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A Concise Introduction to HyperFuzzy, HyperNeutrosophic, HyperPlithogenic, HyperSoft, and HyperRough Sets with Practical Examples

Takaaki Fujita[1]∗, Florentin Smarandache[2],

[1]∗ Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp

[2] University of New Mexico, Gallup Campus, NM 87301, USA. fsmarandache@gmail.com :

Abstract. Various set concepts are widely recognized for effectively a ddressing u ncertainty, i ncluding fuzzy sets, neutrosophic sets, plithogenic sets, rough sets, and soft sets. These concepts have been further extended through hyperstructures (based on powersets) and superhyperstructures (based on n-th powersets, which are sets with repeated power set structures) [5,12–14,32]. These extensions are commonly referred to as HyperUncertain, SuperUncertain, and SuperHyperUncertain Sets/Logics/Probabilities/Statistics, and they were introduced by Smarandache in 2016-2017 [32] as parts of HyperStructures, SuperStructures, and SuperHyperStructures respectively. This paper recalls these concepts along with concrete examples.

Keywords: HyperFuzzy Set, HyperNeutrosophic Set, HyperPlithogenic Set, HyperRough Set, HyperSoft Set

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1. **Introduction**

1.1. *Uncertain Set*

Various types of uncertain sets, such as Fuzzy Set [43], Intuitionistic Fuzzy Set [1], Vague Set [18], Soft Set [22], Rough Set [24], Neutrosophic Set [31], and Plithogenic Set [35], are well-known. For instance:

- A fuzzy set generalizes classical sets by allowing elements to have membership degrees within $[0, 1]$, representing partial truth [43].
- A neutrosophic set extends fuzzy sets by incorporating truth, indeterminacy, and falsity degrees, providing a more nuanced uncertainty model [31].
- A plithogenic set further incorporates multiple attributes, contradictions, and degrees of membership to model complex, contradictory, and multi-valued data [34].
- A soft set is a parameterized family of subsets that models uncertainty by associating attributes with their corresponding approximate elements [22].

- A rough set approximates a set using a pair of lower and upper approximations, capturing uncertainty due to incomplete information [24].

These concepts have been extended to areas like graph theory and topology, leading to various mathematical and applied investigations.

In recent years, these uncertain sets have been generalized into HyperFuzzy, HyperVague, HyperSoft, HyperRough, HyperNeutrosophic, and HyperPlithogenic Sets by utilizing hyperstructures (based on power sets) [5,32]. Furthermore, they have been expanded into SuperHyperFuzzy, SuperHyperVague, SuperHyperSoft, SuperHyperRough, SuperHyperNeutrosophic, and SuperHyperPlithogenic Sets by employing superhyperstructures, which are repeated applications of the power set concept [5].

1.2. *Our Contribution*

This paper revisits the concepts of HyperFuzzy, HyperSoft, HyperRough, HyperNeutrosophic, and HyperPlithogenic Sets, as well as SuperHyperFuzzy, SuperHyperSoft, SuperHyperRough, SuperHyperNeutrosophic, and SuperHyperPlithogenic Sets, as defined in works such as [5, 12–14], providing detailed examples. These sets are also referred to as HyperUncertain Sets and SuperHyperUncertain Sets. This study aims to raise awareness among researchers and contribute to further exploration of HyperUncertain Concepts and SuperHyperUncertain Concepts.

2. **Preliminaries and definitions**

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, and neutrosophic graphs.

2.1. *SuperHyperstructure*

Mathematical structures can be systematically extended into Hyperstructures and SuperHyperstructures using the concepts of the power set and n-th powerset. The n-th powerset represents an iterative extension of the powerset concept, where each iteration generates the powerset of the previous level [17, 29, 37]. The definitions and examples of n-th powersets, hyperstructures, and superhyperstructures are provided below.

Definition 2.1 (Set)**.** [20] A *set* is a collection of distinct, well-defined objects, referred to as *elements*. For any object x, it can be determined whether x is an element of a given set. If x belongs to a set A, this is denoted as $x \in A$. Sets are often represented using curly braces.

Definition 2.2 (Subset)**.** [20] A set A is called a *subset* of a set B if every element of A is also an element of B. This is denoted as $A \subseteq B$. Formally:

$$A \subseteq B \iff (\forall x \in A)(x \in B).$$

If $A \subseteq B$ and $A \neq B$, then A is called a *proper subset* of B, denoted as $A \subset B$.

Definition 2.3 (Empty Set)**.** [20] The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements. Formally:

$$\emptyset = \{x \mid x \neq x\}.$$

For any set A, the empty set is a subset of A:

$$\emptyset \subseteq A.$$

Definition 2.4 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.5 (Powerset)**.** [28] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S, including the empty set and S itself. Formally,

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Example 2.6 (Powerset of a Set)**.** Let $S = \{a, b\}$. The powerset $\mathcal{P}(S)$ is:

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Definition 2.7 (n-th powerset)**.** (cf. [29, 37]) The n-th powerset of H, denoted $P_n(H)$, is defined recursively as:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)) \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset of H, denoted $P_n^*(H)$, is defined as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Example 2.8 (n-th Powerset of a Set)**.** Let $H = \{a, b\}$. The first powerset $\mathcal{P}_1(H)$ is:

$$\mathcal{P}_1(H) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

The second powerset $\mathcal{P}_2(H)$ is the powerset of $\mathcal{P}_1(H)$, given by:

$$\mathcal{P}_2(H) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \ldots, \mathcal{P}_1(H)\}.$$

To establish a formal foundation for the concepts of Hyperstructures and Superhyperstructures, we present the following definitions and propositions.

