NSS

Neutrosophic Sets and Systems, Vol. 80, 2025

University of New Mexico

Trigonometric interval-valued neutrosophic set setting with reciprocal fractional function using a finite weighted method

Murugan Palanikumar[1], Nasreen kausar[2], Tonguc Cagin[3,*]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India; palanimaths86@gmail.com

[2] Department of Mathematics, Faculty of Arts and Science, Balikesir University, 10145 Balikesir, Turkey; kausar.nasreen57@gmail.com

[3] College of Business Administration, American University of the Middle East, Kuwait; tonguc.cagin@aum.edu.kw

*Correspondence: tonguc.cagin@aum.edu.kw

Abstract. This paper presents a new method for creating a trigonometric interval-valued neutrosophic set via reciprocal fractional function. This article will cover the concepts of geometric, generalized weighted averaging, trigonometric interval-valued neutrosophic reciprocal fractional weighted averaging, and generalized weighted geometric. We used an aggregating model to get the weighted average and geometric. To further examine a number of sets with important qualities, the algebraic techniques listed below will be applied.

Keywords: weighted averaging, weighted geometric, generalized weighted averaging, generalized weighted geometric.

---

1. Introduction

Many theories have been proposed to explain uncertainty, such as fuzzy sets (FS) [1], which have a membership degree s (MD) between zero and one. For $\hat{b}, \vec{b} \in [0,1]$ produced an intuitionistic FS (IFS) where each element has two MDs: $0 \leq \hat{b} + \vec{b} \leq 1$ and positive $\hat{b}$ and negative $\vec{b}$ by [2] Atanassov. Yager [3] established the Pythagorean FSs (PFS) concept, which is marked by its MD and non-MD (NMD) with $\hat{b} + \vec{b} \geq 1$ to $\hat{b}^2 + \vec{b}^2 \leq 1$. Numerous studies have examined the use of IFSs and PFSs in various fields. Their information-transmission capabilities are currently restricted. Consequently, the experts were still having trouble interpreting the data in these sets and the associated data. Wang et al. investigated the concept of interval-valued IFS with DOMBI prioritized AOs and its application for trustworthy green supplier selection [4]. Cuong et al. [5] claim that the three primary ideas of the picture FS are positive MD ($\hat{b}$), neutral MD ($\ddot{b}$), and negative MD ($\vec{b}$). In comparison to PFS and IFS, it offers additional benefits. Since $\hat{b}, \ddot{b}, and \vec{b} \in [0,1]$, it has been noted that the picture FS is an enhancement of the IFS that may handle greater inconsistency and $0 \leq \hat{b}+\ddot{b}+\vec{b} \leq 1$. Depending on the image FS description, expert opinions like "yes," "abstain," "no," and "refusal" will be supplied. The SFS for certain AOs was defined by Shahzaib et al. [6] using MADM. With SFS, $0 \leq \hat{b}^2+\ddot{b}^2+\vec{b}^2 \leq 1$ is required rather than $0 \leq \hat{b}+\ddot{b}+\vec{b} \leq 1$. In the beginning, Hussain et al. [7] proposed the concept of an intelligent decision support system for SFS. SFSs and their applications in DM were initially introduced by Rafiq et al. [8]. For instance, a DM problem with a property is $\hat{b}^2+\vec{b}^2 \geq 1$.

On the condition that $0 \leq \hat{b}^3+\vec{b}^3 \leq 1$, Fermatean FS (FFS) was first proposed by Senapati et al. [9] in 2019. Generalized orthopair FSs were initially proposed by Yager [10]. In the q-rung orthogonal pair FS (q-ROFS), both the MD and the NMD have power q; nonetheless, their total always cannot exceed one. Palanikumar et al. [11] introduced the MADM approach for Pythagorean neutrosophic normal interval-valued fuzzy AOs. Xu et al. used IFSs to produce geometric operators, such as weighted, ordered weighted, and hybrid operators [12]. Li and colleagues [13] proposed the use of generalized ordered weighted averaging operators (GOWs) in 2002. The computation of ordered weighted distances using AOs and distance measurements was described by Zeng et al. [14]. Peng et al. studied a basic PFS based on the characteristics of AOs [15]. Dombi AOs were developed by Spherical Fuzzy Ashraf and associates [16]. Additional details on SFSs and T-SFSs are given by Ullah et al. [17, 18]. The idea of interval-valued NSSs was covered by Ibraheem et al. [19] employing a variety of AOs. Type-I extension Diophantine IVNS is discussed by Al-Husban et al. [20]. Palanikumar et al. [21–23] examined a variety of algebraic structures and aggregation approaches with potential applications.

2. AOs based on IVTNRFN

Throughout the section $\propto= 1$. We use IVTNRFWA, IVTNRFWG, GIVTNRFWA, and GIVTNRFWG to describe the AOs. The fractional part of $\angle$, where $\angle$ is a real number, may be expressed as follows if $\mathcal{M}$ is a fractional part function: $\mathcal{M}[\angle] = \lfloor \angle \rfloor = \angle - \lfloor \angle \rfloor$. Alternatively, the difference between a real number and its largest integer value [which is determined via the greatest integer function] can be defined as a fractional part function. If $\angle$ is an integer, then the fractional component of $\angle = 0$. Suppose that $\mathcal{M}[\angle] = \frac{1}{\angle}$ is a reciprocal fractional part function. As is well known, whenever $\angle$ is an integer, its fractional part equals 0. Therefore, $\angle$ must not be an integer in order for $\mathcal{M}[\angle] = \frac{1}{\angle}$ to be defined. With the exception of integers, all real numbers comprise the domain of $\mathcal{M}[\angle] = \frac{1}{\angle}$. We present the concept IVTNRFN and its operations were established and $\tan \pi/2 = \daleth$

Definition 2.1. The IVNRFS $Ⓢ = \Big\lfloor ð, \Big\langle \Big\lfloor \lfloor\lfloor \daleth Z^{tl}\lfloor ð\rfloor, \daleth Z^{tu}\lfloor ð\rfloor\rfloor, \lfloor \daleth Z^{il}\lfloor ð\rfloor, \daleth Z^{iu}\rfloor\lfloor ð\rfloor, \lfloor \daleth Z^{fl}\lfloor ð\rfloor, \daleth Z^{fu}\lfloor ð\rfloor\rfloor\rfloor \Big\rfloor \Big\rangle \Big| ð \in U \Big\rfloor$, where $\daleth Z^{tu}, \daleth Z^{iu}, \daleth Z^{fu} : U \to \lfloor 0, 1\rfloor$ denote the MD, IMD and NMD of $ð \in U$ to Ⓢ, respectively and $0 \leq \lfloor\lfloor \daleth Z^{tu}\rfloor\lfloor ð\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z^{iu}\rfloor\lfloor ð\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z^{fu}\rfloor\lfloor ð\rfloor\rfloor^{\angle-\propto} \leq 1$.
For convenience, $Ⓢ = \Big\langle \lfloor \daleth Z^{tl}, \daleth Z^{tu}\rfloor, \lfloor \daleth Z^{il}, \daleth Z^{iu}\rfloor, \lfloor \daleth Z^{fl}, \daleth Z^{fu}\rfloor \Big\rangle$ is represent a IVTNRFN.

Definition 2.2. Let $Ⓢ = \Big\langle \Big\lfloor \lfloor\lfloor \daleth Z^{tl}, \daleth Z^{tu}\rfloor, \lfloor \daleth Z^{il}, \daleth Z^{iu}\rfloor, \lfloor \daleth Z^{fl}, \daleth Z^{fu}\rfloor\rfloor \Big\rfloor \Big\rangle$
$Ⓢ_1 = \Big\langle \Big\lfloor \lfloor\lfloor \daleth Z_1^{tl}, \daleth Z_1^{tu}\rfloor, \lfloor \daleth Z_1^{il}, \daleth Z_1^{iu}\rfloor, \lfloor \daleth Z_1^{fl}, \daleth Z_1^{fu}\rfloor\rfloor \Big\rfloor \Big\rangle$
and $Ⓢ_2 = \Big\langle \Big\lfloor \lfloor\lfloor \daleth Z_2^{tl}, \daleth Z_2^{tu}\rfloor, \lfloor \daleth Z_2^{il}, \daleth Z_2^{iu}\rfloor, \lfloor \daleth Z_2^{fl}, \daleth Z_2^{fu}\rfloor\rfloor \Big\rfloor \Big\rangle$ be any three IVTNRFNs. Then

