak{N}}(\triangle) + \mathcal{Q}_{\mathfrak{N}}(\triangle) + \mathcal{R}_{\mathfrak{N}}(\triangle) + \mathcal{S}_{\mathfrak{N}}(\triangle) \le 4$.

3. Intuitionistic quadri-partitioned neutrosophic set

We introduce the notions of IQ-PNS and IQ-PNSS, along with a detailed examination of their properties.

Definition 3.1. An IQ-PNS within the universe $\mathcal{U}$ can be expressed as $\mathcal{A} = \{\langle \triangle, \mathcal{P}_{\mathcal{A}}(\triangle), \mathcal{Q}_{\mathcal{A}}(\triangle), \mathcal{R}_{\mathcal{A}}(\triangle), \mathcal{S}_{\mathcal{A}(\triangle)}, \rangle\}$, $\mathcal{P}_{\mathcal{A}}(\triangle) : \mathcal{U} \to [0,1]$, $\mathcal{Q}_{\mathcal{A}}(\triangle) : \mathcal{U} \to [0,1]$, $\mathcal{R}_{\mathcal{A}}(\triangle) : \mathcal{U} \to [0,1]$ and $\mathcal{S}_{\mathcal{A}}(\triangle) : \mathcal{U} \to [0,1]$ denote the degrees of truth, indeterminacy, contradiction, and falsity, respectively, for an element $\triangle \in \mathcal{U}$. These functions must satisfy the following conditions: $0 \le \mathcal{P}_{\mathcal{A}}(\triangle) + \mathcal{S}_{\mathcal{A}(\triangle)} \le 1$ and $0 \le \mathcal{P}_{\mathcal{A}}(\triangle) + \mathcal{Q}_{\mathcal{A}(\triangle)} + \mathcal{R}_{\mathcal{A}(\triangle)} + \mathcal{S}_{\mathcal{A}(\triangle)} \le 3$.

Example 3.2. Let $\mathcal{U} = \{\triangle_1, \triangle_2, \triangle_3\}$ be a set. Then an IQ-PNS is given as, $\mathcal{A} = \{\langle \triangle_1, .2, .9, .7, .8\rangle, \langle \triangle_2, .4, .7, .8, .2\rangle, \langle \triangle_3, 0, .9, .6, .7\rangle\}$.

Definition 3.3. A pair (δ, Λ) is called IQ-PNSS over $\mathcal{U}$, where $\delta : \Lambda \to \mathcal{I}^Q$. Thus, for a parameter $e \in \Lambda$, $\delta(e)$ is an IQ-PNSS.

Example 3.4. Let $\mathcal{U} = \{\triangle_1, \triangle_2, \triangle_3, \triangle_4\}$ represent a group of patients with psychological conditions, and let $\Lambda = \{e_1, e_2, e_3, e_4\}$ denote a set of symptoms corresponding to anxiety (ANX), depression (DEP), mood swings (MS), and difficulty concentrating (DC), respectively. An IQ-PNSS (δ, Λ) be a subsets of $\mathcal{U}$, as diagnosed by a psychologist, based on the information provided in Table 1.

Table 1. Subjects with psychological disorders in IQ-PNSS (δ, Λ) form.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .4, .2, .3, .5\rangle$	$\langle .6, .4, .3, .2\rangle$	$\langle .4, .5, .7, .4\rangle$	$\langle .1, .2, .3, .5\rangle$
$\triangle_2$	$\langle 0, .3, .4, .8\rangle$	$\langle 1, .3, .9, .0\rangle$	$\langle .4, .8, .9, .3\rangle$	$\langle .4, .6, .7, .3\rangle$
$\triangle_3$	$\langle .5, .4, .2, .5\rangle$	$\langle .3, .5, .7, .6\rangle$	$\langle .1, .3, .7, .8\rangle$	$\langle .3, .5, .9, .5\rangle$
$\triangle_4$	$\langle .6, .9, .4, .4\rangle$	$\langle .2, .4, .6, .7\rangle$	$\langle .6, .7, .8, .4\rangle$	$\langle .2, .5, .8, .7\rangle$

Definition 3.5. Let (δ_1, Θ_1) and (δ_2, Θ_2) be two IQ-PNSS over $\mathcal{U}$. Then,
(i) (δ_1, Θ_1) AND (δ_2, Θ_2) is an IQ-PNSS represented as $(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2) = (\delta_\wedge, \Theta_1 \times \Theta_2)$, where $\delta_\wedge(\varrho_1, \varrho_2) = \delta_1(\varrho_1) \cap \delta_2(\varrho_2)$, $\forall\ (\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2$.

$$\delta_\wedge(\varrho_1, \varrho_2) = \Big\langle \min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)), \min(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$$
$$\min(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle)), \max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle), \mathcal{S}_{\delta_2(\varrho_2)}(\triangle))\Big\rangle, \forall\ (\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2;$$

(ii) (δ_1, Θ_1) OR (δ_2, Θ_2) is an IQ-PNSS represented as $(\delta_1, \Theta_1) \vee (\delta_2, \Theta_2) = (\delta_\vee, \Theta_1 \times \Theta_2)$, where $\delta_\vee(\varrho_1, \varrho_2) = \delta_1(\varrho_1) \cup \delta_2(\varrho_2)$, $\forall\ (\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2$.

$$\delta_\vee(\varrho_1, \varrho_2) = \Big\langle \max(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$$
$$\max(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle), \mathcal{S}_{\delta_2(\varrho_2)}(\triangle))\Big\rangle, \forall\ (\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2.$$

Example 3.6. Let $\delta_1 = \langle .1, .4, .3, .9\rangle$ and $\delta_1 = \langle 1, .1, .5, .0\rangle$ be two IQ-PNSS. Then
(i) $\delta_1 \wedge \delta_2 = \langle .1, .1, .3, .9\rangle$;
(ii) $\delta_1 \vee \delta_2 = \langle 1, .4, .5, 0\rangle$.

Definition 3.7. Let (δ_1, Θ_1) and (δ_2, Θ_2) be two IQ-PNSS over $\mathcal{U}$. Then,
(i) (δ_1, Θ_1) union (δ_2, Θ_2) is an IQ-PNSS represented as $(\delta_1, \Theta_1) \Cup (\delta_2, \Theta_2) = (\delta_{\Cup}, \Theta_{\Cup})$, where $\Theta_{\Cup} = \Theta_1 \cup \Theta_2$ and $\forall\ \varrho \in \Theta_{\Cup}$,

$$\delta_{\Cup}(\varrho) = \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \max(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), \right.\right. & \\ \qquad \left.\left. \max(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_1 \cap \Theta_2\right\}. \end{cases}$$

(ii) (δ_1, Θ_1) intersection (δ_2, Θ_2) is a IQ-PNSS represented as $(\delta_1, \Theta_1) \Cap (\delta_2, \Theta_2) = (\delta_{\Cap}, \Theta_{\Cap})$, where $\Theta_{\Cap} = \Theta_1 \cup \Theta_2$ and $\forall\ \varrho \in \Theta_{\Cap}$,

$$\delta_{\Cap}(\varrho) = \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \min(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \min(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), \right.\right. & \\ \qquad \left.\left. \min(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), \max(\mathcal{S}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_1 \cap \Theta_2\right\}. \end{cases}$$

Definition 3.8. The complement of an IQ-PNSS (δ, Λ) can be described as follows:

$$(\delta, \Lambda)^c = \left\{\left\langle \triangle, \mathcal{S}_{\delta(\varrho)}(\triangle), (1 - \mathcal{Q}_{\delta(\varrho)})(\triangle), (1 - \mathcal{R}_{\delta(\varrho)})(\triangle), \mathcal{P}_{\delta(\varrho)}(\triangle)\right\rangle; \text{and } \varrho \in \Lambda\right\}.$$

Example 3.9. Let $\delta = \langle .5, .2, .3, .2\rangle$ be an IQ-PNSS. Then $\delta^c = \langle .2, .8, .7, .5\rangle$.

Theorem 3.10. *Let (δ_1, Θ_1) and (δ_2, Θ_2) be two IQ-PNSS over $\mathcal{U}$. Then,*
(i) $\langle(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2)\rangle^c = (\delta_1, \Theta_1)^c \vee (\delta_2, \Theta_2)^c$;
(ii) $\langle(\delta_1, \Theta_1) \vee (\delta_2, \Theta_2)\rangle^c = (\delta_1, \Theta_1)^c \wedge (\delta_2, \Theta_2)^c$.

Proof. We provide the proof for (i); the proof for (ii) follows similarly.
(i) $(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2) = (\delta_{\wedge}, \Theta_1 \times \Theta_2)$.
$\langle(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2)\rangle^c = (\delta_{\wedge}, \Theta_1 \times \Theta_2)^c$.

$$\begin{aligned} \delta^c_{\wedge}(\varrho_1, \varrho_2) = \big\langle &\max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle), \mathcal{S}_{\delta_2(\varrho_2)}(\triangle)), \max((1 - \mathcal{Q}_{\delta_1(\varrho_1)}(\triangle)), (1 - \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle))), \\ &\max((1 - \mathcal{R}_{\delta_1(\varrho_1)}(\triangle)), (1 - \mathcal{R}_{\delta_2(\varrho_2)}(\triangle))), \\ &\min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle))\big\rangle, \forall(\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2. \end{aligned}$$

$(\delta_1, \Theta_1)^c \vee (\delta_2, \Theta_2)^c = (\delta_\vee, \Theta_1 \times \Theta_2)$.

$$\delta_\vee(\varrho_1, \varrho_2) = \big\langle \max(\mathcal{S}_{\delta_1(\varrho_1)}(\triangle), \mathcal{S}_{\delta_2(\varrho_2)}(\triangle)), \max((1-\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle)), (1-\mathcal{Q}_{\delta_2(\varrho_2)}(\triangle))),$$
$$\max((1-\mathcal{R}_{\delta_1(\varrho_1)}(\triangle)), (1-\mathcal{R}_{\delta_2(\varrho_2)}(\triangle))),$$
$$\min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle))\big\rangle, \forall(\varrho_1, \varrho_2) \in \Theta_1 \times \Theta_2.$$

Thus, $\langle(\delta_1, \Theta_1) \wedge (\delta_2, \Theta_2)\rangle^c = (\delta_1, \Theta_1)^c \vee (\delta_2, \Theta_2)^c$. □

Definition 3.11. Let Θ_1 and Θ_2 be subsets of Λ. The pair (δ_1, Θ_1) is called an intuitionistic quadri-partitioned neutrosophic soft subset (IQ-PNSSS) of (δ_2, Θ_2), denoted by $(\delta_1, \Theta_1) \Subset (\delta_2, \Theta_2)$, iff the below given conditions are satisfied:
(i) $\Theta_1 \subseteq \Theta_2$;
(ii) For each $\varrho \in \Theta_1$, $\delta_1(\varrho)$ is an IQ-PNSSS of $\delta_2(\varrho)$, that is, $\mathcal{P}_{\delta_1(\varrho)}(\triangle) \leq \mathcal{P}_{\delta_2(\varrho)}(\triangle)$, $\mathcal{Q}_{\delta_1(\varrho)}(\triangle) \leq \mathcal{Q}_{\delta_2(\varrho)}(\triangle)$, $\mathcal{R}_{\delta_1(\varrho)}(\triangle) \leq \mathcal{R}_{\delta_2(\varrho)}(\triangle)$, and $\mathcal{S}_{\delta_1(\varrho)}(\triangle) \geq \mathcal{S}\delta_2(\varrho)(\triangle)$.
Additionally, (δ_2, Θ_2) is referred to as an intuitionistic quadri-partitioned neutrosophic soft superset of (δ_1, Θ_1) and is represented as $(\delta_2, \Theta_2) \Supset (\delta_1, \Theta_1)$.

