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The Numerical Applications of (ABM) and (AMM) Numerical Methods on Some Neutrosophic and Dual Problems

Ahmed Salem Heilat
Department of Mathematics, Faculty of Science, Jadara University, P.O. Box 733, Irbid 21110, Jordan

Email: ahmed_heilat@yahoo.com

Abstract:

The objective of this paper is to present two neutrosophic numerical methods for finding the numerical solutions for some neutrosophic differential equations, where a numerical comparison between AMM and ABM methods with numerical tables that clarify some numerical results of the previous two methods applied on some neutrosophic initial value problems IVP. Also, we apply the same methods on dual variables problems, and we will show that they will give us the same numerical results.

Keywords: neutrosophic IVP, neutrosophic numerical analysis, numerical method, numerical approximation.

Introduction

Numerical analysis is a branch of applied mathematics that is concerned with finding numerical solutions to both differential equations and algebraic equations. On the other hand, it may be difficult to find the exact solution to a mathematical problem, so here comes numerical analysis, which plays the primary role through its techniques in finding a number of approximate solutions that are sufficient to deal with the data of the problem [1-4]. The concept of neutrosophic logic and the neutrosophic set was developed to deal with the presence of indeterminacy in scientific data and theses, and sometimes to deal with the lack or conflict of data related to a certain research question [5-7]. In many previous research works [8-10], we find an application of the concepts of numerical analysis and numerical methods in solving Neutrosophic differential and algebraic equations [11-13], where many traditional methods were applied and generalized to a more comprehensive

Neutrosophic version, in order to become useful in dealing with problems that contain an element of indeterminacy [14-16].We note here that issues that contain elements of indeterminacy may appear when studying natural phenomena [17-20], or life issues that contain a lack of information or conflicting data [21-26]. In this research, we are interested in following in the footsteps of researchers in works concerned with neutrosophic numerical analysis [27-33], where we present two neutrosophic numerical methods for finding the numerical solutions for some neutrosophic differential equations, where a numerical comparison between AMM and ABM methods with numerical tables that clarify some numerical results of the previous two methods applied on some neutrosophic initial value problems IVP.

Main Discussion

Definition 2.1 (Adams Bashforth Method of neutrosophic variable) Consider the neutrosophic (IVP):

$$(y_1 + y_2 I)'(x_1 + x_2 I) = f(x_1 + x_2 I, y_1 + y_2 I),$$

so that,

$$(y_1 + y_2 I)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} I\right) = (y_1 + y_2 I)\left(x_i^{(1)} + x_i^{(2)} I\right) + \int_{x_i^{(1)} + x_i^{(2)} I}^{x_{i+1}^{(1)} + x_{i+1}^{(2)} I} f(x_1 + x_2 I, y_1 + y_2 I)\, d(x_1 + x_2 I)$$

$$P_{k-1}(x_1 + x_2 I) = f_i + \frac{\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right)}{h_1 + h_2 I} \nabla f_i + \frac{\left[\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right)\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + I\left(x_2 - x_{i-1}^{(2)}\right)\right]}{2!\,(h_1 + h_2 I)} \nabla^2 f_i + \ldots\ldots$$

$$+ \frac{\left[\left(x_1 - x_i^{(1)}\right) + I(x_2 - x_i^{(2)})\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + I(x_2 - x_{i-1}^{(2)})\right]\ldots\ldots\left[\left(x_1 - x_{i-k+2}^{(1)}\right) + I(x_2 - x_{i-k+2}^{(2)})\right]}{(k-1)!\,(h_1 + h_2 I)^{k-1}} \nabla^{k-1} f_i$$

we put,

$$\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right) = (h_1 + h_2 I)s,$$

then,

$$P_{k-1} = f_i + s\nabla f_i + \frac{1}{2!}\,s(s+1)\nabla^2 f_i + \frac{1}{6}\,s(s+1)(s+2)\nabla^3 f_i + \ldots . + \frac{s(s+1)\ldots(s+k-2)}{(k-1)!}\,\nabla^{k-1} f_i,$$

this means that,

$$(y_1 + y_2 I)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} I\right) = (y_1 + y_2 I)\left(x_i^{(1)} + x_i^{(2)} I\right) + \int_{x_i^{(1)}+x_i^{(2)}I}^{x_{i+1}^{(1)}+x_{i+1}^{(2)}I} [f_i + s\nabla f_i + \ \ldots..]d(x_1 + x_2 I)$$

