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The improper neutrosophic integrals

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Abstract: In light of the expansion of research that dealt with the logic of the neutrosophic and its clear spread at the level of science, it was necessary for us to continue in the series of research that covers the calculus of neutrosophic integration, and this is what led us to study the improper neutrosophic integrals. This study dealt with the types of the improper neutrosophic integrals of the first and second kind. In addition, we discussed the concept of convergent and divergent neutrosophic integrals. As well as providing a set of examples related to these ideas.

Keywords: neutrosophic, integrals, improper, limit, divergent, converges.

1. Introduction and Preliminaries

In contrast to the current logics, Smarandache suggested the Neutrosophic Logic to describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction. Smarandache introduced the concept of neutrosophy as a new school of philosophy [4]. He presented the definition of the standard form of neutrosophic real number [3-5], the Neutrosophic statistics [5-6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1]. A number of studies in the area of integration and differentiation were given by Y. Alhasan [8-11-14], also he presented the definition of the concept of neutrosophic complex numbers and its properties, in addition, he studied the general exponential form of a neutrosophic complex number [2-9]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [12]. The AH isometry was used to study many structures such as conic sections, real analysis concepts, and geometrical surfaces [11-15].

Definition [15]

Let $f: R(I) \rightarrow R(I)$; $f = f(X)$ and $X = x + yI \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable. a neutrosophic real function $f(X)$ written as follows:

$$f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$$

2. Main Discussion

2.1 Improper neutrosophic integrals of the first kind

Definition 1

We will consider the improper neutrosophic integral whose integrands are bound on an infinite interval. Let $f(x,I)$ be a function defined on the infinite interval $[a+bI,+\infty)$, and integrable over any bounded closed interval $[a+bI,k]\subset[a+bI,+\infty)$, then we call $f(x,I)$ over the interval $[a+bI,+\infty)$ the improper neutrosophic integral and write:

$$\int_{a+bI}^{+\infty} f(x,I)\,dx = \lim_{k\to+\infty}\int_{a+bI}^{k} f(x,I)\,dx$$

We call the improper neutrosophic integral convergent when the limit exists, and the neutrosophic integral is of first kind, where it has defined limit, it is the value of the neutrosophic integral. Otherwise the improper neutrosophic integral is divergent (when the improper integral doesn't exist or equal to $\pm\infty$).

Example 1

$$\int_{1+I}^{+\infty}\frac{dx}{x+3+5I} = \lim_{k\to+\infty}\int_{1+I}^{k}\frac{dx}{x+3+5I} = \lim_{k\to+\infty} ln(x+3+5I)|_{1+I}^{k}$$

$$= \lim_{k\to+\infty} ln(x+3+5I)|_{1+I}^{k} = \lim_{k\to+\infty}[ln(k+3+5I) - ln(4+6I)]$$

$$= \lim_{k\to+\infty}[ln(k+3+5I)] - ln(4+6I)$$

by applying:

$$f(X) = f(x+yI) = f(x) + I[f(x+y) - f(x)]$$

we get on:

$$\int_{1+I}^{+\infty}\frac{dx}{x+3+5I} = \lim_{k\to+\infty}\big[ln(k+3) + I[ln(k+3+5) - ln(k+3)]\big] - \big[ln\,4 + I[ln\,10 - ln\,4]\big]$$

$$= \lim_{k\to+\infty}\left[ln(k+3) + I\left[ln\left(\frac{k+3+5}{k+3}\right)\right]\right] - \left[ln\,4 + I\,ln\frac{5}{2}\right]$$

$$= +\infty + I[ln(1)] - \left[ln\,4 + I\,ln\tfrac{5}{2}\right] = +\infty - I\,ln\tfrac{5}{2} = \infty_I$$

(the improper neutrosophic integral diverges)