$$(1)\ Ⓢ_1 \vartriangle Ⓢ_2 = \begin{bmatrix} \sqrt[\angle-\propto]{\begin{matrix}\lfloor\lfloor \daleth Z_1^{tl}\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z_2^{tl}\rfloor\rfloor^{\angle-\propto} \\ -\lfloor\lfloor \daleth Z_1^{tl}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_2^{tl}\rfloor\rfloor^{\angle-\propto}\end{matrix}}, & \sqrt[\angle-\propto]{\begin{matrix}\lfloor\lfloor \daleth Z_1^{tu}\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z_2^{tu}\rfloor\rfloor^{\angle-\propto} \\ -\lfloor\lfloor \daleth Z_1^{tu}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_2^{tu}\rfloor\rfloor^{\angle-\propto}\end{matrix}} \\ \sqrt[\angle-\propto]{\begin{matrix}\lfloor\lfloor \daleth Z_1^{il}\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z_2^{il}\rfloor\rfloor^{\angle-\propto} \\ -\lfloor\lfloor \daleth Z_1^{il}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_2^{il}\rfloor\rfloor^{\angle-\propto}\end{matrix}}, & \sqrt[\angle-\propto]{\begin{matrix}\lfloor\lfloor \daleth Z_1^{iu}\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z_2^{iu}\rfloor\rfloor^{\angle-\propto} \\ -\lfloor\lfloor \daleth Z_1^{iu}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_2^{iu}\rfloor\rfloor^{\angle-\propto}\end{matrix}} \\ \multicolumn{2}{c}{\lfloor\lfloor \daleth Z_1^{fl}\rfloor\rfloor^{\angle-\propto}\lfloor\lfloor \daleth Z_2^{fl}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_1^{fu}\rfloor\rfloor^{\angle-\propto}\lfloor\lfloor \daleth Z_2^{fu}\rfloor\rfloor^{\angle-\propto}} \end{bmatrix},$$

$$(2)\ Ⓢ_1 \odot Ⓢ_2 = \begin{bmatrix} \multicolumn{2}{c}{\lfloor\lfloor \daleth Z_1^{tl}\rfloor\rfloor^{\angle-\propto}\lfloor\lfloor \daleth Z_2^{tl}\rfloor\rfloor^{\angle-\propto}, \lfloor\lfloor \daleth Z_1^{tu}\rfloor\rfloor^{\angle-\propto}\lfloor\lfloor \daleth Z_2^{tu}\rfloor\rfloor^{\angle-\propto},} \\ \sqrt[\angle-\propto]{\begin{matrix}\lfloor\lfloor \daleth Z_1^{il}\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z_2^{il}\rfloor\rfloor^{\angle-\propto} \\ -\lfloor\lfloor \daleth Z_1^{il}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_2^{il}\rfloor\rfloor^{\angle-\propto}\end{matrix}} & \sqrt[\angle-\propto]{\begin{matrix}\lfloor\lfloor \daleth Z_1^{iu}\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z_2^{iu}\rfloor\rfloor^{\angle-\propto} \\ -\lfloor\lfloor \daleth Z_1^{iu}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_2^{iu}\rfloor\rfloor^{\angle-\propto}\end{matrix}} \\ \sqrt[\angle-\propto]{\begin{matrix}\lfloor\lfloor \daleth Z_1^{fl}\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z_2^{fl}\rfloor\rfloor^{\angle-\propto} \\ -\lfloor\lfloor \daleth Z_1^{fl}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_2^{fl}\rfloor\rfloor^{\angle-\propto}\end{matrix}}, & \sqrt[\angle-\propto]{\begin{matrix}\lfloor\lfloor \daleth Z_1^{fl}\rfloor\rfloor^{\angle-\propto} + \lfloor\lfloor \daleth Z_2^{fu}\rfloor\rfloor^{\angle-\propto} \\ -\lfloor\lfloor \daleth Z_1^{fl}\rfloor\rfloor^{\angle-\propto} \cdot \lfloor\lfloor \daleth Z_2^{fu}\rfloor\rfloor^{\angle-\propto}\end{matrix}} \end{bmatrix}$$

$$(3)\ \varrho, Ⓢ = \begin{bmatrix} \sqrt[\angle-\propto]{\propto - \lfloor \propto - \lfloor\lfloor \daleth Z^{tl}\rfloor^{\angle-\propto}\rfloor^{\varrho}}, & \sqrt[\angle-\propto]{\propto - \lfloor \propto - \lfloor\lfloor \daleth Z^{tu}\rfloor^{\angle-\propto}\rfloor^{\varrho}}, \\ \sqrt[\angle-\propto]{\propto - \lfloor \propto - \lfloor\lfloor \daleth Z^{il}\rfloor^{\angle-\propto}\rfloor^{\varrho}}, & \sqrt[\angle-\propto]{\propto - \lfloor \propto - \lfloor\lfloor \daleth Z^{iu}\rfloor^{\angle-\propto}\rfloor^{\varrho}}, \\ \multicolumn{2}{c}{\lfloor\lfloor\lfloor \daleth Z^{fl}\rfloor^{\angle-\propto}\rfloor^{\varrho}, \lfloor\lfloor\lfloor \daleth Z^{fu}\rfloor^{\angle-\propto}\rfloor^{\varrho}} \end{bmatrix}$$

$$(4)\ Ⓢ^{\varrho} = \begin{bmatrix} \multicolumn{2}{c}{\lfloor\lfloor\lfloor \daleth Z^{tl}\rfloor^{\angle-\propto}\rfloor^{\varrho}, \lfloor\lfloor\lfloor \daleth Z^{tu}\rfloor^{\angle-\propto}\rfloor^{\varrho},} \\ \sqrt[\angle-\propto]{\propto - \lfloor \propto - \lfloor\lfloor \daleth Z^{il}\rfloor^{\angle-\propto}\rfloor^{\varrho}}, & \sqrt[\angle-\propto]{\propto - \lfloor \propto - \lfloor\lfloor \daleth Z^{iu}\rfloor^{\angle-\propto}\rfloor^{\varrho}}, \\ \sqrt[\angle-\propto]{\propto - \lfloor \propto - \lfloor\lfloor \daleth Z^{fl}\rfloor^{\angle-\propto}\rfloor^{\varrho}}, & \sqrt[\angle-\propto]{\propto - \lfloor \propto - \lfloor\lfloor \daleth Z^{fu}\rfloor^{\angle-\propto}\rfloor^{\varrho}} \end{bmatrix}.$$

2.1. *IVTNRFWA*

Definition 2.3. Let $Ⓢ_i = \Big\langle \Big\lfloor \lfloor\lfloor \daleth Z_i^{tl}, \daleth Z_i^{tu}\rfloor, \lfloor \daleth Z_i^{il}, \daleth Z_i^{iu}\rfloor, \lfloor \daleth Z_i^{fl}, \daleth Z_i^{fu}\rfloor\rfloor \Big\rfloor \Big\rangle$ be the IVTNRFNs, $W = \lfloor \sharp_1, \sharp_2, ..., \sharp_n\rfloor$ be the weight of $Ⓢ_i$, $\sharp_i \geq 0$ and $\vartriangle_{i\to 1}^{n} \sharp_i = 1$. Then IVTNRFWA $\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n\rfloor = \vartriangle_{i\to 1}^{n} \sharp_i Ⓢ_i$.

Theorem 2.4. *Let* $Ⓢ_i = \Big\langle \Big\lfloor \lfloor\lfloor \daleth Z_i^{tl}, \daleth Z_i^{tu}\rfloor, \lfloor \daleth Z_i^{il}, \daleth Z_i^{iu}\rfloor, \lfloor \daleth Z_i^{fl}, \daleth Z_i^{fu}\rfloor\rfloor \Big\rfloor \Big\rangle$ *be the IVTNRFNs. Then IVTNRFWA*$\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n\rfloor$

$$= \begin{bmatrix} \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_i}}, \\ \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_i}}, \\ \nabla_{i\to 1}^{n} \lfloor\lfloor\lfloor \exists Z_i^{fl} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_i}, \nabla_{i\to 1}^{n} \lfloor\lfloor\lfloor \exists Z_i^{fu} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_i} \end{bmatrix}.$$

