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On the category of neutrosophic submodules

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Abstract. The purpose of this paper is to introduce the important and useful notions of inverse and direct systems in the category of neutrosophic modules and present some of fundamental properties of them.

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1. **Introduction**

In 1965, Zadeh [6] laid the foundation of fuzzy set theory by introducing the concept of degree of membership (or truth) t and defining the fuzzy set. Building on this, Smarandache introduced the term *neutrosophic* to highlight a significant extension of fuzzy logic. The term *neutrosophic* derives from a combination of the French *neutre* (Latin *neuter*) (neutral), and the Greek *sophia* (skill/wisdom) emphasizing knowledge that incorporates neutrality or indeterminacy. This neutrality represents a pivotal distinction between fuzzy and neutrosophic logic. Unlike fuzzy sets, which involve only truth (membership) and falsehood (non-membership) components, neutrosophic sets include an additional, independent component: *indeterminacy* (or *neutrality*). This concept, introduced by Smarandache in 1995 (formally published in 1998), allowed the definition of *neutrosophic set* using three independent components (t, i, f), corresponding to truth, indeterminacy, and falsehood, respectively.

Smarandache's neutrosophic set theory generalizes classical and fuzzy set theories, providing a more flexible and comprehensive framework. It encompasses the notion of interval-valued neutrosophic sets, further broadening its applicability. The versatility of neutrosophic sets has led to their application across diverse fields (as documented on the website `http://fs.gallup.unm.edu/neutrosophy.htm`).

In this paper, we explore the notions of inverse and direct systems within the category of neutrosophic modules, extending the ideas presented in [2]. We also examine their fundamental properties, providing a deeper insight into the theoretical landscape of neutrosophic module systems.

2. Preliminaries

Definition 2.1. Let X be a nonempty set. A *neutrosophic set* [4] A on X is a structure

$$A = \{\langle x, \varsigma_A(x), \varpi_A(x), \nu_A(x)\rangle \;:\; x \in X\}, \tag{1}$$

where $\varsigma_A : X \to [0,1]$ is a truth membership function, $\varpi_A : X \to [0,1]$ is an indeterminate membership function, and $\nu_A : X \to [0,1]$ is a false membership function. The neutrosophic set in (1) will be simply denoted by $A = (\varsigma_A, \varpi_A, \nu_A)$.

Building on the general concept of neutrosophic sets, it is possible to describe integrate it within the context of algebraic structures, especially modules over a ring.

Definition 2.2. Let R be a ring and let F be a R-module. A neutrosophic set $F_1 = (\varsigma_{F_1}, \varpi_{F_1}, \nu_{F_1})$ in F is called a *neutrosophic submodule* of F if the following conditions are satisfied

(1) $\varsigma_{F_1}(0) = 1$,
(2) $\min\{\varsigma_{F_1}(x), \varsigma_{F_1}(y)\} \leq \varsigma_{F_1}(x-y)$ for all $x, y \in F$,
(3) $\varsigma_{F_1}(x) \leq \varsigma_{F_1}(r \cdot x)$ for all $x \in F$ and $r \in R$,
(4) $\varpi_{F_1}(0) = 1$,
(5) $\min\{\varpi_{F_1}(x), \varpi_{F_1}(y)\} \leq \varpi_{F_1}(x-y)$ for all $x, y \in F$,
(6) $\varpi_{F_1}(x) \leq \varpi_{F_1}(r \cdot x)$ for all $x \in F$ and $r \in R$,
(7) $\nu_{F_1}(0) = 0$,
(8) $\nu_{F_1}(x-y) \leq \max\{\nu_{F_1}(x), \nu_{F_1}(y)\}$ for all $x, y \in F$,
(9) $\nu_{F_1}(r \cdot x) \leq \nu_{F_1}(x)$ for all $x \in F$ and $r \in R$.

Definition 2.3. Given an inverse sequence of abelian groups $G_1 \overset{\rho}{\leftarrow} G_2 \overset{\rho}{\leftarrow} G_3 \cdots$ (in short $\{G_i\}$). Denote by Π the direct product of the groups G_i, and consider the map $\delta : \Pi \to \Pi$ defined as

$$\delta(g_1, g_2, \cdots) = (g_1 - \rho g_2, g_2 - \rho g_3, g_3 - \rho g_4, \cdots).$$

The kernel of δ is called the *inverse limit* of the sequence $\{G_i\}$ and we denote it by $\mathcal{L}_{inv}\{G_i\}$.

3. Some properties of neutrosophic modules

In what follows, a neutrosophic submodule $F_1 = (\varsigma, \varpi, \nu)$ of F will be denoted by $(\varsigma, \varpi, \nu)_F$.

Definition 3.1. A function $h : (\varsigma, \varpi, \nu)_F \to (\varsigma', \varpi', \nu')_{F'}$ is called a homomorphism of neutrosophic modules if the conditions $\varsigma_1(h(x)) \geq \varsigma(x)$, $\varpi_1(h(x)) \geq \varpi(x)$ and $\nu_1(h(x)) \leq \nu(x)$ are satisfied.

Observe that, neutrosophic modules and their morphisms form a category which we denote by N-Mod.

Let F,E be two R-modules and let $(\varsigma, \varpi, \nu)_F$ be a neutrosophic submodule of F. Suppose $h : F \to E$ is an R-module homomorphism. We can define a neutrosophic module structure on E *via* the map h has it follows:

$$\varsigma^h(y) = \sup\{\varsigma(x) \ : \ h(x) = y\},$$
$$\varpi^h(y) = \sup\{\varpi(x) \ : \ h(x) = y\},$$
$$\nu^h(y) = \inf\{\nu(x) \ : \ h(x) = y\}.$$

It is clear that $(\varsigma^h, \varpi^h, \nu^h)_E$ is a neutrosophic submodule of E and the function $h : (\varsigma, \varpi, \nu)_F \to (\varsigma^h, \varpi^h, \nu^h)_E$ is a homomorphism of neutrosophic modules.

Conversely, let $(\eta, \theta, \nu)_E$ be a neutrosophic submodule of E and $h : F \to E$ a homomorphism of R-modules. It is possible to define a neutrosophic module structure in F by setting

$$\eta^h(x) = \eta(h(x)),$$
$$\theta^h(x) = \theta(h(x)),$$
$$\nu^h(x) = \nu(h(x)).$$

Hence, $(\eta^h, \theta^h, \nu^h)_F$ is a neutrosophic module and $h : (\eta^h, \theta^h, \nu^h)_F \to (\eta, \theta, \nu)_E$ is a homomorphism of neutrosophic modules.

We state the following lemma.

Lemma 3.2. *Let F and E be R-modules and $h : F \to E$ be a homomorphism of R-modules. The following conditions are true*

(1) *If $(\varsigma, \varpi, \nu)_F$ is a neutrosophic module, then there exist modular grade functions ς^h,ϖ^h,ν^h on E such that for any modular grade function $B = (\alpha, \beta, \gamma)$ on E, $h : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ is a neutrosophic homomorphism if and only if $\alpha \geq \varsigma^h$, $\beta \geq \varpi^h$, $\gamma \leq \nu^h$.*

(2) *If $(\alpha, \beta, \gamma)_E$ is a neutrosophic module, then there exists a modular grade function $(\alpha^h, \beta^h, \gamma^h)$ on F such that for every neutrosophic module $(\varsigma, \varpi, \nu)_F$, $h : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ is a neutrosophic homomorphism if and only if $\varsigma \leq \alpha^h$, $\varpi \leq \beta^h$, $\nu \geq \gamma^h$.*

Lemma 3.3. (1) *Let* $\{F_i\}_{i\in\Delta}$, E *are modules and a family of* R*-homomorphisms* $\eta = \{h_i : F_i \to E\}_{i\in\Delta}$. *If* $\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}\}_{i\in\Delta}$ *is a family of neutrosophic modules, then there exists the smallest grade functions* $\alpha = \varsigma^{\eta} = \varsigma^{\{h_i\}}$, $\beta = \varpi^{\eta} = \varpi^{\{h_i\}}$, $\gamma = \nu^{\eta} = \nu^{\{h_i\}}$ *such that, for all* $i \in \Delta$, $\tilde{h} : (\varsigma_i, \varpi_i, \nu_i)_{F_i} \to (\alpha, \beta, \gamma)_E$ *is a neutrosophic homomorphism.*

(2) *Let* $F, \{E_i\}_{i\in\Delta}$ *be modules and* $B = \{g_i : F \to E_i\}_{i\in\Delta}$ *be a family of* R*-homomorphisms. If* $\{(\alpha_i, \beta_i, \gamma_i)_{E_i}\}_{i\in\Delta}$ *is a family of neutrosophic modules, then there exist the largest grade functions* $\varsigma = \alpha_B = \alpha_{\{h_i\}}$, $\varpi = \beta_B = \beta_{\{h_i\}}$, $\nu = \gamma_B = \gamma_{\{h_i\}}$ *such that, for all* $i \in \Delta$, $h : (\varsigma, \varpi, \nu)_F \to (\alpha_i, \beta_i, \gamma_i)_{E_i}$ *is a neutrosophic homomorphism.*

Proof. (1). Let $\alpha = \varsigma^{\overline{\lambda}} = \bigvee_{\alpha\in\Delta} \mu_{\alpha}^{f_\alpha}, \theta = \xi^{\eta} = \bigvee_{i\in\Delta} \xi_i^{h_i}, \gamma = \nu^{\eta} = \bigwedge_{\alpha\in\Delta} \nu_i^{h_i}$.

