A Comparison between Euler's Method and 4-th Order Runge-Kutta Method For Numerical Solutions of Neutrosophic and Dual Differential Problems

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Abstract:

This paper is dedicated to study some numerical solutions of neutrosophic and dual differential problems through a numerical comparison between neutrosophic Euler's method and 4-th order Runge-Kutta neutrosophic and dual method, where we apply both methods on the same problems and compare the numerical results of solutions and numerical estimations of errors by using numerical tables.

Keywords: Numerical solution, neutrosophic problem, neutrosophic error, numerical estimation

1. Introduction

Numerical analysis is a basic branch of mathematics and is primarily concerned with finding approximate values and approximate solutions to algebraic or differential equations, for which solutions are difficult to find using traditional methods [1-6]. Numerical analysis is closely related to algorithms and computer science [7-10]. The concept of neutrosophic logic and the neutrosophic set was developed to deal with the presence of indeterminacy in scientific data and theses, and sometimes to deal with the lack or conflict of data related to a certain research question [11-15]. In many research papers published recently [16-20], numerical solutions have been studied and found for many equations that contain an element of indeterminacy (neutrosophic), where the effectiveness of many methods in studying these solutions has been compared and classified into independent tables called numerical tables

[21-25]. In this research, we are interested in following in the footsteps of researchers in works concerned with neutrosophic numerical analysis [26-33], where we study some numerical solutions of neutrosophic and dual differential problems through a numerical comparison between neutrosophic Euler's method and 4-th order Runge-Kutta neutrosophic and dual method, where we apply both methods on the same problems and compare the numerical results of solutions and numerical estimations of errors by using numerical tables.

2. Main Discussion

Definition 2.1 Consider the following (IVP) of neutrosophic variables:

$$\begin{cases} (y_1 + y_2 I)' = \dfrac{d(y_1 + y_2 I)^{\emptyset}}{d(x_1 + x_2 I)} = f(x_1 + x_2 I, y_1 + y_2 I) \\ (y_1 + y_2 I)(x_0 + x_0' I) = y_0 + y_0' I \qquad ; x_0, x_0', y_0, y_0' \in \mathbb{R}. \end{cases},$$

The method can be explained by the following equations:

$$(y_1 + y_2 I)(x_1 + x_2 I) = (y_n + y_n' I) + f(x_0 + x_0' I, y_0 + y_0' I)(x_0 + x_0' I,$$

when $x_1 + x_2 I = x_1' + x_2' I$, we have:

$$y_1 + y_1' I = y_0 + y_0' I + (h_1 + h_2 I) f(x_0 + x_0' I, y_0 + y_0' I),$$

The next approximations are:

$y_2 + y_2' I = y_1 + y_1' I + (h_1 + h_2 I) f(x_1 + x_1' I, y_1 + y_1' I)$, at the point $x_1 + x_2 I = x_2 + x_2' I$,

$y_3 + y_3' I = y_2 + y_2' I + (h_1 + h_2 I) f(x_2 + x_2' I, y_2 + y_2' I)$, and $y_{n+1} + y_{n+1}' I = y_n + y_n' I + (h_1 + h_2 I) f(x_n + x_n' I, y_n + y_n' I)$ for all $n = 0,1,2, \dots.$

Definition 2.2 Consider the following (IVP) of neutrosophic variables:

$$(y_1 + y_2 I)' = f(x_1 + x_2 I, y_1 + y_2 I), (y_1 + y_2 I)(x_0 + x_0' I) = y_0 + y_0' I,$$

Let

$$h = (x_1 + x_1' I) - (x_0 + x_0' I) = (x_1 - x_0) + I(x_1' - x_0'),$$

then,

$$(y_1 + y_2 I)(x + x'I + h + h'I)$$
$$= (y_1 + y_2 I)(x + x'I) + (h + h'I)(y_1 + y_2 I)'(x + x'I)$$
$$+ \frac{(h + h'I)^2}{2!}(y_1 + y_2 I)''(x + x'I) + ...$$

Hence,

$$(y_1 + y_2 I)(x + x'I + h + h'I) - (y_1 + y_2 I)(x + x'I)$$
$$= (h + h'I)(y_1 + y_2 I)'(x + x'I) + \frac{(h + h'I)^2}{2!}(y_1 + y_2 I)''(x + x'I) + ...$$

By partial differenation, we get:

$$y' = (y_1 + y_2 I)' = f(x_1 + x_2^* I, y_1 + y_2 I) = f,$$

$$y'' = f_{x_1 + x_2 I} + f . f_{y_1 + y_2 I},$$

$$y''' = f_{(x_1+x_2I)(x_1+x_2I)} + 2ff_{(x_1+x_2I)(y_1+y_2I)} + f^2 f_{(y_1+y_2I)(y_1+y_2I)} + f_{(x_1+x_2I)(y_1+y_2I)} + ff^2_{y_1+y_2I},$$