Definition 2.9 (Classical Structure)**.** (cf. [29, 37]) A *Classical Structure* is a mathematical framework defined on a non-empty set H, equipped with one or more *Classical Operations* that satisfy specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is a positive integer, and H^m denotes the m-fold Cartesian product of H. Common examples include addition and multiplication in algebraic structures such as groups, rings, and fields.

Definition 2.10 (Hyperoperation)**.** (cf. [27, 39–41]) A *hyperoperation* is a generalization of a binary operation where the result of combining two elements is a set, not a single element. Formally, for a set S, a hyperoperation $\circ$ is defined as:

$$\circ : S \times S \to \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the powerset of S.

Definition 2.11 (Hyperstructure)**.** (cf. [17, 29, 37]) A *Hyperstructure* extends the notion of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is the powerset of S, and $\circ$ is an operation defined on subsets of $\mathcal{P}(S)$. Hyperstructures allow for generalized operations that can apply to collections of elements rather than single elements.

Example 2.12 (Hyperstructure)**.** Let $S = \{a, b\}$ be a base set. Its powerset is:

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Define a hyperoperation $\circ$ on $\mathcal{P}(S)$ as follows:

$$X \circ Y = \{x \cup y \mid x \in X, y \in Y\}, \quad \text{for } X, Y \subseteq \mathcal{P}(S).$$

For example:

$$\{a\} \circ \{b\} = \{\{a, b\}\}, \quad \{a\} \circ \emptyset = \{\{a\}\}.$$

This structure $\mathcal{H} = (\mathcal{P}(S), \circ)$ is a hyperstructure where the operation $\circ$ maps pairs of subsets to subsets of the powerset.

Definition 2.13 (SuperHyperOperations)**.** (cf. [37]) Let H be a non-empty set, and let $\mathcal{P}(H)$ denote the powerset of H. The n-th powerset $\mathcal{P}^n(H)$ is defined recursively as follows:

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)), \quad \text{for } k \geq 0.$$

A *SuperHyperOperation* of order (m, n) is an m-ary operation:

$$\circ^{(m,n)} : H^m \to \mathcal{P}^n_*(H),$$

where $\mathcal{P}^n_*(H)$ represents the n-th powerset of H, either excluding or including the empty set, depending on the type of operation:

- If the codomain is $\mathcal{P}^n_*(H)$ excluding the empty set, it is called a *classical-type* (m, n)-*SuperHyperOperation*.
- If the codomain is $\mathcal{P}^n(H)$ including the empty set, it is called a *Neutrosophic* (m, n)-*SuperHyperOperation*.

These SuperHyperOperations are higher-order generalizations of hyperoperations, capturing multi-level complexity through the construction of n-th powersets.

Definition 2.14 (n-Superhyperstructure)**.** (cf. [29,37]) An *n-Superhyperstructure* further generalizes a Hyperstructure by incorporating the n-th powerset of a base set. It is formally described as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ represents an operation defined on elements of $\mathcal{P}_n(S)$. This iterative framework allows for increasingly hierarchical and complex representations of relationships within the base set.

Example 2.15 (SuperHyperstructure)**.** Let $S = \{a, b\}$. The first powerset is:

$$\mathcal{P}_1(S) = \mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

The second powerset is:

$$\mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S)) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \ldots\}.$$

Define a superhyperoperation $\circ$ on $\mathcal{P}_2(S)$ as:

$$X \circ Y = \{\mathcal{P}(x \cup y) \mid x \in X, y \in Y\}, \quad \text{for } X, Y \subseteq \mathcal{P}_2(S).$$

For example:

$$\{\{a\}\} \circ \{\{b\}\} = \{\mathcal{P}(\{a, b\})\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

The structure $\mathcal{SH}_2 = (\mathcal{P}_2(S), \circ)$ is a 2-Superhyperstructure, which incorporates the second powerset and a higher-order operation $\circ$.

3. **Review Result in this paper**

This paper revisits the HyperUncertain and SuperHyperUncertain versions of Fuzzy Sets, Neutrosophic Sets, Plithogenic Sets, Soft Sets, and Rough Sets.

3.1. *Fuzzy Set*

The Fuzzy Set is a well-known concept used to address uncertainty in set theory. These sets can be extended into Hyperfuzzy Sets and SuperHyperfuzzy Sets using hyperstructures and superhyperstructures. The definition is provided below [43].