we get,

$$(y_1 + y_2 I)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} I\right) = (y_1 + y_2 I)\left(x_i^{(1)} + x_i^{(2)} I\right) + \int_0^1 [f_i + s\nabla f_i + \ \ldots..+]d(x_1 + x_2 I) = (y_1 + y_2 I) + (h_1 + h_2 I)(f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\,\nabla^2 f_i + \frac{3}{8}\nabla^3 f_i),$$

thus,

$$\begin{cases} \nabla f_i = f_i - f_{i-1} \\ \nabla^2 f_i = f_i - 2f_{i-1} + f_{i-2} \\ \nabla^3 f_i = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3} \end{cases}$$

thus,

$$y_{i+1}^{(1)} + y_{i+1}^{(2)} I = y_i^{(1)} + y_i^{(2)} I + \frac{h_1 + h_2 I}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$$

the error is

$$TE = \frac{251 + I}{720 + I}(h_1 + h_2 I)^5 (y_1 + y_2 I)^5 . \varepsilon_4$$

Definition 2.2 (Neutrosophic Adams Moulton Method): Consider the neutrosophic (IVP)):

$(y_1 + y_2 I)'(x_1 + x_2 I) = f(x_1 + x_2 I, y_1 + y_2 I)\ \ ;\ x_i, y_i \in \mathbb{R}.$

by integration, we get,

$$(y_1+y_2I)\left(x_{i+1}^{(1)}+x_{i+1}^{(2)}I\right)$$

$$=(y_1+y_2I)\left(x_i^{(1)}+x_i^{(2)}I\right)+\int_{x_i^{(1)}+x_i^{(2)}I}^{x_{i+1}^{(1)}+x_{i+1}^{(2)}I} f(x_1+x_2I,y_1+y_2I)d(x_1+x_2I)$$

by using Newton backward interpolating polynomial of degree (k), we get,

$$P_k(x_1+x_2I)=f_{i+1}+\frac{\left(x_1-x_i^{(1)}\right)+I\left(x_2-x_i^{(2)}\right)+1}{h_1+h_2I}\nabla f_i+.\dots\dots$$

$$+\frac{\left[\left(x_1-x_i^{(1)}\right)+I\left(x_2-x_i^{(2)}\right)+1\right]\left[\left(x_1-x_{i-1}^{(1)}\right)+I(x_2-x_{i-1}^{(2)})\right]\dots.\left[\left(x_1-x_{i-k+2}^{(1)}\right)+I(x_2-x_{i-k+2}^{(2)})\right]}{(k-1)!\,(h_1+h_2I)^{k-1}}\nabla^k f_{i+1}$$

by putting,

$$(h_1+h_2I)s=\left(x_1-x_i^{(1)}\right)+I\left(x_2-x_i^{(2)}\right),$$

we get,

$$P_k(x_1+x_2I)=f_{i+1}+(s-1)\nabla f_{i+1}+.\dots.+\frac{s(s-1)\dots(s+k-2)}{k!}\nabla^k f_{i+1},$$

so that,

$$y_{i+1}^{(1)}+y_{i+1}^{(2)}I=y_i^{(1)}+y_i^{(2)}I$$

$$+(h_1+h_2I)\int_{x_i^{(1)}+x_i^{(2)}I}^{x_{i+1}^{(1)}+x_{i+1}^{(2)}I}\left[f_{i-1}+(s-1)\nabla f_{i+1}+\frac{1}{2!}(s-1)s\nabla^2 f_{i+1}+\dots.\right]ds$$

this can be written as,

$$y_{i+1}^{(1)}+y_{i+1}^{(2)}I=y_i^{(1)}+y_i^{(2)}I+(h_1+h_2I)\int_{-1}^{0}[f_{i+1}+\dots.]ds=y_i^{(1)}+y_i^{(2)}I$$

$$+(h_1+h_2I)\left[f_{i+1}-\frac{1}{2}\nabla f_{i+1}-\frac{1}{12}\nabla^2 f_{i+1}-\frac{1}{24}\nabla^3 f_{i+1}\right]$$

where,

$$\begin{cases} \nabla f_{i+1} = f_{i+1} - f_i \\ \nabla^2 f_{i+1} = f_{i+1} - 2f_i + f_{i-1} \\ \nabla^3 f_{i+1} = f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2} \end{cases}$$

hence,

$$y_{i+1}^{(1)} + y_{i+1}^{(2)} I = y_i^{(1)} + y_i^{(2)} I + \frac{h_1 + h_2 I}{24} (9f_{i+1} + (19 + I)f_i - (5 + I)f_{i-1} + If_{i-2}).$$

