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Isotopic Properties of Neutrosophic Soft Quasigroup and Its Application in Decision-making

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Abstract. A Q-neutrosophic soft quasigroup $(\phi^Q, \mathfrak{A})$ represents a novel mathematical framework designed to address scenarios characterized by indeterminate occurrences. This paper seeks to extend the classical examination of quasigroups to include Q-neutrosophic soft quasigroups. It provides a thorough investigation into isotopism, homomorphism, isomorphism, and the direct product concerning the hybrid concept of neutrosophic soft sets under a non-associative structure. This research expands the concept of the Q-set to a groupoid, which facilitates the development of an algorithm for Q-neutrosophic soft quasigroups. The definitions of isotopism and homomorphism for Q-neutrosophic soft quasigroups are introduced, revealing that every isotope of a Q-neutrosophic soft quasigroup is indeed a Q-neutrosophic soft quasigroup. Furthermore, it is established that the homomorphic image of a Q-neutrosophic soft quasigroup under a quasigroup is not necessarily a Q-neutrosophic soft quasigroup. The findings indicate that the direct product of any two Q-neutrosophic soft quasigroups is also a Q-neutrosophic soft quasigroup. Generalized properties regarding the direct product of Q-neutrosophic soft quasigroups of order n have been established. Additionally, the study presents the necessary and sufficient conditions under which the direct product of two Q-neutrosophic soft subquasigroups, $\psi^Q(a)\times\psi^Q(b)$, contains at least one Q-neutrosophic soft subquasigroup, $\pi_Q(c)$, that is isomorphic to the Q-neutrosophic soft subquasigroup $\pi_Q(a)$. Specifically, $\psi^Q(a) \cong \pi_Q(c) \leq \psi^Q(a)\times\psi^Q(b)$ if and only if a homomorphism exists from $\psi^Q(a)$ to $\psi^Q(b)$. To validate some of these results, examples have been constructed, and a practical application utilizing real-life data has been demonstrated.

Keywords: Isotopism; homomorphism; Isomorphism; Soft set; Neutrosophic set; Quasigroup

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1. Introduction

Suppose that $\hat{G}$ is a non-empty set and the binary operation $(\circ)$ is define on 'Q such that $d \circ w \in \hat{G}$ for all $d, w \in \hat{G}$, the pair $(\hat{G}, \circ)$ is called a *groupoid*. If the equations:

$$\alpha \circ d = \beta \text{ and } w \circ \alpha = \beta$$

has unique solutions $d, w \in \hat{G}$ for all $\alpha, \beta \in \hat{G}$, then $(\hat{G}, \circ)$ is called quasigroup. Suppose there is a unique element $1 \in \hat{G}$ called the identity element such that $1 \circ d = d \circ 1 = d$ for all $d \in \hat{G}$, then $(\hat{G}, \circ)$ is a loop. In this research, we write dw instead of $d \circ w$, and stipulate that $\circ$ has lower priority than juxtaposition among factors to be multiplied. Suppose that d is a fixed element in a quasigroup $(\hat{G}, \circ)$. Then, the left and right translation maps for all $d \in \hat{G}$, written as L_d and R_d respectively are defined by

$$wL_d = d \circ w \qquad \text{and} \qquad wR_d = w \circ d.$$

It is shown that a groupoid $(\hat{G}, \circ)$ is a quasigroup if the left and right translation maps are bijective. Hence, the inverse mappings L_d^{-1} and R_d^{-1} also exist. Thus,

$$d \backslash w = wL_d^{-1} \qquad \text{and} \qquad d/w = dR_w^{-1}.$$

In his publication [20], Molodtsov introduced soft set theory as a mathematical tool for managing uncertainty. Algebra experts have subsequently applied this concept to analyze the structural characteristics of algebraic structures. For example, Oyem delved into the algebraic properties of soft sets within the non-associative structure known as quasigroup [24, 25]. Soft set theory is esteemed for its capacity to accommodate a broad spectrum of information for decision-making in real-world scenarios. Nonetheless, the characterization of membership degrees in neutrosophic sets does not extend to the realm of soft set theory. Consequently, soft set theory may not be suitable for resolving issues involving indeterminate data.

A neutrosophic set, introduced by Smarandache in 1998, is a mathematical framework designed to handle real-life data with indeterminate characteristics. Unlike classical set theory, which deals with precise data, neutrosophic sets are well-suited for addressing problems involving uncertainty and indeterminate. The structure of a neutrosophic set is defined by three distinct membership degrees: true, indeterminate, and false, denoted as T, I, and F, respectively.

In recent times, there has been a surge of interest in the study of neutrosophic sets combined with soft set theory in the field of mathematics [2–6,19,21,26,31]. This combined approach offers a generalized framework for addressing uncertainties and indeterminacy in real-world problems. For instance, the $Q-$neutrosophic soft set extends the concept of the neutrosophic soft set by accommodating two universal sets, allowing for the simultaneous handling of indeterminate memberships from both sets.

Furthermore, researchers in [16] introduced a novel concept of neutrosophic logic in the development of an inspection assignment form for product quality control. Additionally, [14] explored characterizations of separation axioms in neutrosophic topological spaces, while the study in [15] presented the application of neutrosophic methods in operations research for managing corporate

The concept research on a neutrosophic soft set of two universal sets was first introduced in [9]. Various researchers have explored its applications in associative structures such as fields, groups, rings, and modules [7,9]. Oyebo et al. extended the $Q-$neutrosophic soft group to a $Q-$neutrosophic soft quasigroup, which is a generalization of the former. Additionally, Osoba et al. conducted a further investigation into the properties of a $Q-$neutrosophic soft quasigroup and loop [22,23]. For more on recent works in neutrosophic structure, the reader should check [17,18]. It is particularly noteworthy that this research marks the first extension of the Q-set to groupoids and uses it to construct an algorithm for decision-making, which inherently lacks the associative property.

The motivation behind this study is as follows: The work of Abu-Qamar and Hassan [2, 7] reports that the homomorphic images of certain classes of associative algebra—specifically, $Q-$neutrosophic soft fields, $Q-$neutrosophic soft rings, and $Q-$neutrosophic soft groups—remain within their respective categories. This assertion also holds true within classical studies on fields, rings, and groups. Moreover, in classical quasigroup studies, it is well established that the homomorphic image of a quasigroup is not necessarily a quasigroup itself (Pflugfelder, [29]). Additionally, while it is known that the direct product of two groups (or loops) P and H must incorporate a subgroup (or subloop) isomorphic to P—specifically, $P \times \{1\}$—this is not the case in classical quasigroup studies (see Fuval Foguel [12], 2008).

In line with the above motivation, the objectives of the research are as follows:

(1) to carry out the homomorphic characterization of $Q-$NS quasigroups and by extension study their homomorphic images and pre-image.
(2) to examine the direct product of two $Q-$NS quasigroups.
(3) to construct an example of $Q-$NS quasigroups useable to make decision in finance, particularity in stock-exchange related problems

Therefore, with the above objective, this study aims to close some of the existing knowledge gaps by investigating the following research questions:

- is the direct product of two $Q-$neutrosophic soft quasigroups also a $Q-$neutrosophic soft quasigroup?
- does the direct product $(\Psi^Q, A) \times (\Lambda^Q, B)$ of two $Q-$neutrosophic soft quasigroups isomorphic (Ψ^Q, A)?

- is the isotope of $Q-$neutrosophic soft quasigroups also a $Q-$neutrosophic soft quasigroup?
- are the isomorphic images and homomorphic images of $Q-$neutrosophic soft quasigroup also $Q-$neutrosophic soft quasigroup?
- is it possible to create a practical example that can be used to address real-life financial problems requiring decision-making?

This paper introduces a new study area by exploring $Q-$neutrosophic soft set under a non-associative algebra known as quasigroup, expanding on previous research in the field. While many authors have investigated the homomorphism of $Q-$neutrosophic soft sets under associative structures such as groups, rings, modules, and fields, it is worth noting that the homomorphic images of $Q-$neutrosophic soft groups, rings, and fields are indeed $Q-$neutrosophic soft groups, rings, and fields, as demonstrated by the authors in [7]. However, this research shall provide the solution to each research question mentioned above.

Definition 1.1. [13] Let $(\hat{G}, \circ)$ be a groupoid (quasigroup) and $\emptyset \neq H \subseteq \hat{G}$. Then, H is called subgroupoid (subquasigroup) of $\hat{G}$ if $(H, \circ)$ is a groupoid(quasigroup) and it is denoted as $H \leq \hat{G}$. Let S and T be non empty subsets of $\hat{G}$, then $S \circ T = \{s \circ t \mid s \in S, t \in T\}, S/T = \{s/t \mid s \in S, t \in T\}$ and $S \backslash T = \{s \backslash t \mid s \in S, t \in T\}$.

Definition 1.2. [20] Let W be set, a pair $(F, \mathfrak{A})$ is called a soft set if $F : \mathfrak{A} \rightarrow P(W)$, where $P(W)$ is power set of W and $\mathfrak{A}$ is a set of parameters.

Definition 1.3. [28] Let W be a set. A neutrosophic set (NS) is described as
$\phi = \{\langle w, (T_{\phi}(w), I_{\phi}(w), F_{\phi}(w))\rangle : w \in W\}$ such that $T_{\phi}, I_{\phi}, F_{\phi} : W \rightarrow]^{-}0, 1^{+}[$. where T, I, F are the truth, indeterminate and falsity membership degrees respectively.

Definition 1.4. [2] Let Q be a non-empty set, a $Q-$neutrosophic set ϕ^{Q} in W and Q is described in the form
$\phi^{Q} = \{\langle (w, u), T_{\phi^{Q}}(w, u), I_{\phi^{Q}}(w, u), F_{\phi^{Q}}(w, u)\rangle : w \in W, u \in Q\}$, where $T_{\phi^{Q}}, I_{\phi^{Q}}, F_{\phi^{Q}} : W \times Q \rightarrow]^{-}0, 1^{+}[$ are the membership degrees.

Definition 1.5. [21] Let W be a set and $\mathfrak{A}$ be a parameter sets. A neutrosophic soft set $(\phi, \mathfrak{A})$ described as $(\phi, \mathfrak{A}) = \{\langle w, (T_{\phi}(w), I_{\phi}(w), F_{\phi}(w))\rangle : w \in W\}$.

Definition 1.6. [2] Let $\alpha : M \times Q \mapsto N \times Q$ and $\rho : \mathfrak{A} \mapsto \mathfrak{B}$ be two functions, where $\mathfrak{A}$ and $\mathfrak{B}$ are parametric sets for $M \times Q$ and $N \times Q$ respectively. Then, (α, ρ) is called a $Q-$neutrosophic soft mapping or function from the set $M \times Q$ to the set $N \times Q$.