Definition 3.1 (Fuzzy Set)**.** [43] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 3.2 (Fuzzy Set: Temperature Assessment)**.** Consider a universe Y representing daily temperatures in degrees Celsius, ranging from -30 to 50. Define a fuzzy set τ called *Hot*, where:

$$\tau(t) = \begin{cases} 0 & \text{if } t \leq 20, \\ \frac{t-20}{15} & \text{if } 20 < t < 35, \\ 1 & \text{if } t \geq 35. \end{cases}$$

Here, $\tau(t)$ indicates the degree of membership of temperature t in the concept "Hot." Temperatures at or below 20°C are not considered hot ($\tau(t) = 0$), those at or above 35°C are fully hot ($\tau(t) = 1$), and intermediate values transition linearly between 0 and 1.

Definition 3.3 (HyperFuzzy Set)**.** [19, 21, 38] Let X be a non-empty set. A *hyperfuzzy set* over X is defined as a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]) \setminus \{\emptyset\},$$

where $\mathcal{P}([0,1]) \setminus \{\emptyset\}$ represents the family of all non-empty subsets of the interval $[0,1]$.

For each element $x \in X$, $\tilde{\mu}(x)$ assigns a non-empty subset of $[0,1]$, representing the possible membership degrees of x in the hyperfuzzy set. This definition generalizes classical fuzzy sets by allowing the membership degree of each element to be a range (set of values) instead of a single scalar value.

Example 3.4 (HyperFuzzy Set: Interval-Based Uncertainty)**.** Let X be the set of daily temperatures $\{-30, -29, \ldots, 49, 50\}$. Define a hyperfuzzy set $\tilde{\mu}$ called *Hot-Interval*, where:

$$\tilde{\mu}(t) \subseteq [0,1],$$

and each $\tilde{\mu}(t)$ is an interval representing uncertainty about how hot the day is. For instance:

$$\tilde{\mu}(20) = [0, 0.1], \quad \tilde{\mu}(25) = [0.3, 0.5], \quad \tilde{\mu}(35) = [0.9, 1].$$

Thus, rather than assigning a single membership degree, we assign a set of possible degrees, reflecting incomplete or imprecise data on temperature perception.

Definition 3.5 (n-SuperHyperFuzzy Set)**.** [5, 7] Let X be a non-empty set. The n-SuperHyperFuzzy Set is a recursive generalization of fuzzy sets, hyperfuzzy sets, and super-hyperfuzzy sets. It is defined as:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,

$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1])$ is similarly defined for the interval $[0,1]$.
- $\tilde{\mu}_n$ assigns to each element $A \in \tilde{\mathcal{P}}_n(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0,1]$, representing the degrees of membership associated with A at the n-th level.

Example 3.6 (n-SuperHyperFuzzy Set: Multi-Level Uncertainty)**.** Suppose X again represents daily temperatures, and we extend our uncertainty to multiple layers. Let

$$\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X), \quad \tilde{\mathcal{P}}_2(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}(X)), \quad \ldots, \quad \tilde{\mathcal{P}}_n(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{n-1}(X)),$$

where each $\tilde{\mathcal{P}}(\cdot)$ denotes the family of non-empty subsets at each level. Define

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

so that for an element $A \in \tilde{\mathcal{P}}_n(X)$, $\tilde{\mu}_n(A)$ is a non-empty subset of $[0,1]$ capturing multiple layers of uncertainty. Concretely, if A is a second-level set describing "moderately hot" or "extremely hot" temperature ranges, then $\tilde{\mu}_n(A)$ might itself be a family of intervals in $[0,1]$ that represent various degrees of membership for these subcategories. This approach models complex, nested uncertainties about temperature evaluations across several interpretive levels.

3.2. *Neutrosophic Set*

This section provides the definitions of Neutrosophic Set, HyperNeutrosophic Set, and SuperHyperNeutrosophic Set. A Neutrosophic Set models uncertainty using three membership functions: truth (T), indeterminacy (I), and falsity (F), which satisfy:

$$0 \leq T + I + F \leq 3.$$

[4, 16, 30, 31, 42]. These sets can be extended into HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets using hyperstructures and superhyperstructures.

Definition 3.7 (Neutrosophic Set)**.** [30, 31] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 3.8 (Neutrosophic Set: Public Opinion)**.** Imagine a public poll about a new policy, where X is the set of all citizens. For each person $x \in X$, define:

$$T_A(x) = \text{degree of support for the policy},$$

$$I_A(x) = \text{degree of uncertainty},$$

$$F_A(x) = \text{degree of opposition}.$$

Each value lies in $[0,1]$, and we have $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. For instance, if a citizen is mostly supportive yet somewhat undecided, we might have $\big(T_A(x), I_A(x), F_A(x)\big) =$ (0.7, 0.2, 0.1).

Example 3.9 (Neutrosophic Set: Product Quality Evaluation)**.** Let X be a collection of manufactured items. For each product $x \in X$, define:

$$T_A(x) = \text{degree of “high quality”},$$

$$I_A(x) = \text{degree of “uncertain/untested”},$$

$$F_A(x) = \text{degree of “defective”}.$$

Suppose a certain product is partially tested but with ambiguous data. One might assign $\big(T_A(x), I_A(x), F_A(x)\big) =$ (0.5, 0.4, 0.2), indicating moderate confidence in its quality, some unverified aspects, and a small chance of being defective.