(2). Let $\varsigma = \alpha_B = \bigwedge_{i\in\Delta} (\alpha_i)_{h_i}, \varpi = \beta_B = \bigwedge_{i\in\Delta} (\beta_i)_{h_i}, \nu = \gamma_B = \bigvee_{i\in\Delta} (\gamma_i)_{h_i}$. □

We are now in position to define submodule, quotient module, product and co-product operations in the category of neutrosophic modules, by using the above lemma.

If $(\varsigma, \varpi, \nu)_F$ is a neutrosophic module on F and $E \subset F$ is a submodule, then $(\varsigma|_E, \varpi|_E, \nu|_E)_E$ is called a *neutrosophic submodule* of $(\varsigma, \varpi, \nu)_F$.

If $(\varsigma, \varpi, \nu)_F$ is a neutrosophic module and $h_c : F \to F/\sim$ is a canonical homomorphism, then $(\varsigma^{h_c}, \varpi^{h_c}, \nu^{h_c})_{F/\sim}$ is called a *quotient module* of $(\varsigma, \varpi, \nu)_F$.

Hence, for each homomorphism of neutrosophic modules $h : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$, the neutrosophic submodules $(\varsigma|_{\operatorname{Ker} h}, \varpi|_{\operatorname{Ker} h}, \nu|_{\operatorname{Ker} h})_{\operatorname{Ker} h}$ and the neutrosophic quotient module $(\alpha^{\pi}, \beta^{\pi}, \gamma^{\pi})_{E/\operatorname{Im} h}$ are obtained, where $\pi : E \to E/\operatorname{Im} f$ is a canonical homomorphism.

If $\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}\}_{i\in\Delta}$ is a family of neutrosophic modules, then we define the *product* of this family by $(\varsigma_D, \varpi_D, \nu_D)_{\prod_{i\in\Delta} F_i}$, where $D = \left\{\pi_i : \prod_{i\in\Delta} F_i \to F_i\right\}_{i\in\Delta}$ is a family of the usual projection maps. Moreover, the *co-product* of the family is defined as $(\varsigma^H, \varpi^H, \nu^H)_{\sum F_i}$, where $H = \left\{j_i : F_i \to \sum_{i\in\Delta} F_i\right\}_{i\in\Delta}$ is a family of the usual injections.

Mind that, the category of neutrosophic modules has zero objects, sums, products, kernels and cokernels.

Definition 3.4. A functor $\Phi : \Lambda^{op} \to \mathrm{N-Mod}$ ($\Phi : \Lambda \to \mathrm{N-Mod}$), where Λ is a directed set (considered as a category), is called an *inverse* (resp. *direct*) *system* of neutrosophic modules, the limit of Φ is called a *limit* of the inverse (resp. direct) system.

Consider an inverse system of neutrosophic modules

$$(\underline{\varsigma}, \underline{\varpi}, \underline{\nu})_{\underline{F}} = \left\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}, \underline{h_c}_{i'i}\right\}_{i\in\Delta}. \quad (2)$$

Let $D = \left\{\pi_i, \prod_{i\in\Delta} F_i \to F_i\right\}_{i\in\Delta}$ be a family of projections and let $(\varsigma_D, \varpi_D, \nu_D)_{\prod_{i\in\Delta} F_i}$, the direct product of the neutrosophic modules $(\varsigma_i, \varpi_i, \nu_i)_{F_i}$. Then, we obtain a neutrosophic submodule $(\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i}$, where $\varprojlim F_i$ is a limit of an inverse system of modules $\{F_i\}_{i\in\Delta}$.

Theorem 3.5. *Every inverse system* (2) *has a limit in the category* $\mathrm{N-Mod}$*, which is equal to the a neutrosophic submodule* $(\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i}$.

Proof. Our aim is to show that there exists a unique homomorphism of neutrosophic modules $\overline{\theta} : (\alpha, \beta, \gamma)_E \to (\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i}$, which makes the following diagram

$$\begin{tikzcd}
(\alpha, \beta, \gamma)_E \arrow[r, "\overline{\varphi}_i"] \arrow[d, "\overline{\theta}"'] & (\varsigma_i, \varpi_i, \nu_i)_{F_i} \\
(\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i} \arrow[ur] &
\end{tikzcd}$$

commutative. Observe that, for every neutrosophic module $(\alpha, \beta, \gamma)_E$ it is true that $\{\overline{\varphi}_i : (\alpha, \beta, \gamma)_E \to (\varsigma_i, \varpi_i, \nu_i)_{F_i}\}_{i\in\Delta}$ is a family of homomorphism of neutrosophic modules, providing the commutativity of the diagram

$$\begin{tikzcd}
(\alpha, \beta, \gamma)_E \arrow[r, "\overline{\varphi}_i"] \arrow[d, "\overline{\varphi}_{i'}"'] & (\varsigma_i, \varpi_i, \nu_i)_{F_i} \\
(\varsigma_{i'}, \varpi_{i'}, \nu_{i'})_{F_{i'}} \arrow[ur, "\overline{h}^{i,i'}"'] &
\end{tikzcd}$$

Also, $\overline{\pi_i} : (\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i} \to (\varsigma_i, \varpi_i, \nu_i)_{F_i}$ is a canonical projection. We define the map $\theta : E \to \varprojlim F_i$ as $f(x) = \{\varphi_i(x)\}_{i\in\Delta}$, for every $x \in F$, which is a module homomorphism. We show that $\overline{\theta} : (\alpha, \beta, \gamma)_E \to (\varsigma_D|_{\varprojlim F_i}, \varpi_D|_{\varprojlim F_i}, \nu_D|_{\varprojlim F_i})_{\varprojlim F_i}$ is a homomorphism of neutrosophic modules. Since $\overline{\varphi_i} : (\alpha, \beta, \gamma)_F \to (\varsigma_i, \varpi_i, \nu_i)_{F_i}$ is a homomorphism of neutrosophic modules for every $i \in \Delta$, the conditions $\varsigma_i(\varphi_i(x)) \geq \alpha(x)$, $\varpi_i(\varphi_i(x)) \geq \beta(x)$ and $\nu_i(\varphi_i(x)) \leq \gamma(x)$ are satisfied for every $x \in E$. Therefore, we obtain

$$\varsigma_D(\{\varphi_i(x)\}) = \bigwedge_{i\in\Delta} \varsigma_i(\varphi_i(x)) \geq \alpha(x),$$
$$\varpi_D(\{\varphi_i(x)\}) = \bigwedge_{i\in\Delta} \varpi_i(\varphi_i(x)) \geq \beta(x),$$
$$\nu_A(\{\varphi_i(x)\}) = \bigvee_{i\in\Delta} \nu_i(\varphi_i(x)) \leq \gamma(x).$$

Hence, $\overline{\theta}$ is a homomorphism of neutrosophic modules. The uniqueness of θ follows from the uniqueness of $\overline{\theta}$.

It is conspicuous to show that $\varprojlim$ is a functor from the category of inverse system of neutrosophic modules to the category of neutrosophic modules. □

Let us now focus on the problem of exact limit for inverse systems of exact sequences.

Definition 3.6. A sequence

$$\cdots \to (\varsigma_{n-1}, \varpi_{n-1}, \nu_{n-1})_{F_{n-1}} \xrightarrow{\check{h}_{n-1}} (\varsigma_n, \varpi_n, \nu_n)_{F_n} \xrightarrow{\check{h}_n} (\varsigma_{n+1}, \varpi_{n+1}, \nu_{n+1})_{F_{n+1}} \to \cdots$$

of neutrosophic modules is said *neutrosophic exact* if

$$(\varsigma_n|_{\mathrm{Im}\, h_{n-1}}, \varpi_n|_{\mathrm{Im}\, h_{n-1}}, \nu_n|_{\mathrm{Im}\, h_{n-1}}) = (\varsigma_n|_{\mathrm{Ker}\, h_n}, \varpi_n|_{\mathrm{Ker}\, h_n}, \nu_n|_{\mathrm{Ker}\, h_n}), \qquad \text{for every } n \in \mathbb{Z}.$$

Remark 3.7. Observe that, given a sequence of R-modules

$$\cdots \to F_{n-1} \xrightarrow{h_{n-1}} F_n \xrightarrow{h_n} F_{n+1} \to \cdots \tag{3}$$

it is clear that if (3) is exact, then the induced sequence of neutrosophic modules

$$\cdots \to (\varsigma_{n-1}, \varpi_{n-1}, \nu_{n-1})_{F_{n-1}} \xrightarrow{\overline{h_{n-1}}} (\varsigma_n, \varpi_n, \nu_n)_{F_n} \xrightarrow{\overline{h_n}} (\varsigma_{n+1}, \varpi_{n+1}, \nu_{n+1})_{F_{n+1}} \to \cdots \tag{4}$$

is neutrosophic exact, with

$$\varsigma_n = \chi_{\{0\}}, \quad \varpi_n = \chi_{\{0\}}, \quad \nu_n = 1 - (\varsigma_n + \xi_n).$$

Exactness of (4) implies the exactness of (3), since the equality of two neutrosophic sets is just the equality of their respective maps, which implies the equality of their corresponding domains (that is, $\mathrm{Ker}\, h_n = \mathrm{Im}\, h_{n-1}$). On the other hand, the exactness of (3) does not necessarily imply the exactness of (4). Namely, the exactness of R-modules sequences is not derived from the exactness of sequences of a neutrosophic modules.