Definition 1.7. [13] A triple (A, B, C) of bijection from a set M onto a set N is called an isotopism of a groupoid $(M, \circ)$ onto a groupiod $(N, \odot)$ if for all $x, y \in M$, $xA \circ yB = (x \odot y)C$.

2. Properties of Quasigroup under Neutrosophic Soft set

Definition 2.1. Let $(\phi^Q, \mathfrak{A})$ be a $Q-$neutrosophic soft set defined over a quasigroup $(\hat{G}, *)$. Then, $(\phi^Q, \mathfrak{A})$ is called a $Q-$neutrosophic soft subquasigroup over a quasigroup $\hat{G}$ if $\phi^Q(a)$ is a $Q-$neutrosophic soft quasigroup given a mapping $\phi^Q(a) : \hat{G} \times Q \to [0,1]^3$ for all $a \in \mathfrak{A}, v \in Q$.

Definition 2.2. Let $(\phi^Q, \mathfrak{A})$ be a $Q-$neutrosophic soft set defined under a quasigroup $(\hat{G}, \circ, /, \backslash)$. Then, $(\phi^Q, \mathfrak{A})$ is called a $Q-$neutrosphic soft subquasigroup of quasigroup $\hat{G}$ if for all $a \in \mathfrak{A}, r_1, r_2 \in \hat{G}, v \in Q$ satisfies the following conditions

(1) $T_{\phi^Q(a)}(r_1 * r_2, v) \geq \min\{T_{\phi^Q(a)}(r_1, v), T_{\phi^Q(a)}(r_2, v)\}$
(2) $I_{\phi^Q(a)}(r_1 * r_2, v) \leq \max\{I_{\phi^Q(a)}(r_1, v), I_{\phi^Q(a)}(r_1, v)\}$
(3) $F_{\phi^Q(a)}(r_1 * r_2, v) \leq \max\{F_{\phi^Q(a)}(r_1, v), F_{\phi^Q(a)}(r_2, v)\}$

where $* \in \{\circ, /, \backslash\}$.

Example 2.3. Let $\hat{G} = \{i, j, k, l, m, n, o\}$ be quasigroup of order 7 and $A \subseteq E = \hat{G}$ be the parametric set. Given the quasigeoup in Cayley table below.

Define a $Q-$ neutrosophic soft set $(\phi^Q, \mathfrak{A})$ as follows, for all $u \in Q$ and $w, t, z \in \hat{G}, A = \hat{G}$ a

Table 1. Quasigroup of order 7

$\odot$	i	j	k	l	m	n	o
i	i	m	o	n	j	l	k
j	m	j	n	o	i	k	l
k	o	n	k	m	l	j	i
l	n	o	m	k	l	i	j
m	j	i	l	k	m	o	n
n	l	k	j	i	o	n	m
o	k	l	i	j	n	m	o

parameters set with $z = w * t$ and $n \in \mathbb{N}$.

$$T_{\phi^Q(a)}(w \odot t, u) = \begin{cases} 1 - \frac{1}{2n}, & \text{if } z = \{j, k, l, m, n, o\} \\ 1, & \text{otherwise.} \end{cases}$$

$$I_{\phi^Q(a)}(w \odot t, u) = \begin{cases} 0, & \text{if } z = \{j, k, l, m, n, o\} \\ 1 - \frac{1}{2n}, & \text{otherwise.} \end{cases}$$

$$F_{\phi^Q(a)}(w \odot t, u) = \begin{cases} 0, & \text{if } z = \{j, k, l, m, n, o\} \\ 1 - \frac{1}{2n}, & \text{otherwise.} \end{cases}$$

$$T_{\phi^Q(a)}(w/t,u) = \begin{cases} 1-\frac{1}{3n}, & \text{if } z = \{j,k,l,m,n,o\} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\phi^Q(a)}(w/t,u) = \begin{cases} 0, & \text{if } z = \{j,k,l,m,n,o\} \\ 1-\frac{1}{2n}, & \text{otherwise.} \end{cases}$$

$$F_{\phi^Q(a)}(w/t,u) = \begin{cases} 0, & \text{if } z = \{j,k,l,m,n,o\} \\ 1-\frac{1}{2n}, & \text{otherwise.} \end{cases}$$

$$T_{\phi^Q(a)}(w\backslash t,u) = \begin{cases} 1-\frac{1}{2n}, & \text{if } w = \{j,k,l,m,n,o\} \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\phi^Q(a)}(w\backslash t,u) = \begin{cases} 0, & \text{if } z = \{j,k,l,m,n,o\} \\ 1-\frac{1}{2n}, & \text{otherwise.} \end{cases}$$

$$F_{\phi^Q(a)}(w\backslash t,u) = \begin{cases} 0, & \text{if } z = \{j,k,l,m,n,o\} \\ 1-\frac{1}{2n}, & \text{otherwise.} \end{cases}$$

Let consider the operation $'\odot'$ then, $T_{\phi^Q(a)}(w \odot t,u) \geq \min\{T_{\phi^Q(a)}(w,u), T_{\phi^Q(a)}(t,u)\}$. Put $w = j, t = n$, then we have

$$T_{\phi^Q(a)}(j \odot n,u) \geq \min\{T_{\phi^Q(a)}(m,u), T_{\phi^Q(a)}(m,u)\} =$$
$$= 1 - \frac{1}{2n} = 0.5 \in [0,1] \text{ where n=1 } \in \mathbb{N}. \tag{1}$$

On the other hand,

$$\min\{T_{\phi^Q(a)}(j,u), T_{\phi^Q(a)}(n,u)\} =$$
$$\min\{0, 1 - \frac{1}{2n}\} = 0 \in [0,1] \text{ for all } n \in \mathbb{N} \tag{2}$$

Hence, $1 - \frac{1}{2n} \geq 0$ for all $n \in \mathbb{N}$. This is true for membership degree.

3. **Direct Product of $Q-$Neutrosophic Soft quasigroups**

Let (ψ^Q, A) and (ψ^Q, B) be two $Q-$neutrosophic soft quasigroups, then their direct product $(\psi^Q, A)\times(\lambda^Q, B)$ can be constructed in the same way like in $Q-$neutrosophic soft groups, and it will be a $Q-$neutrosophic soft quasigroup. Also if (ψ^Q, A) is a $Q-$neutrosophic soft quasigroup of order n, any direct product of (ψ^Q, A) with another $Q-$neutrosophic soft quasigroup (ψ^Q, B) of order m can form different range of $Q-$neutrosophic soft quasigroups of order nm, which are generalized products variously called extensions, quasidirect products or twisted products Chein et al. [11], which is an attempt to generalize the work in Brucks [10].

It has been established in Theorem 3.2 that if (ψ^Q, A) and (ψ^Q, B) are two $Q-$neutrosophic soft quasigroups over quasigroups $\hat{G}_1$ and $\hat{G}_2$ respectively, then the direct product $(\psi^Q, A) \times (\psi^Q, B)$ is a $Q-$ neutrosophic soft quasigroup.

Definition 3.1. Let (ψ^Q, A) and (ψ, B) be two $Q-$neutrosophic soft sets over a quasigroups $\hat{G}_1$ and $\hat{G}_2$ respectively. The direct product donated by $(\psi^Q, A) \times (\psi^Q, B)$ over $\hat{G}_1 \times \hat{G}_2$ such that $(\phi^Q, A) \times (\psi^Q, B) = (\pi^Q, C)$ is given by $\pi^Q(a,b) = \Big\{\langle((x_1,x_2),q), T_{\phi_Q(a,b)}((x_1,x_2),q), I_{\phi_Q(a,b)}((x_1,x_2),q), F_{\phi_Q(a,b)}((x_1,x_2),q)\rangle : (x_1,x_2) \in \hat{G}_1 \times \hat{G}_2, q \in Q\Big\}$. $T_{\pi_Q(a,b)}((x_1,x_2),q) = \min\{T_{\phi_Q(a)}(x_1,q), T_{\psi_Q(b)}(x_2,q)\}$,
$I_{\pi_Q(a,b)}((x_1,x_2),q) = \max\{I_{\phi_Q(a)}(x_1,q), I_{\psi_Q(b)}(x_2,q)\}$,
$F_{\pi_Q(a,b)}((x_1,x_2),q) = \max\{F_{\phi_Q(a)}(x_1,q), F_{\psi_Q(b)}(x_2,q))\}$.

Theorem 3.2. *Let (ψ^Q, A) and (ψ^Q, B) be two $Q-$neutrosophic soft quasigroups over quasigroups $\hat{G}_1$ and $\hat{G}_2$ respectively. Then $(\psi^Q, A) \times (\psi^Q, B) = (\pi^Q, C)$ is a $Q-$neutrosophic soft quasigroup over $(\hat{G}_1 \times \hat{G}_2)$.*

Proof. Let $(\psi^Q, A) \times (\psi^Q, B) = (\pi_Q, C)$ such that $\psi^Q(a) \times \psi^Q(b) = \pi_Q(a,b)$, for all $a \in A, b \in B$, $(a,b) \in A \times B, (x,y),(z,t) \in \hat{G}_1 \times \hat{G}_2$. The True membership:

$$
\begin{aligned}
&T_{\pi_Q(a,b)}\big((x,y)\times(z,t)),q\big)\\
&= T_{\pi_Q(a,b)}\big((x\odot z, y\odot t),q\big)\\
&= \min\Big\{T_{\phi_Q(a)}\big((x\odot z),q\big), T_{\psi_Q(b)}\big((y\odot t),q\big)\Big\}\\
&\geq \min\Big\{\min\underbrace{\{T_{\phi_Q(a)}(x,q), T_{\phi_Q(a)}(z,q)\}}, \min\underbrace{\{T_{\psi_Q(b)}(y,q), T_{\psi_Q(b)}(t,q)\}}\Big\}\\
&\geq \min\Big\{\min\underbrace{\{T_{\phi_Q(a)}(x,q), T_{\psi_Q(b)}(y,q)\}}, \min\underbrace{\{T_{\phi_Q(a)}(z,q), T_{\psi_Q(b)}(t,q)\}}\Big\}\\
&= \min\Big\{\{T_{\pi_Q(a,b)}((x,y),q), T_{\pi_Q(a,b)}((z,t),q)\Big\}
\end{aligned}
$$

Next:

$$I_{\pi_Q(a,b)}(x,y) \times ((z,t),q)$$
$$I_{\pi_Q(a,b)}(x \odot z),((y \odot t),q)$$
$$= \max\left\{I_{\phi_Q(a)}(x \odot z),q), I_{\psi_Q(b)}((y \odot t),q)\right\}$$
$$\leq \max\left\{\max\underbrace{\{I_{\phi_Q(a)}(x),q), I_{\phi_Q(a)}(z,q)\}}, \max\underbrace{\{I_{\psi_Q(b)}(y,q), I_{\psi_Q(b)}(t,q)\}}\right\}$$
$$\leq \max\left\{\max\underbrace{\{I_{\phi_Q(a)}(x,q), I_{\psi_Q(a)}(y,q)\}}, \max\underbrace{\{I_{\phi_Q(b)}(z,q), I_{\psi_Q(b)}(t,q)\}}\right\}$$
$$= \max\left\{I_{\pi_Q(a,b)}((x,y),q), I_{\pi_Q(a,b)}((z,t),q)\right\}$$