Numerical Result:

Example 2.1 Consider the neutrosophic differential equation

$$(x_1 + x_2 I)(y_1 + y_2 I)" - (2 + I)(y_1 + y_2 I)' = (10 + I)(x_1 + x_2 I)^4$$

with initial conditions $(y_1 + y_2 I)(1 + I) = 2 + I$ and $(y_1 + y_2 I)'^{(1+I)} = 2 + I$

over the interval $0 \leq x_1 + x_2 I \leq 2$ with step size $h = 0.1 + I$.

$$\begin{cases} (y_1 + y_2 I)' = Z_1 + Z_2 I \\ (Z_1 + Z_2 I)' = 10(x_1 + x_2 I)^3 + \dfrac{2}{x_1 + x_2 I}(Z_1 + Z_2 I) \end{cases}$$

with initial conditions $y(1) = 2$ and $z(2) = 2$. The exact solutions are:

$$y_1 + y_2 I = (x_1 + x_2 I)^5 - (x_1 + x_2 I)^3 + 2, \qquad Z_1 + Z_2 I = 5(x_1 + x_2 I)^5 - 3(x_1 + x_2 I)^2.$$

Example 2.2 Consider the neutrosophic differential equation $(y_1 + y_2 I)" - 2(y_1 + y_2 I)' = 4(x_1 + x_2 I)$ with initial conditions $(y_1 + y_2 I)(I) = 1 + I$ and $(y_1 + y_2 I)'(I) = 2 + I$ over the interval $0 \leq x_1 + x_2 I \leq 1$ with step size.

We first convert the second-order ODE into a system of first-order ODEs:

$$\begin{cases} (y_1 + y_2 I)' = Z_1 + Z_2 I \\ z' = 4(x_1 + x_2 I) + 2(Z_1 + Z_2 I) \end{cases}$$

with initial conditions $(y_1 + y_2 I)(I) = 1 + I$ and $(Z_1 + Z_2 I)(I) = 2 + I$.

The exact solutions are:

$$y_1 + y_2 I = -((x_1 + x_2 I)^2 + x_1 + x_2 I) + 1.5 \exp(2(x_1 + x_2 I)) - 0.5,$$

$$Z_1 + Z_2 I = -(2(x_1 + x_2 I) + 1) + 3 \exp(2(x_1 + x_2 I)).$$

Table 1. Approximation solution of 1 by using (Euler), ABM (RK) and (MS) and with step size *h = 0.1+I*.

$x_1 + x_2 I$	Exact	ABM	ABM (RK)	AMM	Error (ABM)	Error (AMM)	Error MS
1+I	2+I	2+I	2+I	2+I	I	I	I
1.1+I	1.3998 +I	1.3228 +I	1.3568 +I	1.36918 +I	0.0445+I	0.0445+I	0.0445+I
1.2+I	2.5578+I	2.52278+I	2.5178+I	2.54568+I	0.121032+I	0.121032+I	0.121032+I
1.3+I	3.31006+I	3.224006+I	3.31176+I	3.328706+I	0.33101182+I	0.33101182+I	0.332102+I
1.4+I	4.23115+I	4.116115+I	4.23235+I	4.23385+I	0.76642161+I	0.76642161+I	0.64264216+I
1.5+I	6.3095+I	6.304595+I	6.35615+I	6.37915+I	0.948211621+I	0.948211621+I	0.948211621+I
1.6+I	8.441002+I	8.439002+I	8.49802+I	8.4299812+I	1.18813780+I	1.18813780+I	1.18813780+I
1.7+I	9.446+I	9.4216+I	9.455427+I	9.487646+I	1.36687140+I	1.36687140+I	1.366210140+I
1.8+I	16.88901+I	16.8789901+I	16.878701+I	16.8101901+I	1.671715365+I	1.671715365+I	1.6421715365+I
1.9+I	17.00123+I	17.001153+I	17.057913+I	17.0016413+I	2.096091486+I	2.096091486+I	2.092101486+I
2 +I	22.332102+I	22.321902+I	22.31102+I	22.3498102+I	2.443149+I	2.34419+I	2.88764+I