Example 3.8. For all $n \in \mathbb{N}$, consider the $\mathbb{Z}$-modules $F_n = \mathbb{Z}$, $F'_n = \mathbb{Z}$, $F''_n = \mathbb{Z}_2$. Then,

$$\underline{F} = (\{F_n\}_{n\in E}, \{p_{n+1n}(m) = 3m\}),$$
$$\underline{F}' = (\{F'_n\}_{n\in E}, \{q_{n+1n}(m) = 3m\}),$$
$$\underline{F}'' = (\{F''_n\}_{n\in E}, \{r_{n+1n}(m) = [m]\})$$

are inverse systems of modules. Hence, we consider the sequence $0 \to \underline{F}' \xrightarrow{h} \underline{F} \xrightarrow{g} \underline{F}'' \to 0$, with $h = \{h_n : F'_n \to F_n \ : \ h_n(m) = 2m\}$ and $g = \{g_n : F_n \to F''_n \ : \ g_n(m) = [m]\}$, which is a short exact sequence of inverse systems of $\mathbb{Z}$-modules. Then, the following sequence $\overline{0} \to (\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \xrightarrow{\overline{h}_n} (\varsigma_n, \varpi_n, \nu_n)_{F_n} \xrightarrow{\overline{g}_n} (\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n} \to \overline{0}$ is a short exact sequence of neutrosophic modules, where $\varsigma_n = (\chi(0))_{F_n}$, $\varpi_n = (\chi(0))_{F_n}$, $\nu_n = 1-(\varsigma_n+\varpi_n)$, $\varsigma'_n = (\chi(0))_{F'_n}$, $\varpi'_n = (\chi(0))_{F'_n}$, $\nu'_n = 1 - (\varsigma'_n + \varpi'_n)$, $\varsigma''_n = (\chi(0))_{F''_n}$, $\varpi''_n = (\chi(0))_{F''_n}$, $\nu''_n = 1 - (\varsigma''_n + \varpi''_n)$. Therefore, the sequence $\overline{0} \to (\underline{\varsigma}', \underline{\varpi}', \underline{\nu}')_{\underline{F}} \xrightarrow{\overline{h_n}} (\underline{\varsigma}, \underline{\varpi}, \underline{\nu})_{\underline{F}} \xrightarrow{\overline{g}} (\underline{\varsigma}'', \underline{\varpi}'', \underline{\nu}'')_{\underline{F}''} \to \overline{0}$ is a short exact sequence of inverse systems of neutrosophic modules. Nevertheless, taking the limit of this sequence, we obtain $0 \to 0 \to 0 \to (\varsigma'', \varpi'', \nu'')_{Z_2} \to 0$, which is not exact.

The limit of inverse system of exact sequence of neutrosophic modules is not exact, hence we need to introduce the notion of derivative functor of inverse limit functor in $\mathrm{N} - \mathrm{Mod}$.

Consider an inverse system of neutrosophic modules as in (2). Define also the R-modules homomorphism $d : \prod\limits_{\alpha} F_\alpha \to \prod\limits_{\alpha} F_\alpha$ as $d(\{x_\alpha\}) = \{x_\alpha - p_{\alpha'\alpha}(x_{\alpha'})\}_{\alpha \prec \alpha'}$. We want to show that d is a homomorphism of neutrosophic modules. Indeed,

$$
\begin{aligned}
\varsigma_D(d(\{x_i\})) &= \varsigma_D(x_i - p_{i'i}(x_{i'})) \\
&= \bigwedge_i \varsigma_i(x_i - p_{i'i}(x_{i'})) \\
&\geq \bigwedge_\alpha \min\{\varsigma_i(x_i), \varsigma_i(p_{i'i}(x_{i'}))\} \\
&\geq \bigwedge_i \min\{\varsigma_i(x_i), \varsigma_i'(x_{i'})\} \ (\because\ \varsigma_i(p_{i'i}(x_{i'})) \geq \varsigma_{i'}(x_{i'})) \\
&= \bigwedge_i (\varsigma_i(x_i) \wedge \varsigma_i'(x_{i'})) \\
&= \bigwedge_i \varsigma_i(x_i) \\
&= \varsigma_D(x_i),
\end{aligned}
$$

$$
\begin{aligned}
\varpi_D(d(\{x_i\})) &= \varpi_D(x_i - p_{i'i}(x_{i'})) \\
&= \bigwedge_i \varpi_i(x_i - p_{i'i}(x_{i'})) \\
&\geq \bigwedge_i \min\{\varpi_i(x_i), \varpi_i(p_{i'i}(x_{i'}))\} \\
&\geq \bigwedge_i \min\{\varpi_i(x_i), \varpi_i'(x_{i'})\} \ (\because\ \varpi_i(p_{i'i}(x_{i'})) \geq \varpi_{i'}(x_{i'})) \\
&= \bigwedge_i (\varpi_i(x_i) \wedge \varpi_i'(x_{i'})) \\
&= \bigwedge_i \varpi_i(x_i) \\
&= \varpi_D(x_i),
\end{aligned}
$$

$$\begin{aligned}
\nu_D(d(\{x_i\})) &= \nu_D(x_i - p_{i'i}(x_{i'})) \\
&= \bigvee_i \nu_i(x_i - p_{i'i}(x_{i'})) \\
&\leq \bigvee_i \max\{\nu_\alpha(x_i), \nu_i(p_{i'i}(x_{i'}))\} \ (\because\ \nu_i(p_{i'i}(x_{i'})) \leq \nu_{i'}(x_{i'})) \\
&\leq \bigvee_i \max\{\nu_i(x_i), \nu i'(x_{i'})\} \\
&= \bigvee_i (\nu_i(x_i) \vee \nu_i'(x_{i'})) \\
&= \bigvee_i \nu_i(x_i) \\
&= \nu_D(x_i).
\end{aligned}$$

This shows that d is a homomorphism of neutrosophic modules. Thus,

$$(\varsigma_D|_{\mathrm{Ker}\,d}, \varpi_D|_{\mathrm{Ker}\,d}, \nu_D|_{\mathrm{Ker}\,d})_{\mathrm{Ker}\,d}, \text{ and } ((\varsigma_D)^p, (\varpi_D)^p, (\nu_D)^p)_{\mathrm{Coker}\,d}$$

are defined.

For inverse system of R-modules $\{F_i, p_{i'i}\}_{i\in\Delta}$, we have $\varprojlim^{(1)} F_i = \prod_i / \operatorname{Im} d$, where $\varprojlim^{(1)}$ is the derivative functor of $\varprojlim$ [3].

If $\pi : \prod_i \to \varprojlim^{(1)} F_i$ is a canonical homomorphism, then we are able to define a neutrosophic module by $((\varsigma_D)^\pi), (\varpi_D)^\pi, (\nu D)^\pi)_{\varprojlim^{(1)} F_i}$.

Definition 3.9. $((\varsigma_D)^\pi, (\varpi_D)^\pi, (\nu_D)^\pi)_{\varprojlim^{(1)} F_i}$ is called the *first derived functor* of the limit of the inverse system of neutrosophic modules (2).

Proposition 3.10. $\varprojlim^{(1)}$ *is a functor.*

Proof. We are done if we show that for each morphism of neutrosophic chain complexes $\check{h} = \left(\rho : H \to D, \left\{\check{h}_j : (\varsigma_{\rho(j)}, \varpi_{\rho(j)}, \nu_{\rho(j)})_{F_{\rho(j)}} \to (i_l, j_l, k_l)_{N_l}\right\}_{j\in H}\right)$, the map $\varprojlim^{(1)}\check{h} : ((\varsigma_D)^\pi, (\varpi_D)^\pi, (\nu_D)^\pi)_{\varprojlim^{(1)} F_i} \to ((\alpha_H)^\pi, (\beta_H)^\pi, (\gamma_H)^\pi)_{\varprojlim^{(1)} N_j}$ is a homomorphism of neutrosophic modules. We have

$$\begin{aligned}
(\varsigma_D)^\pi(x + \operatorname{Im} d) &= \sup_{z\in \operatorname{Im} d} \varsigma_D(x+z) \\
&\leq \sup_{z\in \operatorname{Im} d} \alpha_D(h(x+z)) \\
&= \sup_{z\in \operatorname{Im} d} \alpha_D(h(x)+h(z)) \\
&= \sup_{y=h(z)} \alpha_D(h(x)+y) \\
&\leq \sup_{y\in \operatorname{Im} d} \alpha_D(h(x)+y) \\
&= (\alpha_D)^\pi(\varprojlim^{(1)} h(x + \operatorname{Im} d)),
\end{aligned}$$

$$\begin{aligned}(\varpi_D)^\pi(x+\operatorname{Im} d) &= \sup_{z\in \operatorname{Im} d} \varpi_D(x+z)\\ &\le \sup_{z\in \operatorname{Im} d} \beta_D(h(x+z))\\ &= \sup_{z\in \operatorname{Im} d} \beta_D(f(x)+h(z))\\ &= \sup_{y=h(z)} \beta_D(h(x)+y)\\ &\le \sup_{y\in \operatorname{Im} d} \beta_D(h(x)+y)\\ &= (\beta_D)^\pi(\varprojlim{}^{(1)} h(x+\operatorname{Im} d)),\end{aligned}$$

$$\begin{aligned}(\nu_D)^\pi(x+\operatorname{Im} d) &= \inf_{z\in \operatorname{Im} d} \nu_D(x+z)\\ &\ge \inf_{z\in \operatorname{Im} d} \gamma_H(h(x+z))\\ &= \inf_{z\in \operatorname{Im} d} \gamma_H(h(x)+h(z))\\ &= \inf_{y=h(z)} \gamma_H(h(x)+y)\\ &\ge \inf_{y\in \operatorname{Im} d} \gamma_D(h(x)+y)\\ &= (\gamma_D)^\pi(\varprojlim{}^{(1)} h(x+\operatorname{Im} d)).\end{aligned}$$

Hence, $\varprojlim^{(1)}$ is a functor, as stated. □

We intend to investigate the properties of the functor $\varprojlim^{(1)}$. Therefore, we introduce the category of chain (cochain) complexes of neutrosophic modules. This category is defined in the same lines as in [1].