Proof If $n = 2$, then IVTNRFWA$\lfloor Ⓢ_1, Ⓢ_2 \rfloor = \sharp_1 Ⓢ_1 \vartriangle \sharp_2 Ⓢ_2$, where

$$\sharp_1 Ⓢ_1 = \begin{bmatrix} \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1}}, \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1}}, \\ \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1}}, \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1}}, \\ \lfloor\lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_1}, \lfloor\lfloor\lfloor \exists Z_1^{fu} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_1} \end{bmatrix}$$

$$\sharp_2 Ⓢ_2 = \begin{bmatrix} \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2}}, \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2}}, \\ \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2}}, \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2}} \\ \lfloor\lfloor\lfloor \exists Z_2^{fl} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_2}, \lfloor\lfloor\lfloor \exists Z_2^{fu} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_2} \end{bmatrix}.$$

Now, $\sharp_1 Ⓢ_1 \vartriangle \sharp_2 Ⓢ_2$

$$= \begin{bmatrix} \sqrt[\angle^{-\propto}]{\begin{array}{c} \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1} \right\rceil + \\ \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2} \right\rceil \\ - \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1} \right\rceil \cdot \\ \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2} \right\rceil \end{array}}, & \sqrt[\angle^{-\propto}]{\begin{array}{c} \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1} \right\rceil + \\ \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2} \right\rceil \\ - \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1} \right\rceil \cdot \\ \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2} \right\rceil \end{array}} \\ \sqrt[\angle^{-\propto}]{\begin{array}{c} \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1} \right\rceil + \\ \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2} \right\rceil \\ - \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1} \right\rceil \cdot \\ \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2} \right\rceil \end{array}}, & \sqrt[\angle^{-\propto}]{\begin{array}{c} \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1} \right\rceil + \\ \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2} \right\rceil \\ - \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_1} \right\rceil \cdot \\ \left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\sharp_2} \right\rceil \end{array}} \\ \multicolumn{2}{c}{\lfloor\lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_1} \cdot \lfloor\lfloor\lfloor \exists Z_2^{fl} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_2}, \lfloor\lfloor\lfloor \exists Z_1^{fu} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_1} \cdot \lfloor\lfloor\lfloor \exists Z_2^{fu} \rfloor\rfloor^{\angle^{-\propto}} \rceil^{\sharp_2}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\angle -\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_1} \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_2}}, \\ \sqrt[\angle -\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_1} \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_2}}, \\ \sqrt[\angle -\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_1} \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_2}}, \\ \sqrt[\angle -\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_1} \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_2}}, \\ \lfloor\lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_1} \cdot \lfloor\lfloor\lfloor \exists Z_2^{fl} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_2}, \\ \lfloor\lfloor\lfloor \exists Z_1^{fu} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_1} \cdot \lfloor\lfloor\lfloor \exists Z_2^{fu} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_2} \end{bmatrix}$$

Hence, IVTNRFWA$\lfloor Ⓢ_1, Ⓢ_2 \rfloor$

$$= \begin{bmatrix} \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^{2} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{tl} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^{2} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{tu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \\ \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^{2} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{il} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^{2} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{iu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \\ \nabla_{i\to 1}^{2} \lfloor\lfloor\lfloor \exists Z_i^{fl} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_i}, \nabla_{i\to 1}^{2} \lfloor\lfloor\lfloor \exists Z_i^{fu} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_i} \end{bmatrix}.$$

It valid for $n \geq 3$,
Thus, IVTNRFWA$\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_l \rfloor$

$$= \begin{bmatrix} \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^{l} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{tl} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^{l} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{tu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \\ \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^{l} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{il} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^{l} \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{iu} \rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \\ \nabla_{i\to 1}^{l} \lfloor\lfloor\lfloor \exists Z_i^{fl} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_i}, \nabla_{i\to 1}^{l} \lfloor\lfloor\lfloor \exists Z_i^{fu} \rfloor\rfloor^{\angle -\propto} \rfloor^{\sharp_i} \end{bmatrix}.$$

If $n = l + 1$, then IVTNRFWA $\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_l, Ⓢ_{l+1} \rfloor$

$$= \begin{bmatrix} \sqrt[\angle^{-\propto}]{\begin{array}{c}\triangle_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{tl}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}\right\rceil + \left\lfloor \propto - \left\lfloor \propto - \lfloor Z_{l+1}^{tl}\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_{l+1}}\right\rceil \\ -\nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{tl}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}\right\rceil \cdot \left\lfloor \propto - \left\lfloor \propto - \lfloor Z_{l+1}^{tl}\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_{l+1}}\right\rceil\end{array}}, \\ \sqrt[\angle^{-\propto}]{\begin{array}{c}\triangle_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{tu}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}\right\rceil + \left\lfloor \propto - \left\lfloor \propto - \lfloor Z_{l+1}^{tu}\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_{l+1}}\right\rceil \\ -\nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{tu}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}\right\rceil \cdot \left\lfloor \propto - \left\lfloor \propto - \lfloor Z_{l+1}^{tu}\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_{l+1}}\right\rceil\end{array}}, \\ \sqrt[\angle^{-\propto}]{\begin{array}{c}\triangle_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{il}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}\right\rceil + \left\lfloor \propto - \left\lfloor \propto - \lfloor Z_{l+1}^{il}\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_{l+1}}\right\rceil \\ -\nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{il}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}\right\rceil \cdot \left\lfloor \propto - \left\lfloor \propto - \lfloor Z_{l+1}^{il}\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_{l+1}}\right\rceil\end{array}}, \\ \sqrt[\angle^{-\propto}]{\begin{array}{c}\triangle_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{iu}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}\right\rceil + \left\lfloor \propto - \left\lfloor \propto - \lfloor Z_{l+1}^{iu}\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_{l+1}}\right\rceil \\ -\nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{iu}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}\right\rceil, \left\lfloor \propto - \left\lfloor \propto - \lfloor Z_{l+1}^{iu}\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_{l+1}}\right\rceil\end{array}}, \\ \nabla_{i\to 1}^{l}\lfloor\lfloor\lfloor \daleth Z_i^{fl}\rfloor\rfloor^{\angle^{-\propto}}\rceil^{\sharp_i} \cdot \lfloor\lfloor Z_{l+1}^{fl}\rfloor^{\angle^{-\propto}}\rceil^{\sharp_{l+1}}, \nabla_{i\to 1}^{l}\lfloor\lfloor\lfloor \daleth Z_i^{fu}\rfloor\rfloor^{\angle^{-\propto}}\rceil^{\sharp_i} \cdot \lfloor\lfloor Z_{l+1}^{fu}\rfloor^{\angle^{-\propto}}\rceil^{\sharp_{l+1}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{l+1}\left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{tl}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{l+1}\left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{tu}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}}, \\ \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{l+1}\left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{il}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{l+1}\left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{iu}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}}, \\ \nabla_{i\to 1}^{l+1}\lfloor\lfloor\lfloor \daleth Z_i^{fl}\rfloor\rfloor^{\angle^{-\propto}}\rceil^{\sharp_i}, \nabla_{i\to 1}^{l+1}\lfloor\lfloor\lfloor \daleth Z_i^{fu}\rfloor\rfloor^{\angle^{-\propto}}\rceil^{\sharp_i} \end{bmatrix}.$$

Theorem 2.5. *Let* $Ⓢ_i = \left\langle \left\lfloor \lfloor\lfloor \daleth Z_i^{tl}, \daleth Z_i^{tu}\rfloor, \lfloor \daleth Z_i^{il}, \daleth Z_i^{iu}\rfloor, \lfloor \daleth Z_i^{fl}, \daleth Z_i^{fu}\rfloor\rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then IVTNRFWA* $\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n\rfloor = Ⓢ$ *(idempotency property).*