Next:

$$F_{\pi_Q(a,b)}(x,y) \times ((z,t),q)$$
$$= F_{\pi_Q(a,b)}(x \odot z),((y \odot t),q)$$
$$= \max\{F_{\phi_Q(a)}(x \odot z),q), F_{\psi_Q(b)}((y \odot t),q)\}$$
$$\leq \max\left\{\max\underbrace{\{F_{\phi_Q(a)}(x,q), F_{\phi_Q(a)}(z,q)\}}, \max\underbrace{\{F_{\psi_Q(b)}(y,q), F_{\psi_Q(b)}(t,q)\}}\right\}$$
$$\leq \max\left\{\max\underbrace{\{F_{\phi_Q(a)}(x,q), F_{\psi_Q(b)}(y,q)\}}, \max\underbrace{\{F_{\phi_Q(a)}(z,q), F_{\psi_Q(b)}(t,q)\}}\right\}$$
$$= \max\left\{F_{\pi_Q(a,b)}((x,y),q), F_{\pi_Q(a,b)}((z,t),q)\right\}$$

□

Corollary 3.3. *Let (ψ^Q, A_1) and (ψ^Q, A_2) be two $Q-$ neutrosophis soft quasigroups. Then, $(\psi^Q, A_1) \times (\psi^Q, A_2)$ is a $Q-$ neutrosophis soft quasigroup over $\hat{G}_1 \times \hat{G}_2$.*

Proof. It's easy from Theorem 3.2. □

Definition 3.4. Let (ψ^Q, A_i) be set of $Q-$ neutrosophic soft quasigroups under the quasigroups $\hat{Q}_i$, then $(\psi^Q, A_i)^n = (\psi^Q, A_i^n)$ is called $Q-$ neutrosophis power soft quasigroup $(Q - NPS\hat{G})$ for all $n \in \mathbb{Z}$.

Theorem 3.5. *Let (ψ^Q, A_1) and (ψ^Q, A_2) be two $Q-$ neutrosophis soft quasigroups. Then, $((\psi^Q, A_1) \times (\psi^Q, A_2))^n = (\psi^Q, A_1)^n \times (\psi^Q, A_2)^n$ any natural number $n \in \mathbb{Z}$.*

Proof. Let $n = 2$, then we show that $((\psi^Q, A_1) \times (\psi^Q, A_2))^2 = (\psi^Q, A_1)^2 \times (\psi^Q, A_2)^2$

Let $((\psi^Q, A_1) \times (\psi^Q, A_2))^2 = ((\psi^Q, A_1 \times A_2))^2$ and let $x_i, y_i \in \hat{Q}_i$ where $i = 1,2$ and for $(a_1, a_1) = a_1^2 \in A_1 \times A_1)$ and $(a_2, a_2) = a_2^2 \in A_2 \times A_2)$. Then, the True membership degree is given as $T_{\psi^Q(a_1,a_2)^2}((x_1,x_2)(y_1,y_2))$

$$
\begin{aligned}
&= T_{\psi^Q(a_1,a_2)^2}((x_1y_1,q)(x_2y_2,q)) \\
&= \min\{T_{\psi^Q(a_1,a_2)}((x_1y_1,q),T_{\psi^Q(a_1,a_2)}(x_2y_2,q))\} \\
&= \min\left\{\min\left\{T_{\psi^Q(a_1)}((x_1y_1,q),T_{\psi^Q(a_2)}((x_1y_1,q)\right\},\min\left\{T_{\psi^Q(a_1)}((x_2y_2,q),T_{\psi^Q(a_2)}((x_2y_2,q)\right\}\right\} \\
&= \min\left\{\min\left\{T_{\psi^Q(a_1)}((x_1y_1,q),T_{\psi^Q(a_1)}((x_2y_2,q)\right\},\min\left\{T_{\psi^Q(a_2)}((x_1y_1q),T_{\psi^Q(a_2)}((x_2y_2,q)\right\}\right\} \\
&\geq \min\left\{\min\left[\left\{\min\{T_{\psi^Q(a_1)}((x_1,q),T_{\psi^Q(a_1)}((y_1,q)\},\min\{T_{\psi^Q(a_1)}((x_2,q),T_{\psi^Q(a_1)}((y_2,q)\}\right\}\right],\right. \\
&\qquad \left.\min\left[\left\{\min\{T_{\psi^Q(a_2)}((x_1,q),T_{\psi^Q(a_2)}((y_1,q)\},\min\{T_{\psi^Q(a_2)}((x_2,q),T_{\psi^Q(a_2)}((y_2,q)\}\right\}\right]\right\} \\
&= \min\left\{\left\{\min\{T_{\psi^Q(a_1,a_1)}((x_1y_1,q),T_{\psi^Q(a_1,a_1)}((x_2y_2,q)\},\min\{T_{\psi^Q(a_2,a_2)}(x_1y_1,q),T_{\psi^Q(a_2,a_2)}((x_2y_2,q)\}\right\}\right\} \\
&= \min\left\{T_{\psi^Q(a_1,a_1)^2}((x_1y_1),(x_2y_2)q),T_{\psi^Q(a_2,a_2)^2}((x_1y_1),(x_2y_2)q)\right\}
\end{aligned}
$$

Hence, $((\psi^Q,A_1)\times(\psi^Q,A_2))^2=(\psi^Q,A_1)^2\times(\psi^Q,A_2)^2$.

Similarly, we can show that

$$
\begin{aligned}
((\psi^Q,A_1)\times(\psi^Q,A_2))^3 &= ((\psi^Q,A_1)\times(\psi^Q,A_2))((\psi^Q,A_1)\times(\psi^Q,A_2))^2 \\
&= ((\psi^Q,A_1)\times(\psi^Q,A_2))((\psi^Q,A_1)^2\times(\psi^Q,A_2)^2
\end{aligned}
$$

Since, Mann's multiplication is associative, this mean, it can be show that $((\psi^Q,A_1)\times(\psi^Q,A_2))^3=(\psi^Q,A_1)^3\times(\psi^Q,A_2)^3$. Hence, by induction on a positive integer n, Theorem 3.5 is true for all $n\in\mathbb{Z}$. □

Corollary 3.6. *Let $(\psi^Q,A_1),(\psi^Q,A_2),...,(\psi^Q,A_n)$ be $Q-$ neutrosophis soft quasigroups. Then, $((\psi^Q,A_1)^m\times(\psi^Q,A_2)^m\times...\times(\psi^Q,A_n)^m=(\psi^Q,A_1)^m\times(\psi^Q,A_2)^m\times...\times(\psi^Q,A_n)^m$.*

Proof. It follows from Theorem 3.5. □

Proposition 3.7. *If (ψ^Q,A) has a proper $Q-$neutrosophic soft subquasigroup (ψ^Q,B), then $(\psi^Q,A)\times(\pi_Q,C)$ has the proper $Q-$neutrosophic soft quasigroup $(\psi^Q,B)\times(\pi_Q,C)$.*

Proof. Let $(\psi^Q,A)\times(\pi_Q,C)=(\lambda^Q,D)$ and $(\psi^Q,B)\times(\pi_Q,C)=(\Upsilon_Q,E)$. Then, by Theorem 3.2, (λ^Q,D) and (Υ_Q,E) are $Q-$neutrosophic soft quasigroups. Since (ψ^Q,B) is a proper $Q-$neutrosophic soft subquasigroup of (ψ^Q,A), then (Υ_Q,E) is a proper $Q-$neutrosophic soft subquasigroup of (λ^Q,D). The membership degrees are characterize as $T_{\Upsilon_Q}(x)\leq$

$T_{\lambda^Q}(x), I_{\lambda^Q}(x) \geq I_{\Upsilon_Q}(x), F_{\lambda^Q}(x) \geq F_{\Upsilon_Q}(x)$, are respectively truth membership function, indeterminacy membership function and falsity membership function for all $x \in \hat{G}$ and $q \in Q$. □

4. Isotopism of $Q-$Neutrosophic Soft Quasigroup

Definition 4.1. Let (ψ^Q, A) and (λ^Q, B) be two $Q-$neutrosophic soft sets and under the operation of the sets $M \times Q$ and $N \times Q$ respectively and $\alpha : M \times Q \mapsto N \times Q$ and $\rho : A \mapsto B$ be two functions. Then, the image $(\alpha, \rho)(\psi^Q, A)$ of (ψ^Q, A) under (α, ρ) is a $Q-$ neutrosophic soft set over $N \times Q$. That is,

$$(\alpha, \rho)(\psi^Q, A) = \big(\alpha(\psi^Q), \rho(A)\big) = \big\{b, \alpha(\psi^Q)(b) : b \in \rho(A)\big\} \tag{3}$$

Then, membership degrees are given as for all $n \in N, q \in Q$

$$T_{\alpha_{(\psi^Q)(b)}}(n,q) = \begin{cases} \max_{\alpha(m,q)=(n,q)} \max_{\rho(a)=b}[T_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q) \in \alpha^{-1}(n,q) \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\alpha_{(\psi^Q)(b)}}(n,q) = \begin{cases} \min_{\alpha(m,q)=(n,q)} \min_{\rho(a)=b}[I_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q) \in \alpha^{-1}(n,q) \\ 1, & \text{otherwise.} \end{cases}$$

$$F_{\alpha_{(\psi^Q)(b)}}(n,q) = \begin{cases} \min_{\alpha(m,q)=(n,q)} \min_{\rho(a)=b}[F_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q) \in \alpha^{-1}(n,q) \\ 1, & \text{otherwise.} \end{cases}$$

for all $n \in N$, $q \in Q$.
The preimage $(\alpha, \rho)^{-1}(\lambda^Q, B)$ of (λ^Q, B) under (α, ρ) is a $Q-$ neutrosophic soft set over $M \times Q$ define as

$$(\alpha, \rho)^{-1}(\lambda^Q, B) = \big(\alpha^{-1}(\lambda^Q), \rho(B)\big) = \big\{a, \alpha^{-1}(\psi^Q)(a) : a \in \alpha^{-1}(B)\big\} \tag{4}$$

for all $m \in M$,

$$T_{\alpha^{-1}(\lambda^Q)(a)}(m,q) = T_{\psi^Q[\rho(a)]}(\alpha(n,q))$$
$$, I_{\alpha^{-1}(\lambda^Q)(a)}(m,q) = I_{\psi^Q[\rho(a)]}(\alpha(n,q)),$$
$$F_{\alpha^{-1}(\lambda^Q)(a)}(m,q) = F_{\psi^Q[\rho(a)]}(\alpha(n,q)).$$