Definition 3.12. Let (δ_1, Θ_1) and (δ_2, Θ_2) be IQ-PNSSs, then $(\delta_1, \Theta_1) = (\delta_2, \Theta_2)$ iff $(\delta_1, \Theta_1) \Subset (\delta_2, \Theta_2)$ and $(\delta_2, \Theta_2) \Subset (\delta_1, \Theta_1)$.

4. **Necessity (⊚) and possibility (⊛) operators on IQ-PNSS**

We present ⊚ and ⊛ operators on IQ-PNSS and discuss its properties.

Definition 4.1. If (δ, Λ) is an IQ-PNSS over $\mathcal{U}$ and $\delta : \Lambda \to \mathcal{I}^Q$, then,
(i) the necessity operator (⊚) is represented as,

$$\circledcirc(\delta, \Lambda) = \left\{\langle \triangle, \mathcal{P}_{\circledcirc\delta(\varrho)}(\triangle), \mathcal{Q}_{\circledcirc\delta(\varrho)}(\triangle), \mathcal{R}_{\circledcirc\delta(\varrho)}(\triangle), \mathcal{S}_{\circledcirc\delta(\varrho)}(\triangle)\rangle;\ \varrho \in \Lambda\right\}.$$

Here, $\mathcal{P}_{\circledcirc\delta(\varrho)}(\triangle) = \mathcal{P}_{\delta(\varrho)}(\triangle)$, $\mathcal{Q}_{\circledcirc\delta(\varrho)}(\triangle) = \mathcal{Q}_{\delta(\varrho)}(\triangle)$, $\mathcal{R}_{\circledcirc\delta(\varrho)}(\triangle) = \mathcal{R}_{\delta(\varrho)}(\triangle)$ and $\mathcal{S}_{\circledcirc\delta(\varrho)}(\triangle) = (1 - \mathcal{P}_{\delta(\varrho)}(\triangle))$ denote the degrees of truth, indeterminacy, contradiction and falsity for the object $\triangle$ on the parameter ϱ.

(ii) the possibility operator (⊛) is represented as,

$$\circledast(\delta, \Lambda) = \left\{\langle \triangle, \mathcal{P}_{\circledast\delta(\varrho)}(\triangle), \mathcal{Q}_{\circledast\delta(\varrho)}(\triangle), \mathcal{R}_{\circledast\delta(\varrho)}(\triangle), \mathcal{S}_{\circledast\delta(\varrho)}(\triangle)\rangle;\ \varrho \in \Lambda\right\}.$$

Here, $\mathcal{P}_{\circledast\delta(\varrho)}(\triangle) = (1 - \mathcal{S}_{\delta(\varrho)}(\triangle))$, $\mathcal{Q}_{\circledast\delta(\varrho)}(\triangle) = \mathcal{Q}_{\delta(\varrho)}(\triangle)$, $\mathcal{R}_{\circledast\delta(\varrho)}(\triangle) = \mathcal{R}_{\delta(\varrho)}(\triangle)$ and $\mathcal{S}_{\circledast\delta(\varrho)}(\triangle) = \mathcal{S}_{\delta(\varrho)}(\triangle)$ denote the degrees of truth, indeterminacy, contradiction and falsity for the object $\triangle$ on the parameter ϱ.

Example 4.2. (i) The IQ-PNSS $\circledcirc(\delta, \Lambda)$ for Example 3.2 is presented in Table 2.
(ii) The IQ-PNSS $\circledast(\delta, \Lambda)$ for Example 3.2 is presented in Table 3.

Table 2. Subjects with psychological disorders by utilizing the $\circledcirc$ operator.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .4, .2, .3, .6\rangle$	$\langle .6, .4, .3, .4\rangle$	$\langle .4, .5, .7, .6\rangle$	$\langle .1, .2, .3, .9\rangle$
$\triangle_2$	$\langle 0, .3, .4, 1\rangle$	$\langle 1, .3, .9, .0\rangle$	$\langle .4, .8, .9, ..6\rangle$	$\langle .4, .6, .7, .6\rangle$
$\triangle_3$	$\langle .5, .4, .2, .5\rangle$	$\langle .3, .5, .7, .7\rangle$	$\langle .1, .3, .7, .9\rangle$	$\langle .3, .5, .9, .7\rangle$
$\triangle_4$	$\langle .6, .9, .4, .4\rangle$	$\langle .2, .4, .6, .8\rangle$	$\langle .6, .7, .8, .4\rangle$	$\langle .2, .5, .8, .8\rangle$

Table 3. Subjects with cognitive disorders by utilizing the $\circledast$ operator.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .5, .2, .3, .5\rangle$	$\langle .8, .4, .3, .2\rangle$	$\langle .6, .5, .7, .4\rangle$	$\langle .5, .2, .3, .5\rangle$
$\triangle_2$	$\langle .2, .3, .4, .8\rangle$	$\langle 1, .3, .9, .0\rangle$	$\langle .7, .8, .9, .3\rangle$	$\langle .7, .6, .7, .3\rangle$
$\triangle_3$	$\langle .5, .4, .2, .5\rangle$	$\langle .4, .5, .7, .6\rangle$	$\langle .2, .3, .7, .8\rangle$	$\langle .5, .5, .9, .5\rangle$
$\triangle_4$	$\langle .6, .9, .4, .4\rangle$	$\langle .3, .4, .6, .7\rangle$	$\langle .6, .7, .8, .4\rangle$	$\langle .3, .5, .8, .7\rangle$

Theorem 4.3. *Let (δ_1, Θ_1) and (δ_2, Θ_2) be IQ-PNSSs over $\mathcal{U}$. Then,*
(i) $\circledcirc\langle(\delta_1, \Theta_1) \Cup (\delta_2, \Theta_2)\rangle = \circledcirc(\delta_1, \Theta_1) \Cup \circledcirc(\delta_2, \Theta_2)$;
(ii) $\circledcirc\langle(\delta_1, \Theta_1) \Cap (\delta_2, \Theta_2)\rangle = \circledcirc(\delta_1, \Theta_1) \Cap \circledcirc(\delta_2, \Theta_2)$;
(iii) $\circledcirc\circledcirc(\delta_1, \Theta_1) = \circledcirc(\delta_1, \Theta_1)$.

Proof. We provide the proof for (i); the proof for (ii) follows similarly.

(i) Let $(\delta_1, \Theta_1) \Cup (\delta_2, \Theta_2) = (\delta_{\Cup}, \Theta_{\Cup})$, where $\Theta_{\Cup} = \Theta_1 \cup \Theta_2$, $\forall\ \varrho \in \Theta_{\Cup}$.

Consider,

$$\circledcirc\delta_{\Cup}(\varrho) = \begin{cases} \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_1(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_2(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \max(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), \right.\right. & \\ \quad \left.\left. \max(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), (1 - \max(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_1 \cap \Theta_2\right\}. \end{cases}$$

$$= \begin{cases} \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_1(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_2(\varrho)}(\triangle))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \max(\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{P}_{\delta_2(\varrho)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), \right.\right. & \\ \quad \left.\left. \max(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle)), \min((1 - \mathcal{P}_{\delta_1(\varrho)}(\triangle)), (1 - \mathcal{P}_{\delta_2(\varrho)}(\triangle)))\right\rangle; \right. & \left. \text{if } \varrho \in \Theta_1 \cap \Theta_2\right\}. \end{cases}$$

$$\circledcirc(\delta_1, \Theta_1) = \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_1(\varrho)}(\triangle))\right\rangle; \varrho \in \Theta_1\right\},$$

$$\circledcirc(\delta_2, \Theta_2) = \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), (1 - \mathcal{P}_{\delta_2(\varrho)}(\triangle))\right\rangle; \varrho \in \Theta_2\right\}.$$

Let $\circledcirc(\delta_1, \Theta_1) \Cup \circledcirc(\delta_2, \Theta_2) = (\delta_{\circledcirc\Cup}, \Theta_{\circledcirc\Cup})$, where $\Theta_{\circledcirc\Cup} = \Theta_1 \cup \Theta_2$.

For $\varrho_\in \Theta_{\circledcirc \Cup}$,

$$\delta_{\circledcirc \Cup}(\varrho_)=\begin{cases}\left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho_)}(\triangle), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), (1-\mathcal{P}_{\delta_1(\varrho_)}(\triangle))\right\rangle ; \quad \text{if } \varrho_\in \Theta_1-\Theta_2\right\},\\ \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho_)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), (1-\mathcal{P}_{\delta_2(\varrho_)}(\triangle))\right\rangle ; \quad \text{if } \varrho_\in \Theta_2-\Theta_1\right\},\\ \{\langle \triangle, \max(\mathcal{P}_{\delta_1(\varrho_)}(\triangle), \mathcal{P}_{\delta_2(\varrho_)}(\triangle)), \max(\mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle)),\\ \quad \max(\mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_2(\varrho_)}(\triangle)), \min((1-\mathcal{P}_{\delta_1(\varrho_)}(\triangle)), (1-\mathcal{P}_{\delta_2(\varrho_)}(\triangle)))\rangle;\ \text{if } \varrho_\in \Theta_1\cap\Theta_2\}.\end{cases}$$

Thus, $\circledcirc\langle(\delta_1,\Theta_1)\Cup(\delta_2,\Theta_2)\rangle=\circledcirc(\delta_1,\Theta_1)\Cup\circledcirc(\delta_2,\Theta_2)$.

(iii) $\circledcirc\circledcirc(\delta_1,\Theta_1)=\circledcirc\left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho_)}(\triangle), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), (1-\mathcal{P}_{\delta_1(\varrho_)}(\triangle))\right\rangle ; \varrho_\in\Theta_1\right\}$

$=\left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho_)}(\triangle), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), (1-\mathcal{P}_{\delta_1(\varrho_)}(\triangle))\right\rangle ; \varrho_\in\Theta_1\right\}$

$=\circledcirc(\delta_1,\Theta_1)$. □

Theorem 4.4. *Let (δ_1,Θ_1) and (δ_2,Θ_2) be IQ-PNSSs over $\mathcal{U}$. Then,*

(i) $\circledast\langle(\delta_1,\Theta_1)\Cup(\delta_2,\Theta_2)\rangle=\circledast(\delta_1,\Theta_1)\Cup\circledast(\delta_2,\Theta_2)$;

(ii) $\circledast\langle(\delta_1,\Theta_1)\Cap(\delta_2,\Theta_2)\rangle=\circledast(\delta_1,\Theta_1)\Cap\circledast(\delta_2,\Theta_2)$;

(iii) $\circledast\circledast(\delta_1,\Theta_1)=\circledast(\delta_1,\Theta_1)$.

Proof. We provide the proof for (i); the proof for (ii) follows similarly.