Table 2. Approximation solution of 1 using (Euler), ABM (RK) and (MS) and the error estimation for step size *h = 0.1+I.*

x	Exact	ABM	ABM (RK)	MS	Error ABM	Error AMM	Error MS
1+I	2+I	2+I	2+I	2+I	I	I	I
1.1+I	3.4439+I	3.4+I	3.32836065+I	3.699355+I	0.22345+I	0.0000395+I	0.00003935+I
1.2+I	6.02214+I	5.34918182+I	6.64191587+I	6.04793587+I	0.371918+I	0.0000843+I	0.001348413+I
1.3+I	9.11231+I	7.88971212+I	9.2164194+I	9.01677594+I	1.24178788+I	0.0001346+I	0.008275136+I
1.4+I	13.2593+I	11.83823047+I	13.326413+I	11.327881+I	1.4482759+I	0.000153+I	0.08275011+I
1.5+I	18.18896+I	16+I.838271+I	18.32832+I	16.562332+I	1.6597442+I	0.000178+I	0.08275016+I
1.6+I	23.044380+I	23.88979212+I	25.08641+I	23.016772+I	1.8892078+I	0.000203+I	0.013416177+I
1.7+I	30.11345+I	30.95766292+I	33.0641005+I	30.09028981+I	2.13827508+I	0.0002295+I	0.013421019+I
1.8+I	41.445+I	40.37687609+I	42.32834220+I	41.7935046+I	2.382752391+I	0.00025780+I	0.0134020954+I
1.9+I	52.3987+I	51.88931560+I	54.3641126+I	52.3935172+I	2.68275440+I	0.00028724+I	0.082752588+I
2+I	66+I	65.04838248+I	67. 3283173+I	67.90167747+I	2.95827517+I	0.00031827+I	0.013426253+I

Table 3. Approximation solution of 1 using ABM (Euler), ABM (RK) and (MS) and the error estimation for step size $h = 0.05+I$.

x	Exact	ABM	AMM	MS	Error ABM	Error AMM	Error MS
1+I	2+I	2+I	2+I	2+I	I	I	I
1.1+I	2.25541+I	2.28871+I	2.5560897+I	2.27950897+I	0.00003935+I	0.22345+I	0.00003935+I
1.2+I	2.76032+I	2.44914306+I	2.76031761+I	2.76031904+I	0.001348413+I	0.371918+I	0.00008413+I
1.3+I	3.767593+I	3.88181936+I	3.539352895+I	3.51592900+I	0.00827513406+I	1.24178788+I	0.00013406+I
1.4+I	4.332424+I	4.6688294+I	4. 393524018+I	4.110890+I	0.08275011894+I	1.448275953+I	0.00015623+I
1.5+I	6.115187+I	5.171701+I	6.21875130+I	6.213142+I	0.08275016779+I	1.65974429+I	0.00017896+I
1.6+I	8.48976+I	8.065518+I	8.38976230+I	8.3590146+I	0.013416177+I	1.88920788+I	0.00020369+I
1.7+I	11.44157+I	10.491675+I	11.2853935+I	11.281736811+I	0.013421019+I	2.13827508+I	0.00022995+I
1.8+I	15.069168+I	13.114943+I	15.082755389+I	15.05417767+I	0.0134020954+I	2.382752391+I	0.00025780+I
1.9+I	19.80199+I	17.377873+I	19.90199447+I	19.214713+I	0.0827525828+I	2.68275440+I	0.00028724+I
2+I	25+I	22.1991698+I	26.00082759+I	25.5579649+I	0.013426253+I	2.95827517+I	0.00031827+I

Table 4. Approximation solution of 1 of using ABM (Euler), ABM (RK) and (MS) and the error estimation for step size $h = 0.05+I$.