Definition 4.2. Let the triple $((\alpha, \beta, \gamma)\rho)$ of bijections of a $Q-$neutrosophic soft sets $M \times Q$ onto the set $N \times Q$. If $((\alpha, \beta, \gamma)\rho)$ is an isotopism of the set $(M \times Q, \odot)$ onto a set $(N \times Q, \circ)$, then $((\alpha,\beta,\gamma)\rho)$ is said to be isotope between the two $Q-$ neutrosophic soft quasigroups. The membership degrees are defined by

$$T_{\alpha(\psi^Q)(b)}(n,q)=\begin{cases}\max_{\beta,\gamma(m,q)=(n,q)}\max_{\rho(a)=b}[T_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q)\in\beta^{-1},\gamma^{-1}(n,q)\\ 0, & \text{otherwise.}\end{cases}$$

$$I_{\alpha(\psi^Q)(b)}(n,q)=\begin{cases}\min_{\beta,\gamma(m,q)=(n,q)}\min_{\rho(a)=b}[I_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q)\in\beta^{-1},\gamma^{-1}(n,q)\\ 1, & \text{otherwise.}\end{cases}$$

$$F_{\alpha(\psi^Q)(b)}(n,q)=\begin{cases}\min_{\beta,\gamma(m,q)=(n,q)}\min_{\rho(a)=b}[F_{\phi_{Q(a)}}(m,q)], & \text{if } (m,q)\in\beta^{-1},\gamma^{-1}(n,q)\\ 1, & \text{otherwise.}\end{cases}$$

Definition 4.3. Suppose that the triple $((\alpha,\beta,\gamma)\rho)=(\alpha,\rho)$, where $\alpha=\beta=\gamma$ then (α,ρ) is said to be homomorphism from $(M\times Q,\odot)$ to $(N\times Q,\circ)$. The set

$$(\alpha,\rho)(\psi^Q,A)=\big(\alpha(\psi^Q),\rho(A)\big)=\big\{b,\alpha(\psi^Q)(b):b\in\rho(A)\big\} \tag{5}$$

is called the homomorphic image of $(M\times Q,\odot)$ under (α,ρ).

Suppose that $\alpha: M\times Q\mapsto M\times Q$ and $\rho: A\mapsto A$, then the homomorphism is called endomorphism. It is called isomorphism if (α,ρ) is a bijection. Suppose that $\alpha: M\times Q\mapsto M\times Q$ and $\rho: A\mapsto A$ is a homomorphism, then it called isomorphism.

Theorem 4.4. *Let (ψ^Q,A) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_1$ and $T=\alpha,\beta,\gamma:(\hat{G}_1\times Q,\odot)\to(\hat{G}_2\times Q,\circ)$ be isotopism and $\rho: A\to B$. Then $(T,\rho)(\psi^Q,A)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$.*

Proof. Let $b\in\rho(E)$ the parametric set $s,t\in\hat{G}_2$ and $q\in Q$. Suppose that there exist $x,y\in\hat{G}_1$ such that $(s,q)=\alpha(x,q)$ and $(t,q)=\beta(y,q)$. Let $z\in\hat{G}_1$ such that $(z,q)=((x\odot y),q)$. Then,

$$\begin{aligned}T_{\gamma(\psi^Q)(b)}(s\circ t,q)&=\max_{\alpha(x,q)=(s,q),\beta(y,q)=(t,q)}\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(z,q)]\\ &\geq\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(x\odot y,q)]\\ &\geq\max_{\rho(a)=b}\Big[\min\big\{T_{(\psi^Q)(a)}(x,q),T_{(\psi^Q)(a)}(y,q)\big\}\Big]\\ &\min\big\{[\max_{\rho(a)=b}T_{(\psi^Q)(a)}(x,q)],\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(y,q)]\big\}.\end{aligned}$$

For each $x,y\in\hat{G}_1$, and $q\in Q$, the equality $(s,q)=\alpha(x,q)$ and $(t,q)=\beta(y,q)$ hold for all $s,t\in\hat{G}_2$. Then, this follows the last equality

$$\begin{aligned}&T_{\gamma(\psi^Q)(b)}(s\circ t,q)\\ &\geq\min\big\{\max_{\alpha(x,q)=(s,q)}\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(x,q)],\max_{\beta(y,q)=(t,q)}\max_{\rho(a)=b}[T_{(\psi^Q)(a)}(y,q)]\big\}\\ &=\min\{T_{\alpha(\psi^Q)(b)}(s,q),T_{\beta(\psi^Q)(b)}(t,q).\end{aligned}$$

Let (ψ^Q, B) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_2$ such that $T^{-1}(\psi^Q, B) = (\psi^Q, A)$ and $\rho^{-1}(B) = A$, where $T = (\alpha, \beta, \gamma)$. For all $s, t \in \hat{G}_1$ and $q \in Q$, we have

$$\begin{aligned} T_{\gamma^{-1}(\psi^Q)(a)}(s \odot t, q) &= T_{(\psi^Q)\rho^{-1}(b)}[\alpha(t,q) \circ \beta(s,q)] \\ &\geq \min\{T_{(\psi^Q)\rho^{-1}(b)}\alpha(t,q), T_{(\psi^Q)\rho^{-1}(b)}\beta(t,q)\} \\ &= \min\{T_{\alpha^{-1}(\psi^Q)(a)}(t,q), T_{\beta^{-1}(\psi^Q)(a)}(t,q)\}. \end{aligned}$$

Similarly, we show:
The indeterminate membership degree is given by

$$I_{\gamma(\psi^Q)(b)}(s \circ t, q)$$
$$\geq \max\Big\{ \min_{\alpha(x,q)=(s,q)} \min_{\rho(a)=b} [I_{(\psi^Q)(a)}(x,q)], \min_{\beta(y,q)=(t,q)} \min_{\rho(a)=b} [I_{(\psi^Q)(a)}(y,q)] \Big\}$$
$$= \max\{I_{\alpha(\psi^Q)(b)}(s,q), I_{\beta(\psi^Q)(b)}(t,q)$$

and

$$I_{\gamma^{-1}(\psi^Q)(a)}(s \odot t, q) = \max\{I_{\alpha^{-1}(\psi^Q)(a)}(t,q), I_{\beta^{-1}(\psi^Q)(a)}(t,q)\}.$$

And the falsity membership degree is given as

$$F_{\gamma(\psi^Q)(b)}(s \circ t, q)$$
$$\geq \max\Big\{ \min_{\alpha(x,q)=(s,q)} \min_{\rho(a)=b} [F_{(\psi^Q)(a)}(x,q)], \min_{\beta(y,q)=(t,q)} \min_{\rho(a)=b} [F_{(\psi^Q)(a)}(y,q)] \Big\}$$
$$= \max\{F_{\alpha(\psi^Q)(b)}(s,q), F_{\beta(\psi^Q)(b)}(t,q)$$

$$F_{\gamma^{-1}(\psi^Q)(a)}(s \odot t, q) = \max\{F_{\alpha^{-1}(\psi^Q)(a)}(t,q), F_{\beta^{-1}(\psi^Q)(a)}(t,q)\}$$

□

Example 4.5. Let (ψ^Q, A) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_1$ and $(T, \rho) : (\hat{G}_1 \times Q, \odot) \to (\hat{G}_2 \times Q, \circ)$ be a $Q-$neutrosophic soft isotopism with the parameters $\rho : A \to B$. Let $\hat{G}_1 = \{1, 2, 3, 4\}$ and $\hat{G}_2 = \{a, b, c, d\}$ given in table below.

$\odot$	1	2	3	4
1	2	3	4	1
2	3	2	1	4
3	4	1	2	3
4	1	4	3	2

$\longrightarrow$

$\circ$	a	b	c	d
a	b	d	1	c
b	a	c	b	d
c	d	b	c	a
d	c	a	d	b

TABLE 2. Quasigroup of order 4

Next, construct $Q-$neutrosophic soft set as follows

$$T_{(\psi^Q)(b)}(s\circ t,q)=\begin{cases}0.90, & \text{if } z\in\{a,d,e\}\\ 0, & \text{otherwise.}\end{cases}$$

$$I_{(\psi^Q)(b)}(s\circ t,q)=\begin{cases}0.90, & \text{if } z\in\{a,d,e\}\\ 1, & \text{otherwise.}\end{cases}$$

$$F_{(\psi^Q)(b)}(s\circ t,q)=\begin{cases}0.90, & \text{if } z\in\{a,d,e\}\\ 1, & \text{otherwise.}\end{cases}$$

Let $(T,\rho):(\hat{G}_1\times Q,\odot)\rightarrow(\hat{G}_2\times Q,\circ)$ be isotopism of the $Q-$neutrosophic soft quasigroups defined as

$$\alpha(\psi^Q)(b)=\begin{pmatrix}1&2&3&4\\ a&c&b&d\end{pmatrix},\beta(\psi^Q)(b)=\begin{pmatrix}1&2&3&4\\ b&a&d&c\end{pmatrix}$$

and

$$\gamma(\psi^Q)(b)=\begin{pmatrix}1&2&3&4\\ a&c&b&d\end{pmatrix}$$

where $T=\alpha,\beta,\gamma$ and $\rho(a)=b$. The membership degree is as follows: In particular, let $s=1$, and $t=3$, then the true membership degree is given as

$$\begin{aligned}T_{\gamma(\psi^Q)(b)}(1\odot 3,q) &= \max_{\rho(a)=b}[\{\min T_{\alpha(\psi^Q)(a)}(1,q),T_{\beta(\psi^Q)(a)}(3,q)\}]\\ T_{\gamma(\psi^Q)(b)}(4,q) &\geq \min\{[\max_{\rho(a)=b}T_{\alpha(\psi^Q)(a)}(1,q)],\max_{\rho(a)=b}[T_{\beta(\psi^Q)(a)}(3,q)]\}\\ T_{(\psi^Q)(b)}(d,q) &\geq \min\{T_{(\psi^Q)(b)}(a,q),T_{(\psi^Q)(b)}(d,q)\}.\end{aligned}$$

Consider the LHS, we have $T_{(\psi^Q)(b)}(d,q)=0.90$ since $d\in\{a,d,e\}$. Observing the RHS, we have $\min\{T_{(\psi^Q)(b)}(a,q),T_{(\psi^Q)(b)}(d,q)\}=\min\{0.90,0,90\}=0.90$. Hence, $RHS=LHS$. Indeed, $(T,\rho):(\hat{G}_1\times Q,\odot)\rightarrow(\hat{G}_2\times Q,\circ)$ is a $Q-$neutrosophic soft isotopic quasigroups and $(T,\rho)(\psi^Q,A)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$.
The indeterminate membership degree is by

$$\begin{aligned}I_{\gamma(\psi^Q)(b)}(1\odot 3,q) &= \min_{\rho(a)=b}[\{\max I_{\alpha(\psi^Q)(a)}(1,q),I_{\beta(\psi^Q)(a)}(3,q)\}]\\ I_{\gamma(\psi^Q)(b)}(4,q) &\leq \max\{[\min_{\rho(a)=b}I_{\alpha(\psi^Q)(a)}(1,q)],\min_{\rho(a)=b}[I_{\beta(\psi^Q)(a)}(3,q)]\}\\ I_{(\psi^Q)(b)}(d,q) &\leq \max\{I_{(\psi^Q)(b)}(a,q),I_{(\psi^Q)(b)}(d,q)\}\end{aligned}$$

Consider the LHS of the last equality, $T_{(\psi^Q)(b)}(d,q)=1.0$ since $d\in\{a,d,e\}$ and the RHS, gives $\max\{T_{(\psi^Q)(b)}(a,q),T_{(\psi^Q)(b)}(d,q)\}=\max\{1.0,1.0\}=1.0$. Thus, $RHS=LHS$.