Take:

$$F_1 + F_1' I = f_{x_1 + x_2 I} + f . f_{y_1 + y_2 I},$$

$$F_2 + F_2' I = f_{(x_1+x_2I)(x_1+x_2I)} + 2ff_{(x_1+x_2I)(y_1+y_2I)} + f^2 f_{(x_1+x_2I)(y_1+y_2I)},$$

$$F_3 + F_3' I = f_{(x_1+x_2I)(x_1+x_2I)(x_1+x_2I)} + 3ff_{(x_1+x_2I)(x_1+x_2I)(y_1+y_2I)} + f^2 f_{(y_1+y_2I)(y_1+y_2I)(y_1+y_2I)} + f^2 f_{(y_1+y_2I)(y_1+y_2I)(y_1+y_2I)},$$

hence,

$$\begin{cases} F_1 + F_1' I = y'' \\ F_2 + F_2 I + f_{(y_1+y_2I)}(F_1 + F_1 I) = y''' \\ (F_3 + F_3 I) + f_{(y_1+y_2I)}(F_2 + F_2 I) + 3(F_1 + F_1 I)(f_{(x_1+x_2I)(y_1+y_2I)} + f . f_{(y_1+y_2I)(y_1+y_2I)}) \end{cases},$$

$$(y_1 + y_2 I)(x + x'I + h + h'I) - (y_1 + y_2 I)(x + x'I) = (h_1 + h_2 I)(f) + \frac{(h_1 + h_2 I)^2}{2!}(F_1 + F_1 I) + ...$$

$$K_1 + K_1' I = (h + h'I) f,$$

$$K_2 + K_2' I = (h + h'I) f\big(x + x'I + (m + m'I)(h + h'I), y + y'I + (m + m'I)(K_1 + K_1' I)\big),$$

$$K_3 + K_3' I = (h + h'I) f\big(x + x'I + (n + n'I)(h + h'I), y + y'I + (n + n'I)(K_2 + K_2' I)\big),$$

$$K_4 + K_4' I = (h + h'I) f(x + x'I + (p + p'I)(h + h'I), y + y'I + (p + p'I)(K_3 + K_3' I)),$$

$\Delta_{y_1 + y_2 I} = (y_1 + y_2 I)(x + x'I, y_1 + y_2 I) - (y_1 + y_2 I)(x_1 + x_2 I) = a(K_1 + K_1' I) + b(K_2 + K_2' I) + c(K_3 + K_3' I) + d(K_4 + K_4' I),$

Now, we can use Taylor's expansion for several variables:

$$K_1 + K_1' I = (h + h'I) f,$$

$K_2 + K_2' I = (h + h'I) + \Big[f + (m + m'I)(h + h'I)(F_1 + F_1' I) + \frac{1}{2}(m + m'I)^2 (h + h'I)^2 (F_2 + F_2' I) + \ \ldots\ldots\Big],$

$K_3 + K_3' I = (h + h'I) + \Big[f + (n + n'I)(h + h'I)(F_1 + F_1' I) + \frac{1}{2}(h + h'I)^2 \Big((n + n'I)^2 (F_2 + F_2' I) + 2(m + m'I)(n + n'I) f_y (F_1 + F_1 I)\Big) + \frac{1}{6}(h + h'I)^3 \big((n + n'I)^3 (F_3 + F_3' I) + 3(m + m'I)^2 (n + n'I) f_y F_2 I\big) + 6\,(m + m'I)(n + n'I)^2 (F_1 + F_1 I)\big(F_{xIyI} + f f_{yy}\big) + \ldots\ldots\Big],$

$K_4 + K_4' I = (h + h'I) + \Big[f + (p + p'I)(h + h'I)(F_1 + F_1' I) + \frac{1}{2}(h + h'I)^2 \big[(p + p'I)^2 (F_2 + F_2' I) + 2(n + n'I)(p + p'I) f_y (F_1 + F_1' I)\big] + \frac{1}{6}(h + h'I)^3 \big[(p + p'I)^3 (F_3 + F_3' I) + 3(n + n'I)^2 (p + p'I) f_y (F_2 + F_2' I)\big] + 6\,(n + n'I)(p + p'I)(F_1 + F_1 I)\big(F_{xIyI} + f f_{y^2 I} + \ldots\ldots\big) \ldots.\Big],$

hence,

$-(y_1 + y_2 I)(x + x'I) + (y_1 + y_2 I)\big(x + h + I(x' + h')\big) = a(h + h'I) f + b(h + h'I)[f + (m + m'I)(h + h'I)(F_1 + F_1 I) + \ldots\ldots] + c(h + h'I)\Big[f + (n + n'I)(h + h'I)(F_1 + F_1' I) + \frac{1}{2}(h + h'I)^2 (n + n'I)^2 F_2 I\Big] + \frac{1}{6}(h + h'I)^3 [(p + p'I)^3 (F_3 + F_3' I) + 3(n + n'I)^2 (p + p'I) F_2 I + \ldots\ldots],$