Definition 3.10 (HyperNeutrosophic Set)**.** [5,10,32] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0,1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0,1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \leq T + I + F \leq 3.$$

Example 3.11 (HyperNeutrosophic Set: Ranges of Public Sentiment)**.** Let X be the set of all citizens, and let a *HyperNeutrosophic Set* $\tilde{\mu}$ map each person $x \in X$ to a non-empty subset of $[0,1]^3$. Instead of a single triple (T, I, F), assign a set of possible triples. For example, a person's support for a policy might lie in:

$$\tilde{\mu}(x) = \big\{(0.6, 0.2, 0.2),\ (0.7, 0.1, 0.2)\big\} \subseteq [0,1]^3,$$

reflecting slight uncertainty in whether the person's true stance is $(T = 0.6, I = 0.2, F = 0.2)$ or $(T = 0.7, I = 0.1, F = 0.2)$, with each triple still obeying $0 \leq T + I + F \leq 3$.

Example 3.12 (HyperNeutrosophic Set: Product Quality with Interval Uncertainty)**.** Let X be a set of products, and define $\tilde{\mu}(x) \subseteq [0,1]^3$ for each product x. Suppose a batch of items has variable but uncertain test outcomes. We might assign:

$$\tilde{\mu}(x) = \{(T, I, F) \mid T \in [0.6, 0.8], I \in [0.1, 0.3], F \in [0, 0.2]\},$$

indicating a continuous range of plausible truth, indeterminacy, and falsity values for its quality. All points in that set satisfy $0 \leq T + I + F \leq 3$.

Definition 3.13 (n-SuperHyperNeutrosophic Set)**.** [5, 32] Let X be a non-empty set. An n-SuperHyperNeutrosophic Set is a recursive generalization of Neutrosophic Sets, HyperNeutrosophic Sets, and SuperHyperNeutrosophic Sets. It is defined as:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,

  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ is similarly defined for the unit cube $[0,1]^3$.
- The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for the n-th level subsets of X.

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, and F represent the truth, indeterminacy, and falsity degrees, respectively.

Example 3.14 (n-SuperHyperNeutrosophic Set: Hierarchical Polling)**.** Let X be a population divided into regions, cities, and neighborhoods. Define $\tilde{\mathcal{P}}_1(X)$ as the family of non-empty subsets of X, $\tilde{\mathcal{P}}_2(X)$ as the family of non-empty subsets of $\tilde{\mathcal{P}}_1(X)$, and so on, up to $\tilde{\mathcal{P}}_n(X)$. An n-SuperHyperNeutrosophic Set $\tilde{A}_n$ assigns each nested subset $A \in \tilde{\mathcal{P}}_n(X)$ a set of triples in $[0,1]^3$. For example, at the second level ($n = 2$), a city-wide subset $A \subseteq X$ might map to

$$\tilde{A}_2(A) = \big\{(0.65, 0.20, 0.15),\ (0.70, 0.15, 0.15)\big\},$$

reflecting possible truth (support), indeterminacy, and falsity (opposition) degrees about that city's collective opinion.

Example 3.15 (n-SuperHyperNeutrosophic Set: Multi-Level Product Testing)**.** Consider a company testing components, sub-assemblies, and final products. Define X as all individual components, with $\tilde{\mathcal{P}}_n(X)$ nesting up to n levels (e.g., combining components into sub-assemblies, then products). The map $\tilde{A}_n$ assigns each multi-level subset $A \in \tilde{\mathcal{P}}_n(X)$ a set of (T, I, F) triples in $[0, 1]^3$. If a sub-assembly is tested but still uncertain, we may record:

$$\tilde{A}_n(A) = \{(0.4, 0.5, 0.2), (0.5, 0.4, 0.2)\},$$

allowing multiple possible truth, indeterminacy, and falsity distributions for quality, each adhering to $0 \leq T + I + F \leq 3$.

3.3. *Plithogenic Set*

A Plithogenic Set is a mathematical framework that incorporates multi-valued degrees of appurtenance and contradictions, making it suitable for complex decision-making processes. Various studies have been conducted on Plithogenic Sets [3, 11, 15, 34, 35]. The definition is presented below.

Definition 3.16 (Plithogenic Set)**.** [34,35] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF).*
- $pCF : Pv \times Pv \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF).*

These functions satisfy the following axioms for all $a, b \in Pv$:

(1) *Reflexivity of Contradiction Function*:

$$pCF(a, a) = 0$$

(2) *Symmetry of Contradiction Function*:

$$pCF(a, b) = pCF(b, a)$$

Example 3.17 (Plithogenic Set: Product Evaluation)**.** Consider a universal set S of products, and let $P \subseteq S$ be the subset of products currently sold in a store. Suppose we focus on a single attribute v called "Flavor," with possible values

$$Pv = \{\texttt{Sweet}, \texttt{Sour}, \texttt{Bitter}, \texttt{Salty}\}.$$

Define the Degree of Appurtenance Function

$$pdf : P \times Pv \to [0,1]^1$$

that assigns to each product $x \in P$ and each flavor $\alpha \in Pv$ a membership degree $pdf(x, \alpha) \in [0, 1]$. For example, if product x is a "Lemon Candy," we might have

$$pdf(x, \texttt{Sour}) = 0.8, \quad pdf(x, \texttt{Sweet}) = 0.6, \quad pdf(x, \texttt{Bitter}) = 0.0, \quad pdf(x, \texttt{Salty}) = 0.0.$$