Example 2

$$\int_{0+0I}^{+\infty} e^{-(3+3I)x}\,dx = \lim_{k\to+\infty}\int_{0+0I}^{k} e^{-(3+3I)x}\,dx = \lim_{k\to+\infty}\frac{-1}{3+3I}e^{-(3+3I)x}\Big|_{0+0I}^{k}$$

$$= \frac{-1}{3+3I}\lim_{k\to+\infty}\big[e^{-(3+3I)k} - 1\big]$$

by applying:

$$f(X) = f(x+yI) = f(x) + I[f(x+y) - f(x)]$$

we get on:

$$\int_{0+0I}^{+\infty} e^{-(3+3I)x}\,dx = \frac{-1}{3+3I}\lim_{k\to+\infty}[e^{-3k} + I[e^{-6k} - e^{-3k}] - 1]$$

$$= \frac{1}{3+3I} = \frac{1}{3} - \frac{1}{6}I$$

the improper neutrosophic integral converges.

Definition 2

We will consider the improper neutrosophic integral whose integrands are bound on an infinite interval. Let $f(x,I)$ be a function defined on the infinite interval $(-\infty, c+dI]$, and integrable over any bounded closed interval $[k, c+dI] \subset (-\infty, c+dI]$, then we call $f(x,I)$ over the interval $(-\infty, c+dI]$ the improper neutrosophic integral and write:

$$\int_{-\infty}^{c+dI} f(x,I)\,dx = \lim_{k\to-\infty}\int_{k}^{c+dI} f(x,I)\,dx$$

We call the improper neutrosophic integral convergent when the limit exists, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper neutrosophic integral is divergent (when the improper integral doesn't exist or equal to $\pm\infty$).

Example 3

$$\int_{-\infty}^{0+0I} cos\left(x + \frac{\pi}{3}I\right)dx = \lim_{k\to-\infty}\int_{k}^{0+0I} cos\left(x + \frac{\pi}{3}I\right)dx$$

$$= \lim_{k\to-\infty}\left[sin\left(x + \frac{\pi}{3}I\right)\right]\Big|_{k}^{0+0I} = sin\left(\frac{\pi}{3}I\right) - \lim_{k\to-\infty} sin\left(k + \frac{\pi}{3}I\right)$$

$$= I\frac{\sqrt{3}}{2} - \lim_{k\to-\infty}\left(sin(k) + I\left[sin\left(k + \frac{\pi}{3}\right) - sin(k)\right]\right)$$

(Does not exist, so the improper neutrosophic integral is divergent)

Example 4

$$\int_{-\infty}^{0+0I} (x+2I)e^{x}dx = \lim_{k\to-\infty}\int_{k}^{0+0I} (x+2I)e^{x}dx$$

$$u = (x+2I) \implies du = dx$$
$$v\,dx = e^{x}dx \implies v = e^{x}$$

then:

$$\int_{-\infty}^{0+0I} (x+2I)e^{x}dx = \lim_{k\to-\infty}[(x+2I)e^{x}]|_{k}^{0+0I} - \lim_{k\to-\infty}\int_{k}^{0+0I} e^{x}dx$$

$$= \lim_{k\to-\infty}[(x+2I)e^{x}]|_{k}^{0+0I} - \lim_{k\to-\infty}\int_{k}^{0+0I} e^{x}dx = 2I - \lim_{k\to-\infty}\left((k+2I)e^{k}\right) - \lim_{k\to-\infty} e^{x}|_{k}^{0+0I}$$

$$= 2I - \lim_{k\to-\infty}(ke^k + 2Ie^k) - 1 + \lim_{k\to-\infty} e^k = 2I - 1$$

Theorem 1

$$\int_{1+I}^{+\infty}\frac{dx}{x^p} = \begin{cases}\frac{p}{(1+I)^{p-1}} & if \quad p > 1\\ \text{Divergent} & if \quad p \le 1\end{cases}$$

Example 5

$$\int_{1+I}^{+\infty}\frac{dx}{x^2} = \lim_{k\to+\infty}\int_{1+I}^{k}\frac{dx}{x^2} = \lim_{k\to+\infty}\frac{-2}{x}\Big|_{1+I}^{k}$$

$$= \lim_{k\to+\infty}\left[\frac{-2}{k} + \frac{2}{1+I}\right] = \frac{2}{1+I} = 2 - I$$

the improper neutrosophic integral converges.