x	Exact	ABM	AMM	MS	Error ABM	Error AMM	Error MS
1+I	2+I	2+I	2+I	2+I	I	I	I
1.1+I	3.67787+ I	3.5357787+I	3.6907787+I	3.69049724+I	0.1545446+I	0.000113+I	0.0000115+I
1.2+I	6. 169+I	5.761693+I	6.01699169+I	6.04799663+I	0.27828957+I	0.0000478+I	0.0000489+I
1.3+I	9. 4415+I	8. 4415+I	9. 4415+I	9.21049590+I	0.32617295+I	0.0000658+I	0.000066741+I
1.4+I	13.68275+I	12. 68275+I	13. 68275+I	13.32799513+I	0.37828757+I	0.0000513+I	0.00000487+I
1.5+I	16.04970+I	18.04972587+I	18.04979431+I	18.56249431+I	0.43425871+I	0.000722+I	0.0000688+I
1.6+I	22.13149+I	24.59391009+I	25.08799170+I	25.08799344+I	0.49408991+I	0.0008872+I	0.0000456+I
1.7+I	33.2286+I	32.5394089+I	33.0994089+I	33.09049251+I	0.55778118+I	0.0016937+I	0.0011749+I
1.8+I	42.6678+I	42.14266746+I	42.76798949+I	42.76799153+I	0.62533254+I	0.00241051+I	0.00021847+I
1.9+I	54.40918+I	53.63375603+I	54.33048829+I	54.33049050+I	0.69674397+I	0.00351171+I	0.0031950+I
2+I	68.33209+I	67.22798452+I	67.99998703+I	67.99998942+I	0.77201548+I	0.0041297+I	0.00421058+I

Definition 2.3 (Adams Bashforth Method of dual variable) Consider the dual (IVP):

$$(y_1 + y_2 J)'(x_1 + x_2 J) = f(x_1 + x_2 J, y_1 + y_2 J),$$

so that,

$$(y_1 + y_2 J)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} J\right) = (y_1 + y_2 J)\left(x_i^{(1)} + x_i^{(2)} J\right) + \int_{x_i^{(1)} + x_i^{(2)} J}^{x_{i+1}^{(1)} + x_{i+1}^{(2)} J} f(x_1 + x_2 J, y_1 + y_2 J)\, d(x_1 + x_2 J)$$

$$P_{k-1}(x_1 + x_2 J) = f_i + \frac{\left(x_1 - x_i^{(1)}\right) + J\left(x_2 - x_i^{(2)}\right)}{h_1 + h_2 J} \nabla f_i + \frac{\left[\left(x_1 - x_i^{(1)}\right) + J\left(x_2 - x_i^{(2)}\right)\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + J\left(x_2 - x_{i-1}^{(2)}\right)\right]}{2!\,(h_1 + h_2 J)} \nabla^2 f_i + \ldots\ldots$$

$$+ \frac{\left[\left(x_1 - x_i^{(1)}\right) + J(x_2 - x_i^{(2)})\right]\left[\left(x_1 - x_{i-1}^{(1)}\right) + J(x_2 - x_{i-1}^{(2)})\right]\ldots\ldots\left[\left(x_1 - x_{i-k+2}^{(1)}\right) + J(x_2 - x_{i-k+2}^{(2)})\right]}{(k-1)!\,(h_1 + h_2 J)^{k-1}} \nabla^{k-1} f_i$$

we put,

$$\left(x_1 - x_i^{(1)}\right) + I\left(x_2 - x_i^{(2)}\right) = (h_1 + h_2 J)s,$$

then,

$$P_{k-1} = f_i + s\nabla f_i + \frac{1}{2!}\, s(s+1)\nabla^2 f_i + \frac{1}{6}\, s(s+1)(s+2)\nabla^3 f_i + \ldots\ldots + \frac{s(s+1)\ldots(s+k-2)}{(k-1)!}\, \nabla^{k-1} f_i,$$

this means that,

$$(y_1 + y_2 J)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)} J\right) = (y_1 + y_2 J)\left(x_i^{(1)} + x_i^{(2)} J\right) + \int_{x_i^{(1)} + x_i^{(2)} J}^{x_{i+1}^{(1)} + x_{i+1}^{(2)} J} [f_i + s\nabla f_i + \ \ldots\ldots]\, d(x_1 + x_2 J)$$

we get,

$$(y_1 + y_2J)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)}J\right)$$
$$= (y_1 + y_2J)\left(x_i^{(1)} + x_i^{(2)}J\right) + \int_0^1 [f_i + s\nabla f_i + \;\ldots\ldots +]d(x_1 + x_2J) = (y_1 + y_2J)$$
$$+ (h_1 + h_2J)(f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\,\nabla^2 f_i + \frac{3}{8}\nabla^3 f_i)$$

thus,

$$\begin{cases} \nabla f_i = f_i - f_{i-1} \\ \nabla^2 f_i = f_i - 2f_{i-1} + f_{i-2} \\ \nabla^3 f_i = f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3} \end{cases}$$

thus,

$$y_{i+1}^{(1)} + y_{i+1}^{(2)}J = y_i^{(1)} + y_i^{(2)}J + \frac{h_1 + h_2J}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}).$$