Definition 3.11. A neutrosophic chain complex

$$(\varsigma,\varpi,\nu)_F = \{(\varsigma_n,\varpi_n,\nu_n)_{F_n}, \overline{\partial_n}\}_{n\in\mathbb{Z}}$$

is an object in N-Mod along with a neutrosophic endomorphism $\overline{\partial} : (\varsigma,\varpi,\nu)_F \to (\varsigma,\varpi,\nu)_F$ of degree -1 such that $\overline{\partial\partial} = \overline{0}$.

Definition 3.12. A morphism $\overline{\varphi} : (\varsigma,\varpi,\nu)_F \to (\alpha,\beta,\gamma)_E$ of neutrosophic chain complexes is a morphism $\overline{\varphi} = \{\overline{\varphi}_n : (\varsigma_n,\varpi_n,\nu_n)_{F_n} \to (\alpha_n,\beta_n,\gamma_n)_{E_n}\}$, which has a degree 0 such that $\overline{\varphi_{n-1}} \circ \overline{\partial_n} = \overline{\partial'_n}\overline{\varphi_n}$, where $\overline{\partial}$ denotes the neutrosophic differential in $(\alpha,\beta,\gamma)_E$.

Definition 3.13. Let $(\varsigma, \varpi, \nu)_F = (\varsigma_n, \varpi_n, \nu_n)_{F_n}, \overline{\partial_n}\}_{n\in\mathbb{Z}}$ be a neutrosophic chain complex. The condition $\overline{\partial} \circ \overline{\partial} = \overline{0}$ implies that $\operatorname{Im} \overline{\partial_{n+1}} \subset \operatorname{Ker} \overline{\partial_n}$, for $n \in \mathbb{Z}$. Hence, we can associate a neutrosophic graded module with $(\varsigma, \varpi, \nu)_F$ $H((\varsigma, \varpi, \nu)_F) = \{H_n(\varsigma, \varpi, \nu)_F\}$, where

$$H_n((\varsigma, \varpi, \nu)_F) = \frac{\left(\varsigma_n|_{\operatorname{Ker}\overline{\partial_n}}, \varpi_n|_{\operatorname{Ker}\overline{\partial_n}}, \nu_n|_{\operatorname{Ker}\overline{\partial_n}}\right)_{\operatorname{Ker}\overline{\partial_n}}}{\left(\varsigma_n|_{\operatorname{Im}\overline{\partial_{n+1}}}, \varpi_n|_{\operatorname{Im}\overline{\partial_{n+1}}}, \nu_n|_{\operatorname{Im}\overline{\partial_{n+1}}}\right)_{\operatorname{Im}\overline{\partial_{n+1}}}}.$$

$H((\varsigma, \varpi, \nu)_F)$ is called the *neutrosophic homology module* of the neutrosophic complex $(\varsigma, \varpi, \nu)_F$.

Analogously, one can define cochain neutrosophic complex and neutrosophic cohomology module.

Let $\overline{\varphi}, \overline{\psi} : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ be morphisms of neutrosophic chain complexes.

Definition 3.14. A *neutrosophic homotopy* $\overline{\Sigma} : (\varsigma, \varpi, \nu)_F \to (\alpha, \beta, \gamma)_E$ between $\overline{\varphi}$ and $\overline{\psi}$ is a neutrosophic morphism of degree +1 such that $\overline{\psi} - \overline{\varphi} = \overline{\partial} \circ \overline{\Sigma} + \overline{\Sigma} \circ \overline{\partial}$.

The morphism $\overline{\varphi}$, $\overline{\psi}$ are said to be *neutrosophic homotopic*, if there exists a neutrosophic homotopy between them.

Theorem 3.15. *The neutrosophic homotopy relation is an equivalence relation. Moreover, neutrosophic homology and cohomology modules are invariant with respect of this relation.*

Observe the following neutrosophic cochain complex

$$\overline{0} \to (\varsigma_D, \varpi_D, \nu_D)_{\prod F_\alpha} \xrightarrow{\overline{d}} (\varsigma_D, \varpi_D, \nu_D)_{\prod F_\alpha} \to \overline{0}.$$

Neutrosophic cohomology modules of this complex are $\operatorname{Ker} \overline{d}$ and $\operatorname{Coker} \overline{d}$.

Lemma 3.16. $\varprojlim^{(1)} (\varsigma_i, \varpi_i, \nu_i)_{F_i} = \operatorname{Ker} \overline{d}$ *and* $\varprojlim^{(1)} (\mu_\alpha, \xi_\alpha, \nu_i)_{F_i} = \operatorname{Coker} \overline{d}$.

Proof. Obvious. □

In what follows, we consider the set of natural numbers as index set of inverse systems.

Theorem 3.17. *For each infinite subsequence of the inverse sequence of neutrosophic modules*

$$(F_1, \varsigma_1, \varpi_1, \nu_1)_{F_1} \xleftarrow{\overline{p}_{12}} (F_2, \varsigma_2, \varpi_2, \nu_2)_{F_2} \xleftarrow{\overline{p}_{23}} \cdots,$$

$\varprojlim^{(1)}$ *does not change.*

Proof. Let $S = \{i, j, k, \cdots\}$ be an infinite subsequence of natural numbers. By Lemma 3.16, $\varprojlim^{(1)}$ is defined by the following homomorphism of neutrosophic modules subsequence S

$$\overline{d}' : \left(\bigwedge_{s\in S} \varsigma_s, \bigwedge_{s\in S} \varpi_s, \bigvee_{s\in S} \nu_s\right)_{\prod\limits_{s\in S} F_s} \to \left(\bigwedge_{s\in S} \varsigma_s, \bigwedge_{s\in S} \varpi_s, \bigvee_{s\in S} \nu_s\right)_{\prod\limits_{s\in S} F_s}.$$

We also define two module homomorphisms $h_0, h_1 : \prod_{s\in S} F_s \to \prod_{s\in S} F_s$ as

$$h_0(x_i, x_j, x_k, \cdots) = (p_{1i}(x_i), p_{2i}(x_i), \cdots, p_{i-1i}(x_i), x_i, p_{i+1j}(x_j), \cdots, p_{j-1j}(x_j), x_j, \cdots),$$
$$h_1(x_i, x_j, x_k, \cdots) = (0, 0, \cdots, x_i, 0, \cdots, x_j, 0, \cdots, x_k, 0, \cdots).$$

Moreover, $\left(\bigwedge_{n\in N} \varsigma_n\right)(p_{1i}(x_i), p_{2i}(x_i), \cdots, p_{i-1i}(x_i), x_i, p_{i+1j}(x_j), \cdots, p_{j-1j}(x_j), x_j, \cdots)$

$$\begin{aligned}
&= \varsigma_1(p_{1i}(x_i)) \wedge \cdots \wedge \varsigma_{i-1}(p_{i-1i}(x_i)) \wedge \varsigma_i(x_i) \wedge \varsigma_{i+1}(p_{i+1j}(x_j)) \wedge \cdots \wedge \varsigma_j(x_j) \wedge \cdots \\
&\geq [\mu_i(x_i) \wedge \cdots \wedge \varsigma_i(x_i) \wedge \varsigma_i(x_i)] \wedge [\varsigma_j(x_j) \wedge \cdots \wedge \varsigma_j(x_j)] \wedge \cdots \\
&= \varsigma_i(x_i) \wedge \varsigma_j(x_j) \wedge \cdots \\
&= \bigwedge_{s\in S} \varsigma_s(x_s),
\end{aligned}$$

$\left(\bigwedge_{n\in N} \xi_n\right)(p_{1i}(x_i), p_{2i}(x_i), \cdots, p_{i-1i}(x_i), x_i, p_{i+1j}(x_j), \cdots, p_{j-1j}(x_j), x_j, \cdots)$

$$\begin{aligned}
&= \varpi_1(p_{1i}(x_i)) \wedge \cdots \wedge \varpi_{i-1}(p_{i-1i}(x_i)) \wedge \varpi_i(x_i) \wedge \varpi_{i+1}(p_{i+1j}(x_j)) \wedge \cdots \wedge \varpi_j(x_j) \wedge \cdots \\
&\geq [\varpi_i(x_i) \wedge \cdots \wedge \varpi_i(x_i) \wedge \varpi_i(x_i)] \wedge [\varpi_j(x_j) \wedge \cdots \wedge \varpi_j(x_j)] \wedge \cdots \\
&= \varpi_i(x_i) \wedge \varpi_j(x_j) \wedge \cdots \\
&= \bigwedge_{s\in S} \varpi_s(x_s),
\end{aligned}$$

