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Implication-Based Neutrosophic Finite State Machine over a Finite Group

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Abstract. In this paper, we define and investigate the concept of the implication-based neutrosophic finite state machine (IB-IFSA) over a finite group. Building upon the framework of neutrosophic logic, we introduce the implication-based neutrosophic kernel and the implication-based neutrosophic subsemiautomaton, establishing their formal definitions and properties. Additionally, we explore the structural interplay between these concepts and their implications within algebraic systems.

Keywords: Implication-based neutrosophic finite state machine, implication-based neutrosophic subgroup, neutrosophic kernel, neutrosophic subsemiautomaton, finite group theory, neutrosophic logic.

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1. **Introduction**

In 1965, Zadeh [13] introduced the concept of fuzzy sets, providing a mathematical framework to model uncertainty and imprecision. This idea catalyzed significant developments in various fields, including automata theory. In 1969, Wee [10] proposed the concept of fuzzy automata, extending Zadeh's work to computational models designed for learning and decision-making processes. Later, in 1971, Rosenfeld [5] applied fuzzy set theory to algebraic group structures, which opened new research directions in the study of fuzzy algebraic systems.

The study of fuzzy subgroups, especially fuzzy normal subgroups, further enriched the algebraic applications of fuzzy logic. Notable contributions in this domain include the works of Dib and Hassan [1], Malik et al. [2], and Mukherjee and Bhattacharya [3], which investigated the structural properties of fuzzy groups. Additionally, Asok Kumar [4] explored the products of fuzzy subgroups, providing deeper insights into their composition and interactions.

The introduction of neutrosophic logic by Smarandache [7,8] significantly expanded the theoretical foundation of uncertainty modeling. Neutrosophic sets generalize fuzzy sets by introducing three membership degrees: truth, indeterminacy, and falsity. This enriched framework allows for a more comprehensive representation of incomplete, inconsistent, or indeterminate information, which is especially useful in fields requiring nuanced reasoning under uncertainty.

In this paper, we extend the application of neutrosophic logic to automata theory by defining the implication-based neutrosophic finite state machine (IB-IFSA) over a finite group, also referred to as the implication-based neutrosophic semiautomaton. We introduce and analyze the implication-based neutrosophic kernel and the implication-based neutrosophic subsemiautomaton, exploring their algebraic properties within the context of finite groups. By integrating neutrosophic logic with computational and algebraic systems, this work provides a foundational framework for further research, with potential applications in decision-making models, automated reasoning, and beyond.

2. **Preliminaries**

This section introduces the foundational concepts and definitions of neutrosophic sets, neutrosophic groups, and their properties, forming the mathematical framework for integrating neutrosophic logic with algebraic structures.

Definition 2.1. [9] *Let $\mathbb{U}$ be an initial universe set and $A \subseteq \mathbb{U}$, a* ***neutrosophic set*** *(more precisely, a single valued neutrosophic set) over $\mathbb{U}$ (**SVN-set** for short), denoted by $\widetilde{A} = \langle \mathbb{U}, \mu_A, \sigma_A, \omega_A \rangle$, is a set of the form*

$$\widetilde{A} = \{(u, \mu_A(u), \sigma_A(u), \omega_A(u)) : u \in \mathbb{U}\}$$

where $\mu_A : \mathbb{U} \to I$, $\sigma_A : \mathbb{U} \to I$ and $\omega_A : \mathbb{U} \to I$ are the ***membership function****, the* ***indeterminacy function*** *and the* ***nonmembership function*** *of A, respectively. For every $u \in \mathbb{U}$, $\mu_A(u)$, $\sigma_A(u)$ and $\omega_A(u)$ are said the* ***degree of membership****, the* ***degree of indeterminacy*** *and the* ***degree of nonmembership*** *of u, respectively.*

Definition 2.2. *Let $(\Omega, \cdot)$ be a group, a neutrosophic set $\widetilde{A} = \langle \Omega, \mu_A, \sigma_A, \omega_A \rangle$ over Ω is called a* ***neutrosophic group*** *on Ω, if the following conditions hold:*

(i) $\begin{cases} \mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} \\ \sigma_A(xy) \geq \min\{\sigma_A(x), \sigma_A(y)\} \\ \omega_A(xy) \leq \max\{\omega_A(x), \omega_A(y)\} \end{cases}$, *for every* $x, y \in \Omega$

(ii) $\begin{cases} \mu_A(x^{-1}) \geq \mu_A(x) \\ \sigma_A(x^{-1}) \geq \sigma_A(x) \\ \omega_A(x^{-1}) \leq \omega_A(x) \end{cases}$, *for every* $x \in \Omega$.

Definition 2.3. *Let* $(\Omega, \cdot)$ *be a group, a neutrosophic set* $\widetilde{A} = \langle \Omega, \mu_A, \sigma_A, \omega_A \rangle$ *over* Ω *is called a* ***neutrosophic normal subgroup*** *on* Ω *if:*

$$\begin{cases} \mu_A(xyx^{-1}) \geq \mu_A(y) \\ \sigma_A(xyx^{-1}) \geq \sigma_A(y) \\ \omega_A(xyx^{-1}) \leq \omega_A(y) \end{cases}$$

for every $x, y \in \Omega$.