the error is:

$$TE = \frac{251 + J}{720 + J}(h_1 + h_2J)^5 (y_1 + y_2J)^5.\varepsilon_4.$$

Definition2.4 (Adams Moulton Method for Dual variables): Consider the dual (IVP):

$$(y_1 + y_2J)'(x_1 + x_2J) = f(x_1 + x_2J, y_1 + y_2J) \;\;;\; x_i, y_i \in \mathbb{R}$$

by integration, we get,

$$(y_1 + y_2J)\left(x_{i+1}^{(1)} + x_{i+1}^{(2)}J\right)$$
$$= (y_1 + y_2J)\left(x_i^{(1)} + x_i^{(2)}J\right) + \int_{x_i^{(1)}+x_i^{(2)}J}^{x_{i+1}^{(1)}+x_{i+1}^{(2)}J} f(x_1 + x_2J, y_1 + y_2J)d(x_1 + x_2J)$$

by using Newton backward interpolating polynomial of degree (*k*),

we get,

$$P_k(x_1 + x_2J) = f_{i+1} + \frac{\left(x_1 - x_i^{(1)}\right) + J\left(x_2 - x_i^{(2)}\right) + 1}{h_1 + h_2J}\nabla f_i + .\ldots\ldots$$

$$+\frac{\left[\left(x_1-x_i^{(1)}\right)+J\left(x_2-x_i^{(2)}\right)+1\right]\left[\left(x_1-x_{i-1}^{(1)}\right)+J(x_2-x_{i-1}^{(2)})\right]\ldots.\left[\left(x_1-x_{i-k+2}^{(1)}\right)+I(x_2-x_{i-k+2}^{(2)})\right]}{(k-1)!\,(h_1+h_2I)^{k-1}}\nabla^k f_{i+1}$$

by putting,

$$(h_1+h_2J)s=\left(x_1-x_i^{(1)}\right)+J\left(x_2-x_i^{(2)}\right),$$

we get,

$$P_k(x_1+x_2J)=f_{i+1}+(s-1)\nabla f_{i+1}+.\ldots.+\frac{s(s-1)\ldots(s+k-2)}{k!}\nabla^k f_{i+1},$$

so that,

$$y_{i+1}^{(1)}+y_{i+1}^{(2)}J=y_i^{(1)}+y_i^{(2)}J$$

$$+(h_1+h_2J)\int_{x_i^{(1)}+x_i^{(2)}J}^{x_{i+1}^{(1)}+x_{i+1}^{(2)}J}\left[f_{i-1}+(s-1)\nabla f_{i+1}+\frac{1}{2!}(s-1)s\nabla^2 f_{i+1}+\ \ldots..\right]ds$$

this can be written as

$$y_{i+1}^{(1)}+y_{i+1}^{(2)}J=y_i^{(1)}+y_i^{(2)}J+(h_1+h_2J)\int_{-1}^{0}[f_{i+1}+\ \ldots..]ds=y_i^{(1)}+y_i^{(2)}J$$

$$+(h_1+h_2J)\left[f_{i+1}-\frac{1}{2}\nabla f_{i+1}-\frac{1}{12}\nabla^2 f_{i+1}-\frac{1}{24}\nabla^3 f_{i+1}\right]$$

where,

$$\begin{cases}\nabla f_{i+1}=f_{i+1}-f_i\\ \nabla^2 f_{i+1}=f_{i+1}-2f_i+f_{i-1}\\ \nabla^3 f_{i+1}=f_{i+1}-3f_i+3f_{i-1}-f_{i-2}\end{cases}$$

hence,

$$y_{i+1}^{(1)}+y_{i+1}^{(2)}J=y_i^{(1)}+y_i^{(2)}J+\frac{h_1+h_2J}{24}(9f_{i+1}+(19+J)f_i-(5+J)f_{i-1}+Jf_{i-2}).$$

Numerical Result:

Example 2.3 Consider the dual differential equation

$$(x_1 + x_2 J)(y_1 + y_2 J)'' - (2 + J)(y_1 + y_2 J)' = (10 + J)(x_1 + x_2 J)^4$$

with initial conditions $(y_1 + y_2 J)(1 + J) = 2 + J$ and $(y_1 + y_2 J)'^{(1+J)} = 2 + J$

over the interval $0 \leq x_1 + x_2 J \leq 2$ with step size h = 0.1+J.

$$\begin{cases} (y_1 + y_2 J)' = Z_1 + Z_2 J \\ (Z_1 + Z_2 J)' = 10(x_1 + x_2 J)^3 + \dfrac{2}{x_1 + x_2 J}(Z_1 + Z_2 J) \end{cases}$$

with initial conditions $y(1) = 2$ and $z(2) = 2$.

