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Neutrosophic Linguistic Rhotrix in Managerial Decision Making

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Abstract: The developments in rhotrix theory and their applications are unveiled at recent times. This research work proposes the notion of linguistic rhotrix conjoining the theories of linguistic matrix and rhotrix. The theory of linguistic rhotrix comprising fundamental definitions, operators and classifications is presented in this study. The algorithmic framework of applying linguistic rhotrix to decisioning problem is presented in this work. A managerial decision-making illustration is putforth to validate the proposed decisioning framework based on neutrosophic quantifications of linguistic rhotrix. The results obtained are compared with fuzzy and intuitionistic fuzzy quantifications to demonstrate the efficacy of neutrosophic quantifications of linguistic variables. The results obtained using neutrosophic quantifications are more optimal and reliable in comparisons with the other two quantifications. The proposed decisioning approach based on neutrosophic linguistic rhotrix is more robust and competent in handling intricate decisioning environment.

Keywords: Linguistic Rhotrix, Neutrosophy Linguistic Rhotrix, Decisioning

1. Introduction

Rhotrix is basically termed as rhombus matrix comprising the rhomboidal arrangements of the elements. Ajibade [1] conceptualized the novel theory of rhotrix and researchers are attributing to both the theoretical developments and practical implications of this evolving concept. The properties and theoretical aspects of rhotrix are more in align with matrix theory and some of the significant contributions in this context are discoursed. Sani [2-3] introduced an alternate method for rhotrix multiplication and generalized the row-column multiplication for higher order matrices contributing to the development of rhotrix algebra. Absalom et al [4] discussed the concept of heart-oriented multiplication of rhotrices and its generalization. Mohammed and Sani [5] described the graphical representation of rhotrices as rhomtrees. Mohammed and Balarade [6] presented a revie of rhotrix theory highlighting its inceptions, theoretical concepts and applications. Sharma and Kumar [7] explained the applications of Hadamard rhotrices in designing balanced incomplete block. Usaini and Mohammed [8] focused on rhotrix based eigenvalues and eigen vectors which is more associated with matrices. Mohammed [9] introduced a new expression of rhotrix. Sharma et al [10] discussed the construction of Hadamard codes using Hadamard rhotrices which is a transformation form of Hadamard matrix. The hill cipher is basically a polygraphic substitution cipher based on linear algebra and Kumar [11] extended with rhotrices. Sharma et al [12] discussed a special type of vandermonde rhotrix contributing to algebraic studies. Kaurangini and Aminu [13] discussed Hermitian and skew -Hermitian rhotrices stating the explicit transformation of matrix theory. Anuradha et al [14] applied the theory of rhotrix in solving a coupled linear programming problem.

Isere [15] discussed the natural rhotrix and Mohammed attributed to non-commutative full rhotrix ring. Kavitha et al [16] developed a prediction model for weather forecasting integrating rhotrix optimization based genetic algorithm with interval valued rhotrix representations. Verma et al [17] explored the applications of rhotrix in the construction of balance incomplete block design. Ananda Priya et al [18] contributed to the classification of commutative general rhotrix group. Aminu [19] introduced the theory of fuzzy rhotrix with the integration of fuzzy representations. Ananda Priya et al [20] developed a diagnostic decision model based on neutrosophic based rhotrices. The above contributions present a vivid picture of developments and applications of rhotrix theory based on matrix and its properties. Similar to the aforementioned contributions, this research work extends the theory of linguistic matrix to linguistic rhotrix. Smarandache et al [21] conceptualized the theory of linguistic matrices highlighting the significance of its applications in determining optimal solutions to the realistic problems. Inspired by the works of Smarandache with special reference to linguistic matrix and neutrosophy, this research work contributes to the theoretical development of linguistic rhotrix and extends the same to neutrosophic rhotrix to develop a comprehensive decision-making model. At recent times, the researchers are evolving neutrosophic based decision making models to cater to the dynamic demands of the managerial decision makers. Neutrosophic theory is integrated with multi-criteria decision-making methods [22-23], soft sets[24-25], hypersoft sets [26-27], matrices [28], rough sets [29], graph [30] and algebraic structures [31-32] to formulate robust and reliable decisioning models. The neutrosophic blended decision models are gaining more momentum as the neutrosophic representations facilitate in handling intricate data sets. This has also motivated the authors to exclusively explore the applications of neutrosophic linguistic matrix

The remaining contents of the paper are described as follows, section 2 discusses the theoretical developments of linguistic rhotrix comprising the conceptualization of linguistic rhotrix, operators and classifications of linguistic rhotrix. Section 3 explains the algorithmic framework of linguistic rhotrix applied in decisioning process. Section 4 presents an illustration to the proposed algorithm framework with neutrosophic representations. Section 5 compares the same with both fuzzy and intuitionistic fuzzy based representations. Section 6 concludes the work.

2 Conceptualization of Linguistic Rhotrix

This section explains the theoretical developments of linguistic rhotrix based on the contributions of Smarandache to the theory of linguistic matrix and based on the contributions of Ajibade [1] to the theory of rhotrix. The concept of linguistic matrices and the operators applied are extended to rhotrix theory facilitating the theoretical developments of linguistic rhotrix.

Rhotrix is basically characterized as the rhomboidal arrangement of matrix elements. It is also termed as mathematical arrays comprising square matrix of two and three dimensions. The smallest rhotrix R of order 3 can be defined as.

$$\boldsymbol{R} = \left\langle \begin{matrix} & \mathbf{a} & \\ \mathbf{b} & \mathbf{c} & \mathbf{d} \\ & \mathbf{e} & \end{matrix} \right\rangle \quad \text{where a, b, c, d and e} \in \boldsymbol{R}$$

2.1 Linguistic Rhotrix

Linguistic rhotrix is a kind of rhotrix in which each entry are linguistic variables. The general form of linguistic rhotrix is

$$\mathbf{R_L} = \left\langle \begin{array}{ccccc} & & \mathbf{r_{L11}} & & \\ & \mathbf{r_{L21}} & \mathbf{r_{L11}} & \mathbf{r_{L12}} & \\ \mathbf{r_{L31}} & \mathbf{r_{L21}} & \mathbf{r_{L22}} & \mathbf{r_{L12}} & \mathbf{r_{L13}} \\ & \mathbf{r_{L32}} & \mathbf{r_{L21}} & \mathbf{r_{L23}} & \\ & & \mathbf{r_{L33}} & & \end{array} \right\rangle$$

where $\mathbf{r_{Lij}}$ is a linguistic variable.

2.2 Operators on Linguistic Matrix

The operators of MAX and MIN are applied to a linguistic rhotrix. Let $\mathbf{R_L}$ and $\mathbf{S_L}$ be two linguistic matrices with entries as linguistic variables.

2.2.1 MAX Operator

$$\mathbf{R_L} = \left\langle \begin{array}{ccccc} & & \mathbf{r_{L11}} & & \\ & \mathbf{r_{L21}} & \mathbf{r_{L11}} & \mathbf{r_{L12}} & \\ \mathbf{r_{L31}} & \mathbf{r_{L21}} & \mathbf{r_{L22}} & \mathbf{r_{L12}} & \mathbf{r_{L13}} \\ & \mathbf{r_{L32}} & \mathbf{r_{L22}} & \mathbf{r_{L23}} & \\ & & \mathbf{r_{L33}} & & \end{array} \right\rangle \text{ and } \mathbf{S_L} = \left\langle \begin{array}{ccccc} & & \mathbf{s_{L11}} & & \\ & \mathbf{s_{L21}} & \mathbf{s_{L11}} & \mathbf{s_{L12}} & \\ \mathbf{s_{L31}} & \mathbf{s_{L21}} & \mathbf{s_{L22}} & \mathbf{s_{L12}} & \mathbf{s_{L13}} \\ & \mathbf{s_{L32}} & \mathbf{s_{L22}} & \mathbf{s_{L23}} & \\ & & \mathbf{s_{L33}} & & \end{array} \right\rangle$$