(i) Let $(\delta_1,\Theta_1)\Cup(\delta_2,\Theta_2)=(\delta_\Cup,\Theta_\Cup)$, where $\Theta_\Cup=\Theta_1\cup\Theta_2$, $\forall\ \varrho_\in\Theta_\Cup$.

Consider,

$$\circledast\delta_\Cup(\varrho_)=\begin{cases}\left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_1(\varrho_)}(\triangle)\right\rangle ; \quad \text{if } \varrho_\in\Theta_1-\Theta_2\right\},\\ \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_2(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_2(\varrho_)}(\triangle)\right\rangle ; \quad \text{if } \varrho_\in\Theta_2-\Theta_1\right\},\\ \{\langle \triangle, (1-\min(\mathcal{S}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_2(\varrho_)}(\triangle))), \max(\mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle)),\\ \quad \max(\mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_2(\varrho_)}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_2(\varrho_)}(\triangle))\rangle;\ \text{if } \varrho_\in\Theta_1\cap\Theta_2\}.\end{cases}$$

$$=\begin{cases}\left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_1(\varrho_)}(\triangle)\right\rangle ; \quad \text{if } \varrho_\in\Theta_1-\Theta_2\right\},\\ \left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_2(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_2(\varrho_)}(\triangle)\right\rangle ; \quad \text{if } \varrho_\in\Theta_2-\Theta_1\right\},\\ \{\langle \triangle, \max((1-\mathcal{S}_{\delta_1(\varrho_)}(\triangle)), (1-\mathcal{S}_{\delta_2(\varrho_)}(\triangle))), \max(\mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle)),\\ \quad \max(\mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_2(\varrho_)}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_2(\varrho_)}(\triangle))\rangle;\ \text{if } \varrho_\in\Theta_1\cap\Theta_2\}.\end{cases}$$

$$\circledast(\delta_1,\Theta_1)=\left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_1(\varrho_)}(\triangle)\right\rangle ; \varrho_\in\Theta_1\right\},$$

$$\circledast(\delta_2,\Theta_2)=\left\{\left\langle \triangle, (1-\mathcal{S}_{\delta_2(\varrho_)}(\triangle)), \mathcal{Q}_{\delta_2(\varrho_)}(\triangle), \mathcal{R}_{\delta_1(\varrho_)}(\triangle), \mathcal{S}_{\delta_2(\varrho_)}(\triangle)\right\rangle ; \varrho_\in\Theta_2\right\}.$$

Let $\circledast(\delta_1,\Theta_1)\Cup\circledast(\delta_2,\Theta_2)=(\delta_{\circledast\Cup},\Theta_{\circledast\Cup})$, where $\Theta_{\circledast\Cup}=\Theta_1\cup\Theta_2$.

For $\varrho\in\Theta_{\circledast\Cup}$,

$$\delta_{\circledast\Cup}(\varrho) = \begin{cases} \Big\{\Big\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle)\Big\rangle; & \text{if } \varrho\in\Theta_1-\Theta_2\Big\}, \\ \Big\{\Big\langle \triangle, (1-\mathcal{S}_{\delta_2(\varrho)}(\triangle)), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle)\Big\rangle; & \text{if } \varrho\in\Theta_2-\Theta_1\Big\}, \\ \big\{\big\langle \triangle, \max((1-\mathcal{S}_{\delta_1(\varrho)}(\triangle)), (1-\mathcal{S}_{\delta_2(\varrho)}(\triangle))), \max(\mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle)), & \\ \quad \max(\mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle)), \min(\mathcal{S}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\big\rangle; & \text{if } \varrho\in\Theta_1\cap\Theta_2\big\}. \end{cases}$$

Thus, $\circledast\langle(\delta_1,\Theta_1)\Cup(\delta_2,\Theta_2)\rangle = \circledast(\delta_1,\Theta_1)\Cup\circledast(\delta_2,\Theta_2)$.

$$\text{(iii) } \circledast\circledast(\delta_1,\Theta_1) = \circledast\Big\{\Big\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle)\Big\rangle; \varrho\in\Theta_1\Big\}$$
$$= \Big\{\Big\langle \triangle, (1-\mathcal{S}_{\delta_1(\varrho)}(\triangle)), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle)\Big\rangle; \varrho\in\Theta_1\Big\}$$
$$= \circledast(\delta_1,\Theta_1). \ \square$$

Theorem 4.5. *Let (δ,Λ) be an IQ-PNSS over $\mathcal{U}$. Then,*

(i) $\circledast\circledcirc(\delta,\Lambda) = \circledcirc(\delta,\Lambda)$;

(ii) $\circledcirc\circledast(\delta,\Lambda) = \circledast(\delta,\Lambda)$.

Proof. (i) $\circledast\circledcirc(\delta,\Lambda) = \Big\{\Big\langle \triangle, (1-(1-\mathcal{P}_{\delta(\varrho)}(\triangle))), \mathcal{Q}_{\delta(\varrho)}(\triangle), \mathcal{R}_{\delta(\varrho)}(\triangle), (1-\mathcal{P}_{\delta(\varrho)}(\triangle))\Big\rangle; \varrho\in\Lambda\Big\}$

$$= \Big\{\Big\langle \triangle, \mathcal{P}_{\delta(\varrho)}(\triangle), \mathcal{Q}_{\delta(\varrho)}(\triangle), \mathcal{R}_{\delta(\varrho)}(\triangle), (1-\mathcal{P}_{\delta(\varrho)}(\triangle))\Big\rangle; \varrho\in\Lambda\Big\}$$

$\circledast\circledcirc(\delta,\Lambda) = \circledcirc(\delta,\Lambda)$.

(ii) $\circledcirc\circledast(\delta,\Lambda) = \Big\{\Big\langle \triangle, (1-\mathcal{S}_{\delta(\varrho)}(\triangle)), \mathcal{Q}_{\delta(\varrho)}(\triangle), \mathcal{R}_{\delta(\varrho)}(\triangle), (1-(1-\mathcal{S}_{\delta(\varrho)}(\triangle)))\Big\rangle; \varrho\in\Lambda\Big\}$

$$= \Big\{\Big\langle \triangle, (1-\mathcal{S}_{\delta(\varrho)}(\triangle)), \mathcal{Q}_{\delta(\varrho)}(\triangle), \mathcal{R}_{\delta(\varrho)}(\triangle), \mathcal{S}_{\delta(\varrho)}(\triangle)\Big\rangle; \varrho\in\Lambda\Big\}$$

$\circledcirc\circledast(\delta,\Lambda) = \circledast(\delta,\Lambda)$. $\square$

Theorem 4.6. *Let (δ_1,Θ_1) and (δ_2,Θ_2) be IQ-PNSSs over $\mathcal{U}$. Then*

(i) $\circledcirc\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle = \circledcirc(\delta_1,\Theta_1)\wedge\circledcirc(\delta_2,\Theta_2)$;

(ii) $\circledcirc\langle(\delta_1,\Theta_1)\vee(\delta_2,\Theta_2)\rangle = \circledcirc(\delta_1,\Theta_1)\vee\circledcirc(\delta_2,\Theta_2)$;

(iii) $\circledast\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle = \circledast(\delta_1,\Theta_1)\wedge\circledcirc(\delta_2,\Theta_2)$;

(iv) $\circledast\langle(\delta_1,\Theta_1)\vee(\delta_2,\Theta_2)\rangle = \circledast(\delta_1,\Theta_1)\vee\circledcirc(\delta_2,\Theta_2)$.

Proof. We provide the proofs for (i) and (iii); the proofs for (ii) and (iv) follows similarly.

(i) $\circledcirc\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle$

$$= \big\{\big\langle \min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)), \min(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$$
$$\min(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle)), (1-\min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)))\big\rangle, \forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}$$
$$= \big\{\big\langle \min(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{P}_{\delta_2(\varrho_2)}(\triangle)), \min(\mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle)),$$
$$\min(\mathcal{R}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle)), (\max(1-(\mathcal{P}_{\delta_1(\varrho_1)}(\triangle))), (1-(\mathcal{P}_{\delta_2(\varrho_2)}(\triangle)))))\big\rangle, \forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}.$$

Also, $\circledcirc(\delta_1,\Theta_1) = \big\{\big\langle \triangle, \mathcal{P}_{\delta_1(\varrho_1)}(\triangle), \mathcal{Q}_{\delta_1(\varrho_1)}(\triangle), \mathcal{R}_{\delta_1(\varrho_1)}(\triangle), (1-\mathcal{P}_{\delta_1(\varrho_1)}(\triangle))\big\rangle;\ \varrho_1\in\Theta_1\big\}$,

$\circledcirc(\delta_2,\Theta_2) = \big\{\big\langle \triangle, \mathcal{P}_{\delta_2(\varrho_2)}(\triangle), \mathcal{Q}_{\delta_2(\varrho_2)}(\triangle), \mathcal{R}_{\delta_2(\varrho_2)}(\triangle), (1-\mathcal{P}_{\delta_2(\varrho_2)}(\triangle))\big\rangle;\ \varrho_2\in\Theta_2\big\}$.

Therefore, we have

$$\circledcirc(\delta_1,\Theta_1)\wedge\circledcirc(\delta_2,\Theta_2)$$
$$=\big\{\big\langle\min(\mathcal{P}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{P}_{\delta_{2(\varrho_2)}}(\triangle)),\min(\mathcal{Q}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{Q}_{\delta_{2(\varrho_2)}}(\triangle)),\min(\mathcal{R}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{R}_{\delta_{2(\varrho_2)}}(\triangle)),$$
$$(\max(1-(\mathcal{P}_{\delta_{1(\varrho_1)}}(\triangle))),(1-(\mathcal{P}_{\delta_{2(\varrho_2)}}(\triangle))))\big\rangle,\forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}$$
$$=\circledcirc\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle.$$

(iii) $\circledast\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle$

$$=\big\{\big\langle(1-\max(\mathcal{S}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{S}_{\delta_{2(\varrho_2)}}(\triangle))),\max(\mathcal{Q}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{Q}_{\delta_{2(\varrho_2)}}(\triangle)),$$
$$\max(\mathcal{R}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{R}_{\delta_{2(\varrho_2)}}(\triangle)),\max(\mathcal{S}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{S}_{\delta_{2(\varrho_2)}}(\triangle))\big\rangle,\forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}$$
$$=\big\{\big\langle(\min(1-(\mathcal{S}_{\delta_{1(\varrho_1)}}(\triangle))),(1-(\mathcal{S}_{\delta_{2(\varrho_2)}}(\triangle)))),\max(\mathcal{Q}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{Q}_{\delta_{2(\varrho_2)}}(\triangle)),$$
$$\max(\mathcal{R}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{R}_{\delta_{2(\varrho_2)}}(\triangle)),\max(\mathcal{S}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{S}_{\delta_{2(\varrho_2)}}(\triangle))\big\rangle,\forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}.$$

Also, $\circledast(\delta_1,\Theta_1)=\big\{\big\langle\triangle,(1-\mathcal{S}_{\delta_{1(\varrho_1)}}(\triangle)),\mathcal{Q}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{R}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{S}_{\delta_{1(\varrho_1)}}(\triangle)\big\rangle;\ \varrho_1\in\Theta_1\big\}$,

$\circledast(\delta_2,\Theta_2)=\big\{\big\langle\triangle,(1-\mathcal{S}_{\delta_{2(\varrho_2)}}(\triangle)),\mathcal{Q}_{\delta_{2(\varrho_2)}}(\triangle),\mathcal{R}_{\delta_{2(\varrho_2)}}(\triangle),\mathcal{S}_{\delta_{2(\varrho_2)}}(\triangle)\big\rangle;\ \varrho_2\in\Theta_2\big\}$.