Example 6

$$\int_{1+I}^{+\infty}\frac{dx}{x^3} = \lim_{k\to+\infty}\int_{1+I}^{k}\frac{dx}{x^3} = \lim_{k\to+\infty}\frac{-3}{x^2}\Big|_{1+I}^{k}$$

$$= \lim_{k\to+\infty}\left[\frac{-3}{k^2} + \frac{3}{(1+I)^2}\right] = \frac{3}{(1+I)^2} = \frac{3}{1+3I} = 3 - \frac{9}{4}I$$

the improper neutrosophic integral converges.

Example 7

$$\int_{1+I}^{+\infty}\frac{dx}{x^4} = \lim_{k\to+\infty}\int_{1+I}^{k}\frac{dx}{x^4} = \lim_{k\to+\infty}\frac{-4}{x^3}\Big|_{1+I}^{k}$$

$$= \lim_{k\to+\infty}\left[\frac{-4}{k^3} + \frac{4}{(1+I)^3}\right] = \frac{4}{(1+I)^3} = \frac{4}{1+7I} = 4 - \frac{7}{2}I$$

the improper neutrosophic integral converges.

Example 8

$$\int_{1+I}^{+\infty}\frac{dx}{x} = \lim_{k\to+\infty}\int_{1+I}^{k}\frac{dx}{x} = \lim_{k\to+\infty} ln\,x\Big|_{1+I}^{k}$$

$$= \lim_{k\to+\infty}[ln\,k + ln(1+I)] = \lim_{k\to+\infty} ln\,k - (ln\,1 + I[ln\,2 - ln\,1])$$

$= \infty - I\,ln\,2 = \infty_I$ (the improper neutrosophic integral diverges)

Example 9

$$\int_{2+2I}^{+\infty}\frac{dx}{x(ln\,x)^2} = \lim_{k\to+\infty}\int_{2+2I}^{k}\frac{dx}{x(ln\,x)^2} == \lim_{k\to+\infty}\frac{-1}{ln\,x}\Big|_{2+2I}^{k}$$

$$= \lim_{k\to+\infty}\frac{-1}{ln\,x} + \frac{1}{ln(2+2I)} = \frac{1}{ln\,2 + I[ln\,4 - ln\,2]} = \frac{1}{ln\,2 + I\,ln\,2}$$

$$= \frac{1}{ln\,2} - \frac{ln\,2}{ln\,2\,(2\,ln\,2)} = \frac{1}{ln\,2} - \frac{1}{2\,ln\,2}I$$

(the improper neutrosophic integral converges)

Remark 1
The improper neutrosophic integral of $f(x,I)$ over $(-\infty,\infty)$ is defined as:

$$\int_{-\infty}^{+\infty} f(x,I)\,dx = \int_{-\infty}^{c} f(x,I)\,dx + \int_{c}^{+\infty} f(x,I)\,dx$$

Example 10

$$\int_{-\infty}^{+\infty} \frac{dx}{1+3I+x^2} = \int_{-\infty}^{0+0I} \frac{dx}{1+3I+x^2} + \int_{0+0I}^{+\infty} \frac{dx}{1+3I+x^2}$$

$$= \lim_{k_1\to-\infty} \int_{k_1}^{0+0I} \frac{dx}{1+3I+x^2} + \lim_{k_2\to+\infty} \int_{0+0I}^{k_2} \frac{dx}{1+3I+x^2}$$

$$= \lim_{k_1\to-\infty} \frac{1}{1+I} tan^{-1} \frac{x}{1+I}\bigg|_{k_1}^{0+0I} + \lim_{k_2\to+\infty} \frac{1}{1+I} tan^{-1} \frac{x}{1+I}\bigg|_{0+0I}^{k_2}$$

$$= \lim_{k_1\to-\infty} \left[\frac{1}{1+I} tan^{-1}(0) - \frac{1}{1+I} tan^{-1} \frac{k_1}{1+I}\right] + \lim_{k_2\to+\infty} \left[\frac{1}{1+I} tan^{-1} \frac{k_2}{1+I} - \frac{1}{1+I} tan^{-1}(0)\right]$$