MAX(RL,SL)=

$$\left\langle \begin{array}{ccccc} & & \mathbf{max}(\mathbf{r_{L11}}, s_{L11}) & & \\ & \mathbf{max}(\mathbf{r_{L21}}, s_{L21}) & \mathbf{max}(\mathbf{r_{L11}}, s_{L11}) & \mathbf{max}(\mathbf{r_{L12}}, s_{L12}) & \\ \mathbf{max}(\mathbf{r_{L31}}, s_{L31}) & \mathbf{max}(\mathbf{r_{L21}}, s_{L21}) & \mathbf{max}(\mathbf{r_{L22}}, s_{L22}) & \mathbf{max}(\mathbf{r_{L12}}, s_{L12}) & \mathbf{max}(\mathbf{r_{L13}}, s_{L13}) \\ & \mathbf{max}(\mathbf{r_{L32}}, s_{L32}) & \mathbf{max}(\mathbf{r_{L22}}, s_{L22}) & \mathbf{max}(\mathbf{r_{L23}}, s_{L23}) & \\ & & \mathbf{max}(\mathbf{r_{L33}}, s_{L33}) & & \end{array} \right\rangle$$

Example

Let us consider two linguistic rhotrix with entries as linguistic variables from the set {Low, Medium, High}

$$\mathbf{R_L} = \left\langle \begin{array}{ccccc} & & \mathbf{r_{L11}} & & \\ & \mathbf{r_{L21}} & \mathbf{r_{L11}} & \mathbf{r_{L12}} & \\ \mathbf{r_{L31}} & \mathbf{r_{L21}} & \mathbf{r_{L22}} & \mathbf{r_{L12}} & \mathbf{r_{L13}} \\ & \mathbf{r_{L32}} & \mathbf{r_{L22}} & \mathbf{r_{L23}} & \\ & & \mathbf{r_{L33}} & & \end{array} \right\rangle \text{ and } \mathbf{S_L} = \left\langle \begin{array}{ccccc} & & \mathbf{s_{L11}} & & \\ & \mathbf{s_{L21}} & \mathbf{s_{L11}} & \mathbf{s_{L12}} & \\ \mathbf{s_{L31}} & \mathbf{s_{L21}} & \mathbf{s_{L22}} & \mathbf{s_{L12}} & \mathbf{s_{L13}} \\ & \mathbf{s_{L32}} & \mathbf{s_{L22}} & \mathbf{s_{L23}} & \\ & & \mathbf{s_{L33}} & & \end{array} \right\rangle$$

$$\text{MAX(RL,SL)=} \left\langle \begin{array}{ccccc} & & H & & \\ & \mathbf{H} & \mathbf{M} & \mathbf{H} & \\ \mathbf{H} & \mathbf{M} & \mathbf{H} & \mathbf{M} & \mathbf{M} \\ & \mathbf{M} & \mathbf{H} & \mathbf{H} & \\ & & \mathbf{H} & & \end{array} \right\rangle$$

2.1.2 MIN Operator

MIN(RL,SL)=

$$\left\langle \begin{array}{ccccc} & & \mathbf{min}(\mathbf{r_{L11}}, s_{L11}) & & \\ & \mathbf{min}(\mathbf{r_{L21}}, s_{L21}) & \mathbf{min}(\mathbf{r_{L11}}, s_{L11}) & \mathbf{min}(\mathbf{r_{L12}}, s_{L12}) & \\ \mathbf{min}(\mathbf{r_{L31}}, s_{L31}) & \mathbf{min}(\mathbf{r_{L21}}, s_{L21}) & \mathbf{min}(\mathbf{r_{L22}}, s_{L22}) & \mathbf{min}(\mathbf{r_{L12}}, s_{L12}) & \mathbf{min}(\mathbf{r_{L13}}, s_{L13}) \\ & \mathbf{min}(\mathbf{r_{L32}}, s_{L32}) & \mathbf{min}(\mathbf{r_{L22}}, s_{L22}) & \mathbf{min}(\mathbf{r_{L23}}, s_{L23}) & \\ & & \mathbf{min}(\mathbf{r_{L33}}, s_{L33}) & & \end{array} \right\rangle$$

Example

Let us consider two linguistic rhotrix with entries as linguistic variables from the set {Low, Medium, High}

$$\mathbf{R_L} = \left\langle \begin{array}{ccccc} & & \mathbf{H} & & \\ & \mathbf{H} & \mathbf{M} & \mathbf{M} & \\ \mathbf{M} & \mathbf{H} & \mathbf{L} & \mathbf{L} & \mathbf{H} \\ & \mathbf{L} & \mathbf{H} & \mathbf{M} & \\ & & \mathbf{M} & & \end{array} \right\rangle \quad \text{and} \quad \mathbf{S_L} = \left\langle \begin{array}{ccccc} & & \mathbf{M} & & \\ & \mathbf{M} & \mathbf{H} & \mathbf{L} & \\ \mathbf{L} & \mathbf{M} & \mathbf{M} & \mathbf{H} & \mathbf{L} \\ & \mathbf{M} & \mathbf{H} & \mathbf{L} & \\ & & \mathbf{H} & & \end{array} \right\rangle$$

$$\text{MIN } (R_L,S_L) = \left\langle \begin{array}{ccccc} & & \mathbf{M} & & \\ & \mathbf{M} & \mathbf{M} & \mathbf{L} & \\ \mathbf{L} & \mathbf{M} & \mathbf{L} & \mathbf{L} & \mathbf{L} \\ & \mathbf{L} & \mathbf{H} & \mathbf{L} & \\ & & \mathbf{M} & & \end{array} \right\rangle$$

2.3 Classification of Linguistic Rhotrix

The Linguistic rhotrix is classified based on both the representations of linguistic variable and quantifications of the linguistic variable.

2.3.1 Classification based on Representations of Linguistic Variable

The linguistic rhotrix shall be classified primarily into three types based on the representations of linguistic variable.

(i) Fuzzy Linguistic Rhotrix

A linguistic rhotrix is said to be fuzzy linguistic rhotrix if the linguistic variable is represented in fuzzy forms.

$$\mathbf{R}_{\mathbf{L}_{RF}} = \left\langle \begin{array}{ccccc} & & \mathbf{r}_{\mathbf{L}_F 11} & & \\ & \mathbf{r}_{\mathbf{L}_F 21} & \mathbf{r}_{\mathbf{L}_F 11} & \mathbf{r}_{\mathbf{L}_F 12} & \\ \mathbf{r}_{\mathbf{L}_F 31} & \mathbf{r}_{\mathbf{L}_F 21} & \mathbf{r}_{\mathbf{L}_F 22} & \mathbf{r}_{\mathbf{L}_F 12} & \mathbf{r}_{\mathbf{L}_F 13} \\ & \mathbf{r}_{\mathbf{L}_F 32} & \mathbf{r}_{\mathbf{L}_F 22} & \mathbf{r}_{\mathbf{L}_F 23} & \\ & & \mathbf{r}_{\mathbf{L}_F 33} & & \end{array} \right\rangle$$

Example:

$$\mathbf{R}_{\mathbf{L}_{RF}} = \left\langle \begin{array}{ccccc} & & \boldsymbol{VH} & & \\ & \mathbf{H} & \mathbf{L} & \mathbf{VL} & \\ \mathbf{VH} & \mathbf{VL} & \mathbf{M} & \mathbf{L} & \mathbf{VL} \\ & \mathbf{L} & \mathbf{H} & \mathbf{VH} & \\ & & \mathbf{VL} & & \end{array} \right\rangle$$

(ii) Intuitionistic Linguistic Rhotrix

A linguistic rhotrix is said to be intuitionistic linguistic rhotrix if the linguistic variable is represented in intuitionistic forms.

$$\mathbf{R}_{\mathbf{L}_{RIF}} = \left\langle \begin{array}{ccccc} & & \mathbf{r_{LIF11}} & & \\ & \mathbf{r_{LIF21}} & \mathbf{r_{LIF11}} & \mathbf{r_{LIF12}} & \\ \mathbf{r_{LIF31}} & \mathbf{r_{LIF21}} & \mathbf{r_{LIF22}} & \mathbf{r_{LIF12}} & \mathbf{r_{LIF13}} \\ & \mathbf{r_{LIF32}} & \mathbf{r_{LIF22}} & \mathbf{r_{LIF23}} & \\ & & \mathbf{r_{LIF33}} & & \end{array} \right\rangle$$