Therefore, we have

$$\circledast(\delta_1,\Theta_1)\wedge\circledast(\delta_2,\Theta_2)$$
$$=\big\{\big\langle(\min(1-(\mathcal{S}_{\delta_{1(\varrho_1)}}(\triangle))),(1-(\mathcal{S}_{\delta_{2(\varrho_2)}}(\triangle)))),\max(\mathcal{Q}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{Q}_{\delta_{2(\varrho_2)}}(\triangle)),$$
$$\max(\mathcal{R}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{R}_{\delta_{2(\varrho_2)}}(\triangle)),\max(\mathcal{S}_{\delta_{1(\varrho_1)}}(\triangle),\mathcal{S}_{\delta_{2(\varrho_2)}}(\triangle)),\big\rangle\forall(\varrho_1,\varrho_2)\in\Theta_1\times\Theta_2\big\}$$
$$=\circledcirc\langle(\delta_1,\Theta_1)\wedge(\delta_2,\Theta_2)\rangle.$$

□

5. ⊕ and ⊖ operators on IQ-PNSS

We present the operators (⊕ and ⊖) on IQ-PNSS and discuss its properties.

Definition 5.1. Let (δ_1,Θ_1) and (δ_2,Θ_2) be IQ-PNSSs over $\mathcal{U}$. Then,
(i) the operator ⊕ is represented as $(\delta_1,\Theta_1)\oplus(\delta_2,\Theta_2)=(\delta_\oplus,\Theta_\oplus)$, where $\Theta_\oplus=\Theta_1\cup\Theta_2$. $\forall\ \varrho\in\Theta_\oplus$,

$$\delta_\oplus(\varrho)=\begin{cases}\Big\{\Big\langle\triangle,(\mathcal{P}_{\delta_{1(\varrho)}}(\triangle),\mathcal{Q}_{\delta_{1(\varrho)}}(\triangle),\mathcal{R}_{\delta_{1(\varrho)}}(\triangle),\mathcal{S}_{\delta_{1(\varrho)}}(\triangle))\Big\rangle; & \text{if } \varrho\in\Theta_1-\Theta_2\Big\},\\ \Big\{\Big\langle\triangle,(\mathcal{P}_{\delta_{2(\varrho)}}(\triangle),\mathcal{Q}_{\delta_{2(\varrho)}}(\triangle),\mathcal{R}_{\delta_{1(\varrho)}}(\triangle),\mathcal{S}_{\delta_{2(\varrho)}}(\triangle))\Big\rangle; & \text{if } \varrho\in\Theta_2-\Theta_1\Big\},\\ \Big\{\Big\langle\triangle,\frac{(\mathcal{P}_{\delta_{1(\varrho)}}(\triangle)+\mathcal{P}_{\delta_{2(\varrho)}}(\triangle))}{2},\frac{(\mathcal{Q}_{\delta_{1(\varrho)}}(\triangle)+\mathcal{Q}_{\delta_{2(\varrho)}}(\triangle))}{2}, \\ \quad\frac{(\mathcal{R}_{\delta_{1(\varrho)}}(\triangle)+\mathcal{R}_{\delta_{2(\varrho)}}(\triangle))}{2},\frac{(\mathcal{S}_{\delta_{1(\varrho)}}(\triangle)+\mathcal{S}_{\delta_{2(\varrho)}}(\triangle))}{2}\Big\rangle; & \text{if } \varrho\in\Theta_1\cap\Theta_2\Big\}.\end{cases}$$

(ii) the operator ⊖ is represented as $(\delta_1,\Theta_1)\ominus(\delta_2,\Theta_2)=(\delta_\ominus,\Theta_\ominus)$, where $\Theta_\ominus=\Theta_1\cup\Theta_2$. $\forall\ \varrho\in\Theta_\ominus$,

$$\delta_{\ominus}(\varrho) = \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle; \quad \text{if } \varrho \in \Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle; \quad \text{if } \varrho \in \Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \frac{2*(\mathcal{P}_{\delta_1(\varrho)}(\triangle)*\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle)}, \frac{(\mathcal{Q}_{\delta_1(\varrho)}(\triangle)+\mathcal{Q}_{\delta_2(\varrho)}(\triangle))}{2}, \right.\right. \\ \left.\left. \frac{(\mathcal{R}_{\delta_1(\varrho)}(\triangle)+\mathcal{R}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{2*(\mathcal{S}_{\delta_1(\varrho)}(\triangle)*\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle)}\right\rangle; \quad \text{if } \varrho \in \Theta_1 \cap \Theta_2\right\}. \end{cases}$$

Example 5.2. A psychiatrist has conducted two counseling sessions with his clients, providing data in the IQ-PNSS format. The results from the first session, denoted as (δ_1, Θ_1), are shown in Table 1, while the second session's results, represented as (δ_2, Θ_2), are displayed in Table 4. To assess the combined outcomes of both sessions, we calculate $(\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2)$ and $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2)$, presenting the corresponding findings in Tables 5 and 6, respectively.

TABLE 4. Subjects with psychological disorders in IQ-PNSS (δ_2, Θ_2) form.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .2, .3, .5, .7\rangle$	$\langle .1, 1, 1, .8\rangle$	$\langle .4, .6, .2, .5\rangle$	$\langle .3, .6, .9, .4\rangle$
$\triangle_2$	$\langle .3, .2, .1, .6\rangle$	$\langle .7, .5, .6, .2\rangle$	$\langle .9, .8, .7, .1\rangle$	$\langle .7, .9, 1, .2\rangle$
$\triangle_3$	$\langle .2, .7, .8, .8\rangle$	$\langle .4, .7, .6, .3\rangle$	$\langle .5, .4, .8, .4\rangle$	$\langle .9, .2, .7, .1\rangle$
$\triangle_4$	$\langle .4, .3, .5, .5\rangle$	$\langle .3, .4, .5, .5\rangle$	$\langle .6, .7, .6, .3\rangle$	$\langle .7, .5, .9, .2\rangle$

(i) The IQ-PNSS $(\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2)$ is shown in Table 5.

TABLE 5. Subjects with psychological disorders in IQ-PNSS $(\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2)$ form.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .30, .25, .40, .60\rangle$	$\langle .35, .70, .65, .50\rangle$	$\langle .40, .55, .45, .45\rangle$	$\langle .20, .40, .60, .45\rangle$
$\triangle_2$	$\langle .15, .25, .25, .70\rangle$	$\langle .85, .40, .75, .10\rangle$	$\langle .65, .80, .80, .20\rangle$	$\langle .55, .75, .85, .25\rangle$
$\triangle_3$	$\langle .35, .55, .50, .65\rangle$	$\langle .35, .60, .65, .45\rangle$	$\langle .30, .35, .75, .60\rangle$	$\langle .60, .35, .80, .30\rangle$
$\triangle_4$	$\langle .50, .60, .45, .45\rangle$	$\langle .25, .40, .55, .60\rangle$	$\langle .60, .70, .70, .35\rangle$	$\langle .45, .50, .85, .45\rangle$

(ii) The IQ-PNSS $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2)$ is shown in Table 6.

Proposition 5.3. *Let (δ_1, Θ_1) and (δ_2, Θ_2) be non-empty over $\mathcal{U}$. Then,*
(i) $(\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2) = (\delta_2, \Theta_2) \oplus (\delta_1, \Theta_1)$;
(ii) $[(\delta_1, \Theta_1)^c \oplus (\delta_2, \Theta_2)^c]^c = (\delta_1, \Theta_1) \oplus (\delta_1, \Theta_1)$.

Proof. (i) Proof straightforward.

(ii) Let

$$(\delta_1, \Theta_1) = \left\{\left\langle \triangle, \mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle)\right\rangle; \varrho \in \Theta_1\right\}, \text{and}$$

TABLE 6. Subjects with psychological disorders in IQ-PNSS $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2)$ form.

$\mathcal{U}$	ANX(e_1)	DEP(e_2)	MS(e_3)	DC(e_4)
$\triangle_1$	$\langle .27, .24, .38, .58\rangle$	$\langle .35, .70, .65, .50\rangle$	$\langle .40, .55, .45, .45\rangle$	$\langle .15, .30, .45, .44\rangle$
$\triangle_2$	$\langle .00, .24, .16, .69\rangle$	$\langle .85, .40, .75, .10\rangle$	$\langle .65, .80, .80, .20\rangle$	$\langle .51, .72, .82, .24\rangle$
$\triangle_3$	$\langle .29, .51, .32, .62\rangle$	$\langle .35, .60, .65, .45\rangle$	$\langle .30, .35, .75, .60\rangle$	$\langle .45, .29, .79, .17\rangle$
$\triangle_4$	$\langle .48, .45, .44, .44\rangle$	$\langle .25, .40, .55, .60\rangle$	$\langle .60, .70, .70, .35\rangle$	$\langle .31, .50, .85, .31\rangle$

$(\delta_2, \Theta_2) = \left\{\left\langle \triangle, \mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle)\right\rangle; \varrho\in\Theta_2\right\}$ be IQ-PNSSs.

Then,

$[(\delta_1, \Theta_1)^c \oplus (\delta_2, \Theta_2)^c]$

$$= \begin{cases} \left\{\left\langle \triangle, (\mathcal{S}_{\delta_1(\varrho)}(\triangle), (1-\mathcal{Q}_{\delta_1(\varrho)}(\triangle)), (1-\mathcal{R}_{\delta_1(\varrho)}(\triangle)), \mathcal{P}_{\delta_1(\varrho)}(\triangle))\right\rangle; \quad \text{if } \varrho\in\Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{S}_{\delta_2(\varrho)}(\triangle), (1-\mathcal{Q}_{\delta_2(\varrho)}(\triangle)), (1-\mathcal{R}_{\delta_1(\varrho)}(\triangle)), \mathcal{P}_{\delta_2(\varrho)}(\triangle))\right\rangle; \quad \text{if } \varrho\in\Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \frac{(\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{((1-\mathcal{Q}_{\delta_1(\varrho)}(\triangle))+(1-\mathcal{Q}_{\delta_2(\varrho)}(\triangle)))}{2}, \right.\right. \\ \left.\left. \qquad \frac{((1-\mathcal{R}_{\delta_1(\varrho)}(\triangle))+(1-\mathcal{R}_{\delta_2(\varrho)}(\triangle)))}{2}, \frac{(\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{2}\right\rangle; \quad \text{if } \varrho\in\Theta_1 \cap \Theta_2\right\}. \end{cases}$$

Now consider,

$[(\delta_1, \Theta_1)^c \oplus (\delta_2, \Theta_2)^c]^c$

$$= \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle; \quad \text{if } \varrho\in\Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle; \quad \text{if } \varrho\in\Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \frac{(\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{(\mathcal{Q}_{\delta_1(\varrho)}(\triangle)+\mathcal{Q}_{\delta_2(\varrho)}(\triangle))}{2}, \right.\right. \\ \left.\left. \qquad \frac{(\mathcal{R}_{\delta_1(\varrho)}(\triangle)+\mathcal{R}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{(\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{2}\right\rangle; \quad \text{if } \varrho\in\Theta_1 \cap \Theta_2\right\}. \end{cases}$$

Hence, $[(\delta_1, \Theta_1)^c \oplus (\delta_2, \Theta_2)^c]^c = (\delta_1, \Theta_1) \oplus (\delta_2, \Theta_2)$. □

Proposition 5.4. *Let* (δ_1, Θ_1) *and* (δ_2, Θ_2) *be non-empty over* $\mathcal{U}$. *Then,*
(i) $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2) = (\delta_2, \Theta_2) \ominus (\delta_1, \Theta_1)$;
(ii) $[(\delta_1, \Theta_1)^c \ominus (\delta_2, \Theta_2)^c]^c = (\delta_1, \Theta_1) \ominus (\delta_1, \Theta_1)$.