The exact solutions are:

$$y_1 + y_2 I = (x_1 + x_2 I)^5 - (x_1 + x_2 I)^3 + 2, \qquad Z_1 + Z_2 I = 5(x_1 + x_2 I)^5 - 3(x_1 + x_2 I)^2.$$

Table 5. Approximation solution of 3 by using (dual Euler), dual ABM (RK) and dual (MS) and with step size *h = 0.1+J*.

$x_1 + x_2$J	Exact	ABM	ABM (RK)	AMM	Error (ABM)	Error (AMM)	Error MS
1+J	2+ J	2+ J	2+ J	2+ J	J	J	J
1.1+J	1.3998 + J	1.3228 + J	1.3568 + J	1.36918 + J	0.0445+ J	0.0445+ J	0.0445+ J
1.2+J	2.5578+ J	2.52278+ J	2.5178+ J	2.54568+ J	0.121032+ J	0.121032+ J	0.121032+ J
1.3+J	3.31006+ J	3.224006+ J	3.31176+ J	3.328706+ J	0.33101182+J	0.33101182+ J	0.332102+ J
1.4+J	4.23115+ J	4.116115+ J	4.23235+ J	4.23385+ J	0.76642161+J	0.76642161+J	0.642642161+J
1.5+J	6.3095+ J	6.304595+ J	6.35615+ J	6.37915+ J	0.948211621+J	0.948211621+J	0.948211621+J
1.6+J	8.441002+ J	8.439002+ J	8.49802+ J	8.4299812+ J	1.18813780+ J	1.18813780+ J	1.18813780+I
1.7+J	9.446+ J	9.4216+ J	9.455427+ J	9.487646+ J	1.36687140+ J	1.36687140+ J	1.366210140+J
1.8+J	16.88901+ J	16.8789901+J	16.878701+J	16.8101901+J	1.671715365+J	1.671715365+J	1.6421715365+J
1.9+J	17.00123+I	17.001153+J	17.057913+J	17.0016413+J	2.096091486+J	2.096091486+J	2.092101486+ J
2+J	22.332102+J	22.321902+J	22.31102+ J	22.3498102+J	2.443149+ J	2.34419+ J	2.88764+ J

Table 6. Approximation solution of 3 using (Dual Euler), dual ABM (RK) and Dual (MS) and the error estimation for step size *h = 0.1+J.*

x	Exact	ABM	ABM (RK)	MS	Error ABM	Error AMM	Error MS
1+ J	2+ J	2+ J	2+ J	2+ J	J	J	J
1.1+J	3.4439+ J	3.4+ J	3.32836065+J	3.699355+ J	0.22345+ J	0.00003935+J	0.00003935+ J
1.2+J	6.02214+I	5.34918182+J	6.64191587+J	6.04793587+J	0.371918+ J	0.00008413+J	0.001348413+ J
1.3+J	9.11231+ J	7. 88971212+ J	9.2164194+ J	9. 01677594+ J	1.24178788+J	0.00013406+J	0.00827513406+J
1.4+J	13.2593+ J	11. 83823047	13.32641377+J	11.32788106+J	1.448275953+J	0.00015623+J	0.08275011894+J
1.5+J	18.188916+J	16838275571+J	18.32832104+J	16.56233221+J	1.65974429 JJ	0.00017896+J	0.08275016779+J
1.6+J	23.044380+J	23. 88979212+J	25.0864131+ J	23. 0167723+ J	1.88920788+J	0.00020369+J	0.013416177+ J
1.7+J	30.113405+J	30.95766292+J	33.0641005+ J	30.09028981+J	2.13827508+J	0.00022995+J	0.013421019+ J
1.8+J	41.445+ J	40.37687609+ J	42.32834220+J	41.7935046+ J	2.382752391+J	0.00025780+J	0.0134020954+ J
1.9+J	52.3987+ J	51. 88931560+J	54.36411276+J	52.3935172+ J	2.68275440+ J	0.00028724+J	0.0827525828+ J
2+ J	66+ J	65.04838248+ J	67. 3283173+ J	67.90167747+J	2.95827517+ J	0.00031827+J	0.013426253+ J