thus,

$$(y_1 + y_2 I)\big(x + h + I(x' + h')\big) - (y_1 + y_2 I)(x + x'I)$$
$$= (a + b + c + d)(h + h'I)f$$
$$+ [b(m + m'I) + c(n + n'I) + d(p + p'I)](h + h'I)^2(F_2 + F_2'I)$$
$$+ [b(m + m'I)^2 + c(n + n'I)^2 + d(p + p'I)^2]\frac{(h + h'I)^3(F_2 I)}{2}$$
$$+ [b(m + m'I)^3 + c(n + n'I)^3 + d(p + p'I)^3]\frac{(h + h'I)^4(F_2 I)}{6} + \ldots\ldots,$$

hence,

$$\begin{cases} a + b + c + d = 1 \\ b(m + m'I) + c(n + n'I) + d(p + p'I) = \frac{1}{2} + I \\ b(m + m'I)^2 + c(n + n'I)^2 + d(p + p'I)^2 = \frac{1}{3} + I \\ b(m + m'I)^3 + c(n + n'I)^3 + d(p + p'I)^3 = \frac{1}{4} + I \end{cases},$$

and

$$\begin{cases} c(m + m'I)(n + n'I) + d(n + n'I)(p + p'I) = \frac{1}{6} + I \\ c(m + m'I)^2(n + n'I) + d(n + n'I)^2(p + p'I) = \frac{1}{12} + I \\ c(m + m'I)(n + n'I)^2 + d(n + n'I)(p + p'I)^2 = \frac{1}{8} + I \\ d(m + m'I)(n + n'I)(p + p'I) = \frac{1}{24} \end{cases},$$

so that:

$$\begin{cases} m = \frac{1}{2} \\ n = \frac{1}{2} \\ m' = n' = 1 \end{cases}, \begin{cases} p = 1 \\ p' = 1 \end{cases}, \begin{cases} a = d = \frac{1}{6} \\ b = c = \frac{1}{3} \end{cases}.$$

thus,

$$K_1 + K_1'I = (h + h'I)f(x_0 + x_0'I, y_0 + y_0'I) = df.$$

$$K_2 + K_2'I = (h + h'I)f\left(x_0 + x_0'I + \frac{h+h'I}{2}, y_0 + y_0'I + \frac{K_1+K_1'I}{2}\right),$$

$$K_3 + K_3'I = (h + h'I)f\left(x_0 + x_0'I + \frac{h+h'I}{2}, y_0 + y_0'I + \frac{K_2+K_2'I}{2}\right),$$

$$K_4 + K_4'I = (h + h'I)f(x_0 + x_0'I + h + h'I, y_0 + y_0'I + K_3 + K_3'I)$$

$$\Delta_{y_1+y_2I} = \frac{1}{6}[(K_1 + 2K_2 + 2K_3 + K_4) + I(K_1' + 2K_2' + 2K_3' + K_4')],$$

$$y_1 + y_1'I = y_0 + y_0'I + \frac{1}{6}[(K_1 + 2K_2 + 2K_3 + K_4) + I(K_1' + 2K_2' + 2K_3' + K_4')]\ ,$$

this implies that the neutrosophic 4-th order Runge- Kutta formula is:

$$K_1 + K_1'I = (h + h'I)f(x_n + x_n'I, y_n + y_n'I) = df,$$

$$K_2 + K_2'I = (h + h'I)f\left(x_n + x_n'I + \frac{h+h'I}{2}, y_n + y_n'I + \frac{K_1+K_1'I}{2}\right),$$

$$K_3 + K_3'I = (h + h'I)f\left(x_n + x_n'I + \frac{h+h'I}{2}, y_n + y_n'I + \frac{K_2+K_2'I}{2}\right),$$

$$K_4 + K_4'I = (h + h'I)f(x_n + x_n'I + h + h'I, y_n + y_n'I + K_3 + K_3'I),$$

$$y_{n+1} + y_{n+1}'I = y_n + y_n'I + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) + \frac{1}{6}I(K_1' + 2K_2' + 2K_3' + K_4'),$$

Numerical comparison:

Consider the following neutrosophic problem:

$$(x_1 + x_2I)(y_1 + y_2I)'' - 2(y_1 + y_2I)' = (10 + 10I)(x_1 + x_2I)^4,$$

$$\begin{cases} (y_1 + y_2I)(1 + I) = 2 + I \\ (y_1 + y_2I)'(1 + I) = 2 + I \\ h = 0.1 + 0.1I \\ 0 \le x_1 \le 2, 0 \le x_2 \le 1 \end{cases},$$

hence,

$$\begin{cases} y_1' + y_2'I = z_1 + z_2I \\ z_1' + z_2'I = (10 + I)(x_1 + x_2I)^3 + \dfrac{2}{x_1 + x_2I}(z_1 + z_2I), \end{cases}$$