The Degree of Contradiction Function

$$pCF : Pv \times Pv \to [0,1]^1$$

measures how contradictory two flavor values are. For instance,

$$pCF(\texttt{Sweet}, \texttt{Bitter}) = 0.7$$

and

$$pCF(\texttt{Sweet}, \texttt{Sweet}) = 0,$$

reflecting that identical flavor values are not contradictory (0) and sweet versus bitter has moderate contradiction (0.7). Together, these functions form a Plithogenic Set $\big(P, v, Pv, pdf, pCF\big)$ for product evaluation.

These definitions establish the foundational framework necessary for exploring the HyperPlithogenic Set and the SuperHyperPlithogenic Set. The definitions of the HyperPlithogenic Set and the SuperHyperPlithogenic Set are presented below [5, 7, 15].

Definition 3.18 (HyperPlithogenic Set)**.** [5, 7, 15] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set HPS* over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : (\bigcup_{i=1}^n Pv_i) \times (\bigcup_{i=1}^n Pv_i) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

Example 3.19 (HyperPlithogenic Set: Interval Uncertainty in Product Ratings)**.** Let X be the set of all products in a large online marketplace, and $P \subseteq X$ be the subset of currently active items. Suppose we have attributes $v_1 = \texttt{Quality}$ and $v_2 = \texttt{PriceRange}$. Define:

$$Pv_1 = \{\texttt{Low}, \texttt{Medium}, \texttt{High}\}, \quad Pv_2 = \{\texttt{Budget}, \texttt{Standard}, \texttt{Premium}\}.$$

For each attribute v_i, we introduce a *Hyper Degree of Appurtenance Function*:

$$\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s),$$

where $\tilde{P}([0,1]^s)$ is the set of non-empty subsets of $[0,1]^s$. For instance, for a certain product $x \in P$,

$$\tilde{pdf}_1(x, \texttt{Medium}) = \{[0.4, 0.5], [0.5, 0.7]\},$$

meaning the *Quality* could be in the interval $[0.4, 0.5]$ or $[0.5, 0.7]$ with some uncertainty about how well x meets medium quality. The *PriceRange* might similarly assign sets of possible membership intervals. The *Degree of Contradiction Function* pCF still measures contradictions between any two values from $Pv_1 \cup Pv_2$, for example:

$$pCF(\texttt{High}, \texttt{Budget}) = 0.8,$$

indicating a strong mismatch between "High quality" and "Budget" price. Thus, the tuple

$$\big(P, \{v_i\}_{i=1}^2, \{Pv_i\}_{i=1}^2, \{\tilde{pdf}_i\}_{i=1}^2, pCF\big)$$

constitutes a HyperPlithogenic Set capturing interval-based uncertainties in product ratings.

Definition 3.20 (n-SuperHyperPlithogenic Set)**.** [5,7,15] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.
- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
  (1) Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^n P_{v_i}$,

(2) Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a, b \in \bigcup_{i=1}^{n} P_{v_i}$.

- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

Example 3.21 (n-SuperHyperPlithogenic Set: Multi-Level Inventory Analysis)**.** Consider a retail chain managing products (P_1), their packaging combinations (P_2), and potential distribution bundles (P_3), and so on. Each P_k is formed by taking non-empty subsets of P_{k-1}, leading to

$$P_k = \tilde{\mathcal{P}}(P_{k-1}).$$

Let $V = \{v_1, v_2\}$ be two attributes: $v_1 = \texttt{DemandLevel}$ and $v_2 = \texttt{ShipmentSize}$. Suppose

$$P_{v_1} = \{\texttt{Low}, \texttt{Moderate}, \texttt{High}\}, \quad P_{v_2} = \{\texttt{Small}, \texttt{Medium}, \texttt{Large}\}.$$

At the n-th nesting, each element of P_n might be a complex grouping of lower-level sets. A *Hyper Degree of Appurtenance Function* $\tilde{pdf}_i^{(n)}$ assigns to each (x, a_i) a subset in $[0,1]^s$. For instance, an entire distribution bundle $x \in P_n$ could be associated with

$$\tilde{pdf}_1^{(n)}(x, \texttt{High}) = \{\,(0.7, 0.8)\},$$

indicating it has a strong membership in "High DemandLevel." The *Degree of Contradiction Function*

$$pCF^{(n)} : \bigcup_{i=1}^{2} P_{v_i} \times \bigcup_{i=1}^{2} P_{v_i} \to [0,1]^t$$

remains reflexive and symmetric. For example,

$$pCF^{(n)}(\texttt{High}, \texttt{Low}) = 0.9, \quad pCF^{(n)}(\texttt{Small}, \texttt{Small}) = 0.$$

Putting these pieces together, the n-SuperHyperPlithogenic Set

$$SHPS_n = \left(P_n, V, \{P_{v_i}\}_{i=1}^{2}, \{\tilde{pdf}_i^{(n)}\}_{i=1}^{2}, pCF^{(n)}\right)$$

captures multiple hierarchical levels of inventory grouping, attribute values, and their possible contradictions. This structure helps a large retailer navigate uncertainties in both demand and shipment size across layered distribution networks.