Proof Since $\lfloor \daleth Z_i^{tl}\rfloor = \lfloor \daleth Z^{tl}\rfloor$, $\lfloor \daleth Z_i^{il}\rfloor = \lfloor \daleth Z^{il}\rfloor$ and $\lfloor \daleth Z_i^{fl}\rfloor = \lfloor \daleth Z^{fl}\rfloor$ and $\lfloor \daleth Z_i^{tu}\rfloor = \lfloor \daleth Z^{tu}\rfloor$, $\lfloor \daleth Z_i^{iu}\rfloor = \lfloor \daleth Z^{iu}\rfloor$ and $\lfloor \daleth Z_i^{fu}\rfloor = \lfloor \daleth Z^{fu}\rfloor$ and $\triangle_{i\to 1}^{n} \sharp_i = 1$. Now, IVTNRFWA$\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n\rfloor$

$$= \begin{bmatrix} \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n}\left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{tl}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n}\left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{tu}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}}, \\ \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n}\left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{il}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n}\left\lfloor \propto - \lfloor\lfloor \daleth Z_i^{iu}\rfloor\rfloor^{\angle^{-\propto}}\right\rceil^{\sharp_i}}, \\ \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \daleth Z_i^{fl}\rfloor\rfloor^{\angle^{-\propto}}\rceil^{\sharp_i}, \nabla_{i\to 1}^{n}\lfloor\lfloor\lfloor \daleth Z_i^{fu}\rfloor\rfloor^{\angle^{-\propto}}\rceil^{\sharp_i} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z^{tl}\rfloor^{\angle^{-\propto}}\right\rceil^{\triangle_{i\to 1}^{n}\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z^{tu}\rfloor^{\angle^{-\propto}}\right\rceil^{\triangle_{i\to 1}^{n}\sharp_i}}, \\ \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z^{il}\rfloor^{\angle^{-\propto}}\right\rceil^{\triangle_{i\to 1}^{n}\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \daleth Z^{iu}\rfloor^{\angle^{-\propto}}\right\rceil^{\triangle_{i\to 1}^{n}\sharp_i}}, \\ \lfloor\lfloor\lfloor \daleth Z^{fl}\rfloor^{\angle^{-\propto}}\rceil^{\triangle_{i\to 1}^{n}\sharp_i}, \lfloor\lfloor\lfloor \daleth Z^{fu}\rfloor^{\angle^{-\propto}}\rceil^{\triangle_{i\to 1}^{n}\sharp_i} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\angle -\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z^{tl} \rfloor^{\angle -\propto} \right\rfloor}, \sqrt[\angle -\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z^{tu} \rfloor^{\angle -\propto} \right\rfloor}, \\ \sqrt[\angle -\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z^{il} \rfloor^{\angle -\propto} \right\rfloor}, \sqrt[\angle -\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z^{iu} \rfloor^{\angle -\propto} \right\rfloor}, \\ \lfloor\lfloor \exists Z^{fl} \rfloor^{\angle -\propto}, \lfloor\lfloor \exists Z^{fu} \rfloor^{\angle -\propto} \end{bmatrix}$$
$$= \text{Ⓢ}$$

Theorem 2.6. *Let* $\text{Ⓢ}_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor\rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then* $IVTNRFWA\lfloor \text{Ⓢ}_1, \text{Ⓢ}_2, ..., \text{Ⓢ}_n \rfloor$
where $\underline{\lfloor \exists Z^{tl} \rfloor} = \min\lfloor \exists Z_{ij}^{tl} \rfloor$, $\overline{\lfloor \exists Z^{tl} \rfloor} = \max\lfloor \exists Z_{ij}^{tl} \rfloor$, $\underline{\lfloor \exists Z^{il} \rfloor} = \min\lfloor \exists Z_{ij}^{il} \rfloor$, $\overline{\lfloor \exists Z^{il} \rfloor} = \max\lfloor \exists Z_{ij}^{il} \rfloor$, $\underline{\lfloor \exists Z^{fl} \rfloor} = \min\lfloor \exists Z_{ij}^{fl} \rfloor$, $\overline{\lfloor \exists Z^{fl} \rfloor} = \max\lfloor \exists Z_{ij}^{fl} \rfloor$
and $\underline{\lfloor \exists Z^{tu} \rfloor} = \min\lfloor \exists Z_{ij}^{tu} \rfloor$, $\overline{\lfloor \exists Z^{tu} \rfloor} = \max\lfloor \exists Z_{ij}^{tu} \rfloor$, $\underline{\lfloor \exists Z^{iu} \rfloor} = \min\lfloor \exists Z_{ij}^{iu} \rfloor$, $\overline{\lfloor \exists Z^{iu} \rfloor} = \max\lfloor \exists Z_{ij}^{iu} \rfloor$, $\underline{\lfloor \exists Z^{fu} \rfloor} = \min\lfloor \exists Z_{ij}^{fu} \rfloor$, $\overline{\lfloor \exists Z^{fu} \rfloor} = \max\lfloor \exists Z_{ij}^{fu} \rfloor$ *and where* $1 \leq i \leq n$, $j = 1, 2, ..., i_j$. *Then,*
$\left\langle \underline{\lfloor \exists Z^{tl} \rfloor, \lfloor \exists Z^{tu} \rfloor}, \underline{\lfloor \exists Z^{il} \rfloor, \lfloor \exists Z^{iu} \rfloor}, \overline{\lfloor \exists Z^{fl} \rfloor, \lfloor \exists Z^{fu} \rfloor} \right\rangle$

$$\begin{aligned} &\leq IVTNRFWA\lfloor \text{Ⓢ}_1, \text{Ⓢ}_2, ..., \text{Ⓢ}_n \rfloor \\ &\leq \left\langle \overline{\lfloor \exists Z^{tl}, \exists Z^{tu} \rfloor}, \overline{\lfloor \exists Z^{il}, \exists Z^{iu} \rfloor}, \underline{\lfloor \exists Z^{fl}, \exists Z^{fu} \rfloor} \right\rangle. \end{aligned}$$

(Boundedness property).

Proof Since, $\underline{\lfloor \exists Z^{tl} \rfloor} = \min\lfloor \exists Z_{ij}^{tl} \rfloor$, $\overline{\lfloor \exists Z^{tl} \rfloor} = \max\lfloor \exists Z_{ij}^{tl} \rfloor$ and $\underline{\lfloor \exists Z^{tl} \rfloor} \leq \lfloor \exists Z_{ij}^{tl} \rfloor \leq \overline{\lfloor \exists Z^{tl} \rfloor}$ and $\underline{\lfloor \exists Z^{tu} \rfloor} = \min\lfloor \exists Z_{ij}^{tu} \rfloor$, $\overline{\lfloor \exists Z^{tu} \rfloor} = \max\lfloor \exists Z_{ij}^{tu} \rfloor$ and $\underline{\lfloor \exists Z^{tu} \rfloor} \leq \lfloor \exists Z_{ij}^{tu} \rfloor \leq \overline{\lfloor \exists Z^{tu} \rfloor}$.

Now $\underline{\lfloor \exists Z^{tl} \rfloor}, \underline{\lfloor \exists Z^{tu} \rfloor}$

$$\begin{aligned} &= \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \lfloor \underline{\lfloor \exists Z^{tl} \rfloor} \rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \lfloor \underline{\lfloor \exists Z^{tu} \rfloor} \rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \\ &\leq \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \lfloor\lfloor\lfloor \exists Z_{ij}^{tl} \rfloor\rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \lfloor\lfloor\lfloor \exists Z_{ij}^{tu} \rfloor\rfloor\rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \\ &\leq \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \lfloor \overline{\lfloor \exists Z^{tl} \rfloor} \rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}}, \sqrt[\angle -\propto]{\propto - \nabla_{i\to 1}^n \left\lfloor \propto - \lfloor \overline{\lfloor \exists Z^{tu} \rfloor} \rfloor^{\angle -\propto} \right\rfloor^{\sharp_i}} \\ &= \overline{\lfloor \exists Z^{tl} \rfloor}. \end{aligned}$$

Since, $\underline{\lfloor \exists Z^{il} \rfloor} = \min\lfloor \exists Z_{ij}^{il} \rfloor$, $\overline{\lfloor \exists Z^{il} \rfloor} = \max\lfloor \exists Z_{ij}^{il} \rfloor$ and $\underline{\lfloor \exists Z^{il} \rfloor} \leq \lfloor \exists Z_{ij}^{il} \rfloor \leq \overline{\lfloor \exists Z^{il} \rfloor}$ and $\underline{\lfloor \exists Z^{iu} \rfloor} = \min\lfloor \exists Z_{ij}^{iu} \rfloor$, $\overline{\lfloor \exists Z^{iu} \rfloor} = \max\lfloor \exists Z_{ij}^{iu} \rfloor$ and $\underline{\lfloor \exists Z^{iu} \rfloor} \leq \lfloor \exists Z_{ij}^{iu} \rfloor \leq \overline{\lfloor \exists Z^{iu} \rfloor}$.