Now, consider the following problem:

$$(y_1 + y_2 I)'' - 2(y_1 + y_2 I)' = 4(x_1 + x_2 I),$$

with

$$\begin{cases} (y_1 + y_2 I)(0) = 1 + I \\ (y_1 + y_2 I)'(0) = 2 + I \\ 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{cases}.$$

hence,

$$\begin{cases} (y_1 + y_2 I)'' = z_1 + z_2 I \\ (z_1' + z_2' I) = (4 + I)(x_1 + x_2 I) + (2 + I)(z_1 + z_2 I), \end{cases}$$

with

$$\begin{cases} (y_1 + y_2 I)(0) = 1 + I \\ (z_1 + z_2 I)(0) = 2 + I \end{cases}.$$

Numerical Result:

Example 2.1 Consider the neutrosophic differential equation

$$(x_1 + x_2 I)(y_1 + y_2 I)'' - (2 + I)(y_1 + y_2 I)' = (10 + I)(x_1 + x_2 I)^4,$$

with initial conditions

$$(y_1 + y_2 I)(1 + I) = 2 + I,$$

and

$$(y_1 + y_2 I)'^{(1+I)} = 2 + I,$$

over the interval $0 \le x_1 + x_2 I \le 2$ with step size $h = 0.1 + I$.

$$\begin{cases} (y_1 + y_2 I)' = Z_1 + Z_2 I \\ (Z_1 + Z_2 I)' = 10(x_1 + x_2 I)^3 + \dfrac{2}{x_1 + x_2 I}(Z_1 + Z_2 I)' \end{cases}$$

with initial conditions $y(1) = 2$ and $z(2) = 2$.

The exact solutions are:

$$y_1 + y_2 I = (x_1 + x_2 I)^5 - (x_1 + x_2 I)^3 + 2, \qquad Z_1 + Z_2 I = 5(x_1 + x_2 I)^5 - 3(x_1 + x_2 I)^2.$$

Example 2.2 Consider the neutrosophic differential equation

$$(y_1 + y_2 I)'' - 2(y_1 + y_2 I)' = 4(x_1 + x_2 I),$$

with initial conditions

$$(y_1 + y_2 I)(I) = 1 + I,$$

and

$$(y_1 + y_2 I)'(I) = 2 + I,$$

over the interval $0 \leq x_1 + x_2 I \leq 1$ with step size.

We first convert the second-order ODE into a system of first-order ODEs:

$$\begin{cases} (y_1 + y_2 I)' = Z_1 + Z_2 I \\ z' = 4(x_1 + x_2 I) + 2(Z_1 + Z_2 I)' \end{cases}$$

with initial conditions

$$(y_1 + y_2 I)(I) = 1 + I,$$

and

$$(Z_1 + Z_2 I)(I) = 2 + I.$$

The exact solutions are:

$$y_1 + y_2 I = -((x_1 + x_2 I)^2 + x_1 + x_2 I) + 1.5 \exp(2(x_1 + x_2 I)) - 0.5,$$

$$Z_1 + Z_2 I = -(2(x_1 + x_2 I) + 1) + 3 \exp(2(x_1 + x_2 I)).$$

Table 1. Approximation solution of 1 by using (Euler), ABM (RK) and with step size $h = 0.1 + I$.

$x_1 + x_2 I$	Exact	Euler method	Runge-Kutta	Error (Euler)	Error(Runge-Kutta)
1+I	2+I	2+I	2+I	I	I
1.1+I	1.3998 +I	1.36918 +I	1.3568 +I	0.0445+I	0.0445+I
1.2+I	2.5578+I	2.544168+I	2.54418+I	0.12441032+I	0.1244132+I
1.3+I	3.31006+I	3.324416+I	3.3144176+I	0.331044182+I	0.344101182+I
1.4+I	4.23115+I	4.23385+I	4.23235+I	0.76642161+I	0.76642161+I
1.5+I	6.3095+I	6.378890115+I	6.3889015+I	0.9482188901+I	0.9482889011+I
1.6+I	8.441002+I	8.429889012+I	8.488901802+I	1.18813788901+I	1.18890113780+I
1.7+I	9.446+I	9.48890146+I	9.455427+I	1.36687140+I	1.36687140+I
1.8+I	16.88901+I	16.81889011901+I	16.888901701+I	1.88901715365+I	1.68890115365+I
1.9+I	17.00123+I	17.0016413+I	17.057913+I	2.096091486+I	2.096091486+I
2 +I	22.332102+I	22.3498102+I	22.31102+I	2.443149+I	2.34419+I

Table 2. Approximation solution of 1 using (Euler), ABM (RK) and the error estimation for step size $h = 0.1 + I$.