Theorem 4.6. *Let (ψ^Q,A) be a $Q-$neutrosophis soft quasigroup over a finite quasigroup $\hat{G}_1$ and let $\alpha:(\hat{G}_1\times Q,\odot)\rightarrow(\hat{G}_2\times Q,\circ)$ be a homomorphism such that $\rho:A\rightarrow B$. Then $(\alpha,\rho)(\psi^Q,A)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$.*

Proof. Let $\alpha : (\hat{G_1} \times Q, \odot) \to (\hat{G_2} \times Q, \circ)$ be a $Q-$neutrosophic soft homomorphism and $\rho : A \to B$ by setting $T = \alpha = \beta = \gamma$ in theorem (4.4), we have $(\alpha, \rho)(\psi^Q, A)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$. □

Theorem 4.7. *Let (ψ^Q, A) be a $Q-$neutrosophis soft quasigroup over a finite quasigroup $\hat{G}_1$. If $\alpha : \hat{G}_1 \to \hat{G}_2$ is a $Q-$neutrosophic soft isomorphism. Then, $(\alpha, \rho)((\psi^Q, A))$ is a $Q-$neutrosophis soft quasigroup over $\hat{G}_2$.*

Proof. From theorem (4.4), (α, β, γ) is a $Q-$neutrosophis soft quasigroup isotopism. If $\alpha = \beta = \gamma$, is a $Q-$neutrosophis soft isotopism, then $\alpha(\phi_{Q(a)})$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_2$. □

Theorem 4.8. *Let (ψ^Q, B) be a $Q-$neutrosophic soft quasigroup over a finite quasigroup $\hat{G}_1$ and $\alpha : (\hat{G_1} \times Q, \odot) \to (\hat{G_2} \times Q, \circ)$ be a $Q-$neutrosophic soft homomorphism and $\rho : A \to B$. Then, $(\alpha, \rho)^{-1}(\psi^Q, B)$ is a $Q-$neutrosophic soft quasigroup over $\hat{G}_1$.*

Proof. Let $s, t \in \hat{G}_1, q \in Q$ and $\rho : A \to B$. For any parameter $a \in \rho^{-1}(B)$, we have

$$\begin{aligned} T_{\alpha^{-1}(\psi^Q)(a)}(s \circ t, q) &= T_{(\psi^Q)[\rho(a)]}(\alpha(s \odot t, q)) \\ &= T_{(\psi^Q)[\rho(a)]}(\alpha(s, q)\alpha(t, q)) \\ &\geq \min\{T_{(\psi^Q)[\rho(a)]}(\alpha(s, q)), T_{(\psi^Q)[\rho(a)]}(\alpha(t, q))\} \\ &= \min\{T_{\alpha^{-1}(\psi^Q)(a)}(s, q), T_{\alpha^{-1}(\psi^Q)(a)}(t, q)\}. \end{aligned}$$

Similarly, we shows that

$$I_{\alpha^{-1}(\psi^Q)(a)}(s \circ t, q) = \max\{I_{\alpha^{-1}(\psi^Q)(a)}(s, q), I_{\alpha^{-1}(\psi^Q)(a)}(t, q)\}$$
$$\text{and } F_{\alpha^{-1}(\psi^Q)(a)}(s \circ t, q) = \max\{F_{\alpha^{-1}(\psi^Q)(a)}(s, q), F_{\alpha^{-1}(\psi^Q)(a)}(t, q)\}.$$

□

Definition 4.9. Let (ψ^Q, A) be a $Q-$neutrosophic soft quasigroup and (U, V, W) be an autotopism of a finite quasigroup $(\hat{G}, \odot)$. Then the following hold.

(1) $T_{W(\psi^Q)(a)}(st, q) = \max_{\rho(a)=a}\{T_{U(\psi^Q)(a)}(s, q), T_{V(\psi^Q)(a)}(t, q)\}$.
(2) $I_{W(\psi^Q)(a)}(st, q) = \min_{\rho(a)=a}\{I_{U(\psi^Q)(a)}(s, q), I_{V(\psi^Q)(a)}(t, q)\}$.
(3) $F_{W(\psi^Q)(a)}(st, q) = \min_{\rho(a)=a}\{F_{U(\psi^Q)(a)}(s, q), F_{V(\psi^Q)(a)}(t, q)\}$.

for all $s, t \in \hat{G}, q \in Q$

Lemma 4.10. *Let (ψ^Q, A) be a $Q-$neutrosophic soft quasigroup and $\Gamma = (U, V, W)$ be an autotopism of a finite quasigroup $(\hat{G}, \odot)$. Then*

(1) $(T_{U(\psi^Q)(a)}, T_{V(\psi^Q)(a)}, T_{W(\psi^Q)(a)}) \in AUT(\psi^Q, A)^{(\hat{G},\odot)}$ *for true membership.*
(2) $(I_{U(\psi^Q)(a)}, I_{V(\psi^Q)(a)}, I_{W(\psi^Q)(a)}) \in AUT(\psi^Q, A)^{(\hat{G},\odot)}$ *for unterminate membership.*
(3) $(F_{U(\psi^Q)(a)}, F_{V(\psi^Q)(a)}, F_{W(\psi^Q)(a)}) \in AUT(\psi^Q, A)^{(\hat{G},\odot)}$ *for unterminate membership.*

Proof. It follows from Definition 4.9. □

Theorem 4.11. *Let (ψ^Q, A) be a $Q-$neutrosophis soft quasigroup over a quasigroup $\hat{G}_1$. Let $T_1 = (\alpha_1, \beta_1, \gamma_1)$ and $T_2 = (\alpha_2, \beta_2, \gamma_2)$ be autotopism of $(\hat{G}, \odot)$ and $\rho : A \to A$. Then $(T_1 \circ T_2, \rho)$ is an autotopism of $(\psi^Q, A)^{(\hat{G},\odot)}$.*

Proof. Let Let $T_1 = (\alpha_1, \beta_1, \gamma_1)$ and $T_2 = (\alpha_2, \beta_2, \gamma_2)$ be isotopisms of $(\hat{G}, \odot)$ and $\rho : A \to A$. Then $(\alpha_1, \beta_1, \gamma_1) \circ (\alpha_2, \beta_2, \gamma_2) = (\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2)$ is also an isotopism of $(\hat{G}, \odot)$. To prof this Theorem, it is suffix to show that

$$T_{(\gamma_1 \circ \gamma_2)(\psi^Q)(a)}(x \odot y, q) = \min\{T_{(\alpha_1 \circ \alpha_2)(\psi^Q)(a)}(y, q), T_{(\beta_1 \circ \beta_2)(\psi^Q)(a)}(x, q)\}$$

Let $a \in \rho(E)$ the parametric set, for all $x, y \in \hat{G}$ and $q \in Q$.

$$\begin{aligned}
&\min\{T_{(\psi^Q)(a)}(\alpha_1 \circ \alpha_2)(x, q), T_{(\psi^Q)(a)}(\beta_1 \circ \beta_2)(y, q))\} \\
&\geq \max_{\rho(a)=a} \left[\min \left\{ T_{(\psi^Q)(a)}(\alpha_1 \circ \alpha_2)(x, q), T_{(\psi^Q)(a)(\beta_1 \circ \beta_2)}(y, q) \right\} \right] \\
&= \max_{\rho(a)=a} \left[\min \left\{ T_{\alpha_1(\psi^Q)(a)}[\alpha_2(x, q)], T_{\beta_1(\psi^Q)(a)}[\beta_2(y, q)] \right\} \right] \\
&= \max_{\rho(a)=a} \left[\min \left\{ T_{\alpha_1[\alpha_2(\psi^Q)(a)]}(x, q), T_{\beta_1[\beta_2(\psi^Q)(a)]}(y, q)] \right\} \right] \\
&= \min \left[\left\{ \max_{\rho(a)=a} \left[T_{\alpha_1[\alpha_2(\psi^Q)(a)]}(x, q) \right], \max_{\rho(a)=a} \left[T_{\beta_1[\beta_2(\psi^Q)(a)]}(y, q) \right] \right] \right\} \right].
\end{aligned} \tag{6}$$

Let $T_{\gamma_2(\psi^Q)(a)}(x \odot y, q)] = \min\{T_{\alpha_2(\psi^Q)(a)}(x, q), T_{\beta_2(\psi^Q)(a)}(x, q)\}$.

It follows from equation (6),

$$\begin{aligned}
&\min \left[\left\{ \max_{\rho(a)=a} \left[T_{\alpha_1[\alpha_2(\psi^Q)(a)]}(x, q) \right], \max_{\rho(a)=a} \left[T_{\beta_1[\beta_2(\psi^Q)(a)]}(y, q) \right] \right] \right\} \right] \\
&\qquad = \max_{\rho(a)=a} [T_{\gamma_1(\gamma_2(\psi^Q)}(a))(x \odot y, q)].
\end{aligned} \tag{7}$$

Similarly, it is shows for indeterminate and falsity membership degrees. □

Corollary 4.12. (1) *Let* (ψ^Q, A) *be a* $Q-$*neutrosophis soft quasigroup over a quasigroup* $\hat{G}_1$. *Let* $T = (\alpha, \beta, \gamma)$ *be an autotopism of* $(\hat{G}, \odot)$ *and* $\rho : A \to A$. *Then,*

$$(T_{\alpha^{-1}(\psi^Q)(a)}, T_{\beta^{-1}(\psi^Q)(a)}, T_{\gamma^{-1}\psi^Q)(a)})$$
$$(I_{\alpha^{-1}(\psi^Q)(a)}, I_{\beta^{-1}(\psi^Q)(a)}, I_{\gamma^{-1}(\psi^Q)(a)})$$
$$(F_{\alpha^{-1}(\psi^Q)(a)}, F_{\beta^{-1}(\psi^Q)(a)}, F_{\gamma^{-1}(\psi^Q)(a)})$$

are autotopisms of $(\psi^Q, A)^{(\hat{G}, \odot)}$.