$\left(\bigvee_{n\in N} \nu_n\right)(p_{1i}(x_i), p_{2i}(x_i), \cdots, p_{i-1i}(x_i), x_i, p_{i+1j}(x_j), \cdots, p_{j-1j}(x_j), x_j, \cdots)$

$$\begin{aligned}
&= \nu_1(p_{1i}(x_i)) \vee \cdots \vee \nu_{i-1}(p_{i-1i}(x_i)) \vee \nu_i(x_i) \vee \nu_{i+1}(p_{i+1j}(x_j)) \vee \cdots \vee \nu_j(x_j) \vee \cdots \\
&\geq [\nu_i(x_i) \vee \cdots \vee \nu_i(x_i) \vee \nu_i(x_i)] \vee [\nu_j(x_j) \vee \cdots \vee \nu_j(x_j)] \vee \cdots \\
&= \nu_i(x_i) \vee \nu_j(x_j) \vee \cdots \\
&= \bigvee_{s\in S} \nu_s(x_s)
\end{aligned}$$

$$\begin{aligned}
\left(\bigwedge_{n\in N} \varsigma_n\right)(0, 0, \cdots, x_i, 0, \cdots, x_j, 0, \cdots) &= \varsigma_1(0) \wedge \cdots \wedge \varsigma_i(x_i) \wedge \varsigma_{i+1}(0) \wedge \cdots \wedge \varsigma_j(x_j) \wedge \cdots \\
&= \varsigma_i(x_i) \wedge \varsigma_j(x_j) \wedge \cdots \\
&= \bigwedge_{s\in S} \varsigma_s(x_s)
\end{aligned}$$

$$\begin{aligned}\left(\bigwedge_{n\in N}\varpi_n\right)(0,0,\cdots,x_i,0,\cdots,x_j,0,\cdots) &= \varpi_1(0)\wedge\cdots\wedge\varpi_i(x_i)\wedge\varpi_{i+1}(0)\wedge\cdots\wedge\varpi_j(x_j)\wedge\cdots\\ &= \varpi_i(x_i)\wedge\varpi_j(x_j)\wedge\cdots\\ &= \bigwedge_{s\in S}\varpi_s(x_s)\end{aligned}$$

$$\begin{aligned}\left(\bigvee_{n\in N}\nu_n\right)(0,0,\cdots,x_i,0,\cdots,x_j,0,\cdots) &= \nu_1(0)\vee\cdots\vee\nu_i(x_i)\vee\nu_{i+1}(0)\vee\cdots\vee\nu_j(x_j)\vee\cdots\\ &= \nu_i(x_i)\vee\nu_j(x_j)\vee\cdots\\ &= \bigvee_{s\in S}\nu_s(x_s).\end{aligned}$$

Then, $\check{h}_0,\check{h}_1 : \left(\bigwedge_{s\in S}\varsigma_s, \bigwedge_{s\in S}\varpi_s, \bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s} \to \left(\bigwedge_{n\in N}\varsigma_n, \bigwedge_{n\in N}\varpi_n, \bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}$ are homomorphisms of neutrosophic modules. Observe that, the following diagram is commutative:

$$\begin{array}{ccc}\left(\bigwedge_{s\in S}\varsigma_s, \bigwedge_{s\in S}\varpi_s, \bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s} & \longrightarrow & \left(\bigwedge_{n\in N}\varsigma_n, \bigwedge_{n\in N}\varpi_n, \bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\\ \overline{d}'\big\downarrow & & \big\downarrow\overline{d}\\ \left(\bigwedge_{s\in S}\varsigma_s, \bigwedge_{s\in S}\varpi_s, \bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s} & \longrightarrow & \left(\bigwedge_{n\in N}\varsigma_n, \bigwedge_{n\in N}\varpi_n, \bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\end{array}$$

That is, $\{\check{h}_0,\check{h}_1\}$ are morphisms of neutrosophic cochain complexes. Now, we define two homomorphisms $g_0, g_1 : \prod_{n\in N}F_n \to \prod_{s\in S}F_s$ as $g_0(x_1,x_2,x_3,\cdots) = (x_i,x_j,x_k,\cdots)$, $g_1(x_1,x_2,x_3,\cdots) = (x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}), x_j+p_{jj+1}(x_{j+1})+\cdots+p_{jk-1}(x_{k-1}),\cdots)$. Moreover,

$$\begin{aligned}\Big(\bigwedge_{s\in S}\varsigma_s\Big)(x_i,x_j,x_k,\cdots) &= \varsigma_i(x_i)\wedge\varsigma_j(x_j)\wedge\cdots \geq \bigwedge_{n\in N}\varsigma_n(x_n),\\ \Big(\bigwedge_{s\in S}\varpi_s\Big)(x_i,x_j,x_k,\cdots) &= \varpi_i(x_i)\wedge\xi_j(x_j)\wedge\cdots \geq \bigwedge_{n\in N}\varpi_n(x_n),\\ \Big(\bigvee_{s\in S}\nu_s\Big)(x_i,x_j,x_k,\cdots) &= \nu_i(x_i)\vee\nu_j(x_j)\vee\cdots \leq \bigvee_{n\in N}\nu_n(x_n),\end{aligned}$$

Therefore, we have

$$\left(\bigwedge_{s\in S}\varsigma_s\right)(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}),x_j+\cdots+p_{jk-1}(x_{k-1}),\cdots)$$
$$=\varsigma_i(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}))\wedge\varsigma_j(x_j+\cdots+p_{jk-1}(x_{k-1}))\wedge\cdots$$
$$\geq\min\{\varsigma_i(x_i),\varsigma_i(p_{ii+1}(x_{i+1})),\cdots,\varsigma_i(p_{ij-1}(x_{j-1}))\}\wedge\min\{\varsigma_j(x_j),\cdots,\varsigma_j(p_{jk-1}(x_{k-1}))\}\wedge\cdots$$
$$\geq\min\{\varsigma_i(x_i),\varsigma_{i+1}(x_{i+1}),\cdots,\varsigma_{j-1}(x_{j-1})\}\wedge\min\{\varsigma_j(x_j),\varsigma_{j+1}(x_{j+1}),\cdots,\varsigma_{k-1}(x_{k-1})\}\wedge\cdots$$
$$=\bigwedge_{m\in M}(x_m)$$
$$\geq\bigwedge_{n\in N}(x_n)$$

$$\left(\bigwedge_{s\in S}\varpi_s\right)(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}),x_j+\cdots+p_{jk-1}(x_{k-1}),\cdots)$$
$$=\varpi_i(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}))\wedge\varpi_j(x_j+\cdots+p_{jk-1}(x_{k-1}))\wedge\cdots$$
$$\geq\min\{\varpi_i(x_i),\varpi_i(p_{ii+1}(x_{i+1})),\cdots,\varpi_i(p_{ij-1}(x_{j-1}))\}\wedge\min\{\varpi_j(x_j),\cdots,\xi_j(p_{jk-1}(x_{k-1}))\}\wedge\cdots$$
$$\geq\min\{\varpi_i(x_i),\xi_{i+1}(x_{i+1}),\cdots,\varpi_{j-1}(x_{j-1})\}\wedge\min\{\varpi_j(x_j),\varpi_{j+1}(x_{j+1}),\cdots,\varpi_{k-1}(x_{k-1})\}\wedge\cdots$$
$$=\bigwedge_{m\in M}(x_m)$$
$$\geq\bigwedge_{n\in N}(x_n)$$

$$\left(\bigvee_{s\in S}\nu_s\right)(x_i+p_{ii+1}(x_{i+1})+\cdots+p_{ij-1}(x_{j-1}),x_j+\cdots+p_{jk-1}(x_{k-1}),\cdots)$$
$$\leq\max\{\nu_i(x_i),\nu_i(p_{ii+1}(x_{i+1})),\cdots,\nu_i(p_{ij-1}(x_{j-1}))\}\vee\max\{\nu_j(x_j),\cdots,\nu_j(p_{jk-1}(x_{k-1}))\}\wedge\cdots$$
$$\leq\max\{\nu_i(x_i),\nu_{i+1}(x_{i+1}),\cdots,\nu_{j-1}(x_{j-1})\}\vee\max\{\nu_j(x_j),\nu_{j+1}(x_{j+1}),\cdots,\nu_{k-1}(x_{k-1})\}\wedge\cdots$$
$$=\bigwedge_{m\in M}\nu_n(x_n)$$
$$\leq\bigwedge_{n\in N}\nu_n(x_n).$$