x	Exact	Euler method	Runge-Kutta	Error Euler	ErrorRunge-Kutta
1+I	2+I	2+I	2+I	I	I
1.1+I	3.4439+I	3.4+I	3.699355+I	0.22345+I	0.00003935+I
1.2+I	6.02214+I	5.34911123182+I	6.0471123187+I	0.31123118+I	0.00011231413+I
1.3+I	9.11231+I	7. 88971212+I	9.016711231594+I	1.241123178788+I	0.0011231406+I
1.4+I	13.2593+I	11. 83823047	11.32788106+I	1.448275953+I	0.00015623+I
1.5+I	18.188916+I	16+I.838438571+I	16.5643821+I	1.6543829+I	0.0043896+I
1.6+I	23.044380+I	23. 88979212+I	23. 0167723+I	1.88920788+I	0.00020369+I
1.7+I	30.113405+I	30.95743892+I	30.0943881+I	2.13438508+I	0.0043895+I

1.8+I	41.445+I	40.37687609+I	41.7935046+I	2.382752391+I	0.00025780+I
1.9+I	52.3987+I	51. 8893481960+I	52.3948192+I	2.684819440+I	0.0481924+I
2+I	66+I	65.04838248+I	67.90167747+I	2.95827517+I	0.00031827+I

Table 3. Approximation solution of 2 using ABM (Euler), ABM (RK) and the error estimation for step size $h = 0.05 + I$.

x	**Exact**	**Euler**	**Runge-Kutta**	**Error Euler**	**ErrorRunge-Kutta**
1+I	2+I	2+I	2+I	I	I
1.1+I	2.25541+I	2.2481997+I	2.5691697+I	0.00003935+I	0.22481945+I
1.2+I	2.76032+I	2.7220804+I	2.7669162208061+I	0.00008413+I	0.34819918+I
1.3+I	3.767593+I	3.5154819+I	3.53691652895+I	0.00013406+I	1.248198788+I
1.4+I	4.332424+I	4.11084819+I	4. 393220804018+I	0.00015623+I	1.448275953+I
1.5+I	6.115187+I	6.22208042+I	6.2122080130+I	0.0001691696+I	1.65974429+I
1.6+I	8.48976+I	8.3481946+I	8.3691676230+I	0.0002691669+I	1.88481920788+I
1.7+I	11.44157+I	11.281736811+I	11.2853935+I	0.00022995+I	2.13827508+I
1.8+I	15.069168+I	15.05481967+I	15.0220805389+I	0.00026916780+I	2.3848192391+I
1.9+I	19.80199+I	19.22208013+I	19.9022080447+I	0.000269164+I	2.648195440+I
2+I	25+I	25.5481949+I	26.00082759+I	0.0003691627+I	2.9481927517+I

Table 4. Approximation solution of 2 of using ABM (Euler), ABM (RK) and the error estimation for step size $h = 0.05 + I$.

x	**Exact**	**Euler**	**Runge-Kutta**	**Error ABM**	**Error AMM**
1+I	2+I	2+I	2+I	I	I
1.1+I	3.67787+I	3.69049724+I	3.6907787+I	0.0000115+I	0.000113+I
1.2+I	6. 169+I	6.04799663+I	6.01699169+I	0.0000489+I	0.0000478+I
1.3+I	9. 4415+I	9.21049590+I	9. 4415+I	0.000066741+I	0.0000658+I
1.4+I	13. 68275+I	13.32799513+I	13. 68275+I	0.00000487+I	0.0000513+I
1.5+I	16. 04970+I	18.56249431+I	18. 04979431+I	0.0000688+I	0.000722+I
1.6+I	22.13149+I	25.08799344+I	25.08799170+I	0.0000456+I	0.0008872+I

1.7+I	33.2286+I	33.09049251+I	33.0994089+I	0.0011749+I	0.0016937+I
1.8+I	42.6678+I	42.76799153+I	42.76798949+I	0.00021847+I	0.00241051+I
1.9+I	54.40918+I	54.33049050+I	54.33048829+I	0.0031950+I	0.00351171+I
2+I	68.33209+I	67.99998942+I	67.99998703+I	0.00421058+I	0.0041297+I

Definition 2.3 Consider the following (IVP) of dual variables:

$$\begin{cases} (y_1 + y_2 J)' = \dfrac{d(y_1 + y_2 J)^{\emptyset}}{d(x_1 + x_2 J)} = f(x_1 + x_2 J, y_1 + y_2 J) \\ (y_1 + y_2 J)(x_0 + x_0' J) = y_0 + y_0' J \qquad ; x_0, x_0', y_0, y_0' \in \mathbb{R} \end{cases}.$$