Example:

$$\mathbf{R}_{\mathbf{L}_{RIF}} = \left\langle \begin{array}{ccccc} & & (\mathbf{VH}, \mathbf{VL}) & & \\ & (\mathbf{H}, \mathbf{VL}) & (\mathbf{H}, \mathbf{L}) & (\mathbf{VH}, \mathbf{VH}) & \\ (\mathbf{L}, \mathbf{H}) & (\mathbf{VH}, \mathbf{L}) & (\mathbf{VH}, \mathbf{VH}) & (\mathbf{VL}, \mathbf{VL}) & (\mathbf{VH}, \mathbf{VL}) \\ & (\mathbf{VL}, \mathbf{VL}) & (\mathbf{M}, \mathbf{VL}) & (\mathbf{H}, \mathbf{M}) & \\ & & (\mathbf{VL}, \mathbf{VH}) & & \end{array} \right\rangle$$

(iii) Neutrosophic Linguistic Rhotrix

A linguistic rhotrix is said to be neutrosophic linguistic rhotrix if the linguistic variable is represented in neutrosophic forms.

$$\mathbf{R}_{\mathbf{L}_{RNF}} = \left\langle \begin{array}{ccccc} & & \mathbf{r_{LNF11}} & & \\ & \mathbf{r_{LNF21}} & \mathbf{r_{LNF11}} & \mathbf{r_{LNF12}} & \\ \mathbf{r_{LNF31}} & \mathbf{r_{LNF21}} & \mathbf{r_{LNF22}} & \mathbf{r_{LNF12}} & \mathbf{r_{LNF13}} \\ & \mathbf{r_{LNF32}} & \mathbf{r_{LNF22}} & \mathbf{r_{LNF23}} & \\ & & \mathbf{r_{LNF33}} & & \end{array} \right\rangle$$

Example:

$$\mathbf{R}_{\mathbf{L}_{RNF}} = \left\langle \begin{array}{ccccc} & & (\mathbf{VH}, \mathbf{VL}, \mathbf{VL}) & & \\ & (\mathbf{VH}, \mathbf{L}, \mathbf{M}) & (\mathbf{M}, \mathbf{H}, \mathbf{VL}) & (\mathbf{VH}, \mathbf{VL}, \mathbf{L}) & \\ (\mathbf{VH}, \mathbf{VH}, \mathbf{L}) & (\mathbf{L}, \mathbf{M}, \mathbf{VL}) & (\mathbf{H}, \mathbf{VL}, \mathbf{L}) & (\mathbf{VH}, \mathbf{M}, \mathbf{VL}) & (\mathbf{H}, \mathbf{M}, \mathbf{VL}) \\ & (\mathbf{L}, \mathbf{VH}, \mathbf{VH}) & (\mathbf{H}, \mathbf{VL}, \mathbf{M}) & (\mathbf{VL}, \mathbf{H}, \mathbf{VH}) & \\ & & (\mathbf{VH}, \mathbf{L}, \mathbf{M}) & & \end{array} \right\rangle$$

2.3.2 Classification based on Quantifications of Linguistic Variable

The linguistic rhotrix shall be classified based on the quantifications of linguistic variable. If the linguistic variable is quantified using fuzzy sets, it shall be also referred as fuzzy linguistic rhotrix. Similarly based on the quantifications of linguistic variable, a linguistic rhotrix shall be categorized respectively.

Let us consider a linguistic rhotrix

$$\mathbf{R}_{\mathbf{L}} = \left\langle \begin{array}{ccccc} & & \mathbf{VH} & & \\ & \mathbf{H} & \mathbf{L} & \mathbf{VL} & \\ \mathbf{VH} & \mathbf{VL} & \mathbf{M} & \mathbf{L} & \mathbf{VL} \\ & \mathbf{L} & \mathbf{H} & \mathbf{VH} & \\ & & \mathbf{VL} & & \end{array} \right\rangle$$

(i) **Linguistic Rhotrix with fuzzy quantifications**

A linguistic rhotrix with fuzzy values to quantify the linguistic variables is termed as fuzzy quantified linguistic rhotrix.

$$\mathbf{R}_{\mathbf{L}_{QF}} = \left\langle \begin{array}{ccccc} & & \mathbf{0.94} & & \\ & \mathbf{0.75} & \mathbf{0.25} & \mathbf{0.08} & \\ \mathbf{0.94} & \mathbf{0.08} & \mathbf{0.50} & \mathbf{0.25} & \mathbf{0.08} \\ & \mathbf{0.25} & \mathbf{0.75} & \mathbf{0.94} & \\ & & \mathbf{0.08} & & \end{array} \right\rangle$$

(ii) **Linguistic Rhotrix with intuitionistic quantifications**

A linguistic rhotrix with intuitionistic values to quantify the linguistic variables is termed as intuitionistic quantified linguistic rhotrix.

$$\mathbf{R}_{\mathbf{L}_{QF}} = \left\langle \begin{array}{ccccc} & & (\mathbf{0.75}, \mathbf{1.00}) & & \\ & (\mathbf{0.50}, \mathbf{0.75}) & (\mathbf{0.25}, \mathbf{0.50}) & (\mathbf{0.00}, \mathbf{0.25}) & \\ (\mathbf{0.75}, \mathbf{1.00}) & (\mathbf{0.00}, \mathbf{0.25}) & (\mathbf{0.50}, \mathbf{0.50}) & (\mathbf{0.25}, \mathbf{0.50}) & (\mathbf{0.00}, \mathbf{0.25}) \\ & (\mathbf{0.25}, \mathbf{0.50}) & (\mathbf{0.50}, \mathbf{0.75}) & (\mathbf{0.75}, \mathbf{1.00}) & \\ & & (\mathbf{0.75}, \mathbf{1.00}) & & \end{array} \right\rangle$$

(iii) **Linguistic Rhotrix with neutrosophic quantifications**

A linguistic rhotrix with neutrosophic values to quantify the linguistic variables is termed as neutrosophic quantified linguistic rhotrix.

$$\mathbf{R}_{\mathbf{L}_{QF}} =$$

$$\left\langle \begin{array}{ccccc} & & (\mathbf{0.90}, \mathbf{0.50}, \mathbf{0.50}) & & \\ & (\mathbf{0.75}, \mathbf{0.25}, \mathbf{0.25}) & (\mathbf{0.25}, \mathbf{0.75}, \mathbf{0.75}) & (\mathbf{0.50}, \mathbf{0.95}, \mathbf{0.95}) & \\ (\mathbf{0.90}, \mathbf{0.50}, \mathbf{0.50}) & (\mathbf{0.50}, \mathbf{0.95}, \mathbf{0.95}) & (\mathbf{0.50}, \mathbf{0.50}, \mathbf{0.50}) & (\mathbf{0.25}, \mathbf{0.75}, \mathbf{0.75}) & (\mathbf{0.50}, \mathbf{0.95}, \mathbf{0.95}) \\ & (\mathbf{0.25}, \mathbf{0.75}, \mathbf{0.75}) & (\mathbf{0.75}, \mathbf{0.25}, \mathbf{0.25}) & (\mathbf{0.90}, \mathbf{0.50}, \mathbf{0.50}) & \\ & & (\mathbf{0.50}, \mathbf{0.95}, \mathbf{0.95}) & & \end{array} \right\rangle$$

2.4 Multiplication of Rhotrix

The Heart Based Multiplication of rhotrix is defined as

$$R = \begin{array}{ccc} & \mathbf{a} & \\ \mathbf{b} & \mathbf{h(R)} & \mathbf{d} \\ & \mathbf{e} & \end{array}$$

Then, rows of R can be viewed as (a d), (b e) whereas the columns can be viewed as (a b), (d e). Rhotrix multiplication is then same as matrix row-column multiplication applied in both the coupled matrices. For example,

$$R \circ Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & af + dg & \\ bf + eg & h(R)h(Q) & aj + dk \\ & bj + ek & \end{array} \right\rangle$$

2.5 Heart Based Multiplication of Linguistic rhotrix with Neutrosophic representations

The heart based multiplication of linguistic rhotrices with neutrosophic quantifications is defined as

The i[th] entry of $A_N \circ B_N$ = (max {min($\mathbf{a_i^T}$, $\mathbf{e_b^T}$), min($\mathbf{b_i^T}$, $\mathbf{e_a^T}$)}, max {min($\mathbf{a_i^I}$, $\mathbf{e_b^I}$), min($\mathbf{b_i^I}$, $\mathbf{e_a^I}$)}, min {max($\mathbf{a_i^F}$, $\mathbf{e_b^F}$), max($\mathbf{b_i^F}$, $\mathbf{e_a^F}$)}), for all I, except the heart

Heart of $A_N \circ B_N$ = (min{e_a, e_b}, min{e_a, e_b}, max{e_a, e_b})