3.4. *Soft Set*

A soft set is a mathematical structure that associates subsets of a universal set with parameters, providing a systematic framework for modeling uncertainty [22,23]. A hypersoft set extends the concept of soft sets by mapping Cartesian products of multiple attribute domains to subsets of a universal set [33]. A superhypersoft set further generalizes hypersoft sets by mapping combinations of power sets of attribute domains to subsets of a universal set [2,6,36]. The definitions and concrete examples of these concepts are provided below.

Definition 3.22 (Soft Set). [22, 23] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Example 3.23 (Soft Set: Car Dealership). Let U be the set of all cars in a dealership, and let $A = \{\texttt{Color}, \texttt{Brand}, \texttt{FuelType}\}$ be a set of attributes. Suppose we select $S = \{\texttt{Color}, \texttt{Brand}\} \subseteq A$. A soft set $(\mathcal{F}, S)$ can be defined by specifying, for each attribute in S, the subset of cars having that property:

$$\mathcal{F}(\texttt{Color}) = \{\text{cars that are red}\}, \quad \mathcal{F}(\texttt{Brand}) = \{\text{cars from Toyota}\}.$$

Thus, $\mathcal{F}(\texttt{Color})$ might include any red car in the dealership, and $\mathcal{F}(\texttt{Brand})$ might include every Toyota model on the lot. This soft set captures which cars match each chosen attribute.

Example 3.24 (Soft Set: Student Enrollment). Consider U to be all students at a university, and let $A = \{\texttt{Major}, \texttt{Year}, \texttt{Club}\}$. Suppose we pick $S = \{\texttt{Major}, \texttt{Year}\} \subseteq A$. Define

$$\mathcal{F}(\texttt{Major}) = \{\text{all students in Mathematics}\}, \quad \mathcal{F}(\texttt{Year}) = \{\text{all second-year students}\}.$$

Each attribute in S is a parameter, and $\mathcal{F}(\alpha)$ identifies the subset of students satisfying α. This helps administrators see which students share the same major or academic year.

Definition 3.25 (Hypersoft Set). [33] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair $(G, \mathcal{C})$, where $G : \mathcal{C} \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, \mathcal{C}) = \{(\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Example 3.26 (Hypersoft Set: Apartment Hunting). Let U be the set of all available apartments in a city. Suppose we have three attribute domains:

$$\mathcal{A}_1 = \{\texttt{1-Bedroom}, \texttt{2-Bedroom}\}, \quad \mathcal{A}_2 = \{\texttt{Downtown}, \texttt{Suburb}\}, \quad \mathcal{A}_3 = \{\texttt{LowRent}, \texttt{HighRent}\}.$$

The Cartesian product $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ consists of all 3-tuples specifying (Size, Location, RentLevel). Define $G : \mathcal{C} \to \mathcal{P}(U)$ by:

$$G(\texttt{1-Bedroom}, \texttt{Downtown}, \texttt{LowRent}) = \{\text{all 1-BR downtown apartments with relatively low rent}\}.$$

Each element $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathcal{C}$ maps to the set of apartments matching that combination of features. This hypersoft set enables a real estate agency to group apartments by multiple attributes at once.

Example 3.27 (Hypersoft Set: Course Selection)**.** Let U be all courses offered at a university, and let the attribute domains be

$$\mathcal{A}_1 = \{\texttt{Morning}, \texttt{Afternoon}\}, \quad \mathcal{A}_2 = \{\texttt{Undergraduate}, \texttt{Graduate}\}, \quad \mathcal{A}_3 = \{\texttt{In-Person}, \texttt{Online}\}.$$

The Cartesian product $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ contains all possible combinations of (TimeSlot, Level, Mode). A hypersoft set $(G, \mathcal{C})$ is defined by

$$G(\gamma) \subseteq U,$$

where $G(\gamma)$ is the subset of courses that match each combination γ. For example,

$$G(\texttt{Morning}, \texttt{Undergraduate}, \texttt{Online})$$

could be all online, morning classes offered at the undergraduate level.