Proof. It has been reported in [11] that an autotopism is an isotopism of a quasigroup $(\hat{G}, \odot)$ onto itself. In Theorem 4.4, it was shown that the isotopic image of $Q-$neutrosophic soft quasigroup is also a $Q-$neutrosophic soft quasigroup. So, if the isotopic image has the same operation with the domain under (T, ρ), then second part of Theorem 4.4 gives the desire result. □

Lemma 4.13. *If* $\psi_Q^*(b)$ *is a homomorphic image of* $Q-$ *neutrosophic soft quasigroup* $\psi^Q(a)$ *and* $\psi_Q^*(b) \subseteq \psi^Q(b)$ *is a* $Q-$ *neutrosophic soft quasigroup, then* $\psi_Q^*(b)$ *is a* $Q-$ *neutrosophic soft quasigroup.*

Proof. We established Theorem 4.8 that homomorphic image of a finite $Q-$ neutrosophic soft quasigroup is also a $Q-$ neutrosophic soft quasigroup. Since $\psi_Q^*(b) \leq \psi^Q(b)$ a $Q-$ neutrosophic soft quasigroup, then $\psi_Q^*(b)$ is a $Q-$ neutrosophic soft quasigroup □

Theorem 4.14. *Let* $\psi^Q(a)$ *and* $\psi^Q(b)$ *be* $Q-$ *neutrosophic soft quasigroups. Then,* $\psi^Q(a) \cong \pi_Q(c) \leq \psi^Q(a) \times \psi^Q(b)$ *if and only if there exist a homomorphism* $T : \psi^Q(a) \to \psi^Q(b)$.

Proof. Suppose that $\psi^Q(a) \cong \pi_Q(c) \leq \psi^Q(a) \times \psi^Q(b)$. Then, π is a homomorphism from $\pi_Q(c) \to \psi^Q(a)$ and since $\psi^Q(a) \cong \pi_Q(c)$ there exist a homomorphism $T : \psi^Q(a) \to \psi^Q(b)$.

Conversely, suppose that there exist a homomorphism $T : \psi^Q(a) \to \psi^Q(b)$, then $\psi^Q(a) \cong \{((x, q), T(\psi^Q(a)(x, q))| x \in \psi^Q(a), q \in Q\} \leq \psi^Q(a) \times \psi^Q(b)$ □

Corollary 4.15. $\phi_{Q(a)} \cong \pi_{Q(c)} \leq \phi_{Q(a)} \times \psi_{Q(b)}$ *where* $\phi_{Q(a)}$ *and* $\psi_{Q(b)}$ *are* $Q-$ *neutrosophic soft quasigroups if and only if there* $\psi_{Q(b)}$ *contain a* $Q-$ *neutrosophic soft subquasigroup that is a homomorphic image of* $\phi_{Q(a)}$.

Proof. Using Lemma 4.13 and Theorem 4.14. □

Definition 4.16. Let $\phi_{Q(a)}\times\psi_{Q(b)}$ be a direct product of two $Q-$ neutrosophic soft quasigroups $\phi_{Q(a)}$ and $\psi_{Q(b)}$. A $Q-$ neutrosophic soft subquasigroup $\pi_{Q(c)}$ is call a generalised diagonal subquasigroup if

$$\pi_{Q(c)} = \{(x,q), T(\phi_{Q(c)}(x,q))|x \in \phi_{Q(a)}, q \in Q\} \leq \phi_{Q(a)} \times \psi_{Q(b)}$$

where T is a homomorphism from $\phi_{Q(a)}$ to $\psi_{Q(b)}$.

Theorem 4.17. *$\phi_{Q(a)} \cong \pi_{Q(c)} \leq \phi_{Q(a)} \times \psi_{Q(b)}$ where $\phi_{Q(a)}$ and $\psi_{Q(b)}$ are $Q-$ neutrosophic soft quasigroups if and only if $\phi_{Q(a)} \times \psi_{Q(b)}$ contain a generalised diagonal subquasigroup.*

Proof. By Definition 4.16, $\pi_{Q(c)} \cong \phi_{Q(a)}$. If $\phi_{Q(a)} \cong \pi_{Q(c)} \leq \phi_{Q(a)} \times \psi_{Q(b)}$, then by Lemma 4.13 there exist a homomorphism $T : \phi_{Q(a)} \to \psi_{Q(b)}$. That is

$$\{(x,q), T(\phi_{Q(a)}(x,q))|x \in \phi_{Q(a)}, q \in Q\} \leq \phi_{Q(a)} \times \psi_{Q(b)}$$

is a generalised diagonal subquasigroup $\phi_{Q(a)} \times \psi_{Q(b)}$. □

5. An Application of $Q-$neutrosophic Soft Quasigroups

. In the works of Abu-Qumar et al. [9, 30], algorithms based on Q-neutrosophic soft sets were developed to address real-life problems. Their approach was formulated using arbitrary sets without relying on a binary operation. Additionally, Oyem et al. [25] in 2022 explored the application of soft quasigroups and created an algorithm focused on uniformity and equity. While this method effectively tackles certain real-world issues related to property distribution, it falls short in managing challenges involving indeterminate data. Specifically, it cannot ascertain the degree of membership functions concerning truth, falsity, and indeterminate data.

We propose a generalized method capable of addressing uncertainties and indeterminacies pertinent to two universal sets: Q-set (groupoid) and quasigroup, in relation to real-life scenarios. It is widely acknowledged that decisions in areas such as politics, health, marriage, careers, and stock market investments are crucial, particularly due to the presence of indeterminate factors. Notably, stock market investments are fraught with uncertainties and indeterminacies, as they occur within a virtual market where buyers and sellers trade existing securities. This marketplace, typically facilitated by a government body or institution, involves the trading of shares, stocks, debentures, bonds, futures, options, among others. Given the inherent uncertainties and indeterminacies in this virtual environment, it becomes essential for investors to employ rigorous and analytical mathematical approaches, such as neutrosophic soft quasigroups, to better navigate their choices and determine the most suitable stock exchange companies for investment.

Some results on finite $Q-$neutrosophic soft quasigroup was adopted as guides for setting up an algorithm to describe truth, falsity and indeterminate membership degree. For a finite $Q-$neutrosophic soft quasigroup $(\Psi^Q, \mathfrak{A})$ over $(\hat{G}, \odot)$, the following was adopted:

$$\Psi^Q(a) = \text{ Truth, indeterminate and falsity membership degrees}$$
$$\mathfrak{A} = \text{Parameters (Beneficiaries)}$$
$$Q = \text{ Company's policy}$$
$$\hat{G} = \text{Stock Exchange Companies}$$

Then, $T_{\Phi Q}, I_{\Phi Q}, F_{\Phi Q} : X \times Q \to [0,1]$ defines how the parameters sets $\mathfrak{A}$ will assign its choices to subset in $[0,1]$ such that condition $0 \leq T_{\Phi Q} + I_{\Phi Q} + F_{\Phi Q} \leq 3^+$ hold. Knowing that $\hat{G}$ has a structure (called quasigroup) which determines the interaction of choices of $Q-$neutrosophic soft set $\Psi^Q(a)$ for all $a \in \mathfrak{A}$. To determine the truth, falsity and indeterminate membership degrees, Definition 2.1 and Definition 2.2 are used as guide for constructing an algorithm below.

Step 1 Construct a qroupoid of finite order.

Step 2 Construct a quasigroup of finite order.

Step 3 Construct two $Q-$neutrosophic soft quasigroups $(\Phi^Q, \mathfrak{A})$ and $(\Psi^Q, \mathfrak{B})$ under a finite quasigroup $\hat{G}$.

Step 4 Use Definition 2.2 and Definition 2.1to compute $(\Pi, \mathfrak{C}) = (\Psi^Q, \mathfrak{A}) \odot (\Psi^Q, \mathfrak{B})$ such that $a_i \odot b_j = c_k$ where c_k is unique in $\hat{G}$.

Step 5 From 4, compute $\Pi_{Q(e_i \odot e_j)}$ for all $i \neq j$.

Step 6 Compute the comparison table using the formula: $T_{\Pi^Q(c)} + I_{\Pi^Q(c)} - F_{\Phi^Q(c)}$.

Step 7 Compute the sum of these numerical grades score $S_{(x,q)} \in (\hat{G} \times Q)$ of each object (x, q).

Step 8 Find highest numerical grades corresponding to every pair of parameters on the score table.

Step 9 The best decision is any one of the elements of $V = \max_{(x,q) \in \hat{G} \times Q}(a)(x, q)$.

Example 5.1. Suppose that a newly wedded couple $\mathfrak{A}$ and $\mathfrak{B}$ want to invest their money in stock exchange company. Let $\hat{G} = \{a, b, c, d, e, f, g, h\}$ be the set alternative companies (elements of quasigroup) and $Q = \{q_1, q_2, q_3, q_4, q_5\}$ the company's policies and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ set of parameters where $e_1 =$ trend in earning growth, $e_2 =$ company strength relative to its peers, $e_3 =$ effectiveness of execution leadership skill, $e_4 =$ debt-to-equity ratio in line with industry norms, $e_5 =$ long-term strength and stability and $e_6 =$ rate of dividend. If the couple has to invest according to their choice of parameters $\mathfrak{A} = \{e_1, e_2, e_5\}$ and $\mathfrak{B} = \{e_3, e_4, e_6\}$. The problem is the selection of best stock exchange company which satisfies their criteria. Below are choice of degrees for $(\Psi^Q, \mathfrak{A})$ and $(\Psi^Q, \mathfrak{B})$:

$$(\Psi^Q, \mathfrak{A}) = \begin{cases} \Phi^Q(e_1) &= \{[(a,q_1)0.9,0.7,0.4],[(b,q_2)0.8,0.5,0.7],[(e,q_3)0.9,0.7,0.2],\\ & \quad [(h,q_4)0.9,0.7,0.2]\} \\ \Phi^Q(e_2) &= \{[(a,q_2)0.3,0.5,0.2],[(b,q_3)0.9,0.6,0.4],[(f,q_4)0.1,0.5,0.7],\\ & \quad [(g,q_5)0.6,0.7,0.9]\} \\ \Phi^Q(e_5) &= \{[(b,q_1)0.9,0.6,0.3],[(e,q_3)0.6,0.3,0.7],[(f,q_4)0.5,0.4,0.9],\\ & \quad [(h,5_4)0.5,0.4,0.9]\} \end{cases}$$

$$(\Psi^Q, \mathfrak{B}) = \begin{cases} \Psi^Q(e_3) &= \{[(c,q_1)0.1,0.4,0.5],[(e,q_3)0.7,0.3,0.2],[(f,q_4)0.8,0.3,0.6],\\ & \quad [(g,q_5)0.8,0.3,0.6]]\} \\ \Psi^Q(e_4) &= \{[(a,q_2)0.1,0.6,0.3],[(b,q_3)0.1,0.3,0.5],[(c,q_4)0.7,0.1,0.9],\\ & \quad [(d,q_5)0.7,0.5,0.9]\} \\ \Psi^Q(e_6) &= \{[(c,q_1)0.3,0.4,0.6],[(d,q_2)0.9,0.1,0.6],[(f,q_4)0.6,0.2,0.3],\\ & \quad [(g,q_5)0.5,0.2,0.6]\} \end{cases}.$$

Suppose that the $Q-$NSS $(\Psi^Q, \mathfrak{A})$ describes the relations: "trend in earning growth", "company strength relative to its peers" and "long-term strength and stability" with some rate of dividends and $Q-$NSS $(\Psi^Q, \mathfrak{B})$ describes the relations: "effectiviness of execution leadership skill", "debt-to-equity ratio in line with industry norms" and "price-earning ratio as an indicator of valuation".