2.6 Trace of a rhotrix

The trace of a rhotrix R_n is the sum of its diagonal entries and also the sum of all eigen values of that rhotrix. The trace is denoted by T r and defined as

$$\boldsymbol{Tr(R_n)=\sum_{i=1}^{n} r_{ii}}$$

2.7 Trace of Linguistic rhotrix with Neutrosophic representations

The trace of a linguistic rhotrix of size 'n' is denoted by adding the entries in major vertical axis and it is denoted as $Tr()$

That is for a rhotrix $$\boldsymbol{A_N} = \left\langle \begin{array}{ccc} & (a_1^T, a_1^I, a_1^F) & \\ (a_2^T, a_2^I, a_2^F) & (a_3^T, a_3^I, a_3^F) & (a_4^T, a_4^I, a_4^F) \\ & (a_5^T, a_5^I, a_5^F) & \end{array} \right\rangle$$

Tr($\boldsymbol{A_N}$) = (max{$\boldsymbol{a_1^T}$, $\boldsymbol{a_3^T}$, $\boldsymbol{a_5^T}$},max{$\boldsymbol{a_1^I}$, $\boldsymbol{a_3^I}$, $\boldsymbol{a_5^I}$},min{$\boldsymbol{a_1^F}$, $\boldsymbol{a_3^F}$, $\boldsymbol{a_5^F}$})

3. Algorithmic Framework of Linguistic Rhotrix in Managerial Decisioning

This section presents the steps involved in applying linguistic rhotrix in making optimal managerial decisions using the relations between the decision entities.

Step 1: In this step, the managerial decisioning problem is well defined and the associated entities are selected. Let us assume that the manufacturing companies are preferring sustainable production methods with preference to certain sets of criteria. A linguistic rhotrix CC_L = [cc_{Lij}] of order n, representing the relation between companies and the criteria is formulated with each cc_{Lij} is a linguistic variable indicating the degree of preference of the criteria by the companies.

$$\mathbf{CC_L} = \left\langle \begin{array}{ccccc} & & \mathbf{cc_{L11}} & & \\ & \mathbf{cc_{L21}} & \mathbf{cc_{L11}} & \mathbf{cc_{L12}} & \\ \mathbf{cc_{L31}} & \mathbf{cc_{L21}} & \mathbf{cc_{L22}} & \mathbf{cc_{L12}} & \mathbf{cc_{L13}} \\ & \mathbf{cc_{L32}} & \mathbf{cc_{L22}} & \mathbf{cc_{L23}} & \\ & & \mathbf{cc_{L33}} & & \end{array} \right\rangle$$

Step 2 : Formulate another linguistic rhotrix CM_L = [cm_{Lij}] of order n relating criteria and the sustainable methods chosen for decision. Each of the cells with linguistic variable represents the criteria possessed by the methods opted.

$$\mathbf{CM_L} = \left\langle \begin{array}{ccccc} & & \mathbf{cm_{L11}} & & \\ & \mathbf{cm_{L21}} & \mathbf{cm_{L11}} & \mathbf{cm_{L12}} & \\ \mathbf{cm_{L31}} & \mathbf{cm_{L21}} & \mathbf{cm_{L22}} & \mathbf{cm_{L12}} & \mathbf{cm_{L13}} \\ & \mathbf{cm_{L32}} & \mathbf{cm_{L22}} & \mathbf{cm_{L23}} & \\ & & \mathbf{cm_{L33}} & & \end{array} \right\rangle$$

Step 3: The linguistic variables presented in each of the cells cc_{Lij} and cm_{Lij} are quantified using neutrosophic representations.

$$\mathbf{CC_{NL}} = \left\langle \begin{array}{ccccc} & & (cc^{T}_{L11}, cc^{I}_{L11}, cc^{F}_{L11}) & & \\ & (cc^{T}_{L21}, cc^{I}_{L21}, cc^{F}_{L21}) & (cc^{T}_{L11}, cc^{I}_{L11}, cc^{F}_{L11}) & (cc^{T}_{12}, cc^{I}_{12}, cc^{F}_{12}) & \\ (cc^{T}_{L31}, cc^{I}_{L31}, cc^{F}_{L31}) & (cc^{T}_{L21}, cc^{I}_{L21}, cc^{F}_{L21}) & (cc^{T}_{L22}, cc^{I}_{L22}, cc^{F}_{L22}) & (cc^{T}_{L12}, cc^{I}_{L12}, cc^{F}_{L12}) & (cc^{T}_{13}, cc^{I}_{13}, cc^{F}_{13}) \\ & (cc^{T}_{L32}, cc^{I}_{L32}, cc^{F}_{L32}) & (cc^{T}_{L22}, cc^{I}_{L22}, cc^{F}_{L22}) & (cc^{T}_{L23}, cc^{I}_{L23}, cc^{F}_{L23}) & \\ & & (cc^{T}_{L33}, cc^{I}_{L33}, cc^{F}_{L33}) & & \end{array} \right\rangle$$

$$\mathbf{CM_{NL}} = \left\langle \begin{array}{ccccc} & & (cm^{T}_{L11}, cm^{I}_{L11}, cm^{F}_{L11}) & & \\ & (cm^{T}_{L21}, cm^{I}_{L21}, cm^{F}_{L21}) & (cm^{T}_{L11}, cm_{L11}, cm^{F}_{L11}) & (cm^{T}_{L12}, cm^{I}_{L12}, cm^{F}_{L12}) & \\ (cm^{T}_{L31}, cm^{I}_{L31}, cm^{F}_{L31}) & (cm^{T}_{L21}, cm^{I}_{L21}, cm^{F}_{L21}) & (cm^{T}_{L22}, cm^{I}_{L22}, cm^{F}_{L22}) & (cm^{T}_{L12}, cm^{I}_{L12}, cm^{F}_{L12}) & (cm^{T}_{L13}, cm^{I}_{L13}, cm^{F}_{L13}) \\ & (cm^{T}_{L32}, cm^{I}_{L32}, cm^{F}_{L32}) & (cm^{T}_{L22}, cm^{I}_{L22}, cm^{F}_{L22}) & (cm^{T}_{L23}, cm^{I}_{L23}, cm^{F}_{L23}) & \\ & & (cm^{T}_{L33}, cm^{I}_{L33}, cm^{F}_{L33}) & & \end{array} \right\rangle$$

Step 4 : The heart based multiplication is performed between the neutrosophic linguistic rhotrix to obtain the resultant matrix

$$\mathbf{R_{NL}} = \left\langle \begin{array}{ccccc} & & (d^{T}_{L11}, d^{I}_{L11}, d^{F}_{L11}) & & \\ & (d^{T}_{L21}, d^{I}_{L21}, d^{F}_{L21}) & (d^{T}_{L11}, d^{I}_{L11}, d^{F}_{L11}) & (d^{T}_{L12}, d^{I}_{L12}, d^{F}_{L12}) & \\ (d^{T}_{L31}, d^{I}_{L31}, d^{F}_{L31}) & (d^{T}_{L21}, d^{I}_{L21}, d^{F}_{L21}) & (d^{T}_{L22}, d^{I}_{L22}, d^{F}_{L22}) & (d^{T}_{L12}, d^{I}_{L12}, d^{F}_{L12}) & (d^{T}_{L13}, d^{I}_{L13}, d^{F}_{L13}) \\ & (d^{T}_{L32}, d^{I}_{L32}, d^{F}_{L32}) & (d^{T}_{L22}, d^{I}_{L22}, d^{F}_{L22}) & (d^{T}_{L23}, d^{I}_{L23}, d^{F}_{L23}) & \\ & & (d^{T}_{L33}, d^{I}_{L33}, d^{F}_{L33}) & & \end{array} \right\rangle$$

Step 5: The complement of the neutrosophic linguistic matrices $\mathbf{CC_{NL}}$ and $\mathbf{CM_{NL}}$ is determined and the respective $\mathbf{CC^{S}_{NL}}$ and $\mathbf{CM^{S}_{NL}}$ are obtained by finding the difference between the falsity values and sum of truth and indeterminate values.

Step 6: The score rhotrix is obtained using the difference of $\mathbf{CC^{S}_{NL}}$ and $\mathbf{CM^{S}_{NL}}$ i.e $\mathbf{CC^{S}_{NL}}$ - $\mathbf{CM^{S}_{NL}}$. Based on the maximum values in each of the row, the companies and the methods are matched accordingly.