Notation 2.4. *Let* $\mathbb{U}$ *be a universe set and* $(\Omega, \cdot)$ *a group. In neutrosophic logic, the truth value of a neutrosophic proposition* α *is denoted by* $[\alpha]$. *The following notations are used to describe the neutrosophic logical operations and their corresponding set-theoretical interpretations in this paper:*

$$(\nu \in \widetilde{A}) \equiv \widetilde{A}(\nu),$$
$$(\alpha \wedge \beta) \equiv \min\{[\alpha], [\beta]\},$$
$$(\alpha \to \beta) \equiv \min\{1, 1 - [\alpha] + [\beta]\},$$
$$(\forall \nu\, \alpha(\nu)) \equiv \inf_{\nu \in \mathbb{U}} [\alpha(\nu)],$$
$$(\exists \nu\, \alpha(\nu)) \equiv \sup_{\nu \in \mathbb{U}} [\alpha(\nu)],$$
$$\vDash \alpha \quad \textit{if and only if} \quad [\alpha] = 1 \quad \textit{for all valuations.}$$

This framework establishes the basis for applying neutrosophic logic within the algebraic context analyzed in this study. The truth valuation rules adopted follow the Łukasiewicz system of continuous-valued logic.
Additionally, the concept of λ*-tautology, as introduced by Ying [11], is defined as* $\vDash_\lambda \alpha$ *if and only if* $[\alpha] \geq \lambda$ *for all valuations.*

The following notions in neutrosophic theory are inspired by the frameworks outlined in [12] and [6].

Definition 2.5. *Let* $\widetilde{A} = \langle \Omega, \mu_A, \sigma_A, \omega_A \rangle$ *over* Ω *be a neutrosophic set on a finite group* Ω, *and let* $\lambda \in (0, 1]$ *be a fixed number. If, for any* $\nu_1, \nu_2 \in \Omega$, *the following conditions hold:*

(i) $\vDash_\lambda \big((\nu_1 \in \widetilde{A}) \wedge (\nu_2 \in \widetilde{A}) \to (\nu_1\nu_2 \in \widetilde{A})\big)$,

(ii) $\vDash_\lambda \big((\nu_1 \in \widetilde{A}) \to (\nu_1^{-1} \in \widetilde{A})\big)$

*then $\widetilde{A}$ is called an **implication-based neutrosophic subgroup** of Ω.*

Definition 2.6. *Let $\widetilde{A}$ be an implication-based neutrosophic subgroup of Ω, $\lambda \in (0, 1]$ a fixed number, and $f : \Omega \to \Omega$ a function defined on Ω. Then the **implication-based neutrosophic subgroup** $\widetilde{B}$ of $f(\Omega)$ is defined as:*

$$\vDash_\lambda \left(\exists \nu\{(\nu \in \widetilde{A})\}; \nu \in f^{-1}(\psi)\right) \to (\psi \in \widetilde{B}), \quad \forall \psi \in f(\Omega).$$

Conversely, if $\widetilde{B}$ is an implication-based neutrosophic subgroup of $f(\Omega)$, then the implication-based neutrosophic subgroup $\widetilde{A} = f \circ \widetilde{B}$ in Ω is defined as:

$$\vDash_\lambda \left((f(\nu) \in \widetilde{B}) \to (\nu \in \widetilde{A})\right), \quad \forall \nu \in \Omega,$$

*and is called the **pre-image** of $\widetilde{A}$ under f.*

Definition 2.7. *An implication-based neutrosophic subgroup $\widetilde{A}$ of Ω is called an **implication-based neutrosophic normal subgroup** if:*

$$\vDash_\lambda \left((\nu\psi \in \widetilde{A}) \to (\psi\nu \in \widetilde{A})\right), \quad \forall \nu, \psi \in \Omega,$$

where $\lambda \in]0, 1]$ is a fixed number.

Proposition 2.8. *Let $\widetilde{A} = \langle \Omega, \mu_A, \sigma_A, \omega_A \rangle$ be an implication-based neutrosophic subgroup of a finite group $(\Omega, \cdot)$. Then, for any $\nu \in \Omega$, the following holds:*

$$\vDash_\lambda (\nu \in \widetilde{A}) \to (\epsilon \in \widetilde{A}),$$

where ϵ is the identity element of the group Ω.

From this point onward, let Ω be a finite group with identity element ϵ and $\lambda \in]0, 1]$ a fixed parameter.

3. Implication-Based Neutrosophic Semiautomaton over a Finite Group

The concept of an implication-based neutrosophic semiautomaton combines the algebraic structure of finite groups with the logical framework of neutrosophic logic. This section formalizes the definition of a semiautomaton in this context, highlighting the interactions between group elements, logical variables, and neutrosophic conditions. The proposed model provides a versatile framework for analyzing transitions and behaviors in systems characterized by uncertainty, indeterminacy, and truth variability.