Next, compute step 2 to get a $Q-$neutrosophic soft quasigroup $(\Pi^Q, \mathfrak{C})$ such that $(\Psi^Q, \mathfrak{A}) \odot (\Psi^Q, \mathfrak{B}) = (\Pi^Q, \mathfrak{C})$ under a quasigroup of order 8. Then, $\Pi^Q(e_i \odot e_j) = \Psi^Q(e_i) \odot \Psi^Q(e_j)$, for all $e_i \in \mathfrak{A}, e_j \in \mathfrak{B}$ we have $\mathfrak{A} \odot \mathfrak{B} = \mathfrak{C}$

$$\Pi^Q(e_1 \odot e_3) = \Big\{ \langle [(c,q_1)0.1,0.7,0.5],[(e,q_3)0.7,0.7,0.4],[(f,q_4)0.8,0.7,0.6],[(g,q_5)0.9,0.7,0.6] \\ [(d,q_2)0.1,0.5,0.7],[(f,q_3)0.7,0.5,0.7],[(e,q_5)0.8,0.5,0.7],[(h,q_2)0.8,0.5,0.7] \\ [(g,q_3)0.1,0.7,0.4],[(a,q_4)0.9,0.7,0.2],[(b,q_2)0.8,0.7,0.7],[(c,q_1)0.8,0.7,0.6] \\ [(f,q_4)0.1,0.7,0.5],[(c,q_1)0.9,0.7,0.2],[(d,q_3)0.9,0.7,0.6],[(b,q_5)0.8,0.7,0.6]\rangle \Big\}$$

TABLE 3. Given a quasigroup $(\hat{G}) = \{a, b, c, d, e, f, g, h,\}$ of order 8 representing 8 companies with the method of relative operation of $(\hat{G}, \odot)$ shown in the Cayley table below such that $(ge)h \neq g(eh)$.

$\odot$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	d	c	f	e	h	g
c	c	d	a	b	h	g	e	f
d	d	c	b	a	g	h	f	e
e	e	f	g	h	a	b	c	d
f	f	e	h	g	b	a	d	c
g	g	h	e	f	d	c	a	b
h	h	g	f	e	c	d	b	a

TABLE 4. Given a groupoid $(Q) = \{q_1, q_2, q_3, q_4, q_5\}$ of order 5 representing 5 different policies with their mode of operations $(Q, +)$ is shown in Cayley table below.

$+$	q_1	q_2	q_3	q_4	q_5
q_1	q_1	q_2	q_3	q_4	q_5
q_2	q_2	q_1	q_3	q_5	q_4
q_3	q_3	q_5	q_4	q_2	q_1
q_4	q_4	q_2	q_1	q_3	q_5
q_5	q_5	q_2	q_3	q_1	q_4

$$\begin{aligned}\Pi^Q(e_1 \odot e_4) \quad = \quad & \Big\{\langle[(a, q_5)0.1, 0.7, 0.4], [(b, q_3)0.1, 0.7, 0.5], [(c, q_4)0.7, 0.7, 0.9], [(d, q_1)0.7, 0.7, 0.4] \\ & [(b, q_1)0.1, 0.6, 0.7], [(a, q_3)0.1, 0.5, 0.7], [(d, q_5)0.7, 0.5, 0.9], [(c, q_4)0.7, 0.5, 0.9] \\ & [(e, q_5)0.1, 0.7, 0.3], [(f, q_4)0.1, 0.7, 0.5], [(g, q_2)0.7, 0.7, 0.9], [(h, q_1)0.7, 0.7, 0.9] \\ & [(h, q_2)0.1, 0.7, 0.3], [(g, q_1)0.1, 0.7, 0.5], [(f, q_3)0.7, 0.7, 0.9], [(e, q_5)0.7, 0.7, 0.9]\rangle\Big\}\end{aligned}$$

$$\begin{aligned}\Pi^Q(e_1 \odot e_6) \quad = \quad & \Big\{\langle[(c, q_1)0.3, 0.7, 0.5], [(d, q_2)0.9, 0.7, 0.6], [(f, q_4)0.6, 0.7, 0.4], [(g, q_5)0.5, 0.7, 0.6] \\ & [(d, q_2)0.3, 0.5, 0.7], [(e, q_1)0.8, 0.5, 0.7], [(e, q_5)0.6, 0.5, 0.7], [(h, q_2)0.5, 0.5, 0.7] \\ & [(g, q_3)0.3, 0.7, 0.6], [(h, q_5)0.9, 0.7, 0.6], [(b, q_1)0.6, 0.7, 0.3], [(c, q_3)0.5, 0.7, 0.6] \\ & [(f, q_4)0.3, 0.7, 0.6], [(e, q_2)0.9, 0.7, 0.6], [(d, q_3)0.6, 0.7, 0.3], [(b, q_5)0.5, 0.7, 0.6]\rangle\Big\}\end{aligned}$$

$$\Pi^Q(e_2 \odot e_3) = \Big\{\langle[(c,q_2)0.1,0.5,0.5],[(e,q_3)0.3,0.5,0.2],[(f,q_4)0.8,0.7,0.6],[(g,q_5)0.9,0.7,0.6]$$
$$[(d,q_2)0.1,0.5,0.7],[(f,q_3)0.7,0.5,0.7],[(e,q_5)0.8,0.5,0.7],[(h,q_2)0.8,0.5,0.7]$$
$$[(g,q_3)0.1,0.7,0.4],[(a,q_4)0.9,0.7,0.2],[(b,q_2)0.8,0.7,0.7],[(c,q_1)0.8,0.7,0.6]$$
$$[(f,q_4)0.1,0.7,0.5],[(c,q_1)0.9,0.7,0.2],[(d,q_3)0.9,0.7,0.6],[(b,q_5)0.8,0.7,0.6]\rangle\Big\}$$

$$\Pi^Q(e_2 \odot e_4) = \Big\{\langle[(a,q_1)0.1,0.6,0.3],[(b,q_3)0.1,0.5,0.5],[(c,q_5)0.3,0.5,0.9],[(d,q_4)0.3,0.5,0.9]$$
$$[(b,q_5)0.1,0.6,0.4],[(a,q_4)0.1,0.3,0.5],[(d,q_2)0.7,0.6,0.9],[(c,q_1)0.7,0.6,0.9]$$
$$[(f,q_2)0.1,0.6,0.7],[(e,q_1)0.1,0.5,0.7],[(h,q_3)0.1,0.5,0.9],[(g,q_5)0.1,0.5,0.9]$$
$$[(g,q_2)0.1,0.7,0.9],[(h,q_1)0.1,0.7,0.9],[(e,q_3)0.6,0.7,0.9],[(f,q_5)0.6,0.7,0.9]\rangle\Big\}$$

$$\Pi^Q(e_1 \odot e_6) = \Big\{\langle[(c,q_1)0.3,0.5,0.6],[(d,q_2)0.3,0.5,0.6],[(f,q_5)0.3,0.5,0.3],[(g,q_4)0.3,0.5,0.6]$$
$$[(d,q_1)0.3,0.6,0.6],[(c,q_5)0.9,0.6,0.6],[(e,q_2)0.6,0.6,0.4],[(h,q_1)0.5,0.6,0.6]$$
$$[(h,q_1)0.1,0.5,0.7],[(g,q_2)0.1,0.5,0.7],[a,q_3)0.1,0.5,0.7],[(d,q_5)0.1,0.5,0.7]$$
$$[e,q_5)0.3,0.7,0.9],[(f,q_2)0.6,0.7,0.9],[(c,q_1)0.6,0.7,0.9],[(a,q_4)0.5,0.7,0.9]\rangle\Big\}$$

$$\Pi^Q(e_5 \odot e_3) = \Big\{\langle[(d,q_1)0.1,0.4,0.3],[(f,q_3)0.7,0.6,0.3],[(e,q_4)0.8,0.6,0.6],[(h,q_5)0.8,0.5,0.6]$$
$$[(g,q_3)0.1,0.4,0.7],[(a,q_4)0.6,0.3,0.7],[(b,q_2)0.6,0.3,0.7],[(c,q_4)0.6,0.3,0.7]$$
$$[(h,q_4)0.1,0.4,0.9],[(b,q_1)0.5,0.4,0.9],[a,q_3)0.5,0.9,0.4],[(c,q_5)0.5,0.4,0.9]$$
$$[f,q_5)0.1,0.4,0.9],[(c,q_3)0.5,0.4,0.9],[(d,q_1)0.5,0.4,0.9],[(b,q_4)0.5,0.4,0.9]\rangle\Big\}$$

$$\Pi^Q(e_5 \odot e_4) = \Big\{\langle[(b,q_2)0.1,0.6,0.6],[(a,q_3)0.1,0.6,0.5],[(d,q_4)0.7,0.6,0.9],[(c,q_5)0.8,0.6,0.6]$$
$$[(e,q_5)0.1,0.6,0.7],[(f,q_4)0.1,0.3,0.7],[(g,q_2)0.6,0.3,0.9],[(h,q_1)0.6,0.5,0.9]$$
$$[(a,q_2)0.1,0.6,0.9],[(e,q_1)0.1,0.4,0.9],[h,q_3)0.5,0.4,0.9],[(g,q_5)0.5,0.5,0.9]$$
$$[h,q_2)0.1,0.6,0.9],[(g,q_1)0.1,0.4,0.9],[(f,q_3)0.5,0.4,0.9],[(e,q_5)0.5,0.5,0.9]\rangle\Big\}$$

$$\Pi^Q(e_5 \odot e_6) = \Big\{\langle[(d,q_3)0.3,0.6,0.6],[(c,q_2)0.9,0.6,0.9],[(e,q_4)0.6,0.6,0.3],[(h,q_5)0.5,0.6,0.6]$$
$$[(g,q_3)0.3,0.4,0.7],[(h,q_5)0.6,0.3,0.7],[(b,q_2)0.6,0.3,0.3],[(c,q_1)0.5,0.3,0.7]$$
$$[(h,q_4)0.3,0.4,0.9],[(g,q_2)0.3,0.4,0.9],[a,q_3)0.5,0.4,0.9],[(d,q_5)0.5,0.4,0.9]$$
$$[f,q_5)0.3,0.4,0.9],[(e,q_2)0.5,0.4,0.9],[(d,q_1)0.5,0.4,0.9],[(b,q_5)0.5,0.5,0.9]\rangle\Big\}$$

Nest, We compute the comparison table to obtain the highest numerical grade for each column.