Definition 3.28 (SuperHyperSoft Set)**.** [36] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair $(F, \mathcal{C})$, where:

$$F : \mathcal{C} \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, \mathcal{C}) = \{(\gamma, F(\gamma)) \mid \gamma \in \mathcal{C}, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Example 3.29 (SuperHyperSoft Set: Customized Tour Packages)**.** Let U be the set of all available tourist destinations in a travel agency's database. Suppose we have three distinct attributes:

$$a_1 = \texttt{Sightseeing}, \quad a_2 = \texttt{Adventure}, \quad a_3 = \texttt{Luxury}.$$

Each attribute a_i is linked with a set of possible values A_i. For instance,

$$A_1 = \{\texttt{Museums}, \texttt{HistoricSites}\},\ A_2 = \{\texttt{Hiking}, \texttt{Kayaking}\},\ A_3 = \{\texttt{FiveStarHotels}, \texttt{Resorts}\}.$$

We then consider the power sets $\mathcal{P}(A_1)$, $\mathcal{P}(A_2)$, $\mathcal{P}(A_3)$ and form the Cartesian product

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \mathcal{P}(A_3).$$

A SuperHyperSoft Set $(F, \mathcal{C})$ is defined by a mapping

$$F : \mathcal{C} \to \mathcal{P}(U),$$

where each tuple $\gamma = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \subseteq A_i$ represents a combination of chosen sightseeing options, adventure activities, and luxury services. Then $F(\gamma)$ is the subset of destinations that satisfy the selected combination at once. For instance,

$$F\big(\{\texttt{Museums}, \texttt{HistoricSites}\}, \{\texttt{Hiking}\}, \{\texttt{Resorts}\}\big)$$

could be all travel packages offering museum visits, hiking excursions, and resort accommodations in a single itinerary.

Example 3.30 (SuperHyperSoft Set: Laboratory Testing)**.** Let U be the set of experimental protocols in a research lab. Suppose there are $n = 2$ attributes:

$$a_1 = \texttt{EquipmentUsed}, \quad a_2 = \texttt{SampleType}.$$

Then A_1 might include $\{\texttt{Microscope}, \texttt{Centrifuge}, \texttt{Spectrometer}\}$, and A_2 might include $\{\texttt{BloodSample}, \texttt{TissueSample}\}$. We take the power sets $\mathcal{P}(A_1)$ and $\mathcal{P}(A_2)$:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2).$$

A SuperHyperSoft Set $(F, \mathcal{C})$ assigns to each pair

$$(\alpha_1, \alpha_2) \in \mathcal{P}(A_1) \times \mathcal{P}(A_2)$$

a subset of laboratory protocols in $\mathcal{P}(U)$. For example, if $\alpha_1 = \{\texttt{Microscope}, \texttt{Centrifuge}\}$ and $\alpha_2 = \{\texttt{BloodSample}\}$, then

$$F(\alpha_1, \alpha_2) = \{\text{Protocols requiring a microscope } \textit{and} \text{ centrifuge on blood samples}\}.$$

This approach systematically encodes possible experimental setups and the corresponding protocols.

3.5. *Rough Set*

A Rough Set approximates a subset using lower and upper bounds based on equivalence classes, capturing certainty and uncertainty in membership [24, 26]. The definitions are provided below.

Definition 3.31 (Rough Set Approximation). [25] Let X be a non-empty universe of discourse, and let $R \subseteq X \times X$ be an equivalence relation (or indiscernibility relation) on X. The equivalence relation R partitions X into disjoint equivalence classes, denoted by $[x]_R$ for $x \in X$, where:

$$[x]_R = \{y \in X \mid (x, y) \in R\}.$$

For any subset $U \subseteq X$, the *lower approximation* $\underline{U}$ and the *upper approximation* $\overline{U}$ of U are defined as follows:

(1) *Lower Approximation* $\underline{U}$:

$$\underline{U} = \{x \in X \mid [x]_R \subseteq U\}.$$

The lower approximation $\underline{U}$ includes all elements of X whose equivalence classes are entirely contained within U. These are the elements that *definitely* belong to U.

(2) *Upper Approximation* $\overline{U}$:

$$\overline{U} = \{x \in X \mid [x]_R \cap U \neq \emptyset\}.$$

The upper approximation $\overline{U}$ contains all elements of X whose equivalence classes have a non-empty intersection with U. These are the elements that *possibly* belong to U.

The pair $(\underline{U}, \overline{U})$ forms the *rough set* representation of U, satisfying the relationship:

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

Example 3.32 (Rough Set: Quality Control in a Factory). Suppose X is a collection of manufactured items, and R is an equivalence relation indicating that two items are "indiscernible" if they have the same measured characteristics (e.g., weight and dimensions). For a subset $U \subseteq X$ representing "items potentially defective," the lower approximation $\underline{U}$ contains items whose equivalence classes lie entirely within U (i.e., definitely defective), and the upper approximation $\overline{U}$ includes items whose equivalence classes intersect with U (i.e., possibly defective). If item x shares exactly the same measurements as other defective items, then $[x]_R \subseteq U$ and x is in $\underline{U}$. If item y's measurements partially match a defective class, then $[y]_R \cap U \neq \emptyset$ and y belongs to $\overline{U}$. This helps the factory refine which items definitely or possibly need rechecking or further tests.

The concept of a *HyperRough Set* extends the framework of the HyperSoft Set [33] to Rough Set theory. Furthermore, this has been generalized using superhyperstructures, leading to the notion of the *SuperHyperRough Set*. The formal definition is provided below.

Definition 3.33 (HyperRough Set)**.** [5, 8, 9] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product of these domains as:

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on X, where $[x]_R$ denotes the equivalence class of x under R.