Definition 3.1. *Let $\mathbb{N}_{\mathcal{A}} = \langle A, B, C \rangle$ be an implication-based neutrosophic subgroup of a finite group Ω. An **implication-based neutrosophic semiautomaton** over the finite group $(\Omega, \cdot)$ is a triple $\mathbb{NS} = (\Omega, \Delta, \mathbb{N}_{\mathcal{A}})$, where Δ represents the set of all logical variables.*

Remark 3.2. *The set Δ of the definition above consists of all logical variables used to describe transitions within the semiautomaton. Each variable in Δ encodes a specific logic operation or transition condition.*

Notation 3.3. *If Δ is a set of logical variables, Δ^* denote the set of all finite combinations of these logical variables, including the special element **0**, which represents the null operation or an empty transition.*

Definition 3.4. *Let $\mathbb{NS} = (\Omega, \Delta, \mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the finite group Ω. Define $\mathbb{N}_{\mathcal{A}^*} = \langle A^*, B^*, C^* \rangle$ in $\Omega \times \Delta^* \times \Omega$ such that, for every $\alpha, \beta \in \Omega$, $\xi \in \Delta^*$, and $\omega \in \Delta$, the following conditions hold:*

(i) $\models_\lambda ((\alpha, \boldsymbol{0}, \beta) \in A^*) \to 0$ *(with $\lambda = 0$)*

(ii) $\models_\lambda ((\alpha, \boldsymbol{0}, \beta) \in B^*) \to 0$ *(with $\lambda = 0$)*

(iii) $\models_\lambda ((\alpha, \boldsymbol{0}, \beta) \in C^*) \to 1$

(iv) $\models_\lambda \big(\exists\gamma\{((\beta, \xi, \gamma) \in A^*) \wedge ((\gamma, \omega, \alpha) \in A^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \omega, \alpha) \in A^*)$

(v) $\models_\lambda \big(\exists\gamma\{((\beta, \xi, \gamma) \in B^*) \wedge ((\gamma, \omega, \alpha) \in B^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \omega, \alpha) \in B^*)$

(vi) $\models_\lambda \big(\forall\gamma\{((\beta, \xi, \gamma) \in C^*) \vee ((\gamma, \omega, \alpha) \in C^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \omega, \alpha) \in C^*)$

Theorem 3.5. *Let $\mathbb{NS} = (\Omega, \Delta, \mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the finite group Ω. Then, for all $\alpha, \beta \in \Omega$ and $\xi, \psi \in \Delta^*$, the following hold:*

(1) $\models_\lambda \big(\exists\gamma\{((\beta, \xi, \gamma) \in A^*) \wedge ((\gamma, \psi, \alpha) \in A^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \psi, \alpha) \in A^*)$

(2) $\models_\lambda \big(\exists\gamma\{((\beta, \xi, \gamma) \in B^*) \wedge ((\gamma, \psi, \alpha) \in B^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \psi, \alpha) \in B^*)$

(3) $\models_\lambda \big(\forall\gamma\{((\beta, \xi, \gamma) \in C^*) \vee ((\gamma, \psi, \alpha) \in C^*)\}; \gamma \in \Omega\big) \to ((\beta, \xi \odot \psi, \alpha) \in C^*)$

Proof. Let $\alpha, \beta \in \Omega$ and $\xi, \psi \in \Delta^*$. The proof proceeds by induction on $ord(\psi) = n$.

Base Case ($n = 0$): If $n = 0$, then $\psi = \mathbf{0}$ and $\xi \odot \psi = \xi \odot \mathbf{0} = \xi$. Thus:

$$\begin{aligned}
&\models_\lambda (\exists\gamma\{((\beta, \xi, \gamma) \in A^*) \wedge ((\gamma, \psi, \alpha) \in A^*)\}; \gamma \in \Omega) \\
&\to (\exists\gamma\{((\beta, \xi, \gamma) \in A^*) \wedge ((\gamma, \mathbf{0}, \alpha) \in A^*)\}; \gamma \in \Omega) \\
&\to ((\beta, \xi, \alpha) \in A^*) \\
&\to ((\beta, \xi \odot \psi, \alpha) \in A^*).
\end{aligned}$$

Similarly, the result holds for B^* and C^* using analogous reasoning. Thus, the theorem is valid for $n = 0$.

Inductive Step: Assume that the result holds for any $\zeta \in \Delta^*$ such that $ord(\zeta) = n - 1$ and $n > 0$. Let $\psi \in \Delta^*$ such that $\psi = \zeta \odot \omega$ with $\zeta \in \Delta^*$, $\omega \in \Delta$, and $ord(\zeta) = n - 1$. We need to show the result for ψ.