Now, $\underline{\lfloor \exists Z^{il} \rfloor}, \underline{\lfloor \exists Z^{iu} \rfloor}$

$$
\begin{aligned}
&= \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\underline{\lfloor\exists Z^{il}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}},\ \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\underline{\lfloor\exists Z^{iu}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\\
&\leq \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z^{il}_{ij}\rfloor\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}},\ \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z^{iu}_{ij}\rfloor\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\\
&\leq \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\overline{\lfloor\exists Z^{il}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}},\ \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\overline{\lfloor\exists Z^{iu}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\\
&= \overline{\lfloor\exists Z^{il}\rfloor}, \overline{\lfloor\exists Z^{iu}\rfloor}.
\end{aligned}
$$

Since, $\underline{\lfloor\lfloor\exists Z^{fl}\rfloor^{\angle-\propto}\rfloor} = \min\lfloor\lfloor\exists Z^{fl}_{ij}\rfloor\rfloor^{\angle-\propto}$, $\overline{\lfloor\lfloor\exists Z^{fl}\rfloor^{\angle-\propto}\rfloor} = \max\lfloor\lfloor\exists Z^{fl}_{ij}\rfloor\rfloor^{\angle-\propto}$ and $\underline{\lfloor\lfloor\exists Z^{fl}\rfloor^{\angle-\propto}\rfloor} \leq \lfloor\lfloor\exists Z^{fl}_{ij}\rfloor\rfloor^{\angle-\propto} \leq \overline{\lfloor\lfloor\exists Z^{fl}\rfloor^{\angle-\propto}\rfloor}$ and $\underline{\lfloor\lfloor\exists Z^{fu}\rfloor^{\angle-\propto}\rfloor} = \min\lfloor\lfloor\exists Z^{fu}_{ij}\rfloor\rfloor^{\angle-\propto}$, $\overline{\lfloor\lfloor\exists Z^{fu}\rfloor^{\angle-\propto}\rfloor} = \max\lfloor\lfloor\exists Z^{fu}_{ij}\rfloor\rfloor^{\angle-\propto}$ and $\underline{\lfloor\lfloor\exists Z^{fu}\rfloor^{\angle-\propto}\rfloor} \leq \lfloor\lfloor\exists Z^{fu}_{ij}\rfloor\rfloor^{\angle-\propto} \leq \overline{\lfloor\lfloor\exists Z^{fu}\rfloor^{\angle-\propto}\rfloor}$.

We have,

$$
\begin{aligned}
\underline{\lfloor\lfloor\exists Z^{fl}\rfloor^{\angle-\propto}} &= \nabla_{i\to1}^{n}\underline{\lfloor\lfloor\exists Z^{fl}\rfloor^{\angle-\propto}}\rceil^{\sharp_i}, \nabla_{i\to1}^{n}\underline{\lfloor\lfloor\exists Z^{fu}\rfloor^{\angle-\propto}}\rceil^{\sharp_i}\\
&\leq \nabla_{i\to1}^{n}\lfloor\lfloor\lfloor\exists Z^{fl}_{ij}\rfloor\rfloor^{\angle-\propto}\rceil^{\sharp_i}, \nabla_{i\to1}^{n}\lfloor\lfloor\lfloor\exists Z^{fu}_{ij}\rfloor\rfloor^{\angle-\propto}\rceil^{\sharp_i}\\
&\leq \nabla_{i\to1}^{n}\overline{\lfloor\lfloor\exists Z^{fl}\rfloor^{\angle-\propto}}\rceil^{\sharp_i}, \nabla_{i\to1}^{n}\overline{\lfloor\lfloor\exists Z^{fu}\rfloor^{\angle-\propto}}\rceil^{\sharp_i}\\
&= \overline{\lfloor\lfloor\exists Z^{fl}\rfloor^{\angle-\propto}\rfloor}, \overline{\lfloor\lfloor\exists Z^{fu}\rfloor^{\angle-\propto}\rfloor}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\times\begin{bmatrix}
\begin{bmatrix}\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\underline{\lfloor\exists Z^{tl}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}-\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\underline{\lfloor\exists Z^{il}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}\\
+1-\left\lfloor\nabla_{i\to1}^{n}\lfloor\lfloor\overline{\lfloor\exists Z^{fl}\rfloor}\rfloor^{\angle-\propto}\rceil^{\sharp_i}\right\rceil^{2}\end{bmatrix}\\
+\begin{bmatrix}\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\underline{\lfloor\exists Z^{tu}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}-\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\underline{\lfloor\exists Z^{iu}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}\\
-\left\lfloor\nabla_{i\to1}^{n}\lfloor\lfloor\overline{\lfloor\exists Z^{fu}\rfloor}\rfloor^{\angle-\propto}\rceil^{\sharp_i}\right\rceil^{2}\end{bmatrix}
\end{bmatrix}\\
\leq\ &\frac{1}{2}\times\begin{bmatrix}
\begin{bmatrix}\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\lfloor\lfloor\exists Z^{tl}_{ij}\rfloor\rfloor\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}-\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\lfloor\exists Z^{il}_{ij}\rfloor\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}+\\
+1-\left\lfloor\nabla_{i\to1}^{n}\lfloor\lfloor\lfloor\exists Z^{fl}_{ij}\rfloor\rfloor^{\angle-\propto}\rceil^{\sharp_i}\right\rceil^{2}\end{bmatrix}\\
+\begin{bmatrix}\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\lfloor\lfloor\exists Z^{tu}_{ij}\rfloor\rfloor\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}-\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\lfloor\exists Z^{iu}_{ij}\rfloor\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}+\\
-\left\lfloor\nabla_{i\to1}^{n}\lfloor\lfloor\lfloor\exists Z^{fu}_{ij}\rfloor\rfloor^{\angle-\propto}\rceil^{\sharp_i}\right\rceil^{2}\end{bmatrix}
\end{bmatrix}\\
\leq\ &\frac{1}{2}\times\begin{bmatrix}
\begin{bmatrix}\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\overline{\lfloor\exists Z^{tl}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}-\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\overline{\lfloor\exists Z^{il}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}+\\
+1-\left\lfloor\nabla_{i\to1}^{n}\lfloor\underline{\lfloor\exists Z^{fl}\rfloor}\rfloor^{\angle-\propto}\rceil^{\sharp_i}\right\rceil^{2}\end{bmatrix}\\
+\begin{bmatrix}\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\overline{\lfloor\exists Z^{tu}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}-\left\lfloor\sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor\propto-\lfloor\overline{\lfloor\exists Z^{iu}\rfloor}\rfloor^{\angle-\propto}\right\rceil^{\sharp_i}}\right\rceil^{2}+\\
-\left\lfloor\nabla_{i\to1}^{n}\lfloor\underline{\lfloor\exists Z^{fu}\rfloor}\rfloor^{\angle-\propto}\rceil^{\sharp_i}\right\rceil^{2}\end{bmatrix}
\end{bmatrix}.
\end{aligned}
$$

Hence,
$\Big\langle \underline{\lfloor \exists Z^{tl} \rfloor, \lfloor \exists Z^{tu} \rfloor}, \underline{\lfloor \exists Z^{il} \rfloor, \lfloor \exists Z^{iu} \rfloor}, \overline{\lfloor \exists Z^{fl} \rfloor, \lfloor \exists Z^{fu} \rfloor} \Big\rangle$
$\leq IVTNRFWA\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n \rfloor$
$\leq \langle \overline{\lfloor \exists Z^{tl} \rfloor, \lfloor \exists Z^{tu} \rfloor}, \overline{\lfloor \exists Z^{il} \rfloor, \lfloor \exists Z^{iu} \rfloor}, \underline{\lfloor \exists Z^{fl} \rfloor, \lfloor \exists Z^{fu} \rfloor} \rangle$.

Theorem 2.7. *Let* $Ⓢ_i = \Big\langle \Big\lfloor \lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \rfloor \Big\rfloor \Big\rangle$ *and* $W_i = \langle \lfloor \lfloor \exists Z^{tl}_{h_{ij}}, \exists Z^{tu}_{h_{ij}} \rfloor, \lfloor \exists Z^{il}_{h_{ij}}, \exists Z^{iu}_{h_{ij}} \rfloor, \lfloor \exists Z^{fl}_{h_{ij}}, \exists Z^{fu}_{h_{ij}} \rfloor \rfloor \rangle$, *be the IVTNRFWAs. For any* i, *if there is* $\lfloor \exists Z^{tu}_{t_{ij}} \rfloor^2 \leq \lfloor \exists Z^{tu}_{h_{ij}} \rfloor^2$ *and* $\lfloor \exists Z^{iu}_{t_{ij}} \rfloor^2 \leq \lfloor \exists Z^{iu}_{h_{ij}} \rfloor^2$ *and* $\lfloor \exists Z^{fu}_{t_{ij}} \rfloor^2 \geq \lfloor \exists Z^{fu}_{h_{ij}} \rfloor^2$ *or* $Ⓢ_i \leq W_i$. *Prove that* $IVTNRFWA\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n \rfloor \leq IVTNRFWA\lfloor W_1, W_2, ..., W_n \rfloor$, *where* $\lfloor i = 1, 2, ..., n \rfloor; \lfloor j = 1, 2, ..., i_j \rfloor$ *(monotonicity property).*