The method can be explained by the following equations:

$$(y_1 + y_2 J)(x_1 + x_2 J) = (y_n + y_n' J) + f(x_0 + x_0' J, y_0 + y_0' J)(x_0 + x_0' J),$$

when,

$$x_1 + x_2 J = x_1' + x_2' J,$$

we have:

$$y_1 + y_1' J = y_0 + y_0' J + (h_1 + h_2 J) f(x_0 + x_0' J, y_0 + y_0' J),$$

The next approximations are:

$$y_2 + y_2' J = y_1 + y_1' J + (h_1 + h_2 J) f(x_1 + x_1' J, y_1 + y_1' J),$$

at the point

$$x_1 + x_2 J = x_2 + x_2' J,$$

$$y_3 + y_3' J = y_2 + y_2' J + (h_1 + h_2 J) f(x_2 + x_2' J, y_2 + y_2' J),$$

and

$$y_{n+1} + y_{n+1}' J = y_n + y_n' J + (h_1 + h_2 J) f(x_n + x_n' J, y_n + y_n' J) \text{ for all } n = 0,1,2, \dots..$$

Definition 2.4 Consider the following (IVP) of dual variables:

$$(y_1 + y_2J)' = f(x_1 + x_2J, y_1 + y_2J), (y_1 + y_2J)(x_0 + x_0'J) = y_0 + y_0'J.$$

let

$$h = (x_1 + x_1'J) - (x_0 + x_0'J) = (x_1 - x_0) + J(x_1' - x_0'),$$

then,

$$(y_1 + y_2J)(x + x'J + h + h'J) = (y_1 + y_2J)(x + x'J) + (h + h'J)(y_1 + y_2J)'(x + x'J) + \frac{(h+h'J)^2}{2!}(y_1 + y_2J)''(x + x'I) + ...,$$

hence,

$$(y_1 + y_2J)(x + x'J + h + h'J) - (y_1 + y_2J)(x + x'J)$$
$$= (h + h'J)(y_1 + y_2J)'(x + x'J) + \frac{(h + h'J)^2}{2!}(y_1 + y_2J)''(x + x'J) + ...$$

by partial differenation, we get:

$$y' = (y_1 + y_2J)' = f(x_1 + x_2^*J, y_1 + y_2J) = f,$$

$$y'' = f_{x_1+x_2J} + f.f_{y_1+y_2J},$$

$$y''' = f_{(x_1+x_2J)(x_1+x_2J)} + 2ff_{(x_1+x_2J)(y_1+y_2J)} + f^2f_{(y_1+y_2J)(y_1+y_2J)} + f_{(x_1+x_2J)(y_1+y_2J)} + ff_{y_1+y_2J}^2,$$

take,

$$F_1 + F_1'J = f_{x_1+x_2J} + f.f_{y_1+y_2J},$$

$$F_2 + F_2'J = f_{(x_1+x_2J)(x_1+x_2J)} + 2ff_{(x_1+x_2J)(y_1+y_2J)} + f^2f_{(x_1+x_2J)(y_1+y_2J)},$$

$$F_3 + F_3'J = f_{(x_1+x_2J)(x_1+x_2J)(x_1+x_2J)} + 3ff_{(x_1+x_2J)(x_1+x_2J)(y_1+y_2J)} + f^2f_{(y_1+y_2J)(y_1+y_2J)(y_1+y_2J)} + f^2f_{(y_1+y_2J)(y_1+y_2J)(y_1+y_2J)},$$

hence,

$$\begin{cases} F_1 + F_1'J = y'' \\ F_2 + F_2J + f_{(y_1+y_2I)}(F_1 + F_1J) = y''' \\ (F_3 + J) + f_{(y_1+y_2J)}(F_2 + F_2J) + 3(F_1 + J)(f_{(x_1+x_2J)(y_1+y_2J)} + f.f_{(y_1+y_2J)(y_1+y_2J)}) \end{cases},$$

$(y_1 + y_2J)(x + x'J + h + h'J) - (y_1 + y_2J)(x + x'J) = (h_1 + h_2J)(f) + \frac{(h_1+h_2J)^2}{2!}(F_1 + F_1J) + \ldots$

$$K_1 + K_1'J = (h + h'J)f,$$

$$K_2 + K_2'J = (h + h'J)f\big(x + x'J + (m + m'J)(h + h'J), y + y'J + (m + m'J)(K_1 + K_1'I)\big),$$

$$K_3 + K_3'J = (h + h'J)f\big(x + x'J + (n + n'J)(h + h'J), y + y'J + (n + n'J)(K_2 + K_2'I)\big),$$

$$K_4 + K_4'J = (h + h'J)f(x + x'J + (p + p'J)(h + h'J), y + y'J + (p + p'J)(K_3 + K_3'I)),$$

$$\Delta_{y_1+y_2I} = (y_1 + y_2J)(x + x'J, y_1 + y_2J) - (y_1 + y_2J)(x_1 + x_2J) = a(K_1 + K_1'J) + b(K_2 + K_2'J) + c(K_3 + K_3'J) + d(K_4 + K_4'J),$$

now, we can use Taylor's expansion for several variables:

$$K_1 + K_1'J = (h + h'J)f,$$

$$K_2 + K_2'J = (h + h'J) + \Big[f + (m + m'J)(h + h'J)(F_1 + F_1'J) + \frac{1}{2}(m + m'J)^2(h + h'J)^2(F_2 + F_2'J) + \ldots\Big],$$