$$= \lim_{k_1\to-\infty} \left[-\frac{1}{1+I} tan^{-1} \left(1-\frac{1}{2}I\right) k_1\right] + \lim_{k_2\to+\infty} \left[\frac{1}{1+I} tan^{-1} \left(1-\frac{1}{2}I\right) k_2\right]$$

$$= \lim_{k_1\to-\infty} \left[-\frac{1}{1+I}\left(tan^{-1}(k_1) + I\left[tan^{-1}\left(\frac{1}{2}k_1\right) - tan^{-1}(k_1)\right]\right)\right] + \lim_{k_2\to+\infty} \left[\frac{1}{1+I}\left(tan^{-1}(k_2) + I\left[tan^{-1}\left(\frac{1}{2}k_1\right) - tan^{-1}(k_1)\right]\right)\right]$$

$$= -\frac{1}{1+I}\left(\frac{-\pi}{2} + I\left[\frac{-\pi}{2} + \frac{\pi}{2}\right]\right) + \frac{1}{1+I}\left(\frac{\pi}{2} - I\left[\frac{\pi}{2} - \frac{\pi}{2}\right]\right)$$

$$= \frac{\pi}{1+I} = \pi - \frac{\pi}{2}I$$

the improper neutrosophic integral converges.

2.2 Improper neutrosophic integrals of the second kind

Definition 3
If $f(x,I)$ is continuous on the interval $[a+bI, c+dI]$ and not defined at $c+dI$, then the improper neutrosophic integral of $f(x,I)$ over the interval $[a+bI, c+dI]$ is defined as:

$$\int_{a+bI}^{c+dI} f(x,I)\,dx = \lim_{k\to c+dI^-} \int_{a+bI}^{k} f(x,I)\,dx$$

The improper neutrosophic integral converges when the limit is existing, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper neutrosophic integral diverges (when the improper neutrosophic integral doesn't exist or equal to $\pm\infty$).

Example 11

$$\int_{0+0I}^{1+I} \frac{1}{1+I-x}\,dx = \lim_{k\to 1+I^-} \int_{0+0I}^{k} \frac{1}{1+I-x}\,dx$$

$$= \lim_{k\to 1+I^-} ln(1+I-x)\big|_{0+0I}^{k} = \lim_{k\to 1+I^-} ln(1+I-k) - ln(1+I)$$

$$= \lim_{k\to 1+I^-} ln(1+I-k) - ln(1+I) = -\infty - I\,ln\,2 = -\infty_I$$

So, the improper neutrosophic integral diverges

Example 12

$$\int_{0+0I}^{2-I} \frac{dx}{\sqrt{4-3I-x^2}} = \lim_{k\to 2-I^-} \int_{0+0I}^{k} \frac{dx}{\sqrt{4-3I-x^2}}$$

$$= \lim_{k\to 2-I^-} sin^{-1}\left(\frac{x}{2-I}\right)\Big|_{0+0I}^{k} = \lim_{k\to 2-I^-} sin^{-1}\left(\frac{x}{2-I}\right) - sin^{-1}(0)$$

$$= sin^{-1}\left(\frac{2-I}{2-I}\right) = sin^{-1}(1) = \frac{\pi}{2}$$

The improper neutrosophic integral converges.

Definition 4

If $f(x,I)$ is continuous on the interval $[a+bI, c+dI]$, except for a singular point (discontinuity) $a+bI$, then the improper neutrosophic integral of $f(x,I)$ over the interval $[a+bI, c+dI]$ is defined as:

$$\int_{a+bI}^{c+dI} f(x,I)\,dx = \lim_{k\to a+bI^+} \int_{k}^{c+dI} f(x,I)\,dx$$

The improper neutrosophic integral converges when the limit is existing, and the integral is of first kind, where it has defined limit, it is the value of the integral. Otherwise the improper neutrosophic integral diverges (when the improper neutrosophic integral doesn't exist or equal to $\pm\infty$).