Proof For any i, $\lfloor \exists Z^{tl}_{t_{ij}} \rfloor^2 \leq \lfloor \exists Z^{tl}_{h_{ij}} \rfloor^2$.
Therefore, $\propto - \lfloor \lfloor \exists Z^{tl}_{t_i} \rfloor \rfloor^2 \geq \propto - \lfloor \lfloor \exists Z^{tl}_{h_i} \rfloor \rfloor^2$.
Hence, $\nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{tl}_{t_i} \rfloor \big\rfloor^2 \Big\rfloor^{\sharp_i} \geq \nabla^n_{i \to 1} \Big\lfloor \propto - \lfloor \lfloor \exists Z^{tl}_{h_i} \rfloor \rfloor^2 \Big\rfloor^{\sharp_i}$
and $\sqrt[\angle - \propto]{\propto - \nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{tl}_{t_i} \rfloor \big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}} \leq \sqrt[\angle - \propto]{\propto - \nabla^n_{i \to 1} \Big\lfloor \propto - \Big\lfloor \lfloor \exists Z^{tl}_{h_i} \rfloor \Big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}}$.
Similarly, $\lfloor \exists Z^{tu}_{t_{ij}} \rfloor^2 \leq \lfloor \exists Z^{tu}_{h_{ij}} \rfloor^2$.
Therefore, $\propto - \lfloor \lfloor \exists Z^{tu}_{t_i} \rfloor \rfloor^2 \geq \propto - \lfloor \lfloor \exists Z^{tu}_{h_i} \rfloor \rfloor^2$.
Hence, $\nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{tu}_{t_i} \rfloor \big\rfloor^2 \Big\rfloor^{\sharp_i} \geq \nabla^n_{i \to 1} \Big\lfloor \propto - \lfloor \lfloor \exists Z^{tu}_{h_i} \rfloor \rfloor^2 \Big\rfloor^{\sharp_i}$
and $\sqrt[\angle - \propto]{\propto - \nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{tu}_{t_i} \rfloor \big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}} \leq \sqrt[\angle - \propto]{\propto - \nabla^n_{i \to 1} \Big\lfloor \propto - \Big\lfloor \lfloor \exists Z^{tu}_{h_i} \rfloor \Big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}}$.
For any i, $\lfloor \exists Z^{il}_{t_{ij}} \rfloor^{\angle - \propto} \leq \Big\lfloor \lfloor \exists Z^{il}_{h_{ij}} \rfloor \Big\rfloor^{\angle - \propto}$.
Therefore, $\propto - \big\lfloor \lfloor \exists Z^{il}_{t_i} \rfloor \big\rfloor^{\angle - \propto} \geq \propto - \big\lfloor \lfloor \exists Z^{il}_{h_i} \rfloor \big\rfloor^{\angle - \propto}$.
Hence, $\nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{il}_{t_i} \rfloor \big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i} \geq \nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{il}_{h_i} \rfloor \big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}$.
This implies that $\sqrt[\angle - \propto]{\propto - \nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{il}_{t_i} \rfloor \big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}} \leq \sqrt[\angle - \propto]{\propto - \nabla^n_{i \to 1} \Big\lfloor \propto - \Big\lfloor \lfloor \exists Z^{il}_{h_i} \rfloor \Big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}}$.

Similarly, for any i, $\lfloor \exists Z^{iu}_{t_{ij}} \rfloor^{\angle - \propto} \leq \Big\lfloor \lfloor \exists Z^{iu}_{h_{ij}} \rfloor \Big\rfloor^{\angle - \propto}$.
Therefore, $\propto - \big\lfloor \lfloor \exists Z^{iu}_{t_i} \rfloor \big\rfloor^{\angle - \propto} \geq \propto - \big\lfloor \lfloor \exists Z^{iu}_{h_i} \rfloor \big\rfloor^{\angle - \propto}$.
Hence, $\nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{iu}_{t_i} \rfloor \big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i} \geq \nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{iu}_{h_i} \rfloor \big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}$.
This implies that
$\sqrt[\angle - \propto]{\propto - \nabla^n_{i \to 1} \Big\lfloor \propto - \big\lfloor \lfloor \exists Z^{iu}_{t_i} \rfloor \big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}} \leq \sqrt[\angle - \propto]{\propto - \nabla^n_{i \to 1} \Big\lfloor \propto - \Big\lfloor \lfloor \exists Z^{iu}_{h_i} \rfloor \Big\rfloor^{\angle - \propto} \Big\rfloor^{\sharp_i}}$.

For any i, $\Big\lfloor \lfloor \exists Z^{fl}_{t_{ij}} \rfloor \Big\rfloor^2 \geq \Big\lfloor \lfloor \exists Z^{fl}_{h_{ij}} \rfloor \Big\rfloor^2$ and $\Big\lfloor \lfloor \exists Z^{fl}_{t_{ij}} \rfloor \Big\rfloor^{\angle - \propto} \geq \Big\lfloor \lfloor \exists Z^{fl}_{h_{ij}} \rfloor \Big\rfloor^{\angle - \propto}$.
Therefore, $\propto - \frac{\Big\lfloor \nabla^n_{i \to 1} \lfloor \exists Z^{fl}_{t_{ij}} \rfloor \Big\rfloor^{\angle - \propto}}{2} \leq \propto - \frac{\Big\lfloor \nabla^n_{i \to 1} \lfloor \exists Z^{fl}_{h_{ij}} \rfloor \Big\rfloor^{\angle - \propto}}{2}$.
Similarly, for any i,

$\left\lfloor \lfloor \exists Z_{t_{ij}}^{fu} \rfloor \right\rfloor^2 \geq \left\lfloor \lfloor \exists Z_{h_{ij}}^{fu} \rfloor \right\rfloor^2$ and $\left\lfloor \lfloor \exists Z_{t_{ij}}^{fu} \rfloor \right\rfloor^{\angle-\propto} \geq \left\lfloor \lfloor \exists Z_{h_{ij}}^{fu} \rfloor \right\rfloor^{\angle-\propto}$.

Therefore, $-\left\lfloor \nabla_{i\to1}^{n} \lfloor \exists Z_{t_{ij}}^{fu} \rfloor \right\rfloor^{\angle-\propto} \leq -\left\lfloor \nabla_{i\to1}^{n} \lfloor \exists Z_{h_{ij}}^{fu} \rfloor \right\rfloor^{\angle-\propto}$.

Hence,

$$\frac{1}{2}\times\left[\begin{array}{c} \left[\begin{array}{c} \left\lfloor \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_{ti}^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \right\rfloor^2 - \left\lfloor \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_{t_i}^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \right\rfloor^2 \\ +\propto-\left\lfloor \nabla_{i\to1}^{n}\lfloor\lfloor\exists Z_{t_i}^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^2 \end{array}\right] \\ +\left[\begin{array}{c} \left\lfloor \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_{ti}^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \right\rfloor^2 - \left\lfloor \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_{t_i}^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \right\rfloor^2 \\ -\left\lfloor \nabla_{i\to1}^{n}\lfloor\lfloor\exists Z_{t_i}^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^2 \end{array}\right] \end{array}\right]$$

$$\leq \frac{1}{2}\times\left[\begin{array}{c} \left[\begin{array}{c} \left\lfloor \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_{hi}^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \right\rfloor^2 - \left\lfloor \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_{h_i}^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \right\rfloor^2 \\ +\propto-\left\lfloor \nabla_{i\to1}^{n}\lfloor\lfloor\exists Z_{h_i}^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^2 \end{array}\right] \\ +\left[\begin{array}{c} \left\lfloor \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_{hi}^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \right\rfloor^2 - \left\lfloor \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_{h_i}^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \right\rfloor^2 \\ -\left\lfloor \nabla_{i\to1}^{n}\lfloor\lfloor\exists Z_{h_i}^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^2 \end{array}\right] \end{array}\right].$$

Hence, IVTNRFWA$\lfloor ⓢ_1, ⓢ_2, ..., ⓢ_n \rfloor \leq IVTNRFWA\,\lfloor W_1, W_2, ..., W_n \rfloor$.

2.2. *IVTNRFWG*

Definition 2.8. Let $ⓢ_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor\rfloor \right\rfloor \right\rangle$ be the IVTNRFNs. Then NRFWG $\lfloor ⓢ_1, ⓢ_2, ..., ⓢ_n \rfloor = \nabla_{i\to1}^{n} ⓢ_i^{\sharp_i}$.