Proof. (i) Consider,

$$(\delta_1, \Theta_1)\ominus(\delta_2, \Theta_2) = \begin{cases} \left\{\left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle))\right\rangle; \quad \text{if } \varrho\in\Theta_1 - \Theta_2\right\}, \\ \left\{\left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle))\right\rangle; \quad \text{if } \varrho\in\Theta_2 - \Theta_1\right\}, \\ \left\{\left\langle \triangle, \frac{2*(\mathcal{P}_{\delta_1(\varrho)}(\triangle)*\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle)}, \frac{(\mathcal{Q}_{\delta_1(\varrho)}(\triangle)+\mathcal{Q}_{\delta_2(\varrho)}(\triangle))}{2}, \right.\right. \\ \left.\left. \qquad \frac{(\mathcal{R}_{\delta_1(\varrho)}(\triangle)+\mathcal{R}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{2*(\mathcal{S}_{\delta_1(\varrho)}(\triangle)*\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle)}\right\rangle; \quad \text{if } \varrho\in\Theta_1 \cap \Theta_2\right\}. \end{cases}$$

$$= \begin{cases} \left\{ \left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle)) \right\rangle ; \quad \text{if } \varrho \in \Theta_1 - \Theta_2 \right\}, \\ \left\{ \left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle)) \right\rangle ; \quad \text{if } \varrho \in \Theta_2 - \Theta_1 \right\}, \\ \left\{ \left\langle \triangle, \frac{2*(\mathcal{P}_{\delta_2(\varrho)}(\triangle)*\mathcal{P}_{\delta_1(\varrho)}(\triangle))}{\mathcal{P}_{\delta_2(\varrho)}(\triangle)+\mathcal{P}_{\delta_1(\varrho)}(\triangle)}, \frac{(\mathcal{Q}_{\delta_2(\varrho)}(\triangle)+\mathcal{Q}_{\delta_1(\varrho)}(\triangle))}{2}, \right.\right. \\ \left.\left. \frac{(\mathcal{R}_{\delta_2(\varrho)}(\triangle)+\mathcal{R}_{\delta_1(\varrho)}(\triangle))}{2}, \frac{2*(\mathcal{S}_{\delta_2(\varrho)}(\triangle)*\mathcal{S}_{\delta_1(\varrho)}(\triangle))}{\mathcal{S}_{\delta_2(\varrho)}(\triangle)+\mathcal{S}_{\delta_1(\varrho)}(\triangle)} \right\rangle ; \quad \text{if } \varrho \in \Theta_1 \cap \Theta_2 \right\}. \end{cases}$$

Hence, $(\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2) = (\delta_2, \Theta_2) \ominus (\delta_1, \Theta_1)$.

(ii) Consider,

$(\delta_1, \Theta_1)^c \ominus (\delta_2, \Theta_2)^c$

$$= \begin{cases} \left\{ \left\langle \triangle, (\mathcal{S}_{\delta_1(\varrho)}(\triangle), (1-\mathcal{Q}_{\delta_1(\varrho)}(\triangle)), (1-\mathcal{R}_{\delta_1(\varrho)}(\triangle)), \mathcal{P}_{\delta_1(\varrho)}(\triangle)) \right\rangle ; \quad \text{if } \varrho \in \Theta_1 - \Theta_2 \right\}, \\ \left\{ \left\langle \triangle, (\mathcal{S}_{\delta_2(\varrho)}(\triangle), (1-\mathcal{Q}_{\delta_2(\varrho)}(\triangle)), (1-\mathcal{R}_{\delta_2(\varrho)}(\triangle)), \mathcal{P}_{\delta_2(\varrho)}(\triangle)) \right\rangle ; \quad \text{if } \varrho \in \Theta_2 - \Theta_1 \right\}, \\ \left\{ \left\langle \triangle, \frac{2*(\mathcal{S}_{\delta_1(\varrho)}(\triangle)*\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle)}, \frac{((1-\mathcal{Q}_{\delta_1(\varrho)}(\triangle))+(1-\mathcal{Q}_{\delta_2(\varrho)}(\triangle)))}{2}, \right.\right. \\ \left.\left. \frac{((1-\mathcal{R}_{\delta_1(\varrho)}(\triangle))+(1-\mathcal{R}_{\delta_2(\varrho)}(\triangle)))}{2}, \frac{2*(\mathcal{P}_{\delta_1(\varrho)}(\triangle)*\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle)} \right\rangle ; \quad \text{if } \varrho \in \Theta_1 \cap \Theta_2 \right\}. \end{cases}$$

Then,

$[(\delta_1, \Theta_1)^c \ominus (\delta_2, \Theta_2)^c]^c$

$$= \begin{cases} \left\{ \left\langle \triangle, (\mathcal{P}_{\delta_1(\varrho)}(\triangle), \mathcal{Q}_{\delta_1(\varrho)}(\triangle), \mathcal{R}_{\delta_1(\varrho)}(\triangle), \mathcal{S}_{\delta_1(\varrho)}(\triangle)) \right\rangle ; \quad \text{if } \varrho \in \Theta_1 - \Theta_2 \right\}, \\ \left\{ \left\langle \triangle, (\mathcal{P}_{\delta_2(\varrho)}(\triangle), \mathcal{Q}_{\delta_2(\varrho)}(\triangle), \mathcal{R}_{\delta_2(\varrho)}(\triangle), \mathcal{S}_{\delta_2(\varrho)}(\triangle)) \right\rangle ; \quad \text{if } \varrho \in \Theta_2 - \Theta_1 \right\}, \\ \left\{ \left\langle \triangle, \frac{2*(\mathcal{P}_{\delta_1(\varrho)}(\triangle)*\mathcal{P}_{\delta_2(\varrho)}(\triangle))}{\mathcal{P}_{\delta_1(\varrho)}(\triangle)+\mathcal{P}_{\delta_2(\varrho)}(\triangle)}, \frac{(\mathcal{Q}_{\delta_1(\varrho)}(\triangle)+\mathcal{Q}_{\delta_2(\varrho)}(\triangle))}{2}, \right.\right. \\ \left.\left. \frac{(\mathcal{R}_{\delta_1(\varrho)}(\triangle)+\mathcal{R}_{\delta_2(\varrho)}(\triangle))}{2}, \frac{2*(\mathcal{S}_{\delta_1(\varrho)}(\triangle)*\mathcal{S}_{\delta_2(\varrho)}(\triangle))}{\mathcal{S}_{\delta_1(\varrho)}(\triangle)+\mathcal{S}_{\delta_2(\varrho)}(\triangle)} \right\rangle ; \quad \text{if } \varrho \in \Theta_1 \cap \Theta_2 \right\}. \end{cases}$$

Hence, $[(\delta_1, \Theta_1)^c \ominus (\delta_2, \Theta_2)^c]^c = (\delta_1, \Theta_1) \ominus (\delta_2, \Theta_2)$. □

6. Δ_μ, $\Delta_{\mu,\nu}$ and $\phi_{\mu,\nu}$ operators on IQ-PNSS

We present the operators Δ_μ, $\Delta_{\mu,\nu}$ and $\delta_{\mu,\nu}$ on IQ-PNSS and discuss some of its properties.

Definition 6.1. Let $\mu \in [0,1]$. Then the operator $\Delta_\mu(\delta, \Lambda)$ is represented as,

$$\begin{aligned} \Delta_\mu(\delta, \Lambda) = \big\{\big\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \\ \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\mu)\big(1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big)\big\rangle; \varrho \in \Lambda\big\} \\ = \big\{\big\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \\ \big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\mu)\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\big\rangle; \varrho \in \Lambda\big\}, \\ \text{where } \pi_{\psi(\varrho)}(\triangle) = (1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)). \end{aligned}$$

Proposition 6.2. *Let $\mu, \nu \in [0,1]$ and $\mu \leq \nu$. Then for every IQ-PNSS (δ, Λ) the following hold:*
(i) $\Delta_\mu(\delta, \Lambda) \Subset \Delta_\nu(\delta, \Lambda)$*;*
(ii) $\Delta_0(\delta, \Lambda) = \circledcirc(\delta, \Lambda)$*;*
(iii) $\Delta_1(\delta, \Lambda) = \circledast(\delta, \Lambda)$.

Proof. (i) $\Delta_\mu(\delta, \Lambda) = \big\{\big\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),$
$\big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\mu)\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\big\rangle; \varrho{\in}\Lambda\big\},$
and $\mathcal{S}_\nu(\delta, \Lambda) = \big\{\big\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),$
$\big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\nu)\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\big\rangle; \varrho{\in}\Lambda\big\}.$

Since $\mu \leq \nu$, we have

$$(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu(\pi_{\psi(\varrho)}(\triangle))) \leq (\mathcal{P}_{\psi(\varrho)}(\triangle) + \nu(\pi_{\psi(\varrho)}(\triangle))).$$

Also, $(1-\nu) \leq (1-\mu)$, we have

$$(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\nu)(\pi_{\psi(\varrho)}(\triangle))) \leq (\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-\mu)(\pi_{\psi(\varrho)}(\triangle))).$$

Hence, $\Delta_\mu(\delta, \Lambda) \Subset \Delta_\nu(\delta, \Lambda)$.

(ii) Consider, $\mu = 0$

$$\begin{aligned}\Delta_0(\delta, \Lambda) = \big\{\big\langle\ \triangle, &\big(\mathcal{P}_{\psi(\varrho)}(\triangle) + 0\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),\\ &\big(\mathcal{S}_{\psi(\varrho)}(\triangle) + (1-0)\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\big\rangle; \varrho{\in}\Lambda\big\}\\ =&\big\{\big\langle\ \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \big(\mathcal{Q}_{\psi(\varrho)}(\triangle)\big), \big(\mathcal{R}_{\psi(\varrho)}(\triangle)\big), \big(1-\mathcal{P}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho{\in}\Lambda\big\}\\ =&\circledcirc(\delta, \Lambda).\end{aligned}$$

Hence, $\Delta_0(\delta, \Lambda) = \circledcirc(\delta, \Lambda)$.