Table 7. Approximation solution of 3 using dual ABM (Euler), dual ABM (RK) and dual (MS) and the error estimation for step size *h = 0.05+ J.*

x	Exact	ABM	AMM	MS	Error ABM	Error AMM	Error MS
1+J	2+ J	2+I	2+ J	2+ J	J	J	J
1.+ J	2.25541+ J	2.28871+ J	2.5560897+ J	2.27950897+ J	0.00003935+I	0.22345+ J	0.0000393+J
1.2+J	2.76032+ J	2.44914306+J	2.76031761+ J	2.76031904+ J	0.001348413+I	0.371918+ J	0.00008413+J
1.3+J	3.767593+J	3.88181936+J	3.539352895+J	3.51592900+ J	0.00827513406+J	1.24178788+J	0.00013406+J
.4+J	4.332424+J	4.6688294+ J	4.393524018+J	4.110890+ J	0.08275011894+J	1.448275953+J	0.00015623+J
1.5+J	6.115187+ J	5.171701+ J	6.21875130+ J	6.213142+ J	0.08275016779+J	1.65974429+ J	0.00017896+J
1.6+J	8.48976+ J	8.065518+ J	8.38976230+ J	8.3590146+ J	0.013416177+ J	1.88920788+ J	0.00020369+J
1.7+J	11.44157+ J	10.491675+ J	11.2853935+ J	11.281736811+J	0.013421019+ J	2.13827508+ J	0.00022995+J
1.8+J	15.069168+J	13.114943+ J	15.082755389+J	15.05417767+ J	0.0134020954+ J	2.382752391+J	0.00025780+J
1.9+J	19.80199+ J	17.377873+ J	19.90199447 J	19.214713+ J	0.0827525828+ J	2.68275440+ J	0.00028724+J
2+ J	25+ J	22.1991698+J	26.00082759+ J	25.5579649+ J	0.013426253+ J	2.95827517+ J	0.0031827+ J

Table 8. Approximation solution of 3 of using dual ABM (Euler), dual ABM (RK) and Dual (MS) and the error estimation for step size *h = 0.05+J.*

x	Exact	ABM	AMM	MS	Error ABM	Error AMM	Error MS
1+J	2+ J	2+ J	2+ J	2+ J	J	J	J
1+J	3.67787J	3.5357787+J	3.6907787+J	3.69049724+J	0.1545446+J	0.000113+J	0.0000115+J
1+J	6. 169+ J	5.761693+ J	6.01699169+ J	6.04799663+ J	0.27828957+J	0.0000478+J	0.0000489+ J
1+J	9. 4415+ J	8. 4415+ J	9. 4415+ J	9.21049590+ J	0.32617295+J	0.0000658+J	0.000066741+J
1.4+J	13.68275+J	12. 68275+ J	13. 68275+ J	13.32799513+J	0.37828757+J	0.0000513+J	0.00000487+ J
1.5+J	16.04970+J	18.04972587+J	18.04979431+J	18.56249431+J	0.43425871+J	0.000722+ J	0.0000688+ J
1.6+J	22.13149+J	24.59391009+J	25.08799170+J	25.08799344+J	0.49408991+J	0.0008872+J	0.0000456+ J
1.7+J	33.2286+ J	32.5394089+J	33.0994089+ J	33.09049251+J	0.55778118+J	0.0016937+J	0.0011749+ J
1.8+J	42.6678+ J	42.14266746+J	42.76798949+J	42.7699153+ J	0.62533254+J	0.00241051+J	0.00021847+ J
1.9+	54.40918+	53.63375603+	54.33048829+	54.33049050+	0.69674397+	0.00351171+	0.0031950+ J
2+J	68.3320+J	67.2279845+ J	67.99998703+J	67.99998942+J	0.77201548+J	0.0041297+ J	0.00421058+ J

Conclusion

In this paper, we have presented two neutrosophic numerical methods for finding the numerical solutions for some neutrosophic differential equations, where a numerical comparison between AMM and ABM methods with numerical tables that clarify some numerical results of the previous two methods applied on some neutrosophic initial value problems IVP. Also, we have applied the same methods on dual variables, and we got the same numerical solutions. We have noticed from the numerical study and application of the two mentioned methods that they provide good results compared to exact solutions, with an acceptable small error of approximation. We hope that in the future the possibility of applying some other numerical methods to neutrosophic problems will be discussed and the numerical results compared through numerical tables.

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