Corollary 2.9. *Let* $ⓢ_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor\rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then IVTNRFWG* $\lfloor ⓢ_1, ⓢ_2, ..., ⓢ_n \rfloor$

$$= \left[\begin{array}{c} \nabla_{i\to1}^{n}\lfloor\lfloor\lfloor\exists Z_i^{tl}\rfloor\rfloor^{\angle-\propto}\rfloor^{\sharp_i}, \nabla_{i\to1}^{n}\lfloor\lfloor\lfloor\exists Z_i^{tu}\rfloor\rfloor^{\angle-\propto}\rfloor^{\sharp_i} \\ \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_i^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_i^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \\ \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_i^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle-\propto]{\propto-\nabla_{i\to1}^{n}\left\lfloor \propto-\lfloor\lfloor\exists Z_i^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \end{array}\right].$$

Corollary 2.10. *Let* $ⓢ_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor\rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs and all are equal. Then* $NRFWG\lfloor ⓢ_1, ⓢ_2, ..., ⓢ_n \rfloor = ⓢ$.

2.3. *Generalized IVTNRFWA (GIVTNRFWA)*

Definition 2.11. Let $ⓢ_i = \left\langle \left\lfloor \lfloor\lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor\rfloor \right\rfloor \right\rangle$ be the IVTNRFN. Then GIVTNRFWA $\lfloor ⓢ_1, ⓢ_2, ..., ⓢ_n \rfloor = \left\lfloor \triangle_{i\to1}^{n} \sharp_i ⓢ_i^{\varrho} \right\rfloor^{1/\varrho}$.

Theorem 2.12. *Let* $Ⓢ_i = \left\langle \left\lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then GIVTNRFWA* $\lfloor Ⓢ_1, Ⓢ_2, ..., Ⓢ_n \rfloor$

$$= \left\lfloor \begin{array}{c} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{tl} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}}, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{tu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}}, \\ \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{il} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}}, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{iu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}}, \\ \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor \lfloor \exists Z_i^{fl} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}}} \right\rfloor^{\sharp_i} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{1/\angle^{-\propto}}}, \\ \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor \lfloor \exists Z_i^{fu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}}} \right\rfloor^{\sharp_i} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{1/\angle^{-\propto}}} \end{array} \right\rfloor .$$

Proof To illustrate this, we may first show that,

$$\Delta_{i\to 1}^{n} \sharp_i Ⓢ_i^{\angle^{-\propto}} =$$

$$\left\lfloor \begin{array}{c} \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{tl} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{tu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \\ \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{il} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{iu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \\ \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor \lfloor \exists Z_i^{fl} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}}} \right\rfloor^{\sharp_i}, \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor \lfloor \exists Z_i^{fu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}}} \right\rfloor^{\sharp_i} \end{array} \right\rfloor .$$

Put $n = 2$, $\sharp_1 Ⓢ_1 \vartriangle \sharp_2 Ⓢ_2$

$$
= \left[\begin{array}{c}
\sqrt[\angle^{-\propto}]{\begin{array}{c} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} + \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \\ - \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \cdot \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_2^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \end{array}} \\
, \sqrt[\angle^{-\propto}]{\begin{array}{c} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} + \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \\ - \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \cdot \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_2^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \end{array}}, \\
\sqrt[\angle^{-\propto}]{\begin{array}{c} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} + \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \\ - \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \cdot \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_2^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \end{array}} \\
, \sqrt[\angle^{-\propto}]{\begin{array}{c} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} + \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \\ - \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \cdot \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_2^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_1}} \right\rceil^{\angle^{-\propto}} \end{array}} \\
\left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{fl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}}} \right\rceil^{\sharp_1}, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{fl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}}} \right\rceil^{\sharp_1} \\
, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_1^{fu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}}} \right\rceil^{\sharp_1}, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_2^{fu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}}} \right\rceil^{\sharp_1}
\end{array} \right]
$$

$$
= \left[\begin{array}{c}
\sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{2} \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{2} \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_i}} \\
\sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{2} \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{il} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_i}}, \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{2} \left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{iu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}} \right\rceil^{\sharp_i}} \\
\nabla_{i\to 1}^{2} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{fl} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}}} \right\rceil^{\sharp_i}, \nabla_{i\to 1}^{2} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{fu} \rfloor\rfloor^{\angle^{-\propto}} \right\rceil^{\angle^{-\propto}}} \right\rceil^{\sharp_i}
\end{array} \right].
$$

Hence,

$$\triangle_{i\to 1}^{l} \sharp_i Ⓢ_i^{\varrho} = \begin{bmatrix} \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \\ \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \\ \nabla_{i\to 1}^{l}\left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}, \nabla_{i\to 1}^{l}\left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i} \end{bmatrix}.$$

If $n = l+1$, then $\triangle_{i\to 1}^{l} \sharp_i Ⓢ_i^{\varrho} + \sharp_{l+1} Ⓢ_{l+1}^{\varrho} = \triangle_{i\to 1}^{l+1} \sharp_i Ⓢ_i^{\varrho}$.
Now, $\triangle_{i\to 1}^{l} \sharp_i Ⓢ_i^{\varrho} + \sharp_{l+1} Ⓢ_{l+1}^{\varrho} = \sharp_1 Ⓢ_1^{\varrho} \vartriangle \sharp_2 Ⓢ_2^{\varrho} \vartriangle ... \vartriangle \sharp_l Ⓢ_l^{\varrho} \vartriangle \sharp_{l+1} Ⓢ_{l+1}^{\varrho}$

$$= \begin{bmatrix} \sqrt[\angle-\propto]{\begin{array}{l} \left\lfloor \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_i^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^{\angle-\propto} + \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \left\lfloor \lfloor Z_{l+1}^{tl}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right\rfloor^{\angle-\propto} \\ - \left\lfloor \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_i^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^{\angle-\propto} \cdot \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \left\lfloor \lfloor Z_{l+1}^{tl}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right\rfloor^{\angle-\propto} \end{array}} \\ , \sqrt[\angle-\propto]{\begin{array}{l} \left\lfloor \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_i^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^{\angle-\propto} + \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \left\lfloor \lfloor Z_{l+1}^{tu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right\rfloor^{\angle-\propto} \\ - \left\lfloor \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_i^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^{\angle-\propto} \cdot \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \left\lfloor \lfloor Z_{l+1}^{tu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right\rfloor^{\angle-\propto} \end{array}} \\ \sqrt[\angle-\propto]{\begin{array}{l} \left\lfloor \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_i^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^{\angle-\propto} + \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \left\lfloor \lfloor Z_{l+1}^{il}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right\rfloor^{\angle-\propto} \\ - \left\lfloor \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_i^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^{\angle-\propto} \cdot \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \left\lfloor \lfloor Z_{l+1}^{il}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right\rfloor^{\angle-\propto} \end{array}} \\ , \sqrt[\angle-\propto]{\begin{array}{l} \left\lfloor \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_i^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^{\angle-\propto} + \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \left\lfloor \lfloor Z_{l+1}^{iu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right\rfloor^{\angle-\propto} \\ - \left\lfloor \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_i^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}\right\rfloor^{\angle-\propto} \cdot \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \left\lfloor \lfloor Z_{l+1}^{iu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_1}}\right\rfloor^{\angle-\propto} \end{array}} \\ \nabla_{i\to 1}^{l}\left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i} \cdot \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \lfloor Z_{l+1}^{fl}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_1} \\ , \nabla_{i\to 1}^{l}\left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i} \cdot \left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \lfloor Z_{l+1}^{fu}\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_1} \end{bmatrix}$$

$$\triangle_{i\to 1}^{l+1} \sharp_i Ⓢ_i^{\angle-\propto} = \begin{bmatrix} \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l+1}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l+1}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{tu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \\ \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l+1}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{il}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}}, \sqrt[\angle-\propto]{\propto - \nabla_{i\to 1}^{l+1}\left\lfloor \propto - \left\lfloor \lfloor\lfloor \exists Z_1^{iu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}\right\rfloor^{\sharp_i}} \\ \nabla_{i\to 1}^{l+1}\left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{fl}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i}, \nabla_{i\to 1}^{l+1}\left\lfloor \sqrt[\angle-\propto]{\propto - \left\lfloor \propto - \lfloor\lfloor \exists Z_i^{fu}\rfloor\rfloor^{\angle-\propto}\right\rfloor^{\angle-\propto}}\right\rfloor^{\sharp_i} \end{bmatrix}.$$

$$\left\lfloor \triangle_{i\to 1}^{l+1} \sharp_i \circledS_i^{\varrho} \right\rfloor^{1/\varrho} =$$

$$\begin{bmatrix} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{l+1} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{tl} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}}, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{l+1} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{tu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}} \\ \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{l+1} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{il} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}}, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{l+1} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{iu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}} \\ \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \nabla_{i\to 1}^{l+1} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor \lfloor \exists Z_i^{fl} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}}} \right\rfloor^{\sharp_i} \right\rfloor^{2} \right\rfloor^{1/\angle^{-\propto}}}, \\ \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \nabla_{i\to 1}^{l+1} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor \lfloor \exists Z_i^{fu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}}} \right\rfloor^{\sharp_i} \right\rfloor^{2} \right\rfloor^{1/\angle^{-\propto}}} \end{bmatrix}$$

Theorem 2.13. *If all* $\circledS_i = \left\langle \left\lfloor \lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \rfloor \right\rfloor \right\rangle$ *and all are equal. Then* $GIVTNRFWA\lfloor \circledS_1, \circledS_2, ..., \circledS_n \rfloor = \circledS$.