Thus, $\bar{g}_0,\bar{g}_1:\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\to\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}$ are homomorphisms of neutrosophic modules. Define now $D:\prod_{n\in N}F_n\to\prod_{n\in N}F_n$ as

$$\begin{aligned}D(x_1,x_2,x_3,\cdots)=\big(&x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}),x_2+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}),\cdots,\\&x_{i-1},0,x_{i+1}+p_{i+1i+2}(x_{i+2})+\cdots+p_{i+1j-1}(x_{j-1}),x_{i+2}+\cdots+p_{i+2j-1}(x_{j-1}),0,\cdots\big)\end{aligned}$$

which is a module homomorphism. Hence, we have

$$\left(\bigwedge_{n\in N}\varsigma_n\right)(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}),x_2+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$\begin{aligned}
&=\varsigma_1(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}))\wedge\varsigma_2(x_2)+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}))\wedge\cdots\\
&\wedge\varsigma_{i-1}(x_{i-1})\wedge\varsigma_i(0)\wedge\varsigma_{i+1}(x_{i+1}+p_{i+1i+2}(x_i+2)+\cdots+p_{i+1j}-1(x_{j-1}))\wedge\cdots\\
&\geq\min\{\varsigma_1(x_1),\varsigma_1(p_{12}(x_2)),\cdots,\varsigma_1(p_{1i-1}(x_{i-1}))\}\wedge\\
&\min\{\varsigma_2(x_2),\varsigma_2(p_{23}(x_3)),\cdots,\varsigma_2(p_{2i-1}(x_i-1))\}\wedge\varsigma_{i-1}(x_{i-1})\wedge\\
&\min\{\varsigma_{i+1}(x_{i+1}),\varsigma_{i+1}(p_{i+1i+2}(x_{i+2})),\cdots,\varsigma_{i+1}(p_{i+1j}-1(x_{j-1}))\}\wedge\cdots\\
&\geq\min\{\varsigma_1(x_1),\varsigma_2(x_2),\cdots,\varsigma_{i-1}(x_{i-1})\}\wedge\min\{\varsigma_2(x_2),\varsigma_3(x_3),\cdots,\varsigma_{i-1}(x_{i-1})\}\wedge\\
&\varsigma_{i-1}(x_{i-1})\wedge\varsigma_{i+1}(x_{i+1})\wedge\cdots\\
&=\bigwedge_{k=1}^{i-1}\varsigma_k(x_k)\wedge\bigwedge_{k=2}^{i-1}\mu_k(x_k)\wedge\cdots\\
&=\bigwedge_{n\in N}\varsigma_n(x_n),
\end{aligned}$$

$$\left(\bigwedge_{n\in N}\xi_n\right)(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}),x_2+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$\begin{aligned}
&=\varpi_1(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}))\wedge\varpi_2(x_2)+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}))\wedge\cdots\\
&\wedge\varpi_{i-1}(x_{i-1})\wedge\varpi_i(0)\wedge\varpi_{i+1}(x_{i+1}+p_{i+1i+2}(x_i+2)+\cdots+p_{i+1j}-1(x_{j-1}))\wedge\cdots\\
&\geq\min\{\varpi_1(x_1),\varpi_1(p_{12}(x_2)),\cdots,\varpi_1(p_{1i-1}(x_{i-1}))\}\wedge\\
&\min\{\varpi_2(x_2),\varpi_2(p_{23}(x_3)),\cdots,\xi_2(p_{2i-1}(x_i-1))\}\wedge\varpi_{i-1}(x_{i-1})\wedge\\
&\min\{\varpi_{i+1}(x_{i+1}),\varpi_{i+1}(p_{i+1i+2}(x_{i+2})),\cdots,\varpi_{i+1}(p_{i+1j}-1(x_{j-1}))\}\wedge\cdots\\
&\geq\min\{\varpi_1(x_1),\xi_2(x_2),\cdots,\varpi_{i-1}(x_{i-1})\}\wedge\min\{\varpi_2(x_2),\xi_3(x_3),\cdots,\varpi_{i-1}(x_{i-1})\}\wedge\\
&\varpi_{i-1}(x_{i-1})\wedge\varpi_{i+1}(x_{i+1})\wedge\cdots\\
&=\bigwedge_{k=1}^{i-1}\varpi_k(x_k)\wedge\bigwedge_{k=2}^{i-1}\varpi_k(x_k)\wedge\cdots\\
&=\bigwedge_{n\in N}\varpi_n(x_n),
\end{aligned}$$

$$\left(\bigvee_{n\in N}\nu_n\right)(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}),x_2+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}),\cdots,x_{i-1},0,\cdots)$$

$$\begin{aligned}
&=\nu_1(x_1+p_{12}(x_2)+\cdots+p_{1i-1}(x_{i-1}))\vee\nu_2(x_2)+p_{23}(x_3)+\cdots+p_{2i-1}(x_{i-1}))\\
&\vee\cdots\vee\nu_{i-1}(x_{i-1})\vee\nu_i(0)\vee\nu_{i+1}(x_{i+1}+p_{i+1i+2}(x_i+2)+\cdots+p_{i+1j}-1(x_{j-1}))\vee\cdots\\
&\geq\max\{\nu_1(x_1),\nu_1(p_{12}(x_2)),\cdots,\nu_1(p_{1i-1}(x_{i-1}))\}\vee\\
&\max\{\nu_2(x_2),\nu_2(p_{23}(x_3)),\cdots,\nu_2(p_{2i-1}(x_i-1))\}\vee\nu_{i-1}(x_{i-1})\vee\\
&\max\{\nu_{i+1}(x_{i+1}),\nu_{i+1}(p_{i+1i+2}(x_{i+2})),\cdots,\nu_{i+1}(p_{i+1j}-1(x_{j-1}))\}\vee\cdots\\
&\geq\max\{\nu_1(x_1),\nu_2(x_2),\cdots,\nu_{i-1}(x_{i-1})\}\vee\\
&\max\{\nu_2(x_2),\nu_3(x_3),\cdots,\nu_{i-1}(x_{i-1})\}\vee\nu_{i-1}(x_{i-1})\vee\nu_{i+1}(x_{i+1})\vee\cdots\\
&=\bigvee_{k=1}^{i-1}\nu_k(x_k)\vee\bigvee_{k=2}^{i-1}\nu_k(x_k)\vee\cdots\\
&=\bigvee_{n\in N}\nu_n(x_n).
\end{aligned}$$

Therefore, $\overline{D}:\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\to\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}$ is a homomorphism of neutrosophic modules. By some calculation, One can show that $\overline{D}$ is a neutrosophic chain homotopy between $\check{h_0}\circ\overline{g_0}$ and $\check{h_1}\circ\overline{g_1}$ homomorphisms. Then, the following cohomology modules of neutrosophic cochain complexes are quasi isomorphic (see [1]).

$$\overline{0}\to\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\xrightarrow{\overline{d}}\left(\bigwedge_{n\in N}\varsigma_n,\bigwedge_{n\in N}\varpi_n,\bigvee_{n\in N}\nu_n\right)_{\prod_{n\in N}F_n}\to\overline{0}$$

$$\overline{0}\to\left(\bigwedge_{s\in S}\varsigma_s,\bigwedge_{s\in S}\varpi_s,\bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s}\xrightarrow{\overline{d}}\left(\bigwedge_{s\in S}\varsigma_s,\bigwedge_{s\in S}\varpi_s,\bigvee_{s\in S}\nu_s\right)_{\prod_{s\in S}F_s}\to\overline{0}$$

By the fact that $\varprojlim^{(1)}$ is the first cohomology module, we are done. □

Remark 3.18. Since $\varprojlim(\varsigma_n,\varpi_n,\nu_n)_{F_n}=\operatorname{Ker}\overline{d}$ and $p_{n+1n}(x_{n+1})=x_n$ are satisfied for each $\{x_n\}\in\varprojlim F_n$, one obtains

$$\begin{aligned}
\varsigma_n(x_n)&=\varsigma_n(p_{n+1n}(x_{n+1}))\geq\varsigma_{n+1}(x_{n+1}),\\
\varpi_n(x_n)&=\varpi_n(p_{n+1n}(x_{n+1}))\geq\varpi_{n+1}(x_{n+1}),\\
\nu_n(x_n)&=\nu_n(p_{n+1n}(x_{n+1}))\leq\nu_{n+1}(x_{n+1}),
\end{aligned}$$

that is, for each $\{x_n\}\in\operatorname{Ker}\overline{d}$, $\{\varsigma_n(x_n)\}$ is a decreasing sequence, $\{\varpi_n(x_n)\}$ is a decreasing sequence and $\{\nu_n(x_n)\}$ is an increasing sequence.

Theorem 3.19. *For every $\{x''_n\} \in \operatorname{Ker}\overline{d}$, if $\lim_{n\to\infty} \varsigma''_n(x''_n) = 0$ or $\lim_{n\to\infty} \varpi''_n(x''_n) = 0$ or $\lim_{n\to\infty} \nu''_n(x''_n) = 1$ and the following diagram*

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (\varsigma'_2, \varpi'_2, \nu'_2)_{F'_2} & \to & (\varsigma_2, \varpi_2, \nu_2)_{F_2} & \to & (\varsigma''_2, \varpi''_2, \nu''_2)_{F''_2} & \to & 0 \\ 0 & \to & (\varsigma'_1, \varpi'_1, \nu'_1)_{F'_1} & \to & (\varsigma_1, \varpi_1, \nu_1)_{M_1} & \to & (\varsigma''_1, \varpi''_1, \nu''_1)_{F''_1} & \to & 0 \end{array}$$

is a short exact sequence of inverse system of neutrosophic modules, then the sequence

$$\overline{0} \to \varprojlim(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim(\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n} \to \varprojlim(\varsigma'_n, \varpi'_n, \nu'_n)_{M_n}$$
$$\to \varprojlim(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim(\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n} \to \overline{0},$$

is exact.