(iii) Consider, $\mu = 1$

$$\begin{aligned}\Delta_1(\delta, \Lambda) = \big\{\big\langle\ \triangle, &\big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),\\ &\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho{\in}\Lambda\big\}\\ =&\big\{\big\langle\ \triangle, \big(1-\mathcal{S}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \mathcal{S}_{\psi(\varrho)}(\triangle)\big\rangle; \varrho{\in}\Lambda\big\}\\ =&\circledcirc(\delta, \Lambda).\end{aligned}$$

Hence, $\Delta_1(\delta, \Lambda) = \circledast(\delta, \Lambda)$. □

Remark 6.3. Operators $\circledcirc$ and $\circledast$ are an extension of the operator Δ_μ.

Definition 6.4. Let $\mu, \nu \in [0,1]$ and $\mu + \nu \leq 1$. Then the operator $\Delta_{\mu,\nu}(\delta, \Lambda)$ is represented as,

$$\Delta_{\mu,\nu}(\delta,\Lambda) = \big\{\big\langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle),$$
$$\big(\mathcal{S}_{\psi(\varrho)}(\triangle) + \nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\big\rangle; \varrho \in \Lambda\big\},$$
$$\text{where } \pi_{\psi(\varrho)}(\triangle) = (1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)).$$

Theorem 6.5. *Let $\mu, \nu, \chi \in [0,1]$ and $\mu + \nu \leq 1$. Then for every IQ-PNSS (δ, Λ) the following hold:*
(i) $\Delta_{\mu,\nu}(\delta, \Lambda)$ is a IQ-PNSS;
(ii) If $0 \leq \chi \leq \mu$ then $\Delta_{\chi,\nu}(\delta, \Lambda) \Subset \Delta_{\mu,\nu}(\delta, \Lambda)$;
(iii) If $0 \leq \chi \leq \nu$ then $\Delta_{\mu,\nu}(\delta, \Lambda) \Subset \mathcal{S}_{\mu,\chi}(\delta, \Lambda)$;
(iv) $\Delta_{\mu}(\delta, \Lambda) = \Delta_{\mu,(1-\mu)}(\delta, \Lambda)$;
(v) $\circledcirc(\delta, \Lambda) = \Delta_{0,1}(\delta, \Lambda)$;
(vi) $\circledast(\delta, \Lambda) = \Delta_{1,0}(\delta, \Lambda)$;
(vii) $(\mathcal{S}_{\mu,\nu}(\delta, \Lambda)^c)^c = (\mathcal{S}_{\nu,\mu}(\delta, \Lambda))$.

Proof. (i) Consider,

$$\frac{(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu(\pi_{\psi(\varrho)}(\triangle)))}{2} + (\mathcal{Q}_{\psi(\varrho)}(\triangle)) + (\mathcal{R}_{\psi(\varrho)}(\triangle)) + \frac{(\mathcal{S}_{\psi(\varrho)}(\triangle) + \nu(\pi_{\psi(\varrho)}(\triangle)))}{2}$$
$$= \frac{(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mathcal{S}_{\psi(\varrho)}(\triangle)}{2} + (\mathcal{Q}_{\psi(\varrho)}(\triangle)) + (\mathcal{R}_{\psi(\varrho)}(\triangle)) + (\mu + \nu)\frac{(\pi_{\psi(\varrho)}(\triangle))}{2}$$
$$\leq \frac{(\mathcal{P}_{\psi(\varrho)}(\triangle) + \mathcal{S}_{\psi(\varrho)}(\triangle)}{2} + (\mathcal{Q}_{\psi(\varrho)}(\triangle)) + (\mathcal{R}_{\psi(\varrho)}(\triangle)) + \frac{(1 - \mathcal{P}_{\psi(\varrho)}(\triangle) - \mathcal{S}_{\psi(\varrho)}(\triangle)}{2}$$
$$\leq \frac{1}{2} + 2 < 3. \qquad (\text{Since}, \mu + \nu \leq 1 \text{ and } \mathcal{Q}_{\psi(\varrho)}(\triangle) + \mathcal{R}_{\psi(\varrho)}(\triangle) \leq 2)$$

Hence, $\Delta_{\mu,\nu}(\delta, \Lambda)$ is an IQ-PNSS.

(ii) Consider,

$$\Delta_{\chi,\nu}(\delta,\Lambda) = \big\{\big\langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)+\chi\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle)+\nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\big\rangle; \varrho \in \Lambda\big\},$$

$$\Delta_{\mu,\nu}(\delta,\Lambda) = \big\{\big\langle \triangle, \big(\mathcal{P}_{\psi(\varrho)}(\triangle)+\mu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \big(\mathcal{S}_{\psi(\varrho)}(\triangle)+\nu\big(\pi_{\psi(\varrho)}(\triangle)\big)\big)\big\rangle; \varrho \in \Lambda\big\}.$$

Now

$$(\mathcal{P}_{\psi(\varrho)}(\triangle) + \chi(\pi_{\psi(\varrho)}(\triangle)) \leq (\mathcal{P}_{\psi(\varrho)}(\triangle) + \mu(\pi_{\psi(\varrho)}(\triangle)). \qquad (\text{Since}, \chi \leq \mu)$$

Hence, $\Delta_{\chi,\nu}(\delta, \Lambda) \Subset \Delta_{\mu,\nu}(\delta, \Lambda)$.

(iii) Same like proof (ii).

(iv) Consider,

$$\Delta_{\mu,1-\mu}(\delta,\Lambda) =\{\langle \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle)+\mu(\pi_{\psi(\varrho)}(\triangle))), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (\mathcal{S}_{\psi(\varrho)}(\triangle)+1-\mu(\pi_{\psi(\varrho)}(\triangle)))\rangle; \varrho\in\Lambda\} =\Delta_{\mu}(\delta,\Lambda).$$

Hence, $\Delta_{\mu}(\delta,\Lambda)=\Delta_{\mu,(1-\mu)}(\delta,\Lambda)$.

(v) Let $\mu=0$ and $\nu=1$,

$$\begin{aligned}\Delta_{0,1}(\delta,\Lambda) &=\{\langle \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle)), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (\mathcal{S}_{\psi(\varrho)}(\triangle)+(\pi_{\psi(\varrho)}(\triangle)))\rangle; \varrho\in\Lambda\}\\ &=\{\langle \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle)), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (1-\mathcal{P}_{\psi(\varrho)}(\triangle)))\rangle; \varrho\in\Lambda\}\\ &=\circledcirc(\delta,\Lambda).\end{aligned}$$

Hence, $\circledcirc(\delta,\Lambda)=\Delta_{0,1}(\delta,\Lambda)$.

(vi)Let $\mu=1$ and $\nu=0$,

$$\begin{aligned}\Delta_{1,0}(\delta,\Lambda) &=\{\langle \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle)+(\pi_{\psi(\varrho)}(\triangle))), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (\mathcal{S}_{\psi(\varrho)}(\triangle))\rangle; \varrho\in\Lambda\}\\ &=\{\langle \triangle, (1-\mathcal{S}_{\psi(\varrho)}(\triangle)), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (\mathcal{S}_{\psi(\varrho)}(\triangle)))\rangle; \varrho\in\Lambda\}\\ &=\circledast(\delta,\Lambda).\end{aligned}$$

Hence, $\circledast(\delta,\Lambda)=\Delta_{1,0}(\delta,\Lambda)$.

(vii) Consider,

$$\Delta_{\mu,\nu}(\delta,\Lambda)^c = \{\langle \triangle, (\mathcal{S}_{\psi(\varrho)}(\triangle)+\mu(\pi_{\psi(\varrho)}(\triangle))), (1-\mathcal{Q}_{\psi(\varrho)}(\triangle)), (1-\mathcal{R}_{\psi(\varrho)}(\triangle)), (\mathcal{P}_{\psi(\varrho)}(\triangle)+\nu(\pi_{\psi(\varrho)}(\triangle)))\rangle; \varrho\in\Lambda\}.$$

$$\begin{aligned}(\mathcal{S}_{\mu,\nu}(\delta,\Lambda)^c)^c &=\{\langle \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle)+\nu(\pi_{\psi(\varrho)}(\triangle))), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (\mathcal{S}_{\psi(\varrho)}(\triangle)+\mu(\pi_{\psi(\varrho)}(\triangle)))\rangle; \varrho\in\Lambda\}\\ &=(\Delta_{\nu,\mu}(\delta,\Lambda)).\end{aligned}$$

Hence, $(\Delta_{\mu,\nu}(\delta,\Lambda)^c)^c=(\Delta_{\nu,\mu}(\delta,\Lambda))$. □

Remark 6.6. If $\mu+\nu=1$, then $\Delta_{\mu,\nu}(\delta,\Lambda)=\Delta_{\mu}(\delta,\Lambda)$.

Definition 6.7. Let $\mu,\nu\in[0,1]$ and $\mu+\nu\leq 1$. Then the operator $\phi_{\mu,\nu}(\delta,\Lambda)$ is represented as,

$$\phi_{\mu,\nu}(\delta,\Lambda)=\{\langle \triangle, \mu(\mathcal{P}_{\psi(\varrho)}(\triangle)), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu(\mathcal{S}_{\psi(\varrho)}(\triangle))\rangle; \varrho\in\Lambda\}.$$

Theorem 6.8. *Let $\mu,\nu,\chi\in[0,1]$. Then for every IQ-PNSS (ϕ,Λ) the following hold:*
(i) $\phi_{\mu,\nu}(\delta,\Lambda)$ is an IQ-PNSS;
(ii) If $\mu\leq\chi$ then $\phi_{\mu,\nu}(\delta,\Lambda)\Subset\phi_{\chi,\nu}(\delta,\Lambda)$;
(iii) If $\nu\leq\chi$ then $\phi_{\mu,\nu}(\delta,\Lambda)\Supset\phi_{\mu,\chi}(\delta,\Lambda)$;
(iv) If $\delta\in[0,1]$ then $\phi_{\mu,\nu}(\phi_{\chi,\delta}(\delta,\Lambda))=\phi_{\mu\chi,\nu\delta}(\delta,\Lambda)=\phi_{\chi,\delta}(\mathcal{Q}_{\mu,\nu}(\delta,\Lambda))$;
(v)$(\phi_{\mu,\nu}(\delta,\Lambda)^c)^c=(\mathcal{Q}_{\nu,\mu}(\delta,\Lambda))$.

Proof. (i) Proof straightforward.

(ii) Consider,

$$\phi_{\mu,\nu}(\delta,\Lambda) = \big\{\big\langle\ \triangle, \mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\},$$

$$\phi_{\chi,\nu}(\delta,\Lambda) = \big\{\big\langle\ \triangle, \chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\}.$$

Since, $\mu \leq \chi$, $\mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big) \leq \chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big)$.

Hence, $\phi_{\mu,\nu}(\delta,\Lambda) \Subset \phi_{\chi,\nu}(\delta,\Lambda)$.

(iii) Consider,

$$\phi_{\mu,\nu}(\delta,\Lambda) = \big\{\big\langle\ \triangle, \mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\},$$

$$\phi_{\mu,\chi}(\delta,\Lambda) = \big\{\big\langle\ \triangle, \mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \chi\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\}.$$

Since, $\nu \leq \chi$, $\nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big) \leq \chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big)$.

Hence, $\phi_{\mu,\nu}(\delta,\Lambda) \Supset \phi_{\mu,\chi}(\delta,\Lambda)$.