A *HyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns a subset $F(a) \subseteq X$ to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$.
- For each $a \in J$, the rough set $(\underline{F(a)}, \overline{F(a)})$ is defined as:

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

The *lower approximation* $\underline{F(a)}$ represents the set of elements in X whose equivalence classes are entirely contained within $F(a)$, while the *upper approximation* $\overline{F(a)}$ includes elements whose equivalence classes have a non-empty intersection with $F(a)$.

Additionally, the following properties hold:

- $\underline{F(a)} \subseteq \overline{F(a)}$ for all $a \in J$.
- If $F(a) = \varnothing$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

Example 3.34 (HyperRough Set: Customer Segmentation With Composite Attributes)**.** Let X be the set of customers at a retail chain, and define attributes $T_1 = \texttt{AgeGroup}$ (e.g., "Under30," "30-60," "Over60") and $T_2 = \texttt{SpendingLevel}$ (e.g., "Low," "Medium," "High"). Their domains are J_1 and J_2, and the Cartesian product $J = J_1 \times J_2$ lists all pairs $(\texttt{AgeGroup}, \texttt{SpendingLevel})$. Let R be an equivalence relation on X such that two customers are indiscernible if they have the same purchase history pattern. A *HyperRough Set* is formed by $F : J \to \mathcal{P}(X)$. For instance,

$$F(\texttt{Under30}, \texttt{High}) = \{\text{customers under 30 who might spend at a high level}\}.$$

Its rough approximations $\underline{F(a)}, \overline{F(a)}$ identify which customers definitely or possibly belong to that group, based on equivalence classes. This approach refines marketing strategies by clarifying which customers are unequivocally in a target demographic versus those whose purchase patterns only partially fit.

Definition 3.35 (n-SuperHyperRough Set)**.** [5, 8, 9] Let X be a non-empty finite universe, and let $T_1, T_2, \ldots, T_n$ be n distinct attributes with respective domains $J_1, J_2, \ldots, J_n$. For each attribute T_i, let $\mathcal{P}(J_i)$ denote the power set of J_i. Define the set of all possible attribute value combinations as the Cartesian product of these power sets:

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n).$$

Let $R \subseteq X \times X$ be an equivalence relation on X, where $[x]_R$ denotes the equivalence class of x under R.

An *n-SuperHyperRough Set* over X is a pair (F, J), where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns a subset $F(A) \subseteq X$ to each attribute value combination $A = (A_1, A_2, \ldots, A_n) \in J$, where $A_i \subseteq J_i$ for all i.
- For each $A \in J$, the rough set $(\underline{F(A)}, \overline{F(A)})$ is defined as:

$$\underline{F(A)} = \{x \in X \mid [x]_R \subseteq F(A)\}, \quad \overline{F(A)} = \{x \in X \mid [x]_R \cap F(A) \neq \emptyset\}.$$

The *lower approximation* $\underline{F(A)}$ represents the set of elements in X whose equivalence classes are entirely contained within $F(A)$, while the *upper approximation* $\overline{F(A)}$ includes elements whose equivalence classes have a non-empty intersection with $F(A)$.

Properties:

- $\underline{F(A)} \subseteq \overline{F(A)}$ for all $A \in J$.
- If $F(A) = \varnothing$, then $\underline{F(A)} = \overline{F(A)} = \varnothing$.
- If $F(A) = X$, then $\underline{F(A)} = \overline{F(A)} = X$.
- For any $A, B \in J$:

$$\underline{F(A \cap B)} \subseteq \underline{F(A)} \cap \underline{F(B)}, \quad \overline{F(A \cup B)} \supseteq \overline{F(A)} \cup \overline{F(B)}.$$

Example 3.36 (n-SuperHyperRough Set: Multi-Level Data Analysis)**.** Consider a scenario where X is a set of documents in a large digital library. Let $T_1, T_2, \ldots, T_n$ be attributes, such as `Topic`, `Author`, `PublicationYear`, etc., each with a domain J_i. For each i, $\mathcal{P}(J_i)$ is the power set of possible attribute values. The Cartesian product

$$J = \mathcal{P}(J_1) \times \mathcal{P}(J_2) \times \cdots \times \mathcal{P}(J_n)$$

lists all combinations of chosen topics, authors, publication years, and so on. Define an equivalence relation R where two documents are indiscernible if they share the same metadata (e.g., identical classification tags). An n-SuperHyperRough Set (F, J) assigns each combination $A = (A_1, A_2, \ldots, A_n)$ a subset $F(A) \subseteq X$, and obtains rough approximations $\underline{F(A)}$ and $\overline{F(A)}$. For instance, if

$$A_1 = \{\texttt{Mathematics}, \texttt{Physics}\}, \quad A_2 = \{\texttt{Smith}, \texttt{Johnson}\},$$

then

$$F(A) = \{\text{documents concerning Mathematics or Physics written by Smith or Johnson}\}.$$

Its lower approximation consists of documents whose equivalence classes lie entirely in that set, and its upper approximation includes all documents whose equivalence classes intersect with it. This multi-level approach enables more nuanced filtering, revealing which documents definitely or possibly match complex criteria across multiple attributes.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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