For A^* w have that:

$$\vDash_\lambda (\exists\gamma\{((\beta,\xi,\gamma)\in A^*)\wedge((\gamma,\psi,\alpha)\in A^*)\};\gamma\in\Omega)$$
$$\to (\exists\gamma\{((\beta,\xi,\gamma)\in A^*)\wedge((\gamma,\zeta\odot\omega,\alpha)\in A^*)\};\gamma\in\Omega)$$
$$\to (\exists\gamma\{((\beta,\xi,\gamma)\in A^*)\wedge(\exists\delta\{((\gamma,\zeta,\delta)\in A^*)\wedge((\delta,\omega,\alpha)\in A^*)\};\delta\in\Omega)\};\gamma\in\Omega)$$
$$\to (\exists\delta\{((\beta,\xi\odot\zeta,\delta)\in A^*)\wedge((\delta,\omega,\alpha)\in A^*)\};\delta\in\Omega)$$
$$\to ((\beta,\xi\odot\zeta\odot\omega,\alpha)\in A^*)$$
$$\to ((\beta,\xi\odot\psi,\alpha)\in A^*).$$

For B^* and C^* the proofs follow similar steps, substituting B^* and C^* for A^*, with appropriate adjustments for logical operations.
So, by using the Induction Principle, the result holds for all $\psi\in\Delta^*$ with $ord(\psi)=n$ and this completes the proof. □

Definition 3.6. *Let $\mathbb{N}_{\mathcal{A}}=\langle A,B,C\rangle$ be an implication-based neutrosophic subgroup over a finite group $(\Omega,\cdot)$, and let $\mathbb{NS}=(\Omega,\Delta,\mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the same finite group Ω. A neutrosophic subset $\langle\iota,\overline{A},\overline{B},\overline{C}\rangle$ of the group Ω is called an **implication-based neutrosophic subsemiautomaton** of $\mathbb{NS}$ if the following conditions hold:*

(i) $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ *is an implication-based neutrosophic subgroup of* Ω.
(ii) $\vDash_\lambda \big(((\alpha,\xi,\beta)\in A)\wedge(\alpha\in\overline{A})\big)\to(\beta\in\overline{A})$.
(iii) $\vDash_\lambda \big(((\alpha,\xi,\beta)\in B)\wedge(\alpha\in\overline{B})\big)\to(\beta\in\overline{B})$.
(iv) $\vDash_\lambda \big((\beta\in\overline{C})\big)\to\big(((\alpha,\xi,\beta)\in C)\vee(\alpha\in\overline{C})\big)$,

for all $\alpha,\beta\in\Omega$ and $\xi\in\Delta$.

Theorem 3.7. *Let $\mathbb{NS}=\big(\Omega,\Delta,\mathbb{N}_{\tilde{\mathcal{A}}}\big)$ be an implication-based neutrosophic semiautomaton over the finite group $(\Omega,\cdot)$. Let $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ be an implication-based neutrosophic subgroup of Ω. Then $\mathbb{N}_{\tilde{\mathcal{A}}}$ is an implication-based neutrosophic subsemiautomaton of $\mathbb{NS}$ if and only if the following conditions hold:*

(1) $\vDash_\lambda \big(((\alpha,\xi,\beta)\in A^*)\wedge(\alpha\in\overline{A})\big)\to(\beta\in\overline{A})$
(2) $\vDash_\lambda \big(((\alpha,\xi,\beta)\in B^*)\wedge(\alpha\in\overline{B})\big)\to(\beta\in\overline{B})$
(3) $\vDash_\lambda \big((\beta\in\overline{C})\big)\to\big(((\alpha,\xi,\beta)\in C^*)\vee(\alpha\in\overline{C})\big)$

for all $\alpha,\beta\in\Omega$ and $\xi\in\Delta^*$.

Proof. Let $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ be an implication-based neutrosophic subsemiautomaton of $\mathbb{NS}$. Let $\alpha,\beta\in\Omega$ and $\xi\in\Delta^*$. We prove the theorem by induction on $ord(\xi)=n$.

Base Case: If $n = 0$, then $\xi = \mathbf{0}$. For A^*:

$$\begin{aligned}\vDash_\lambda \big(((\alpha,\xi,\beta)\in A^*)\wedge(\alpha\in\overline{A})\big) &\to \big(((\alpha,\mathbf{0},\beta)\in A^*)\wedge(\alpha\in\overline{A})\big)\\ &\to \big(0\in A^*\big)\wedge(\alpha\in\overline{A})\\ &\to (\beta\in\overline{A}).\end{aligned}$$

Analogously, the same logic applies to B^*, while for $\overline{C}$, we have:

$$\vDash_\lambda (\beta\in\overline{C}) \to ((\alpha,\mathbf{0},\beta)\in C^*)\vee(\alpha\in\overline{C}) \to ((\alpha,\xi,\beta)\in C^*)\vee(\alpha\in\overline{C}).$$

Thus, the base case holds.