Proof. For an inverse system of neutrosophic modules $\{(\varsigma_n, \varpi_n, \nu_n)_{F_n}\}_{n\in N}$ one has that

$$C' = \overline{0} \xrightarrow{\overline{0}} (\varsigma'_A, \varpi'_A, \nu'_A)_{\prod_{n\in N} F_n} \xrightarrow{\overline{d}} (\varsigma'_A, \varpi'_A, \nu'_A)_{\prod_{n\in N} F_n} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

is a cochain complex of neutrosophic modules.

$$\begin{aligned} H^0(C) &= \varprojlim((\varsigma_n, \varpi_n, \nu_n)_{F_n})_{F_n}, \\ H^1(C) &= \varprojlim^{(1)}((\varsigma_n, \varpi_n, \nu_n)_{F_n})_{F_n}, \\ H^k(C) &= 0, k \geq 2 \end{aligned} \tag{5}$$

are neutrosophic cohomology modules of this complex. analogously, for the inverse system of modules $\{(\varsigma'_n, \varpi'_n, \nu'_n)_{M'_n})\}$ and $\{(\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n})\}$, we can establish the following neutrosophic cochain complexes

$$C' = \overline{0} \xrightarrow{\overline{0}} (\varsigma'_A, \varpi'_A, \nu'_A)_{\prod_{n\in N} F'_n} \xrightarrow{\overline{d}} (\varsigma'_A, \varpi'_A, \nu'_A)_{\prod_{n\in N} F'_n} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

$$C'' = \overline{0} \xrightarrow{\overline{0}} (\varsigma''_A, \varpi''_A, \nu''_A)_{\prod_{n\in N} F''_n} \xrightarrow{\overline{d}} (\varsigma''_A, \varpi''_A, \nu''_A)_{\prod_{n\in N} F''_n} \xrightarrow{\overline{0}} \overline{0} \xrightarrow{\overline{0}} \cdots$$

Observe that, cohomology modules of this complexes have the form of (5). By the hypothesis of this theorem, the following sequence

$$\overline{0} \to C' \to C \to C'' \to \overline{0}$$

is a short exact sequence of cochain complexes of neutrosophic modules. Mind that in general, the following sequence of cohomology modules of this sequence

$$0 \to H^0(C') \to H^0(C) \to H^0(C'') \xrightarrow{\overline{\partial}} H^1(C) \to H^1(C'') \to H^2(C'') \to \cdots$$

is not exact. This is so, since $\overline{\partial}$ is often not a homomorphism of neutrosophic modules. By the fact that $H^0(C'') = \operatorname{Ker} d''$ and $\lim_{n\to\infty} \varsigma''_n(x''_n) = 0$ ($\lim_{n\to\infty} \varpi''_n(x''_n) = 0$, $\lim_{n\to\infty} \nu''_n(x''_n) = 1$), the grade function $\varsigma''(\varpi'', \nu'')$ of a neutrosophic module $(\varsigma'', \varpi'', \nu'')_{H^0(c'')}$ is equal to the grade function

$\varsigma''(\varpi'', \nu'')$ is equal to the grade function indicated in (2). Thus, $\overline{\partial}$ is a homomorphism of neutrosophic modules. This means that the sequence of neutrosophic homology modules

$$0 \to H^0(C') \to H^0(C) \to H^0(C'') \xrightarrow{\overline{\partial}} H^1(C) \to H^1(C'') \to H^2(C') \to \cdots$$

is exact. By (5), we obtain the following exact sequence of neutrosophic modules $\overline{0} \to \varprojlim(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim(\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n} \to \varprojlim(\varsigma'_n, \varpi'_n, \nu'_n)_{F_n} \to \varprojlim(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim(\varsigma''_n, \varpi''_n, \nu''_n)_{F''_n} \to \overline{0}$. □

Let us now investigate the necessary conditions for which the derivative functor $\varprojlim^{(1)}$ is equal to zero.

Theorem 3.20. *Let the following be an inverse systems of neutrosophic modules*

$$(\varsigma_1, \varpi_1, \nu_1)_{F_1} \xleftarrow{\overline{\varphi_1}} (\varsigma_2, \varpi_2, \nu_2)_{F_2} \xleftarrow{\overline{\varphi_2}} \cdots . \tag{6}$$

If every homomorphisms $\overline{\varphi_n}$ is a neutrosophic epimorphism, then $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} = 0$.

Proof. The proof follows from the fact that

$$\overline{d} : \prod_{n=1}^{\infty}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \prod_{n=1}^{\infty}(\varsigma_n, \varpi_n, \nu_n)_{F_n}$$

is a neutrosophic epimorphism. □

Definition 3.21. Consider the inverse system of neutrosophic modules (6). If there exists $m \geq n$, for every integer n, such that for all $i \geq m$

$$\mathrm{Im}((\varsigma_i, \varpi_i, \nu_i)_{M_i} \to (\varsigma_n, \varpi_n, \nu_n)_{F_n}) = \mathrm{Im}((\varsigma_m, \varpi_m, \nu_m)_{F_m} \to (\varsigma_n, \varpi_n, \nu_n)_{F_n})$$

then it is said that the inverse system (6) satisfies the *Mittag-Leffler condition*.

Theorem 3.22. *If the inverse system in* (6) *satisfies the Mittag-Leffler condition, then* $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} = 0$.

Proof. Let us denote $F'_n = \mathrm{Im}\, \varphi^i_n$, for large i. By the hypothesis of the theorem, the homomorphism $\varphi_{n|F'_{n+1}}$ carries the module F'_{n+1} to F'_n. Then, $\varphi_{n|F'_{n+1}}$ is an epimorphism. Thus, for large i, the homomorphisms

$$\overline{\varphi_n} : (\varsigma_n|_{F'_{n+1}}, \varpi_n|_{F'_{n+1}}, \nu_n|_{F'_{n+1}})_{F'_{n+1}} \to (\varsigma_n|_{F'_n}, \varpi_n|_{F'_n}, \nu_n|_{F'_n})_{F'_n}$$

are epimorphisms. Therefore, by Theorem 3.20, we have $\varprojlim^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} = 0$. Now, $\varsigma'_n = \varsigma_n|_{F'_n}, \varpi'_n = \varpi_n|_{F'_n}, \nu'_n = \nu_n|_{F'_n}$. Observe the following sequence of the inverse system of neutrosophic quotient modules

$$(\widetilde{\varsigma_1}, \widetilde{\varpi_1}, \widetilde{\nu_1})_{F_1/F'_1} \leftarrow (\widetilde{\varsigma_2}, \widetilde{\varpi_2}, \widetilde{\nu_2})_{F_2/F'_2} \leftarrow \cdots . \tag{7}$$

For every n, there exists $m > n$ such that the homomorphism $F_m/F'_m \to F_n/F'_n$ is a zero homomorphism. Then, $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} = 0$. This implies that the limit of inverse system in (7) is equal to 0. Hence, $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} = 0$. Now, let us see on the following short exact sequence of inverse systems in the category of N-Mod

$$\overline{0} \to \{(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n}\} \to \{(\varsigma_n, \varpi_n, \nu_n)_{F_n}\} \to \{(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n}\} \to \overline{0}. \tag{8}$$

Granting that $(F_n/F'_n) = 0$. By utilizing Theorem 3.20 to the sequence (8), we obtain the following exact sequence

$$\begin{aligned} &\overline{0} \to \varprojlim{}^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} \to \\ &\varprojlim{}^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} \to \overline{0}. \end{aligned} \tag{9}$$

Since $\varprojlim^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} = \overline{0}$, $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} = \overline{0}$ and $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n/F'_n} = \overline{0}$, respectively. Sequence (9) becomes

$$\overline{0} \to \varprojlim{}^{(1)}(\varsigma'_n, \varpi'_n, \nu'_n)_{F'_n} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \overline{0} \to \overline{0} \to \varprojlim{}^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} \to \overline{0} \to \overline{0}.$$

This proves that $\varprojlim^{(1)}(\varsigma_n, \varpi_n, \nu_n)_{F_n} = \overline{0}$. □

4. Direct system of neutrosophic modules

In this section we present some basic properties of direct systems of neutrosophic modules. Let

$$(\overline{\varsigma}, \overline{\varpi}, \overline{\nu})_{\overline{F}} = \left\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}, \overline{p}^{i'i}\right\}_{i \in \Delta} \tag{10}$$

be a direct system of neutrosophic modules, where $(\varsigma^B, \varpi^B, \nu^B)_{\bigoplus_i F_i}$ is a neutrosophic module and $\pi : \bigoplus_i F_i \to \varinjlim_i F_i$ a canonical epimorphism. Moreover, one can consider the neutrosophic module $\left((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\right)_{\varinjlim_i F_i}$.

Theorem 4.1. *Every direct system in the representation* (10) *has a limit in the category of* N-Mod *which is equal to the neutrosophic module* $\left((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\right)_{\varinjlim_i F_i}$.