TABLE 5. Compute $T_{\Pi_Q}(c)) + I_{\Pi_Q}(c) - F_{\Pi_Q}(c)$ for company a

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(a, q_1)	-	-	-	-	0.4	-	-	-	-
(a, q_2)	-	-	-	-	-	-	-	-0.2	-
(a, q_3)	-	-	-	-	-	-0.1	1.0	0.2	0.0
(a, q_4)	1.4	-0.1	-	1.4	-0.1	0.3	0.2	-	-
(a, q_5)	-	-	-	-	-	-	-	-	-

TABLE 6. Compute $T_{\Pi_Q}(c)) + I_{\Pi_Q}(c) - F_{\Pi_Q}(c)$ for company b

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(b, q_1)	-	0.0	1.0	-	-	-	0.0	-	-
(b, q_2)	8.0	-	-	8.0	-	-	0.2	0.1	0.6
(b, q_3)	-	-	-	-	0.1	-	-	-	-
(b, q_4)	-	-	-	-	-	-	0.0	-	-
(b, q_5)	0.9	-	0.6	0.9	0.3	-	-	-	0.1

TABLE 7. Compute $T_{\Pi Q}(c)) + I_{\Pi Q}(c) - F_{\Pi Q}(c)$ for company c

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(c, q_1)	0.3,0.9,1.4	0.5	0.5	0.9	0.4	0.2, 0.4	-	-	0.1
(c, q_2)	-	-	-	0.1	-	-	-	-	0.6
(c, q_3)	-	-	0.6	1.4	-	-	0.0	-	-
(c, q_4)	-	0.3, 0.5	-	-	-	-	0.2	-	-
(c, q_5)	-	-	-	-	-0.1	0.9	0.0	0.8	-

TABLE 8. Compute $T_{\Pi Q}(c)) + I_{\Pi Q}(c) - F_{\Pi Q}(c)$ for company d

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(d, q_1)	-	1.0	-	-	-	0.3	0.0. 0.2	-	0.0
(d, q_2)	-0.1	1.0	1.0	-0.1	0.4	0.2	-	-	-
(d, q_3)	0.1 -	0.1, 1.0	1.0	-	-	-	-	-	0.3
(d, q_4)	-	-	-	-	-0.1	-	-	0.4	-
(d, q_5)	-	0.3	-	-	-	-0.1	-	-	0.0

TABLE 9. Compute $T_{\Pi Q}(c)) + I_{\Pi Q}(c) - F_{\Pi Q}(c)$ for company e

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(e, q_1)	-	-	0.6	-	-0.1	-	-	-0.4	0.0
(e, q_2)	-	-	1.0	-	-	0.8	-	-	0.0
(e, q_3)	1.0	-	-	0.3	-	-	-	-	-
(e, q_4)	-	-	-	-	-	-	0.8	-	0.8
(e, q_5)	0.6	0.5	0.4	0.6	0.5	0.1	-	-	0.1,0.0

TABLE 10. Compute $T_{\Pi Q}(c)) + I_{\Pi Q}(c) - F_{\Pi Q}(c)$ for company f

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(f, q_1)	-	-	0.6	-	-	-	-	-	-
(f, q_2)	-	-	-	-	0.0	0.4	-	-	-
(f, q_3)	0.5	-	-	0.5	-	-	0.4	0.0	-
(f, q_4)	0.9,0.3	0.3	0.9, 0.4	0.9, 0.3	-	-	-	-	-0.3
(f, q_5)	-	-	-	-	0.5	0.5	-0.4	0.1	-0.2

TABLE 11. Compute $T_{\Pi Q}(c)) + I_{\Pi Q}(c) - F_{\Pi Q}(c)$ for company g

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(g, q_1)	-	-	-	-	-	-	-	-0.7	-
(g, q_2)	-	0.5	-	-	-0.1	-0.1	-	0.0	-0.2
(g, q_3)	0.4	-	0.4	0.4	-	-	0.2	-	0.0
(g, q_4)	-	-	-	-	-	0.2	-	-	-
(g, q_5)	1.0	-	0.5	1.0	-0.3	-	-	-	-

TABLE 12. Compute $T_{\Pi Q}(c)) + I_{\Pi Q}(c) - F_{\Pi Q}(c)$ for company h

$(\hat{G} \times Q)$	(e_1e_3)	(e_1e_4)	(e_1e_6)	(e_2e_3)	(e_2e_4)	(e_2e_6)	(e_5e_3)	(e_5e_4)	(e_5e_6)
(h, q_1)	-	0.5	-	-	-0.1	0.5, -0.1	-	0.2	-
(h, q_2)	0.6	-	0.3	0.6	-	-	-	-0.3	-
(h, q_3)	-	-	-	-	-0.3	-	0.0	-	
(h, q_4)	-	-	-	-	-	-	0.4	-	-0.2
(h, q_5)	-	-	1.0	-	-	-	0.7	-	-0.5, 0.2

Compute the sum of these numerical grades score $S_{(x,q)} \in (\hat{G} \times Q)$ of each object $\Pi_{Q(e_i \odot e_j)}(x, q_k)$ for all $k = 1, ..., 5$).

TABLE 13. Compute $\sum_{\Pi Q_{(e_i \odot e_j)}}(x, q_k)$ for all k=1,..., 5

$\hat{G} \times Q$	Total	$\hat{G} \times Q$	Total	$\hat{G} \times Q$	Total	$\hat{G} \times Q$	Total	$\hat{G} \times Q$	Total
(a, q_1)	0.4	(b, q_1)	1.0	(c, q_1)	7.6	(d, q_1)	1.5	(e, q_1)	0.1
(a, q_2)	-0.2	(b, q_2)	2.5	(c, q_2)	o.5	(d, q_2)	1.4	(e, q_2)	1.8
(a, q_3)	1.1	(b, q_3)	0.1	(c, q_3)	2.0	(d, q_3)	2.5	(e, q_3)	1.3
(a, q_4)	3.1	(b, q_4)	0.0	(c, q_4)	1.0	(d, q_4)	0.3	(e, q_4)	1.6
(a, q_5)	-	(b, q_5)	2.6	(c, q_5)	1.6	(d, q_5)	0.2	(e, q_5)	2.8

$\hat{G} \times Q$	Total	$\hat{G} \times Q$	Total	$\hat{G} \times Q$	Total
(f, q_1)	0.6	(g, q_1)	-0.7	(h, q_1)	1.0
(f, q_2)	0.4	(g, q_2)	0.1	(h, q_2)	1.2
(f, q_3)	1.4	(g, q_3)	1.4	(h, q_3)	-0.3
(f, q_4)	3.7	(g, q_4)	0.2	(h, q_4)	0.4
(f, q_5)	0.5	(g, q_5)	2.2	(h, q_5)	0.4

Now, selecting the highest numerical value from company "a, b, c, d, e, f, g, h" with their respective policy

(1) $(a, q_4) = 3.1 > (a, q_3) = 1.1 > (a, q_1) = 0.4 > (a, q_2) = -0.2 > (a, q_5) - invalid$
(2) $(b, q_5) = 2.6 > (b, q_2) = 2.5 > (b, q_1) = 1.0 > (b, q_3) = 0.1 > (b, q_4) = 0.0$
(3) $(c, q_1) = 7.6 > (c, q_3) = 2.0 > (c, q_5) = 1.6 > (c, q_4) = 1.0 > (c, q_2) = 0.5$
(4) $(d, q_3) = 2.5 > (d, q_1) = 1.5 > (d, q_2) = 1.4 > (d, q_4) = 0.3 > (d, q_5) = 0.2$
(5) $(e, q_5) = 2.8 > (e, q_2) = 1.8 > (e, q_4) = 1.6 > (e, q_3) = 1.3 > (e, q_1) = 0.1$
(6) $(f, q_4) = 3.7 > (f, q_3) = 1.4 > (f, q_1) = 0.6 > (f, q_5) = 0.5 > (f, q_2) = 0.4$
(7) $(g, q_5) = 2.2 > (g, q_3) = 1.4 > (g, q_2) = 0.1 > (g, q_4) = 0.2 > (g, q_1) = -0.7$
(8) $(h, q_2) = 1.2 > (h, q_1) = 1.0 > (h, q_4) = 0.4 = (h, q_5) = 0.4 > (h, q_3) = -0.3$

Select the highest numerical value from each company with their respectively policies q_k for $k = 1, ..., 5$ in table 13
$(c, q_1) = 7.6 > (f, q_4) = 3.7 > (a, q_3) = 3.1 > (b, q_5) = 2.6 > (d, q_3) = 2.5 = (e, q_5) > (g, q_5) = 2.2 > (h, q_2) = 1.2$. Hence, company 'c' with the policy q_1 is the best to invest in.

6. Discussion

In contrast to previous findings on $Q-$neutrosophic soft sets, this work does not assume the associative law, thereby providing a new perspective. It has been demonstrated that the exploration of $Q-$neutrosophic soft quasigroups offers a more comprehensive understanding of the properties of $Q-$neutrosophic soft groups, compared to existing generalizations in $Q-$neutrosophic soft sets. This research also showcases the superiority of non-associativity as a better generalization in comparison to its counterpart in the existing literature. It has been established that the discoveries in $Q-$NS group theory can be extended to $Q-$NS quasigroups, thus paving the way for further research in this area. These findings not only expand the scope of $Q-$NS sets theory and classical quasigroup theory but also contribute to the development of a new theoretical framework. Overall, this work has opened up new avenues of exploration and is a significant improvement upon existing.

7. Conclusions

This paper explores the concepts of isotopism, homomorphism, and isomorphism within the framework of $Q-$neutrosophic soft quasigroups, along with an examination of the direct product of these quasigroups. It addresses two relevant questions and provides appropriate solutions. An extension of the $Q-$set to a groupoid was employed to develop an algorithm for decision-making. However, it is important to note that the characteristics of the homomorphic image and preimage of a $Q-$neutrosophic soft quasigroup do not necessarily form a $Q-$neutrosophic soft quasigroup. This paper outlines the necessary and sufficient conditions for this situation and tackles a specific issue regarding whether the direct product of two $Q-$neutrosophic soft quasigroups contains a $Q-$neutrosophic soft subquasigroup isomorphic to (ψ^Q, A).

Future research may delve into the properties of isotopy and homomorphism in quasigroups, with the potential to extend these concepts to $Q-$neutrosophic soft n-ary quasigroups, thereby generalizing the notion of $Q-$neutrosophic soft quasigroups. Additionally, it may involve constructing an algorithm using the hybrid model of $Q-$neutrosophic soft n-ary quasigroups for decision-making purposes.

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