(iv) Consider,

$$\begin{aligned}\phi_{\mu,\nu}(\phi_{\chi,\delta}(\delta,\Lambda)) &= \big\{\big\langle\ \triangle, \mu\chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\delta\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\}\\ &= \phi_{\mu\chi,\nu\delta}(\delta,\Lambda).\\ \phi_{\chi,\delta}(\phi_{\mu,\nu}(\delta,\Lambda)) &= \big\{\big\langle\ \triangle, \chi\mu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \delta\nu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\}\\ &= \big\{\big\langle\ \triangle, \mu\chi\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \nu\delta\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\}\\ &= \phi_{\mu\chi,\nu\delta}(\delta,\Lambda)).\end{aligned}$$

Hence, $\phi_{\mu,\nu}(\phi_{\chi,\delta}(\delta,\Lambda)) = \mathcal{Q}_{\mu\chi,\nu\delta}(\delta,\Lambda) = \phi_{\chi,\delta}(\phi_{\mu,\nu}(\delta,\Lambda))$.

(v) Consider,

$$\begin{aligned}(\delta,\Lambda)^c &= \big\{\big\langle\ \triangle, \big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big), (1-\mathcal{Q}_{\psi(\varrho)}(\triangle)), (1-\mathcal{R}_{\psi(\varrho)}(\triangle)), \big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\}.\\ \phi_{\mu,\nu}(\delta,\Lambda)^c &= \big\{\big\langle\ \triangle, \mu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big), (1-\mathcal{Q}_{\psi(\varrho)}(\triangle)), (1-\mathcal{R}_{\psi(\varrho)}(\triangle)), \nu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\}.\\ (\phi_{\mu,\nu}(\delta,\Lambda)^c)^c &= \big\{\big\langle\ \triangle, \nu\big(\mathcal{P}_{\psi(\varrho)}(\triangle)\big), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), \mu\big(\mathcal{S}_{\psi(\varrho)}(\triangle)\big)\big\rangle; \varrho\in\Lambda\big\}\\ &= \phi_{\nu,\mu}(\delta,\Lambda).\end{aligned}$$

Hence, $(\phi_{\mu,\nu}(\delta,\Lambda)^c)^c = \phi_{\nu,\mu}(\delta,\Lambda)$. □

7. Concentration ($\mathcal{CO}$) and dilation ($\mathcal{DO}$) operators on IQ-PNSS

We present ($\mathcal{CO}$) and ($\mathcal{DO}$) on IQ-PNSS and discuss its properties.

Definition 7.1. Let (δ,Λ) be an IQ-PNSS over $\mathcal{U}$. Then,

(i) the $\mathcal{CO}$ of (δ,Λ) is represented as,

$$\mathcal{C}(\delta,\Lambda) = \big\{\big\langle\ \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^2, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2)\big\rangle; \varrho\in\Lambda\big\};$$

(ii) the $\mathcal{DO}$ of (δ,Λ) is represented as,

$$\mathcal{D}(\delta,\Lambda)=\big\{\big\langle\ \triangle,((\mathcal{P}_{\psi(\varrho)}(\triangle),)^{\frac{1}{2}},\mathcal{Q}_{\psi(\varrho)}(\triangle),\mathcal{R}_{\psi(\varrho)}(\triangle),(1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle;\varrho\in\Lambda\big\}.$$

Proposition 7.2. *Let $\mathcal{U}$ denote a non-empty set and (δ,Λ) be an IQ-PNSS over $\mathcal{U}$.*
(i) If $\pi_{\psi(\varrho)}(\triangle)=0$, then $\pi_{\mathcal{C}\psi(\varrho)}(\triangle)=0$ iff $\mathcal{P}_{\psi(\varrho)}(\triangle)=0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle)=1$;
(ii) $\circledcirc[\mathcal{C}(\psi,\Lambda)]=\mathcal{C}[\circledcirc(\psi,\Lambda)]$ iff $\mathcal{P}_{\psi(\varrho)}(\triangle)=0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle)=1$;
(iii) $\circledast[\mathcal{C}(\psi,\Lambda)]=\mathcal{C}[\circledast(\psi,\Lambda)]$ iff $\mathcal{S}_{\psi(\varrho)}(\triangle)=0$ or $\mathcal{S}_{\psi(\varrho)}(\triangle)=1$.

Proof. (i) If $\pi_{\psi(\varrho)}(\triangle)=0\Leftrightarrow 1-\mathcal{P}_{\psi(\varrho)}(\triangle)-\mathcal{S}_{\psi(\varrho)}(\triangle)=0$.

$$\begin{aligned}\mathcal{C}(\delta,\Lambda)&=\big\{\big\langle\ \triangle,((\mathcal{P}_{\psi(\varrho)}(\triangle))^2,\mathcal{Q}_{\psi(\varrho)}(\triangle),\mathcal{R}_{\psi(\varrho)}(\triangle),1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2)\big\rangle;\varrho\in\Lambda\big\}\\&=\big\{\big\langle\ \triangle,((\mathcal{P}_{\psi(\varrho)}(\triangle))^2,\mathcal{Q}_{\psi(\varrho)}(\triangle),\mathcal{R}_{\psi(\varrho)}(\triangle),1-\mathcal{P}^2_{\psi(\varrho)}(\triangle))\big\rangle;\varrho\in\Lambda\big\}.\end{aligned}$$

If $\pi_{\mathcal{C}\psi(\varrho)}(\triangle)=0\Leftrightarrow 1-\mathcal{P}_{\psi(\varrho)}(\triangle)-(1-\mathcal{P}^2_{\psi(\varrho)}(\triangle))=0$.
Then $\mathcal{P}_{\psi(\varrho)}(\triangle)(1-\mathcal{P}_{\psi(\varrho)}(\triangle))=0\Leftrightarrow\mathcal{P}_{\psi(\varrho)}(\triangle)=0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle)=1$.

(ii)

$$\circledcirc[\mathcal{C}(\psi,\Lambda)]=\big\{\big\langle\ \triangle,(\mathcal{P}_{\psi(\varrho)}(\triangle),\mathcal{Q}_{\psi(\varrho)}(\triangle),\mathcal{R}_{\psi(\varrho)}(\triangle),1-\mathcal{P}_{\psi(\varrho)}(\triangle))\big\rangle;\varrho\in\Lambda\big\}.\tag{1}$$

Also,

$$\begin{aligned}\mathcal{C}[\circledcirc(\psi,\Lambda)]&=\big\{\big\langle\ \triangle,((\mathcal{P}_{\psi(\varrho)}(\triangle))^2,\mathcal{Q}_{\psi(\varrho)}(\triangle),\mathcal{R}_{\psi(\varrho)}(\triangle),1-(1-(1-\mathcal{P}_{\psi(\varrho)}(\triangle)))^2)\big\rangle;\varrho\in\Lambda\big\}\\&=\big\{\big\langle\ \triangle,((\mathcal{P}_{\psi(\varrho)}(\triangle))^2,\mathcal{Q}_{\psi(\varrho)}(\triangle),\mathcal{R}_{\psi(\varrho)}(\triangle),1-\mathcal{P}^2_{\psi(\varrho)}(\triangle))\big\rangle;\varrho\in\Lambda\big\}.\end{aligned}\tag{2}$$

By solving (1) and (2), we get

$$\begin{aligned}\circledcirc[\mathcal{C}(\psi,\Lambda)]=\mathcal{C}[\circledcirc(\psi,\Lambda)]&\Leftrightarrow 1-\mathcal{P}_{\psi(\varrho)}(\triangle)=1-\mathcal{P}^2_{\psi(\varrho)}(\triangle).\\&\Leftrightarrow\mathcal{P}_{\psi(\varrho)}(\triangle)(1-\mathcal{P}_{\psi(\varrho)}(\triangle))=0.\\&\Leftrightarrow\mathcal{P}_{\psi(\varrho)}(\triangle)=0\text{ or }\mathcal{P}_{\psi(\varrho)}(\triangle)=1.\end{aligned}$$

(iii)

$$\circledast[\mathcal{C}(\psi,\Lambda)]=\big\{\big\langle\ \triangle,(1-(1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2),\mathcal{Q}_{\psi(\varrho)}(\triangle),\mathcal{R}_{\psi(\varrho)}(\triangle),1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2)\big\rangle;\varrho\in\Lambda\big\}.\tag{3}$$

Also,

$$\mathcal{C}[\circledast(\psi,\Lambda)]=\big\{\big\langle\ \triangle,((1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2,\mathcal{Q}_{\psi(\varrho)}(\triangle),\mathcal{R}_{\psi(\varrho)}(\triangle),1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2)\big\rangle;\varrho\in\Lambda\big\}.\tag{4}$$

By solving (3) and (4), we get

$$\begin{aligned}\circledast[\mathcal{C}(\psi,\Lambda)] = \mathcal{C}[\circledast(\psi,\Lambda)] &\Leftrightarrow 1-(1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2) = 1-\mathcal{S}_{\psi(\varrho)}(\triangle).\\ &\Leftrightarrow (1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2 = 1-\mathcal{S}_{\psi(\varrho)}(\triangle).\\ &\Leftrightarrow \mathcal{S}_{\psi(\varrho)}(\triangle)(1-\mathcal{S}_{\psi(\varrho)}(\triangle)) = 0.\\ &\Leftrightarrow \mathcal{S}_{\psi(\varrho)}(\triangle) = 0 \text{ or } \mathcal{S}_{\psi(\varrho)}(\triangle) = 1.\end{aligned}$$

□

Proposition 7.3. *Let $\mathcal{U}$ denote a non-empty set and (δ,Λ) be an IQ-PNSS over $\mathcal{U}$.*
(i) If $\pi_{\psi(\varrho)}(\triangle) = 0$, then $\pi_{\mathcal{D}\psi(\varrho)}(\triangle) = 0$ iff $\mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$;
(ii) $\circledcirc[\mathcal{D}(\psi,\Lambda)] = \mathcal{D}[\circledcirc(\psi,\Lambda)]$ iff $\mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$;
(iii) $\circledast[\mathcal{D}(\psi,\Lambda)] = \mathcal{D}[\circledast(\psi,\Lambda)]$ iff $\mathcal{S}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{S}_{\psi(\varrho)}(\triangle) = 1$.

Proof. If $\pi_{\psi(\varrho)}(\triangle) = 0 \Leftrightarrow 1-\mathcal{P}_{\psi(\varrho)}(\triangle)-\mathcal{S}_{\psi(\varrho)}(\triangle) = 0$.

$$\begin{aligned}\mathcal{D}(\delta,\Lambda) =& \big\{\big\langle\, \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}\\ =& \big\{\big\langle\, \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}.\end{aligned}$$

If $\pi_{\mathcal{D}\psi(\varrho)}(\triangle) = 0 \Leftrightarrow 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} - 1 - (\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} = 0 \Leftrightarrow (\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} = (\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}$.
Then $\mathcal{P}_{\psi(\varrho)}(\triangle)(1-\mathcal{P}_{\psi(\varrho)}(\triangle)) = 0 \Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle) = 0$ or $\mathcal{P}_{\psi(\varrho)}(\triangle) = 1$.