$$K_3 + K_3'J = (h + h'J) + \Big[f + (n + n'J)(h + h'J)(F_1 + F_1'J) + \frac{1}{2}(h + h'J)^2\Big((n + n'J)^2(F_2 + J) + 2(m + m'J)(n + n'J)f_y(F_1 + F_1J)\Big) + \frac{1}{6}(h + h'J)^3\big((n + n'J)^3(F_3 + F_3'J) + 3(m + m'J)^2(n + n'J)f_yF_2J\big) + 6\,(m + m'J)(n + n'J)^2(F_1 + F_1J)\big(F_{xIyI} + ff_{yy}\big) + \ldots\Big],$$

$$K_4 + K_4'J = (h + h'J) + \Big[f + (p + p'J)(h + h'J)(F_1 + F_1'J) + \frac{1}{2}(h + h'J)^2\big[(p + p'J)^2(F_2 + F_2'J) + 2(n + n'J)(p + p'J)f_y(F_1 + F_1'J)\big] + \frac{1}{6}(h + h'J)^3\big[(p + p'J)^3(F_3 + F_3'J) + 3(n + n'J)^2(p + p'J)f_y(F_2 + F_2'J)\big] + 6\,(n + n'J)(p + p'J)(F_1 + F_1J)\big(F_{xIyI} + ff_{y^2I} + \ldots\big) \ldots\Big],$$

hence,

$$-(y_1+y_2J)(x+x'J)+(y_1+y_2J)\big(x+h+J(x'+h')\big)=a(h+h'J)f+b(h+h'J)[f+(m+m'J)(h+h'J)(F_1+F_1J)+.\dots.]+c(h+h'J)\Big[f+(n+n'J)(h+h'J)(F_1+F_1'J)+\frac{1}{2}(h+h'J)^2(n+n'J)^2F_2J\Big]+\frac{1}{6}(h+h'J)^3[(p+p'J)^3(F_3+F_3'J)+3(n+n'J)^2(p+p'J)F_2J+\dots..],$$

thus,

$$(y_1+y_2J)\big(x+h+J(x'+h')\big)-(y_1+y_2J)(x+x'J)=(a+b+c+d)(h+h'J)f+[b(m+m'J)+c(n+n'J)+d(p+p'J)](h+h'J)^2(F_2+F_2'I)+[b(m+m'J)^2+c(n+n'J)^2+d(p+p'J)^2]\frac{(h+h'J)^3(F_2J)}{2}+[b(m+m'J)^3+c(n+n'J)^3+d(p+p'J)^3]\frac{(h+h'J)^4(F_2I)}{6}+.\dots..,$$

so that,

$$\begin{cases} m=\frac{1}{2} \\ n=\frac{1}{2} \\ m'=n'=1 \end{cases},\begin{cases} p=1 \\ p'=1 \end{cases},\begin{cases} a=d=\frac{1}{6} \\ b=c=\frac{1}{3} \end{cases},$$

thus,

$$K_1+K_1'J=(h+h'J)f(x_0+x_0'J,y_0+J)=df.$$

$$K_2+K_2'J=(h+h'J)f\left(x_0+x_0'J+\frac{h+h'J}{2},y_0+y_0'J+\frac{K_1+K_1'J}{2}\right),$$

$$K_3+K_3'J=(h+h'J)f\left(x_0+x_0'J+\frac{h+h'J}{2},y_0+y_0'J+\frac{K_2+K_2'J}{2}\right),$$

$$K_4+K_4'J=(h+h'J)f(x_0+x_0'J+h+h'J,y_0+y_0'J+K_3+K_3'J),$$

Numerical comparison:

Consider the following dual problem:

$$(x_1+x_2J)(y_1+y_2J)"-2(y_1+y_2J)'=(10+10J)(x_1+x_2J)^4,$$

$$\begin{cases}(y_1+y_2I)(1+J)=2+J\\(y_1+y_2J)'(1+I)=2+J\\h=0.1+0.1J\\0\le x_1\le 2,0\le x_2\le 1\end{cases},$$

hence,

$$\begin{cases}y_1'+y_2'J=z_1+z_2J\\z_1'+z_2'J=(10+J)(x_1+x_2J)^3+\dfrac{2}{x_1+x_2J}(z_1+z_2J).\end{cases}$$

now, consider the following problem:

$$(y_1+y_2J)"-2(y_1+y_2J)'=4(x_1+x_2J),$$

with

$$\begin{cases}(y_1+y_2J)(0)=1+J\\(y_1+y_2J)'(0)=2+J\\0\le x_1\le 1\\0\le x_2\le 1\end{cases}.$$

hence,

$$\begin{cases}(y_1+y_2J)"=z_1+z_2J\\(z_1'+z_2'J)=(4+J)(x_1+x_2J)+(2+J)(z_1+z_2J)'\end{cases}$$

with

$$\begin{cases}(y_1+y_2J)(0)=1+J\\(z_1+z_2J)(0)=2+J\end{cases}.$$

Table 1. Approximation solution of the example by using (Euler), ABM (RK) and with step size $h = 0.1+J$.