Inductive Step: Assume the result is true for all $\psi\in\Delta^*$ such that $ord(\psi) = n-1$ and $n > 0$. Let $\xi = \psi\odot\omega$ where $\omega\in\Delta$. For A^*:

$$\begin{aligned}\vDash_\lambda \big(((\alpha,\xi,\beta)\in A^*)\wedge(\alpha\in\overline{A})\big) &\to \big(((\alpha,\psi\odot\omega,\beta)\in A^*)\wedge(\alpha\in\overline{A})\big)\\ &\to \big(\exists\gamma\{((\alpha,\psi,\gamma)\in A^*)\wedge((\gamma,\omega,\beta)\in A)\};\gamma\in\Omega\big)\wedge(\alpha\in\overline{A})\\ &\to \big(\exists\gamma\{(\gamma\in\overline{A})\wedge((\gamma,\omega,\beta)\in A)\};\gamma\in\Omega\big)\\ &\to (\beta\in\overline{A}).\end{aligned}$$

The case for B^* is analogous, while for $\overline{C}$, we have that:

$$\begin{aligned}\vDash_\lambda (\beta\in\overline{C}) &\to \big(\forall\gamma\{((\gamma,\omega,\beta)\in C)\vee(\gamma\in\overline{C})\};\gamma\in\Omega\big)\\ &\to \big(\forall\gamma\{((\alpha,\psi,\gamma)\in C^*)\vee((\gamma,\omega,\beta)\in C)\};\gamma\in\Omega\big)\vee(\alpha\in\overline{C})\\ &\to ((\alpha,\xi,\beta)\in C^*)\vee(\alpha\in\overline{C}).\end{aligned}$$

Thus, by induction, the forward implication has been proven for all $ord(\xi) = n$, showing that if $\mathbb{N}_{\tilde{\mathcal{A}}}$ satisfies the conditions of an implication-based neutrosophic kernel, then the stated properties hold for A^*, B^*, and C^*.
Since the converse is trivial, the theorem is completely proved. □

Definition 3.8. *Let $\mathbb{N}_{\mathcal{A}} = \langle A, B, C\rangle$ be an implication-based neutrosophic subgroup over a finite group $(\Omega,\cdot)$, and let $\mathbb{NS} = (\Omega,\Delta,\mathbb{N}_{\mathcal{A}})$ be an implication-based neutrosophic semiautomaton over the finite group $(\Omega,\cdot)$. A neutrosophic subset $\langle\iota,\overline{A},\overline{B},\overline{C}\rangle$ of the group Ω is called an* ***implication-based neutrosophic kernel*** *of $\mathbb{NS}$ if the following conditions are satisfied:*

(i) $\mathbb{N}_{\tilde{\mathcal{A}}} = \langle\overline{A},\overline{B},\overline{C}\rangle$ *is an implication-based neutrosophic normal subgroup of* Ω.
(ii) $\vDash_\lambda \big(((\beta\kappa,\xi,\alpha)\in A)\wedge((\beta,\xi,\gamma)\in A)\wedge(\kappa\in\overline{A})\big)\to(\alpha\gamma^{-1}\in\overline{A})$.
(iii) $\vDash_\lambda \big(((\beta\kappa,\xi,\alpha)\in B)\wedge((\beta,\xi,\gamma)\in B)\wedge(\kappa\in\overline{B})\big)\to(\alpha\gamma^{-1}\in\overline{B})$.
(iv) $\vDash_\lambda \big((\alpha\gamma^{-1}\in\overline{C})\big)\to\big(((\beta\kappa,\xi,\alpha)\in C)\vee((\beta,\xi,\gamma)\in C)\vee(\kappa\in\overline{C})\big)$,

for all $\alpha,\beta,\gamma,\kappa\in\Omega$ and $\xi\in\Delta$.

Theorem 3.9. *Let* $\mathbb{NS} = (\Omega, \Delta, \mathbb{N}_{\mathbb{A}})$ *be an implication-based neutrosophic semiautomaton over the finite group* $(\Omega, \cdot)$ *and* $\mathbb{N}_{\tilde{\mathbb{A}}} = \langle \overline{A}, \overline{B}, \overline{C} \rangle$ *be an implication-based neutrosophic subgroup of* Ω*. Then* $\mathbb{N}_{\tilde{\mathbb{A}}}$ *is an implication-based neutrosophic kernel of* $\mathbb{NS}$ *if and only if the following conditions hold:*

(1) $\models_\lambda \big(((\beta\kappa, \xi, \alpha) \in A^*) \wedge ((\beta, \xi, \gamma) \in A^*) \wedge (\kappa \in \overline{A})\big) \rightarrow (\alpha\gamma^{-1} \in \overline{A})$

(2) $\models_\lambda \big(((\beta\kappa, \xi, \alpha) \in B^*) \wedge ((\beta, \xi, \gamma) \in B^*) \wedge (\kappa \in \overline{B})\big) \rightarrow (\alpha\gamma^{-1} \in \overline{B})$

(3) $\models_\lambda \big((\alpha\gamma^{-1} \in \overline{C})\big) \rightarrow \big(((\beta\kappa, \xi, \alpha) \in C^*) \vee ((\beta, \xi, \gamma) \in C^*) \vee (\kappa \in \overline{C})\big)$

for all $\alpha, \beta, \gamma, \kappa \in \Omega$ *and* $\xi \in \Delta^*$.

Proof. Let $\mathbb{N}_{\tilde{\mathbb{A}}} = \langle \overline{A}, \overline{B}, \overline{C} \rangle$ be an implication-based neutrosophic kernel of $\mathbb{NS}$. We prove the result by induction on $ord(\xi) = n$.