4. Illustration using Neutrosophic Quantifications of Linguistic Rhotrix

Let us consider a managerial decision-making situation where the companies have to choose the sustainable production methods based on criteria. Assume that there are seven companies and seven criteria. Out of these, four companies $(\mathbf{A_1}, \mathbf{A_3}, \mathbf{A_5}, \mathbf{A_7})$ give high preference to four criteria in specific $(\mathbf{T_2}, \mathbf{T_3}, \mathbf{T_5}, \mathbf{T_6})$ while the remaining three companies $(\mathbf{A_2}, \mathbf{A_4}, \mathbf{A_6})$ prefer three other criteria $(\mathbf{T_1}, \mathbf{T_4}, \mathbf{T_7})$. The linguistic rhotrix CC_L representing the relation between the companies and criteria is as follows.

$$CC = \left\langle \begin{array}{ccccccccccccc} & & & & & & \mathbf{A_1T_2} & & & & & & \\ & & & & \mathbf{A_3T_2} & & \mathbf{A_2T_1} & & \mathbf{A_1T_3} & & & & \\ & & \mathbf{A_5T_2} & & \mathbf{A_4T_1} & & \mathbf{A_3T_3} & & \mathbf{A_2T_4} & & \mathbf{A_1T_5} & & \\ \mathbf{A_7T_2} & & \mathbf{A_6T_1} & & \mathbf{A_5T_3} & & \mathbf{A_4T_4} & & \mathbf{A_3T_5} & & \mathbf{A_2T_7} & & \mathbf{A_1T_6} \\ & & \mathbf{A_7T_3} & & \mathbf{A_6T_4} & & \mathbf{A_5T_5} & & \mathbf{A_4T_7} & & \mathbf{A_3T_6} & & \\ & & & & \mathbf{A_7T_5} & & \mathbf{A_6T_7} & & \mathbf{A_5T_6} & & & & \\ & & & & & & \mathbf{A_7T_6} & & & & & & \end{array} \right\rangle$$

The respective linguistic rhotrix with linguistic variables representing the degrees of preference of the criteria by the companies is

$$CC_L = \left\langle \begin{array}{ccccccc} & & & \boldsymbol{VL} & & & \\ & & \mathbf{M} & \mathbf{VL} & \mathbf{VH} & & \\ & \mathbf{M} & \mathbf{VL} & \boldsymbol{L} & \mathbf{M} & \mathbf{VL} & \\ \mathbf{VL} & \mathbf{VH} & \mathbf{M} & \mathbf{VH} & \mathbf{H} & \boldsymbol{VH} & \mathbf{H} \\ & \mathbf{M} & \mathbf{VL} & \mathbf{VH} & \mathbf{M} & \boldsymbol{L} & \\ & & \boldsymbol{L} & \mathbf{VL} & \mathbf{VH} & & \\ & & & \mathbf{VH} & & & \end{array} \right\rangle$$

The equivalent linguistic rhotrix with fuzzy, intuitionistic fuzzy and neutrosophic quantifications are presented below using the values in Table 1

$$CC_{NL} = \left\langle \begin{array}{ccccccc} & & & (0.5,0.95,0.95) & & & \\ & & (0.5,0.5,0.5) & (0.5,0.95,0.95) & (0.9,0.5,0.5) & & \\ & (0.5,0.5,0.5) & (0.5,0.95,0.95) & (0.25,0.75,0.75) & (0.5,0.5,0.5) & (0.5,0.95,0.95) & \\ (0.5,0.95,0.95) & (0.9,0.5,0.5) & (0.5,0.5,0.5) & (0.9,0.5,0.5) & (0.75,0.25,0.25) & (0.9,0.5,0.5) & (0.75,0.25,0.25) \\ & (0.5,0.5,0.5 & (0.5,0.95,0.95) & (0.9,0.5,0.5) & (0.5,0.5,0.5) & (0.25,0.75,0.75) & \\ & & (0.25,0.75,0.75) & (0.5,0.95,0.95) & (0.9,0.5,0.5) & & \\ & & & (0.9,0.5,0.5) & & & \end{array} \right\rangle$$

Table 1: Quantification of the Linguistic Variable

Linguistic Variable	**Notation**	**Fuzzy Quantification**	**Intuitionistic Fuzzy Quantification**	**Neutrosophic Quantification**
Very Low	VL	0.2	(0.1,0.9)	(.25,.75,.75)
Very Low	L	0.4	(0.3,0.7)	(.5,.95,.95)
Moderate	M	0.6	(0.5,0.5)	(.5,.5,.5)
High	H	0.8	(0.8,0.2)	(.75,.25,.25)
Very High	VH	1	(0.9,0.1)	(.9,.5,.5)

Let us construct another linguistic rhotrix relating the methods and criteria. Let us assume that four methods namely ($\mathbf{M_1}, \mathbf{M_3}$,$\mathbf{M_5}, \mathbf{M_7}$) are known for the criteria ($\mathbf{A_2}, \mathbf{A_4}$,$\mathbf{A_5}, \mathbf{A_6}$),while the other three methods ($\mathbf{M_2}, \mathbf{M_4}$,$\mathbf{M_6}$) are known for the three criteria ($\mathbf{A_1}, \mathbf{A_3}$,$\mathbf{A_7}$).

$$CM= \left\langle \begin{array}{c} \mathbf{M_1A_2} \\ \mathbf{M_3A_2} \quad \mathbf{M_2A_1} \quad \mathbf{M_1A_4} \\ \mathbf{M_5A_2} \quad \mathbf{M_4A_1} \quad \mathbf{M_3A_4} \quad \mathbf{M_2A_3} \quad \mathbf{M_1A_5} \\ \mathbf{M_7A_2} \quad \mathbf{M_6A_1} \quad \mathbf{M_5A_4} \quad \mathbf{M_4A_3} \quad \mathbf{M_3A_5} \quad \mathbf{M_2A_7} \quad \mathbf{M_1A_6} \\ \mathbf{M_7A_4} \quad \mathbf{M_6A_3} \quad \mathbf{M_5A_5} \quad \mathbf{M_4A_7} \quad \mathbf{M_3A_6} \\ \mathbf{M_7A_5} \quad \mathbf{M_6A_7} \quad \mathbf{M_5A_6} \\ \mathbf{M_7A_6} \end{array} \right\rangle$$

The respective linguistic rhotrix with linguistic variables representing the degrees of fulfilment of the criteria by the methods is

$$CM_L = \left\langle \begin{array}{c} \boldsymbol{VH} \\ \mathbf{VH} \quad \mathbf{VL} \quad \mathbf{VH} \\ \mathbf{VL} \quad \mathbf{M} \quad \boldsymbol{L} \quad \mathbf{M} \quad \mathbf{VH} \\ \mathbf{VL} \quad \mathbf{VH} \quad \mathbf{M} \quad \mathbf{VL} \quad \mathbf{M} \quad \boldsymbol{H} \quad \mathbf{H} \\ \mathbf{VL} \quad \mathbf{M} \quad \mathbf{VH} \quad \mathbf{M} \quad \boldsymbol{L} \\ \boldsymbol{H} \quad \mathbf{VL} \quad \mathbf{H} \\ \boldsymbol{H} \end{array} \right\rangle$$

The equivalent linguistic rhotrix with neutrosophic quantifications is presented below using the values in Table 1.