2.4. *Generalized IVT NRFWG (GIVT NRFWG)*

Definition 2.14. Let $\circledS_i = \left\langle \left\lfloor \lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \rfloor \right\rfloor \right\rangle$ be the IVT-NRFNs. Then GIVTNRFWG $\lfloor \circledS_1, \circledS_2, ..., \circledS_n \rfloor = \frac{1}{\varrho} \left\lfloor \nabla_{i\to 1}^{n} \lfloor \varrho \circledS_i \rfloor^{\sharp_i} \right\rfloor$.

Corollary 2.15. *Let* $\circledS_i = \left\langle \left\lfloor \lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \rfloor \right\rfloor \right\rangle$ *be the IVTNRFNs. Then* $GIVTNRFWG\lfloor \circledS_1, \circledS_2, ..., \circledS_n \rfloor$

$$= \begin{bmatrix} \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor \lfloor \exists Z_i^{tl} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}}} \right\rfloor^{\sharp_i} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{1/\angle^{-\propto}}}, \\ \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \left\lfloor \nabla_{i\to 1}^{n} \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \left\lfloor \propto - \lfloor \lfloor \exists Z_i^{tu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}}} \right\rfloor^{\sharp_i} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{1/\angle^{-\propto}}} \\ \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{il} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}}, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{iu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}} \\ \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{fl} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}}, \left\lfloor \sqrt[\angle^{-\propto}]{\propto - \nabla_{i\to 1}^{n} \left\lfloor \propto - \left\lfloor \lfloor \lfloor \exists Z_i^{fu} \rfloor \rfloor^{\angle^{-\propto}} \right\rfloor^{\angle^{-\propto}} \right\rfloor^{\sharp_i}} \right\rfloor^{1/\angle^{-\propto}} \end{bmatrix}.$$

Corollary 2.16. *If all* $\circledS_i = \left\langle \left\lfloor \lfloor \lfloor \exists Z_i^{tl}, \exists Z_i^{tu} \rfloor, \lfloor \exists Z_i^{il}, \exists Z_i^{iu} \rfloor, \lfloor \exists Z_i^{fl}, \exists Z_i^{fu} \rfloor \rfloor \right\rfloor \right\rangle$ *are equal. Then* $GIVTNRFWG\lfloor \circledS_1, \circledS_2, ..., \circledS_n \rfloor = \circledS$.

3. **Conclusion:**

This work presents novel weighted operators, such as geometric and averaging operators. Boundedness, idempotency, commutativity, associativity, and monotonicity are some of the characteristics of these operators. In order to describe the weighted vector, we looked at a number of conventional measures. A wide range of aggregation operator conditions have been investigated. Some findings have been made after a few aggregating techniques for these IVT-NRFNs have been examined.

Acknowledgments: The authors would like to thank the Editor-InChief and the anonymous referees for their various suggestions and helpful comments that have led to the improved in the quality and clarity version of the paper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. L. A. Zadeh, Fuzzy sets, Information and control, 8(3), (1965), 338-353.
2. K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems, 20(1), (1986), 87–96.
3. R. R. Yager, Pythagorean membership degree s in multi criteria decision-making, IEEE Trans. Fuzzy Systems, 22, (2014), 958–965.
4. Wang, P., Zhu, B., Yu, Y., Ali, Z., & Almohsen, B. Complex intuitionistic fuzzy DOMBI prioritized aggregation operators and their application for resilient green supplier selection. Facta Universitatis, Series: Mechanical Engineering, 21(3), (2023), 339-357.
5. B.C. Cuong and V. Kreinovich, Picture fuzzy sets a new concept for computational intelligence problems, in Proceedings of 2013 Third World Congress on Information and Communication Technologies (WICT 2013), IEEE, (2013), 1–6.
6. S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani and T. Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision making problems, Journal of Intelligent and Fuzzy Systems, 36, (2019), 2829–284.
7. Hussain, A., & Ullah, K. An Intelligent Decision Support System for Spherical Fuzzy Sugeno-Weber Aggregation Operators and Real-Life Applications. Spectrum of Mechanical Engineering and Operational Research, 1(1), (2024), 177-188.
8. M. Rafiq, S. Ashraf, S. Abdullah, T. Mahmood, S. Muhammad, The cosine similarity measures of spherical fuzzy sets and their applications in decision making. Journal of Intelligent & Fuzzy Systems, 36(6), 2019, 6059-6073.
9. T. Senapati, R.R. Yager, Fermatean, fuzzy sets. J. Ambient Intell. Humaniz. Comput. 11, (2020), 663-674.
10. R.R. Yager, Generalized Orthopair Fuzzy Sets. IEEE Trans Fuzzy Syst 25(5), 2016, 1222-1230.
11. M Palanikumar, K Arulmozhi, C Jana, Multiple attribute decision-making approach for Pythagorean neutrosophic normal interval-valued fuzzy aggregation operators, Computational and Applied Mathematics, 41(3), (2022)
12. Z. Xu, R.R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, Int. J. Gen. Syst. 35, (2006), 417–433.
13. D.F. Li, Multi-attribute decision making method based on generalized OWA operators with intuitionistic fuzzy sets, Expert Syst. Appl. 37, (2010), 8673–8678.
14. S. Zeng, W. Sua, Intuitionistic fuzzy ordered weighted distance operator, Knowl. Based Syst. 24, (2011), 1224–1232.
15. X. Peng, H. Yuan, Fundamental properties of Pythagorean fuzzy aggregation operators, Fundam. Inform. 147, (2016), 415–446.

16. S. Ashraf, S. Abdullah, T. Mahmood, Spherical fuzzy Dombi aggregation operators and their application in group decision making problems, J. Amb. Intell. Hum. Comput. 11, (2020), 2731–2749.
17. K. Ullah, H. Garg, T. Mahmood, N. Jan, Z. Ali, Correlation coefficients for T-spherical fuzzy sets and their applications in clustering and multi-attribute decision making, Soft Comput. 24, (2020), 1647–1659.
18. K. Ullah, T. Mahmood, H. Garg, Evaluation of the performance of search and rescue robots using T-spherical fuzzy hamacher aggregation operators, Int. J. Fuzzy Syst. 22, (2020), 570–582.
19. Abu, Ibraheem. , Al-Husban, Abdallah. , J., Lejo. , J., Jamil. , Palanikumar, M., Balaji, G..(2024) Selection process real-life application for new type interval-valued neutrosophic sets using various aggregation operators. International Journal of Neutrosophic Science, 23(4) 136 153.
20. J., Lejo. , Damrah, Sadeq. , M., Mutaz. , Al-Husban, Abdallah. , Palanikumar, M.. (2024). Type-I extension Diophantine neutrosophic interval valued soft set in real life applications for a decision making. International Journal of Neutrosophic Science, 24(4), 151-164.
21. M. Palanikumar, K. Arulmozhi, A Iampan, Multi criteria group decision making based on VIKOR and TOPSIS methods for Fermatean fuzzy soft with aggregation operators, ICIC Express Letters 16 (10), (2022), 1129–1138.
22. M. Palanikumar, K. Arulmozhi, MCGDM based on TOPSIS and VIKOR using Pythagorean neutrosophic soft with aggregation operators, Neutrosophic Sets and Systems, (2022), 538–555.
23. M. Palanikumar, N. Kausar, H. Garg, A. Iampan, S. Kadry, M. Sharaf, Medical robotic engineering selection based on square root neutrosophic normal interval-valued sets and their aggregated operators, AIMS Mathematics, 8(8), (2023), 17402–17432.

Received: Oct 10, 2024. Accepted: Feb 3, 2025