Base Case: If $n = 0$, then $\xi = \mathbf{0}$. For A^* we have:

$$\begin{aligned}
\models_\lambda &\big(((\beta\kappa, \xi, \alpha) \in A^*) \wedge ((\beta, \xi, \gamma) \in A^*) \wedge (\kappa \in \overline{A})\big) \\
&\rightarrow \big(((\beta\kappa, \mathbf{0}, \alpha) \in A^*) \wedge ((\beta, \mathbf{0}, \gamma) \in A^*) \wedge (\kappa \in \overline{A})\big) \\
&\rightarrow \big(0 \wedge 0 \wedge (\kappa \in \overline{A})\big) \\
&\rightarrow (\alpha\gamma^{-1} \in \overline{A}).
\end{aligned}$$

Similarly, for B^* the argument follows the same reasoning, while for $\overline{C}$ we have that:

$$\begin{aligned}
\models_\lambda \big((\alpha\gamma^{-1} \in \overline{C})\big) &\rightarrow 1 \vee 1 \vee (\kappa \in \overline{C}) \\
&\rightarrow ((\beta\kappa, \mathbf{0}, \alpha) \in C^*) \vee ((\beta, \mathbf{0}, \gamma) \in C^*) \vee (\kappa \in \overline{C}) \\
&\rightarrow ((\beta\kappa, \xi, \alpha) \in C^*) \vee ((\beta, \xi, \gamma) \in C^*) \vee (\kappa \in \overline{C}).
\end{aligned}$$

Thus, the base case holds.

Inductive Step: Assume the result holds for all $\psi \in \Delta^*$ such that $ord(\psi) = n - 1$. Let $\xi = \psi \odot \omega$ where $\omega \in \Delta$. For A^* we have:

$$\begin{aligned}
&\models_\lambda ((\beta\kappa, \xi, \alpha) \in A^*) \wedge ((\beta, \xi, \gamma) \in A^*) \wedge (\kappa \in \overline{A}) \\
&\rightarrow ((\beta\kappa, \psi \odot \omega, \alpha) \in A^*) \wedge ((\beta, \psi \odot \omega, \gamma) \in A^*) \wedge (\kappa \in \overline{A}) \\
&\rightarrow (\exists\zeta\{((\beta\kappa, \psi, \zeta) \in A^*) \wedge ((\zeta, \omega, \alpha) \in A)\}; \zeta \in \Omega) \wedge (\exists\eta\{((\beta, \psi, \eta) \in A^*) \wedge \\
&\qquad ((\eta, \omega, \gamma) \in A)\}; \gamma \in \Omega) \wedge (\kappa \in \overline{A})
\end{aligned}$$

$$\rightarrow \big(\exists\eta\{\exists\zeta\{((\beta\kappa,\psi,\alpha)\in A^*)\wedge((\alpha,\omega,\alpha)\in A)\wedge(\kappa\in\overline{A})\wedge((\beta,\psi,\eta)\in A^*)\wedge ((\eta,\omega,\gamma)\in A)\};\zeta\in\Omega\};\eta\in\Omega)$$

$$\rightarrow \big(\exists\eta\{\exists\zeta\{((\beta\kappa,\psi,\alpha)\in A^*)\wedge((\beta,\psi,\eta)\in A^*)\wedge(\kappa\in\overline{A})\wedge((\zeta,\omega,\alpha)\in A)\wedge ((\eta,\omega,\gamma)\in A)\};\zeta\in\Omega\};\eta\in\Omega)$$

$$\rightarrow \big(\exists\eta\{\exists\zeta\{(\zeta\eta^{-1}\in\overline{A})\wedge((\zeta,\omega,\alpha)\in A)\wedge((\eta,\omega,\gamma)\in A)\};\zeta\in\Omega\};\eta\in\Omega\big)$$

$$\rightarrow \big(\exists\eta\{\exists\zeta\{((\eta,\omega,\gamma)\in A)\wedge(((\eta.\eta^{-1})\alpha,\omega,\alpha)\in A)\wedge(\alpha\eta^{-1}\in\overline{A})\};\zeta\in\Omega\};\eta\in\Omega\big)$$

$$\rightarrow \big(\exists\eta\{\exists\zeta\{((\eta,\omega,\gamma)\in A)\wedge((\eta.(\alpha\eta^{-1}),\omega,\alpha)\in A)\wedge(\zeta\eta^{-1}\in\overline{A})\};\zeta\in\Omega\};\eta\in\Omega\big)$$

$$\rightarrow \big(\exists\eta\{\exists\zeta\{((\eta.(\zeta\eta^{-1}),\omega,\alpha)\in A)\wedge((\eta,\omega,\gamma)\in A)\wedge(\zeta\eta^{-1}\in\overline{A})\};\zeta\in\Omega\};\eta\in\Omega\big)$$

$$\rightarrow (\alpha\gamma^{-1}\in\overline{A})$$

Analogously for B^*, while for $\overline{C}$ we have that:

$$\vDash_\lambda (\alpha\gamma^{-1}\in\overline{C})$$

$$\rightarrow \big(\forall\eta\{\forall\zeta\{((\eta.(\zeta\eta^{-1}),\omega,\alpha)\in C)\vee((\eta,\omega,\gamma)\in C)\vee(\zeta\eta^{-1}\in\overline{C})\};\zeta\in\Omega\};\eta\in\Omega\big)$$

$$\rightarrow \big(\forall\eta\{\forall\zeta\{((\eta,\omega,\gamma)\in C)\vee((\eta.(\zeta\eta^{-1}),\omega,\alpha)\in C)\vee(\zeta\eta^{-1}\in\overline{C})\};\zeta\in\Omega\};\eta\in\Omega\big)$$

$$\rightarrow \big(\forall\eta\{\forall\zeta\{((\eta,\omega,\gamma)\in C)\vee(((\eta.\eta^{-1})\zeta,\omega,\alpha)\in C)\vee(\zeta\eta^{-1}\in\overline{C})\};\zeta\in\Omega\};\eta\in\Omega\big)$$

$$\rightarrow \big(\forall\eta\{\forall\zeta\{(\zeta\eta^{-1}\in\overline{C})\vee((\zeta,\omega,\alpha)\in C)\vee((\eta,\omega,\gamma)\in C)\};\zeta\in\Omega\};\eta\in\Omega\big)$$

$$\rightarrow \big(\forall\eta\{\forall\zeta\{((\beta\kappa,\psi,\alpha)\in C^*)\vee((\beta,\psi,\eta)\in C^*)\vee(\kappa\in\overline{C})\vee((\zeta,\omega,\alpha)\in C)\vee ((\eta,\omega,\gamma)\in C)\};\zeta\in\Omega\};\eta\in\Omega)$$

$$\rightarrow \big(\forall\eta\{\forall\zeta\{((\beta\kappa,\psi,\alpha)\in C^*)\vee((\zeta,\omega,\alpha)\in C)\vee(\kappa\in\overline{C})\vee((\beta,\psi,\eta)\in C^*)\vee ((\eta,\omega,\gamma)\in C)\};\zeta\in\Omega\};\eta\in\Omega)$$

$$\rightarrow (\forall\zeta\{((\beta\kappa,\psi,\alpha)\in C^*)\vee((\zeta,\omega,\alpha)\in C)\};\zeta\in\Omega)\vee(\forall\eta\{((\beta,\psi,\eta)\in C^*)\vee ((\eta,\omega,\gamma)\in C)\};\eta\in\Omega)\vee(\kappa\in\overline{C})$$

$$\rightarrow ((\beta\kappa,\psi\odot\omega,\alpha)\in C^*)\vee((\beta,\psi\odot\omega,\gamma)\in C^*)\vee(\kappa\in\overline{C})$$

$$\rightarrow ((\beta\kappa,\xi,\alpha)\in C^*)\vee((\beta,\xi,\gamma)\in C^*)\vee(\kappa\in\overline{C})$$

Thus, the result holds for $ord(\xi)=n$ and so, by applying the Principle of Induction, the forward implication has been established.

Since the reverse implication is straightforward, the proof is complete. □

Theorem 3.10. *Let $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ be an implication-based neutrosophic kernel of the implication-based neutrosophic semiautomaton $\mathbb{NS}=(\Omega,\Delta,\mathbb{N}_{\mathcal{A}})$ over the finite group $(\Omega,\cdot)$. Then $\mathbb{N}_{\tilde{\mathcal{A}}}$ is an implication-based neutrosophic subsemiautomaton if and only if the following conditions are satisfied:*

(1) $\models_\lambda \big(((\epsilon,\xi,\alpha)\in A)\wedge(\epsilon\in\overline{A})\big)\to(\alpha\in\overline{A})$

(2) $\models_\lambda \big(((\epsilon,\xi,\alpha)\in B)\wedge(\epsilon\in\overline{B})\big)\to(\alpha\in\overline{B})$

(3) $\models_\lambda \big(\alpha\in\overline{C}\big)\to\big(((\epsilon,\xi,\alpha)\in C)\vee(\epsilon\in\overline{C})\big)$

for all $\alpha\in\Omega$ and $\xi\in\Delta$, where ϵ is the identity element of the finite group $(\Omega,\cdot)$.

Proof. Let $\mathbb{N}_{\tilde{\mathcal{A}}}=\langle\overline{A},\overline{B},\overline{C}\rangle$ be an implication-based neutrosophic kernel of $\mathbb{NS}$.
Assume $\mathbb{N}_{\tilde{\mathcal{A}}}$ satisfies:

(i) $\models_\lambda \big(((\epsilon,\xi,\alpha)\in A)\wedge(\epsilon\in\overline{A})\big)\to(\alpha\in\overline{A})$,

(ii) $\models_\lambda \big(((\epsilon,\xi,\alpha)\in B)\wedge(\epsilon\in\overline{B})\big)\to(\alpha\in\overline{B})$,

(iii) $\models_\lambda \big((\alpha\in\overline{C})\big)\to\big(((\epsilon,\xi,\alpha)\in C)\vee(\epsilon\in\overline{C})\big)$,

for all $\alpha\in\Omega$ and $\xi\in\Delta$.