x	Exact	Euler	Runge-Kutta	Error Euler	Error RK
1+ J	2+ J	2+ J	2+ J	J	J
1.1+J	3.67787+J	3.5357787+ J	3.6907787+ J	0.15454464+J	0.000113+ J
1.2+J	6. 169+ J	5.761693+ J	6.01699169+J	0.27828957+J	0.0000478+J
1.3+J	9. 4415+ J	8. 4415+ J	9. 4415+ J	0.32617295+J	0.0000658+J
1.4+J	13.68275+J	12. 68275+ J	13. 68275+ J	0.37828757+J	0.0000513+JJ
1.5+J	16.04970+J	18.04972587+J	18.04979431+J	0.43425871+J	0.000722+ J
1.6+J	22.13149+J	24.59391009+J	25.08799170+J	0.49408991+J	0.0008872+ J

1.7+J	33.2286+ J	32.5394089+ J	33.0994089+ J	0.55778118+J	0.0016937+ J
1.8+J	42.6678+ J	42.14266746+J	42.76798949+J	0.62533254+J	0.00241051+J
1.9+	54.40918+	53.63375603+	54.33048829+	0.69674397+	0.00351171+
2+J	68.33209+J	67.22798452+J	67.99998703+J	0.77201548+J	0.0041297+J

Table 2. Approximation solution of the example using (Euler), ABM (RK) and the error estimation for step size $h = 0.9 + J$.

x	**Exact**	**Euler**	**Runge-Kutta**	**Error Euler**	**Error RK**
1+ J	2+ J	2+I	2+ J	J	J
1.1+J	2.25541+ J	2.28871+ J	2.5560897+ J	0.00003935+I	0.22345+ J
1.2+J	2.76032+ J	2.44914306+J	2.76031761+ J	0.001348413+I	0.371918+ J
1.3+J	3.767593+ J	3.88181936+J	3.539352895+ J	0.00827513406+J	1.24178788+ J
1.4+J	4.332424+ J	4.6688294+ J	4. 393524018+ J	0.08275011894+J	1.448275953+J
1.5+J	6.115187+ J	5.171701+ J	6.21875130+ J	0.08275016779+J	1.65974429+ J
1.6+J	8.48976+ J	8.065518+ J	8.38976230+ J	0.013416177+ J	1.88920788+ J
1.7+J	11.44157+ J	10.491675+ J	11.2853935+ J	0.013421019+ J	2.13827508+ J
1.8+J	15.069168+J	13.114943+ J	15.082755389+J	0.0134020954+J	2.382752391+J
1.9+J	19.80199+ J	17.377873+ J	19.90199447 J	0.0827525828+J	2.68275440+J
2+ J	25+ J	22.1991698+J	26.00082759+J	0.013426253+ J	2.95827517+ J

Table 3. Approximation solution of the example using ABM (Euler), ABM (RK) and the error estimation for step size $h = 0.08 + J$.

x	**Exact**	**Euler**	**Runge-Kutta**	**Error Euler**	**Error RK**
1+ J	2+ J	2+ J	2+ J	J	J
1.1+J	3.67787+ J	3.5357787+ J	3.6907787+ J	0.15454464+J	0.000113+ J
1.2+J	6. 169+ J	5.761693+ J	6.01699169+J	0.27828957+J	0.0000478+J
1.3+J	9. 4415+ J	8. 4415+ J	9. 4415+ J	0.32617295+J	0.0000658+J
1.4+J	13.68275+J	12. 68275+ J	13. 68275+ J	0.37828757+J	0.0000513+J
1.5+J	16.04970+J	18.04972587+J	18.04979431+J	0.43425871+J	0.000722+ J
1.6+J	22.13149+J	24.59391009+J	25.08799170+J	0.49408991+J	0.0008872+J
1.7+J	33.2286+J	32.5394089+ J	33.0994089+ J	0.55778118+J	0.0016937+J
1.8+J	42.6678+J	42.14266746+J	42.76798949+J	0.62533254+J	0.00241051+J

1.9+	54.40918+	53.63375603+	54.33048829+	0.69674397+	0.00351171+
2+J	68.33209+J	67.22798452+J	67.99998703+J	0.77201548+J	0.0041297+ J

Conclusion

In this paper we have studied some numerical solutions of neutrosophic differential problems through a numerical comparison between neutrosophic Euler's method and 4-th order Runge-Kutta neutrosophic method, where we applied both methods on the same problems and compare the numerical results of solutions and numerical estimations of errors by using numerical tables. We have noticed from the numerical study and application of the two mentioned methods that they provide good results compared to exact solutions, with an acceptable small error of approximation. We hope that in the future the possibility of applying some other numerical methods to neutrosophic problems will be discussed and the numerical results compared through numerical tables.

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