Proof. It suffices to demonstrate that, there exists a unique homomorphism of neutrosophic modules $\overline{\psi} : \left((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\right)_{\varinjlim_i F_i} \to (\alpha, \beta, \gamma)_N$, making the diagram

$$\begin{tikzcd} (\varsigma_i, \varpi_i, \nu_i)_{F_i} \arrow[r, "\overline{\varphi}_i"] \arrow[d, "\overline{\pi}_i"'] & (\alpha, \beta, \gamma)_E \\ \left((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\right)_{\varinjlim_i F_i} \arrow[ur, "\overline{\psi}"'] & \end{tikzcd}$$

commutative, where $\overline{\varphi} = \{\overline{\varphi}_i : (\varsigma_i, \varpi_i, \nu_i)_{F_i} \to (\alpha, \beta, \gamma)_E\}_{i\in\Delta}$ is a family of homomorphisms of neutrosophic modules providing the commutativity of the diagram

$$\begin{tikzcd}
(\varsigma_i, \varpi_i, \nu_i)_{F_i} \arrow[r, "\overline{\varphi}_i"] \arrow[d, "\overline{p}^{i,i'}"'] & (\alpha, \beta, \gamma)_E \\
(\varsigma_{i'}, \varpi_{i'}, \nu_{i'})_{F_{i'}} \arrow[ur] &
\end{tikzcd}$$

$\overline{l_i} : (\varsigma_i, \varpi_i, \nu_i)_{F_i} \to (\varsigma^B, \varpi^B, \nu^B)_{\bigoplus\limits_i F_i}$ are usual injections and $\pi_i = \pi \circ l_i$. For every $x \in \varinjlim\limits_{i} F_i$, there exists $x_i \in F_i$ such that $\pi_i(x_i) = x$. If $\pi_{i'}(x_{i'}) = x$ for each $x_{i'} \in F_{i'}$, then $\varphi_{i'}(x_{i'})$ is equal to $\varphi_i(x_i)$. We define the homomorphism $\psi : \varinjlim\limits_{i} F_i \to E$ by $\psi(x) = \varphi_i(x_i)$. We want to show that $\overline{\psi}$ is a homomorphism of neutrosophic modules. For each $x \in \varinjlim\limits_{i} F_i$, let $\pi \circ l_i(x_i) = x$. Here,

$$\begin{aligned}
(\varsigma^B)^\pi(x) &= \sup(\bigvee_i \varsigma_i)(x) = \sup\{\bigvee_i \varsigma_i(x) : \pi_i(x_i) = x\},\\
(\varpi^B)^\pi(x) &= \sup(\bigvee_i \varpi_i)(x) = \sup\{\bigvee_i \varpi_i(x) : \pi_i(x_i) = x\},\\
(\nu^B)^\pi(x) &= \inf(\bigwedge_\alpha \nu_i)(x) = \inf\{\bigwedge_i \nu_i(x) : \pi_i(x_i) = x\}.
\end{aligned}$$

Therefore,

$$\alpha(\psi(x)) = i(\varphi_i(x_i)) \geq \varsigma_i(x_i), \quad \beta(\psi(x)) = \beta(\varphi_i(x_i)) \geq \varpi_i(x_i), \quad \gamma(\psi(x)) = \alpha(\varphi_i(x_i)) \leq \nu_i(x_i).$$

Since this inequality is satisfied for each x_i such that $\pi_i(x_i) = x$, we have $\alpha(\psi(x)) \geq (\varsigma^B)^\pi(x)$, $\beta(\psi(x)) \geq (\varpi^B)^\pi(x)$, $\gamma(\psi(x)) \geq (\nu^B)^\pi(x)$. From the definition of $\overline{\psi}$, it is obvious that the above diagram is commutative. We can easily show that $\varinjlim$ is a functor from the category of direct systems of neutrosophic modules to the category of neutrosophic modules. □

We can now focus on the problem of the exact direct limit of exact sequences of direct systems of neutrosophic modules. Let

$$\begin{aligned}
\overline{F} &= \left\{(\varsigma_i, \varpi_i, \nu_i)_{F_i}, \overline{p}^{i'i}\right\}_{i\in\Delta},\\
\overline{F}' &= \left\{(\varsigma_i', \varpi_i', \nu_i')_{F_i'}, \overline{p}^{i'i}\right\}_{i\in\Delta},\\
\overline{F}'' &= \left\{(\varsigma_i'', \varpi_i'', \nu_i'')_{F_i''}, \overline{p}^{i'i}\right\}_{i\in\Delta},
\end{aligned}$$

be direct systems of neutrosophic modules, and let

$$\overline{F}' \xrightarrow{\check{h}} \overline{F} \xrightarrow{\overline{g}} \overline{F}'' \tag{11}$$

be an exact sequence.

The following holds.

Theorem 4.2. *Direct limit of the exact sequence* (11),

$$\varinjlim_{i}(\varsigma_i', \varpi_i', \nu_i')_{F_i'} \to \varinjlim_{i}(\varsigma_i, \varpi_i, \nu_i)_{F_i} \to \varinjlim_{i}(\varsigma_i'', \varpi_i'', \nu_i'')_{F_i''},$$

is exact.

Proof. Since the sequence (11) is exact, then the ordinary sequence of R-modules $F_i' \to F_i \to F_i''$ is exact for every $i \in \Delta$. Hence, the sequence $\{F_i'\}_i \xrightarrow{\{h_i\}} \{F_i\}_i \xrightarrow{\{g_i\}} \{F_i''\}_i$ is an exact sequence of direct system of ordinary modules. Then, taking the limit one obtains the sequence

$$\varinjlim_{i} F_\alpha' \xrightarrow{\varinjlim_{i} h_i} \varinjlim_{\alpha} F_i \xrightarrow{\varinjlim_{\alpha} g_i} \varinjlim_{i} F_i'', \tag{12}$$

which is also exact. Therefore, the following sequence of neutrosophic modules

$$\big((\varsigma'^B)^\pi, (\varpi'^B)^\pi, (\nu'^B)^\pi\big)_{\varinjlim_{i} F_i'} \xrightarrow{\overline{\varinjlim_{i} h_i}} \big((\varsigma^B)^\pi, (\varpi^B)^\pi, (\nu^B)^\pi\big)_{\varinjlim_{i} F_i} \xrightarrow{\overline{\varinjlim_{i} g_i}} \big((\varsigma''^B)^\pi, (\varpi''^B)^\pi, (\nu''^B)^\pi\big)_{\varinjlim_{i} F_i''}$$

fulfills the relations

$$(\varsigma^B)^\pi|_{\mathrm{Im}\varinjlim_{i} h_i} = (\varsigma^B)^\pi|_{\mathrm{Ker}\varinjlim_{i} g_i},$$
$$(\varpi^B)^\pi|_{\mathrm{Im}\varinjlim_{i} h_i} = (\varpi^B)^\pi|_{\mathrm{Ker}\varinjlim_{i} g_i},$$
$$(\nu^B)^\pi|_{\mathrm{Im}\varinjlim_{i} h_i} = (\nu^B)^\pi|_{\mathrm{Ker}\varinjlim_{i} g_i},$$

which are true, since sequence (12) is exact. □

Corollary 4.3. *The direct limit functor preserves monomorphism and epimorphism in the category of neutrosophic modules.*

Let us see on the direct system of chain complexes. Let I be a directed set, for every $i \in I$ suppose

$$C(i) = \Big\{(\varsigma_n^{(i)}, \varpi_n^{(i)}, \nu_n^{(i)})_{F_n^{(i)}}, \overline{\partial}_n : (\varsigma_n(i), \varpi_n(i), \nu_n(i))_{F_n(i)} \to (\varsigma_{n-1}(i), \varpi_{n-1}(i), \nu_{n-1}(i))_{F_{n-1}(i)}\Big\}_n$$

is a chain complex of neutrosophic modules and for every $i < j$, let $\check{h}_{ij} : C(i) \to C(j)$ be a morphism of chain complexes and $\{C(i), \check{h}_{ij}\}$ be a direct system of these chain complexes.

Theorem 4.4. *The limit of homology modules of direct system of chain complexes of neutrosophic modules is quasi isomorphic to the homology modules of the limit of this direct system, i.e.,* $H_n\Big(\varinjlim_{i} C(i)\Big) \simeq \varinjlim_{i} H_n(C(i))$.

Proof. It is proved by using Corollary 4.3. Therefore, we have

$$\begin{aligned}
\varinjlim_{i} H_n(C(i)) &= \varinjlim_{i} \left(\widetilde{\varsigma_n}(i), \widetilde{\varpi_n}(i), \widetilde{\nu_n}(i)\right)_{\operatorname{Ker}\overline{\partial}_n(i)|_{\operatorname{Im}\overline{\partial}_{n+1}(i)}} \\
&\approx \varinjlim_{i} \left(\varsigma_n|_{\operatorname{Ker}\overline{\partial}_n(i)}, \varpi_n|_{\operatorname{Ker}\overline{\partial}_n(i)}, \nu_n|_{\operatorname{Ker}\overline{\partial}_n(i)}\right)_{\operatorname{Ker}\overline{\partial}_n(i)}\Big|_{\varinjlim_{i} \left(\varsigma_n|_{\operatorname{Ker}\overline{\partial}_n(i)}, \varpi_n|_{\operatorname{Ker}\overline{\partial}_n(i)}, \nu_n|_{\operatorname{Ker}\overline{\partial}_n(i)}\right)_{\operatorname{Ker}\overline{\partial}_n(i)}} \\
&\approx \operatorname{Ker}\varinjlim_{i}\overline{\partial}_n(i)|\varinjlim_{i}\overline{\partial}_n(i) \\
&= H_n\left(\varinjlim_{i} C(i)\right).
\end{aligned}$$

□

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