CM_{NL} =

$$\left\langle \begin{array}{c} (\mathbf{0.9, 0.5, 0.5}) \\ (\mathbf{0.9, 0.5, 0.5}) \quad (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{0.9, 0.5, 0.5}) \\ (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{0.5, 0.5, 0.5}) \quad (\mathbf{0.25, 0.75, 0.75}) \quad (\mathbf{0.5, 0.5, 0.5}) \quad (\mathbf{0.9, 0.5, 0.5}) \\ (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{0.9, 0.5, 0.5}) \quad (\mathbf{0.5, 0.5, 0.5}) \quad (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{0.5, 0.5, 0.5}) \quad (\mathbf{0.75, 0.25, 0.25}) \quad (\mathbf{0.75, 0.25, 0.25}) \\ (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{0.5, 0.5, 0.5}) \quad (\mathbf{0.9, 0.5, 0.5}) \quad (\mathbf{0.5, 0.5, 0.5}) \quad (\mathbf{0.25, 0.75, 0.75}) \\ ((\mathbf{0.75, 0.25, 0.25} \quad (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{0.75, 0.25, 0.25}) \\ (\mathbf{0.75, 0.25, 0.25}) \end{array} \right\rangle$$

By applying the heart-based multiplication of two neutrosophic quantified linguistic rhotrices CC_{NL} and CM_{NL} we get

C_N=

$$\left\langle \begin{array}{c} (\mathbf{0.9, 0.95, .5}) \\ (\mathbf{0.9, 0.5, 0.5}) \quad (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{0.9, 0.5, 0.5}) \\ (\mathbf{0.5, 0.5, 0.95}) \quad (\mathbf{0.5, 0.0, 0.0}) \quad (\mathbf{.25, .75, .75}) \quad (\mathbf{0.5, 0.5, 0.5}) \quad (\mathbf{.9, .95, .5}) \\ (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{0.9, 0.5, 0.5}) \quad (\mathbf{0.5, 0.5, 0.5}) \quad (\mathbf{0.5, 0.5, 0.95}) \quad (\mathbf{0.75, 0.5, 0.25}) \quad (\mathbf{.75, .5, .5}) \quad (\mathbf{.75, .25, .5}) \\ (\mathbf{0.5, 0.5, 0.95}) \quad (\mathbf{.5, .95, .5}) \quad (\mathbf{.9, .5, .5}) \quad (\mathbf{.5, .5, .5}) \quad (\mathbf{.25, .75, .75}) \\ (\mathbf{0.75, 0.75, .5}) \quad (\mathbf{0.5, 0.95, 0.95}) \quad (\mathbf{.75, .5, .5}) \\ (\mathbf{.9, .5, .25}) \end{array} \right\rangle$$

The complement of CC_{NL} and CM_{NL} is also determined

$\boldsymbol{CC_{NL}^{C}}$=

$$\left\langle \begin{array}{c} (0.5,0.05,0.05) \\ (0.5,0.5,0.5) \quad (0.95,0.05,0.05) \quad (0.1,0.5,0.5) \\ (0.5,0.5,0.5) \quad (0.5,0.05,0.05) \quad (0.75,0.25,0.25) \quad (0.5,0.5,0.5) \quad (0.5,0.05,0.05) \\ (0.5,0.05,0.05) \quad (0.1,0.5,0.5) \quad (0.5,0.5,0.5) \quad (0.1,0.5,0.5) \quad (0.25,0.75,0.75) \quad (0.1,0.5,0.5) \quad (0.25,0.75,0.75) \\ (0.5,0.5,0.5) \quad (0.5,0.05,0.05) \quad (0.1,0.5,0.5) \quad (0.5,0.5,0.5) \quad (0.75,0.25,0.25) \\ (0.75,0.25,0.25) \quad (0.5,0.05,0.05) \quad (0.1,0.5,0.5) \\ (0.1,0.5,0.5) \end{array} \right\rangle$$

CM_{NL}^{C} =

$$\left\langle \begin{array}{c} (0.1,0.5,0.5) \\ (0.1,0.5,0.5) \quad (0.5,0.05,0.05) \quad (0.1,0.5,0.5) \\ (0.5,0.95,0.95) \quad (0.5,0.5,0.5) \quad (0.25,0.75,0.75) \quad (0.5,0.5,0.5) \quad (0.9,0.5,0.5) \\ (0.5,0.05,0.05) \quad (0.1,0.5,0.5) \quad (0.5,0.5,0.5) \quad (0.5,0.05,0.05) \quad (0.5,0.5,0.5) \quad (0.25,0.75,0.75) \quad (0.25,0.75,0.75) \\ (0.5,0.05,0.05) \quad (0.5,0.5,0.5) \quad (0.1,0.5,0.5) \quad (0.5,0.5,0.5) \quad (0.75,0.25,0.25) \\ ((0.25,0.75,0.75) \quad (0.5,0.05,0.05) \quad (0.25,0.75,0.75) \\ (0.25,0.75,0.75) \end{array} \right\rangle$$

By applying the heart-based multiplication of two neutrosophic quantified linguistic $CC_{NL}^{C}$and CM_{NL}^{C} we get

D_{NL}=

$$\left\langle \begin{array}{c} (.5,.5,.05) \\ (.5,.5,.5) \quad (.5,.05,.05) \quad (.1,.5,.5) \\ (.5,.5,.5) \quad (.5,0,0) \quad (.5,.5,.25) \quad (.5,.5,.25) \quad (.5,.5,.05) \\ (.5,.05,.05) \quad (.1,.5,.5) \quad (.5,.5,.5) \quad (0.1,0.05,0.5) \quad (.5,.75,.5) \quad (.1,.5,.05) \quad (.25,.5,.75) \\ (.5,.05,.5) \quad (.5,.5,.05) \quad (.1,.5,.5) \quad (.5,.5,.5) \quad (.5,.25,.25) \\ (.5,.5,.25) \quad (.5,.05,.05) \quad (.1,.5,.5) \\ (.1,.5,0) \end{array} \right\rangle$$

Then, by using step 5 of the aforementioned algorithm, the modified rhotrices are obtained

$$C_{NL}^{S} = \left\langle \begin{array}{c} 1.35 \\ 0.90 \quad 0.50 \quad 0.90 \\ 0.05 \quad 0.50 \quad 0.25 \quad 0.50 \quad 0.90 \\ 0.50 \quad 0.90 \quad 0.50 \quad 0.05 \quad 1.00 \quad 0.75 \quad 0.50 \\ 0.05 \quad 0.95 \quad 0.90 \quad 0.50 \quad 0.25 \\ 1.00 \quad 0.50 \quad 0.75 \\ 1.15 \end{array} \right\rangle$$

$$D_{NL}^{S} = \left\langle \begin{array}{c} 0.95 \\ 0.5 \quad 0.5 \quad 0.1 \\ 0.5 \quad 0.5 \quad 0.75 \quad 0.75 \quad 0.95 \\ 0.5 \quad 0.1 \quad 0.5 \quad -0.35 \quad 0.1 \quad 0.55 \quad 0 \\ 0.05 \quad 0.95 \quad 0.1 \quad 0.75 \quad 0.5 \\ 0.75 \quad 0.5 \quad 0.1 \\ 0.6 \end{array} \right\rangle$$

Using step 6, the score rhotrix is obtained.

The score rhotrix is $\left\langle \begin{array}{ccccccc} & & & 0.400 & & & \\ & & 0.400 & 0.000 & 0.800 & & \\ & -0.45 & 0.000 & -0.50 & -0.25 & -0.05 & \\ 0.000 & 0.800 & 0.000 & 0.400 & 0.250 & 0.200 & 0.500 \\ & 0.000 & 0.000 & 0.800 & 0.000 & -0.25 & \\ & & 0.250 & 0.000 & 0.650 & & \\ & & & 0.550 & & & \end{array} \right\rangle$

The neutrosophic score values of the matchings between the methods and the companies are presented graphically in Fig. 1.

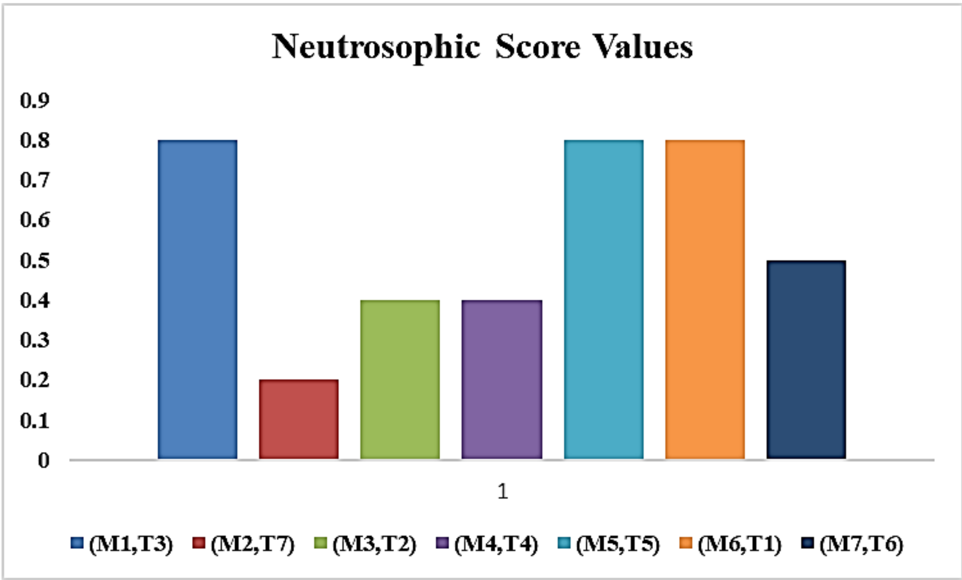

Fig. 1 Neutrosophic Score Values

The optimal matching results of the methods and the companies based on neutrosophic quantified linguistic rhotrix is presented in the below Table 2 and the graphical representation of the neutrosophic values is presented in Fig.1.