Example 13

the neutrosophic integral is improper and the integrand neutrosophic function is not defined at $2+2I$:

$$\int_{1+I}^{4+4I} \frac{1}{\sqrt[3]{(x-2-2I)^2}}\,dx = \int_{1+I}^{2+2I} \frac{1}{\sqrt[3]{(x-2-2I)^2}}\,dx + \int_{2+2I}^{4+4I} \frac{1}{\sqrt[3]{(x-2-2I)^2}}\,dx$$

$$= \int_{1+I}^{2+2I} (x-2-2I)^{-2/3}\,dx + \int_{2+2I}^{4+4I} (x-2-2I)^{-2/3}\,dx$$

$$= \lim_{k_1\to 2+2I^-} \int_{1+I}^{k_1} (x-2-2I)^{-2/3}\,dx + \lim_{k_2\to 2+2I^+} \int_{k_2}^{4+4I} (x-2-2I)^{-2/3}\,dx$$

$$= \lim_{k_1\to 2+2I^-} \sqrt[3]{x-2-2I}\Big|_{1+I}^{k_1} + \lim_{k_2\to 2+2I^+} \sqrt[3]{x-2-2I}\Big|_{k_2}^{4+4I}$$

$$= \lim_{k_1 \to 2+2I^-} \sqrt[3]{k_1 - 2 - 2I} - \sqrt[3]{1 + I - 2 - 2I} + \sqrt[3]{4 + 4I - 2 - 2I} - \lim_{k_2 \to 2+2I^+} \sqrt[3]{k_2 - 2 - 2I}$$

$$= \sqrt[3]{1 + I} + \sqrt[3]{2 + 2I} = 1 + I\left[\sqrt[3]{2} - 1\right] + \sqrt[3]{2} + I\left[\sqrt[3]{4} - \sqrt[3]{2}\right]$$

$$= 1 + \sqrt[3]{2} + I\left[\sqrt[3]{4} - 1\right]$$

the improper neutrosophic integral converges.

Remark 2

If $f(x, I)$ is a continuous neutrosophic function over $[a + bI, c + dI]$ except value on in $(a + bI, c + dI)$, then the improper neutrosophic integral of $f(x, I)$ over interval $[a + bI, c + dI]$ is defined as:

$$\int_{a+bI}^{c+dI} f(x, I)\, dx = \int_{a+bI}^{k} f(x, I)\, dx + \int_{k}^{c+dI} f(x, I)\, dx$$

Where k is any real number. The improper neutrosophic integral is convergent if the first term and second term on the right side of the various integral (both of them) are convergent, while the integral is divergent if either term (at least one or both) are divergent.

Example 14

$$\int_{-3-I}^{3+I} \frac{dx}{\sqrt{9 + 7I - x^2}}\, dx = \int_{-3-I}^{0+0I} \frac{dx}{\sqrt{9 + 7I - x^2}} + \int_{0+0I}^{3+I} \frac{dx}{\sqrt{9 - 7I - x^2}}$$

$$= \lim_{k_1 \to -3-I^+} \int_{k_1}^{0+0I} \frac{dx}{\sqrt{9 + 7I - x^2}} + \lim_{k_2 \to 3+I^-} \int_{0+0I}^{k_2} \frac{dx}{\sqrt{9 + 7I - x^2}}$$

$$= \lim_{k_1 \to -3-I^+} sin^{-1}\left(\frac{x}{3 + I}\right)\Big|_{k_1}^{0+0I} + \lim_{k_2 \to 3+I^-} sin^{-1}\left(\frac{x}{3 + I}\right)\Big|_{0+0I}^{k_2}$$

$$= sin^{-1}(0) - \lim_{k_1 \to -3-I^+} sin^{-1}\left(\frac{x}{3 + I}\right) + \lim_{k_2 \to 3+I^-} sin^{-1}\left(\frac{x}{3 + I}\right) - sin^{-1}(0)$$

$$= -sin^{-1}\left(\frac{-3 - I}{3 + I}\right) + sin^{-1}\left(\frac{3 + I}{3 + I}\right) = -sin^{-1}(-1) + sin^{-1}(1) = \pi$$

the improper neutrosophic integral converges.

3. Conclusions

In this paper, we introduced the improper neutrosophic integrals, by providing several definitions that clarify the concept of the improper neutrosophic integrals, and we obtained several results for these integrals, as the improper neutrosophic integral can be convergent, divergent.

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