(ii)

$$\circledcirc\,[\mathcal{D}(\psi,\Lambda)] = \big\{\big\langle\, \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}. \tag{5}$$

Also,

$$\begin{aligned}\mathcal{D}[\circledcirc(\psi,\Lambda)] &= \big\{\big\langle\, \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-(1-\mathcal{P}_{\psi(\varrho)}(\triangle)))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}\\ &= \big\{\big\langle\, \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}.\end{aligned} \tag{6}$$

By solving (5) and (6), we get

$$\begin{aligned}\circledcirc[\mathcal{D}(\psi,\Lambda)] = \mathcal{D}[\circledcirc(\psi,\Lambda)] &\Leftrightarrow 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} = 1-(\mathcal{P}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}.\\ &\Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle)(1-\mathcal{P}_{\psi(\varrho)}(\triangle)) = 0.\\ &\Leftrightarrow \mathcal{P}_{\psi(\varrho)}(\triangle) = 0 \text{ or } \mathcal{P}_{\psi(\varrho)}(\triangle) = 1.\end{aligned}$$

(iii)

$$\circledast\,[\mathcal{D}(\psi,\Lambda)] = \big\{\big\langle\, \triangle, (1-(1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle)))^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}. \tag{7}$$

Also,

$$\mathcal{D}[\circledast(\psi,\Lambda)] = \big\{\big\langle\, \triangle, (1-\mathcal{S}_{\psi(\varrho)}(\triangle)^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), 1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})\big\rangle; \varrho\in\Lambda\big\}. \tag{8}$$

By solving (7) and (8), we get

$$\circledast[\mathcal{D}(\psi,\Lambda)] = \mathcal{D}[\circledast(\psi,\Lambda)] \Leftrightarrow 1-(\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}} = 1-(\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}.$$
$$\Leftrightarrow \mathcal{S}_{\psi(\varrho)}(\triangle)(1-\mathcal{S}_{\psi(\varrho)}(\triangle)) = 0.$$
$$\Leftrightarrow \mathcal{S}_{\psi(\varrho)}(\triangle) = 0 \text{ or } \mathcal{S}_{\psi(\varrho)}(\triangle) = 1.$$

□

Proposition 7.4. *For any IQ-PNSS (ψ,Λ), $\mathcal{C}(\psi,\Lambda) \Subset (\psi,\Lambda) \Subset \mathcal{D}(\psi,\Lambda)$.*

Proof. We have

$$(\psi,\Lambda) = \left\{\left\langle \triangle, \mathcal{P}_{\delta(\varrho)}(\triangle), \mathcal{Q}_{\delta(\varrho)}(\triangle), \mathcal{R}_{\delta(\varrho)}(\triangle), \mathcal{S}_{\delta(\varrho)}(\triangle)\right\rangle;\ \varrho\in\Lambda\right\}.$$

$$\mathcal{C}(\delta,\Lambda) = \{\langle\ \triangle, (\mathcal{P}_{\psi(\varrho)}(\triangle), \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\delta(\varrho)}(\triangle), (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2))\rangle; \varrho\in\Lambda\}.$$

Since, $\mathcal{S}_{\delta(\varrho)}(\triangle) \in [0,1], (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^2) \geq \mathcal{S}_{\delta(\varrho)}(\triangle)$.

$$\text{Hence, } \mathcal{C}(\delta,\Lambda) \Subset (\psi,\Lambda). \tag{9}$$

$$\mathcal{D}(\delta,\Lambda) = \{\langle\ \triangle, ((\mathcal{P}_{\psi(\varrho)}(\triangle),)^{\frac{1}{2}}, \mathcal{Q}_{\psi(\varrho)}(\triangle), \mathcal{R}_{\psi(\varrho)}(\triangle), (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}}))\rangle; \varrho\in\Lambda\}.$$

Since, $\mathcal{P}_{\psi(\varrho)}(\triangle), \mathcal{S}_{\psi(\varrho)}(\triangle) \in [0,1]$,
$\mathcal{P}_{\psi(\varrho)}(\triangle) \leq (\mathcal{P}_{\psi(\varrho)}(\triangle),)^{\frac{1}{2}}$, $\mathcal{S}_{\psi(\varrho)}(\triangle) \geq (1-(1-\mathcal{S}_{\psi(\varrho)}(\triangle))^{\frac{1}{2}})$.

$$\text{Hence, } (\psi,\Lambda) \Subset \mathcal{D}(\delta,\Lambda). \tag{10}$$

By solving (9) and (10), we get $\mathcal{C}(\psi,\Lambda) \Subset (\psi,\Lambda) \Subset \mathcal{D}(\psi,\Lambda)$. □

8. Conclusion

We have discussed the concepts of Intuitionistic Quadri-Partitioned Neutrosophic Sets (IQ-PNS) and Intuitionistic Quadri-Partitioned Neutrosophic Soft Sets (IQ-PNSS), thoroughly examining their essential properties. Our primary aim has been to tackle the challenges that experts encounter when managing truth, indeterminacy, and falsity values under defined constraints. The framework we propose not only offers a robust approach to addressing these complexities but also holds potential for future applications involving hypersoft sets, paving the way for further research and exploration in this area.

References

1. Abdel-Basset, M.; Mohamed, M.; Elhoseny, M. Cosine similarity measures of bipolar neutrosophic set for diagnosis of bipolar disorder diseases. Artif. Intell. Med. 2019.
2. Anita Shanthi, S.; Thillaigovindan, N.; Vadivel Naidx, J. Application of Interval Valued Intuitionistic Fuzzy Soft Sets of Root Type in Decision Making. ICTACT J. Soft Comput. 2016, 6, 1224-1230.
3. Atanassov, K.T. Intuitionistic Fuzzy Sets. Fuzzy Sets Syst. 1986, 20, 87-96.
4. Atanassov, K.T. New Operations Defined Over the Intuitionistic Fuzzy Sets. Fuzzy Sets Syst. 1994, 61, 137-142.
5. Beg, I.; Tabasam, R. Group Decision Making Using Comparative Linguistic Expression Based on Hesitant Intuitionistic Fuzzy Sets. Appl. Appl. Math. 2015, 10, 1082-1092.
6. Bhowmik, M.; Pal, M. Intuitionistic Neutrosophic Set. J. Inf. Comput. Sci. 2008, 4, 142-152.
7. Broumi, S.; Smarandache, F. Several Similarity Measures of Neutrosophic Sets. Neutrosophic Sets Syst. 2013, 1, 54-62.
8. Broumi, S.; Smarandache, F. Intuitionistic Neutrosophic Soft Set. J. Inf. Comput. Sci. 2013, 8, 130-140.
9. Broumi, S.; Talea, M.; Smarandache, F.; Bakali, A. Single Valued Neutrosophic Graphs: Degree, Order and Size. IEEE Int. Conf. Fuzzy Syst. 2016, 2444-2451.
10. Chahhtterjee, R.; Majumdar, P.; Samanta, S.K. Similarity Measures in Neutrosophic Sets-I. Fuzzy Multi-Criteria Decision-Making Using Neutrosophic Sets. Stud. Fuzziness Soft Comput. 2019, 369, 249-294.
11. Chatterjee, R.; Majumdar, P.; Samanta, S.K. On Some Similarity Measures and Entropy on Quadri partitioned Single Valued Neutrosophic Sets. J. Int. Fuzzy Syst. 2016, 30, 2475-2485.
12. Chicaiza, C.; Paspuel, O.; Cuesta, P.; Hernandez, S. Neutrosophic Psychology for Emotional Intelligence Analysis in Students of the Autonomous University of Los Andes, Ecuador. Neutrosophic Sets Syst. 2020, 34, 1-8.
13. Chinnadurai, V.; Smarandache, F.; Bobin, A. Multi-Aspect Decision-Making Process in Equity Investment Using Neutrosophic Soft Matrices. Neutrosophic Sets Syst. 2020, 31, 224-241.
14. Chinnadurai, V.; Bobin, A. Multiple-Criteria Decision Analysis Process by Using Prospect Decision Theory in Interval-Valued Neutrosophic Environment. CAAI Trans. Intell. Technol. 2020, 5, 209-221.
15. Chinnadurai, V.; Bobin, A. Simplified Intuitionistic Neutrosophic Soft Set and its Application on Diagnosing Psychological Disorder by Using Similarity Measure. Applications and Applied Mathematics: An International Journal (AAM), 2021, 16(1),605-630.
16. Chinnadurai, V.; Bobin, A. Interval Valued Intuitionistic Neutrosophic Soft Set and its Application on Diagnosing Psychiatric Disorder by Using Similarity Measure. Neutrosophic Sets and Systems, 2021, 41, 215-245.
17. Christianto, V.; Smarandache, F. A Review of Seven Applications of Neutrosophic Logic: In Cultural Psychology, Economics Theorizing, Conflict Resolution, Philosophy of Science, etc. J-Multidiscip. Sci. J. 2019, 2, 128-137.
18. Hamidi, M.; Borumand Saeid, A. Achievable Single-Valued Neutrosophic Graphs in Wireless Sensor Networks. New Math. Nat. Comput. 2018, 14, 157-185.
19. Hashim, R.M.; Gulistan, M.; Smarandache, F. Applications of Neutrosophic Bipolar Fuzzy Sets in HOPE Foundation for Planning to Build a Children Hospital with Different Types of Similarity Measures. Symmetry 2018, 10, 331.
20. Jianming, Z.; Khan, M.; Gulistan, M.; Ali, A. Applications of Neutrosophic Cubic Sets in Multi-Criteria Decision-Making. Int. J. Uncertain. Quantif. 2017, 7, 377-394.
21. Lix, D.; Lix, G.; Lix, Z. Some Similarity Measures of Neutrosophic Sets Based on the Euclidean Distance and Their Application in Medical Diagnosis. Comput. Math. Methods Med. 2018, 2018, 1-9.

22. Majid Khan, M.; Gulistan, M.; Yaqoob, N.; Smarandache, F. Neutrosophic Cubic Einstein Geometric Aggregation Operators with Application to Multi-Criteria Decision Making Method. Symmetry 2019, 11, 247.
23. Molodtsov, D. Soft Set Theory - First Results. Comput. Math. Appl. 1999, 37, 19-31.
24. Saranya, S.; Vigneshwaran, M.; Jafari, S. C # Application to Deal with Neutrosophic g μ - Closed Sets in Neutrosophic Topology. Appl. Appl. Math. 2020, 15, 226-239.
25. Shahzadi, G.; Akram, M.; Borumand Saeid, A. An Application of Single-Valued Neutrosophic Sets in Medical Diagnosis. Neutrosophic Sets Syst. 2017, 18, 80-88.
26. Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Am. Res. Press. 1999.
27. Smarandache, F. Neutropsychic Personality: A Mathematical Approach to Psychology (Third Enlarged Edition). 2018.
28. Smarandache, F. Degree of Dependence and Independence of the Subcomponents of Fuzzy Set and Neutrosophic Set. Neutrosophic Sets Syst. 2016, 11, 95-97.
29. Smarandache, F. Extension of Soft Set to Hypersoft Set, and Then to Plithogenic Hypersoft Set. Neutrosophic Sets Syst. 2018, 22, 168-170.
30. Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single-Valued Neutrosophic Sets. Rev. Air Force Acad. 2010, 1, 10-14.
31. Zadeh, L.A. Fuzzy Sets. Inf. Control. 1965, 8, 338-353.

Received: Oct 2, 2024. Accepted: Jan 20, 2025