Table 2: Neutrosophic Results

Methods	M1	M2	M3	M4	M5	M6	M7
Companies	T3	T7	T2	T4	T5	T1	T6

From the results presented in Table 2, one can make a better inference on the matching of the methods to the demands and nature of the companies considered for decisioning. Since each company is different in its functioning, the optimal match of the methods with the companies appears to be more optimal and pragmatic.

5. Comparative Analysis with Fuzzy and Intuitionistic Fuzzy Quantifications

5.1 Decisioning with Fuzzy Quantification based Linguistic Rhotrix

The illustration discussed in section 3 with neutrosophic representations is discussed with fuzzy representations to make a comparative analysis of the results.

Using the fuzzy quantification presented in Table 1 the respective rhotrices CC_F and CM_F are obtained.

$$CC_F = \left\langle \begin{array}{ccccccc} & & & 0.2 & & & \\ & & 0.6 & 0.2 & 1.0 & & \\ & 0.6 & 0.2 & 0.4 & 0.6 & 0.2 & \\ 0.2 & 1.0 & 0.6 & 1.0 & 1.0 & 1.0 & 0.8 \\ & 0.6 & 0.2 & 0.1 & 0.6 & 0.4 & \\ & & 0.6 & 0.2 & 1.0 & & \\ & & & 1.0 & & & \end{array} \right\rangle$$

$$CM_F = \left\langle \begin{array}{ccccccc} & & & 0.8 & & & \\ & & 1.0 & 0.2 & 0.6 & & \\ & 0.2 & 1.0 & 0.4 & 0.6 & 1.0 & \\ 0.2 & 1.0 & 0.6 & 0.2 & 1.0 & 0.8 & 0.8 \\ & 0.2 & 0.6 & 1.0 & 0.6 & 0.4 & \\ & & 0.8 & 0.2 & 0.8 & & \\ & & & 0.4 & & & \end{array} \right\rangle$$

The heart based multiplication of CC_F and CM_F is

$$C_F = \left\langle \begin{array}{ccccccc} & & & 0.8 & & & \\ & & 1.0 & 1.0 & 0.6 & & \\ & 0.2 & 1.0 & 0.6 & 0.6 & 1.0 & \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 0.8 & 0.8 \\ & 0.2 & 0.6 & 1.0 & 0.6 & 0.6 & \\ & & 0.8 & 1.0 & 1.0 & & \\ & & & 0.4 & & & \end{array} \right\rangle$$

Build the complement matrix CC_F^C and CM_F^C of CC_F and CM_F

$$\boldsymbol{CC_F^C} = \left\langle \begin{array}{ccccccc} & & & 0.8 & & & \\ & & 0.4 & 0.8 & 0.0 & & \\ & 0.4 & 0.8 & 0.6 & 0.4 & 0.8 & \\ 0.8 & 0.0 & 0.4 & 0.0 & 0.0 & 0.0 & 0.2 \\ & 0.4 & 0.8 & 0.0 & 0.4 & 0.6 & \\ & & 0.4 & 0.8 & 0.0 & & \\ & & & 0.0 & & & \end{array} \right\rangle$$

$$\boldsymbol{CM_F^c} = \left\langle \begin{array}{ccccccc} & & & 0.2 & & & \\ & & 0.0 & 0.8 & 0.4 & & \\ & 0.8 & 0.0 & 0.6 & 0.4 & 0.0 & \\ 0.8 & 0.0 & 0.4 & 0.8 & 0.0 & 0.2 & 0.2 \\ & 0.8 & 0.4 & 0.0 & 0.4 & 0.6 & \\ & & 0.2 & 0.8 & 0.2 & & \\ & & & 0.6 & & & \end{array} \right\rangle$$

The heart - based multiplication of $\boldsymbol{CC_F^C}$ and $\boldsymbol{CM_F^C}$ is

$$D_F = \left\langle \begin{array}{ccccccc} & & & 0.8 & & & \\ & & 0.4 & 0.8 & 0.4 & & \\ & 0.8 & 0.8 & 0.6 & 0.4 & 0.8 & \\ 0.8 & 0.0 & 0.4 & 0.8 & 0.0 & 0.2 & 0.2 \\ & 0.8 & 0.8 & 0.0 & 0.4 & 0.6 & \\ & & 0.4 & 0.8 & 0.2 & & \\ & & & 0.6 & & & \end{array} \right\rangle$$

The score rhotrix is

$$\left\langle \begin{array}{ccccccc} & & & 0.00 & & & \\ & & 0.60 & 0.20 & 0.20 & & \\ & -0.6 & 0.20 & 0.00 & 0.20 & 0.20 & \\ 0.20 & 1.00 & 0.60 & 0.20 & 1.00 & 0.60 & 0.00 \\ & -0.60 & -0.20 & 1.00 & 0.20 & 0.00 & \\ & & 0.40 & 0.20 & 0.80 & & \\ & & & -0.20 & & & \end{array} \right\rangle$$

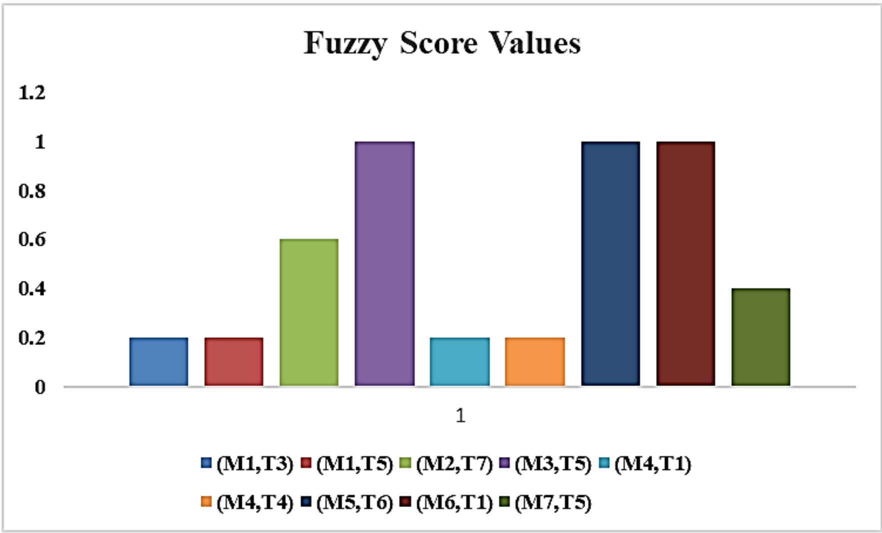

Fig. 2 Fuzzy Score Values

The optimal matching results of the methods and the companies based on fuzzy quantified linguistic rhotrix is presented in the below Table 3 and the fuzzy score values are presented graphically in Fig. 2.

Table 3: Fuzzy Results

Methods	M1	M2	M3	M4	M5	M6	M7
Companies	T3, T5	T7	T5	T1, T4	T5	T1	T5

From the results presented in Table 3, it is inferred that, no method is matched with company T2, also overlapping matchings arises in case of fuzzy quantifications. The decisioning results are not optimal for instance, the company T5 has several options in adopting the methods, in this case, the fuzzy based quantifications are not facilitating the companies to adopt the most matching method subjected to their requirements.

5.2 Decisioning with Intuitionistic Fuzzy Quantification based Linguistic Rhotrix

The illustration discussed in section 3 with neutrosophic representations is discussed with intuitionistic fuzzy representations to make a comparative analysis of the results.