Using these conditions, for A^*, we have:

$$\models_\lambda \big(((\alpha,\xi,\beta)\in A)\wedge(\alpha\in\overline{A})\big)\to\big((\beta\in\overline{A})\big).$$

Similarly, for B^*, we deduce:

$$\models_\lambda \big(((\alpha,\xi,\beta)\in B)\wedge(\alpha\in\overline{B})\big)\to(\beta\in\overline{B}).$$

Finally for $\overline{C}$, we conclude:

$$\models_\lambda \big((\beta\in\overline{C})\big)\to\big(((\alpha,\xi,\beta)\in C)\vee(\alpha\in\overline{C})\big).$$

Thus, $\mathbb{N}_{\tilde{\mathcal{A}}}$ satisfies the definition of an implication-based neutrosophic subsemiautomaton.

Conversely, suppose $\mathbb{N}_{\tilde{\mathcal{A}}}$ is an implication-based neutrosophic subsemiautomaton. Then:

$$\models_\lambda \big(((\epsilon,\xi,\alpha)\in A)\wedge(\epsilon\in\overline{A})\big)\to(\alpha\in\overline{A}),$$

$$\models_\lambda \big(((\epsilon,\xi,\alpha)\in B)\wedge(\epsilon\in\overline{B})\big)\to(\alpha\in\overline{B}),$$

$$\models_\lambda \big((\alpha\in\overline{C})\big)\to\big(((\epsilon,\xi,\alpha)\in C)\vee(\epsilon\in\overline{C})\big).$$

Thus, $\mathbb{N}_{\tilde{\mathcal{A}}}$ satisfies the necessary conditions and the theorem is completely proved. □

4. Conclusions

This study has explored the extension of neutrosophic logic to the algebraic framework of finite groups through the introduction of implication-based neutrosophic subgroups. The analysis has focused on key properties of these subgroups, including their connection to the identity element and their behavior under group operations. Additionally, the concept of implication-based neutrosophic normal subgroups has been introduced, providing a systematic approach to studying these structures within group theory.

The results obtained generalize existing theories of fuzzy subgroups and demonstrate the capacity of neutrosophic logic to incorporate varying degrees of truth, indeterminacy, and falsity into algebraic reasoning. By parametrizing these notions with λ, the proposed framework allows for a refined and adjustable characterization of subgroup properties, enriching the theoretical understanding of algebraic systems under uncertainty.

Potential directions for future research include the extension of these methods to more complex algebraic structures, such as rings and fields, as well as the development of computational techniques for their practical application. Furthermore, exploring the interaction between neutrosophic logic and other branches of mathematics may reveal additional insights and broaden the scope of this approach in both theoretical and applied contexts.

References

[1] Dib K.A., Hassan A.A.M. The fuzzy normal subgroup. *Fuzzy Sets and Systems*, **98**, pp. 393-402, 1998.

[2] Malik D.S., Mordeson J.N., Nair P. Fuzzy normal subgroups into fuzzy subgroups. *Journal of the Korean Mathematical Society*, **29**, pp. 1-8, 1992.

[3] Mukherjee N.P., Bhattacharya P. Fuzzy normal subgroups and fuzzy cosets. *Information Sciences*, **34**, pp. 225-239, 1984.

[4] Ray A.K. On product of fuzzy subgroups. *Fuzzy Sets and Systems*, **105**, pp. 181-183, 1999.

[5] Rosenfeld A. Fuzzy groups. *Journal of Mathematical Analysis and Applications*, **35**, pp. 512-517, 1971.

[6] Selvarathi M., Spinneli A. Implication-based fuzzy normal subgroup of a finite group. *International Journal of Applied Engineering Research*, **10**(80), pp. 5-8, 2015.

[7] Smarandache F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. *American Research Press*, Rehoboth, IFM, 1999.

[8] Smarandache F. Neutrosophic set—a generalization of the fuzzy set. *International Journal of Pure and Applied Mathematics*, **24**(3), pp. 287-297, 2005.

[9] Wang H., Smarandache F., Zhang Y.Q., Sunderraman R. Single Valued Neutrosophic Sets. *Technical Sciences and Applied Mathematics*, pp. 10-14, 2012.

[10] Wee K. William G. A formation of fuzzy automata and its application as a model of learning systems. *IEEE Transactions on Systems Science and Cybernetics*, **3**, pp. 215-223, 1969.

[11] Ying M. A new approach for fuzzy topology (i). *Fuzzy Sets and Systems*, **39**, pp. 303-321, 1997.

[12] Yuan Y.R., Zhang C. Generalized fuzzy groups and many-valued implications. *Fuzzy Sets and Systems*, **138**, pp. 205-211, 2003.

[13] Zadeh L.A. Fuzzy sets. *Information Control*, **8**, pp. 338-353, 1965.

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