Using the intuitionistic fuzzy quantification presented in Table 1 the respective rhotrices CC_{IF} and CM_{IF} are obtained.

$$CC_{IF} = \left\langle \begin{array}{ccccccc} & & & (.1,.9) & & & \\ & & (.5,.5) & (.1,.9) & (.9,.1) & & \\ & (.5,.5) & (.1,.9) & (.3,.7) & (.5,.5) & (.1,.9) & \\ (.1,.9) & (.9,.1) & (.5,.5) & (.9,.1) & (.9,.1) & (.9,.1) & (.8,.2) \\ & (.5,.5) & (.1,.9) & (.9,.1) & (.5,.5) & (.3,.7) & \\ & & (.3,.7) & (.1,.9) & (.9,.1) & & \\ & & & (.9,.1) & & & \end{array} \right\rangle$$

$$CM_{IF} = \left\langle \begin{array}{ccccccc} & & & (.8,.2) & & & \\ & & (.9,.1) & (.1,.9) & (.5,.5) & & \\ & (.1,.9) & (.9,.1) & (.3,.7) & (.5,.5) & (.9,.1) & \\ (.1,.9) & (.9,.1) & (.5,.5) & (.1,.9) & (.9,.1) & (.8,.2) & (.8,.2) \\ & (.1,.9) & (.5,.5) & (.9,.1) & (.5,.5) & (.3,.7) & \\ & & (.8,.2) & (.1,.9) & (.8,.2) & & \\ & & & (.3,.7) & & & \end{array} \right\rangle$$

The heart-based multiplication of CC_F and CM_F is

$$C_F = \left\langle \begin{array}{ccccccc} & & & 0.8 & & & \\ & & 0.9 & 0.1 & 0.9 & & \\ & 0.5 & 0.9 & 0.3 & 0.5 & 0.9 & \\ 0.1 & 0.9 & 0.5 & 0.9 & 0.9 & 0.9 & 0.8 \\ & 0.5 & 0.5 & 0.9 & 0.5 & 0.3 & \\ & & 0.8 & 0.1 & 0.9 & & \\ & & & 0.9 & & & \end{array} \right\rangle$$

Build the complement matrix $CC_F{}^C$ and $CM_F{}^C$ of CC_F and CM_F

$$CC_{IF}^{C} = \left\langle \begin{array}{ccccccc} & & & (.9,.1) & & & \\ & & (.5,.5) & (.9,.1) & (.1,.9) & & \\ & (.5,.5) & (.9,.1) & (.7,.3) & (.5,.5) & (.9,.1) & \\ (.9,.1) & (.1,.9) & (.5,.5) & (.1,.9) & (.1,.9) & (.1,.9) & (.2,.8) \\ & (.5,.5) & (.9,.1) & (.1,.9) & (.5,.5) & (.7,.3) & \\ & & (.7,.3) & (.9,.1) & (.1,.9) & & \\ & & & (.1,.9) & & & \end{array} \right\rangle$$

$$CM_{IF}^{c} = \left\langle \begin{array}{c} (.2,.8) \\ (.1,.9)\quad(.9,.1)\quad(.5,.5) \\ (.9,.1)\quad(.1,.9)\quad(.7,.3)\quad(.5,.5)\quad(.1,.9) \\ (.9,.1)\quad(.1,.9)\quad(.5,.5)\quad(.9,.1)\quad(.1,.9)\quad(.2,.8)\quad(.2,.8) \\ (.9,.1)\quad(.5,.5)\quad(.1,.9)\quad(.5,.5)\quad(.7,.3) \\ (.2,.8)\quad(.9,.1)\quad(.2,.8) \\ (.7,.3) \end{array} \right\rangle$$

The heart-based multiplication of CC_{IF}^{C} and CM_{IF}^{C} is

$$D_{IF} = \left\langle \begin{array}{c} 0.9 \\ 0.5\quad 0.9\quad 0.5 \\ 0.9\quad 0.9\quad 0.7\quad 0.5\quad 0.9 \\ 0.9\quad 0.1\quad 0.5\quad 0.9\quad 0.1\quad 0.2\quad 0.2 \\ 0.9\quad 0.9\quad 0.1\quad 0.5\quad 0.7 \\ 0.7\quad 0.9\quad 0.2 \\ 0.7 \end{array} \right\rangle$$

The score rhotrix is

$$\left\langle \begin{array}{c} -0.1 \\ 0.4\quad -0.8\quad 0.4 \\ -0.4\quad 0.0\quad -0.4\quad 0.0\quad 0.0 \\ -0.8\quad 0.8\quad 0.0\quad 0.0\quad 0.8\quad 0.7\quad 0.6 \\ -0.4\quad -0.4\quad 0.0\quad 0.0\quad -0.4 \\ 0.1\quad -0.8\quad 0.7 \\ 0.2 \end{array} \right\rangle$$

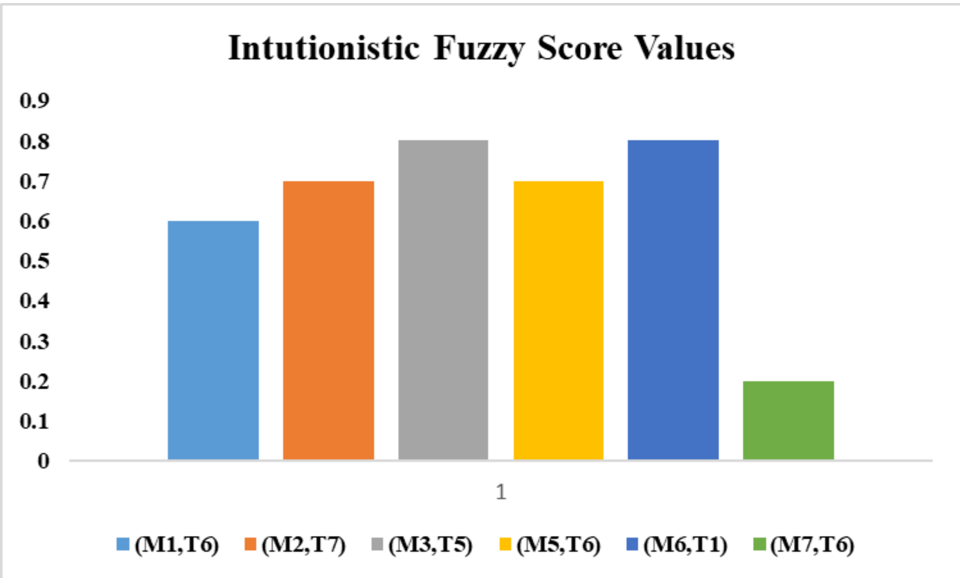

Fig. 3 Intuitionistic Fuzzy Score Values

The optimal matching results of the methods and the companies based on intuitionistic fuzzy quantified linguistic rhotrix is presented in the below Table 4 and the fuzzy score values are presented graphically in Fig. 3.

Table 3: Intuitionistic Fuzzy Results

Methods	M1	M2	M3	M4	M5	M6	M7
Companies	T6	T7	T5	-	T6	T1	T6

The table 3 clearly demonstrates the shortcomings of intuitionistic fuzzy quantifications in determining the optimal matching of the methods and the companies. For instance, the method M4 is not matched to any of the companies whereas the company T6 has three options of the methods to be chosen. Also, the companies T2, T3 and T4 are not assigned with any of the methods. This result does not facilitate in resolving the conflicts in decisioning.

5.3 Discussions

The managerial decisioning problem on optimal matching of the companies with the methods is discussed under three cases using linguistic rhotrix based on the quantifications of fuzzy, intuitionistic and neutrosophic respectively. Both fuzzy and intuitionistic results are not optimal as overlapping of the matchings take place and, in some instances, it is not possible to assign a method for the companies such as T2 under fuzzy and T2, T3 and T4 under intuitionistic fuzzy. However, on other hand the matchings based on neutrosophic is more compatible and comprehensive with each method is assigned to different companies. The result fulfills the objective of the decisioning problem of determining an optimal matching solution. The drawbacks of fuzzy and intuitionistic based quantifications are well handled by the neutrosophic representations. On comparing the results under all the three cases, it is inferred that the neutrosophic based quantifications are more competent in handling intricacies in data representations.

6. Conclusion

This research work proposes the novel concept of linguistic rhotrix and demonstrates the applications of neutrosophic quantified linguistic rhotrix in managerial decisioning. The comparisons of the results obtained using both fuzzy, intuitionistic fuzzy and neutrosophic based quantifications yield same results demonstrating the consistency of the proposed algorithmic framework. This research work shall be extended by considering quantifications using different kinds of sets, also, hypersoft set representations of criterion values shall be used in evolving new kind of linguistic rhotrices. The proposed kind of linguistic rhotrix decisioning approach shall be extended and discussed with neutrosophic hypersoft sets for facilitating the decision makers in designing furthermore optimum solutions to intricate managerial decisioning involving several interdependent entities characterized by uncertainty and indeterminacy.

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