Neutrosophic augmented Lagrange multipliers method Nonlinear Programming Problems Constrained by Inequalities

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Abstract: Operations research uses scientific methods that take the language of mathematics as a basis and uses the computer, without which it would not be possible to achieve numerical solutions to the problems raised. It is concerned with applying scientific methods to complex issues in the management and direction of large systems in various fields and private and governmental businesses in peace and war in politics, management, economics, planning and implementation and in all aspects of life. In issues that require sound solutions, when solutions are numerous and options are multiple, we need a decision based on sound scientific foundations that take into account all the circumstances and changes that the decision maker may face during the course of work, and help him make a decision that leaves nothing to chance or luck. For this reason, operations research has provided many methods that help us transform life issues into mathematical models whose optimal solution is the appropriate decision. The nonlinear programming method is one of the most important methods presented by operations research because most problems, when modeled, result in a nonlinear model, which prompted many students and researchers to search for appropriate scientific methods to solve these models. One of the most important and most widely used of these methods is the Lagrange multipliers method.

Keywords: Operations research; nonlinear programming; Lagrange multipliers; neutrosophic science; neutrosophic nonlinear programming; neutrosophic augmented Lagrange multipliers method.

1. Introduction

Operations research is the applied aspect of mathematics and is one of the most important modern sciences that deals with practical issues and meets the desire and demand of decision-makers to obtain ideal decisions through the methods it provides that are appropriate for all issues. We know that operations research methods depend on the data attached to each issue and this data is values that were set through observation and its accuracy depends on who collects it and is appropriate for working conditions similar to the conditions in which this data was collected. Before the emergence of neutrosophic science, we relied on this data to study real issues and accept the results as they are, but after the emergence of neutrosophic science, the science that caused a major revolution in all fields and proved its ability to give results that enjoy a margin of freedom and suit all conditions, many researchers interested in scientific development have turned to reformulating many scientific concepts using the concepts of this science. We now have neutrosophic numbers - neutrosophic groups - neutrosophic probabilities - neutrosophic statistics - neutrosophic differentiation - neutrosophic integration - neutrosophic linear programming - neutrosophic dynamic programming - neutrosophic simulation - neutrosophic transport models… [1-11, 16-18].

Nonlinear programming is one of the most important methods presented by operations research because most practical problems lead to nonlinear models. Therefore, in a previous study [12], we formulated some basic concepts of nonlinear programming using the concepts of neutrosophic science. In the two studies [13,14], we presented a neutrosophic study of the graphical method and the Lagrange multipliers method in the case of constraints of the type of equality methods used to find the optimal solution for nonlinear models. In this study, we present a formulation of the augmented Lagrange multipliers method that is used to find the optimal solution for nonlinear models restricted by inequality constraints using the concepts of neutrosophic science [15]. That is, we will take the data of the problem under study as neutrosophic numbers with the following standard formula $N = a + bI$ where $a \; and \; b$ are real or complex coefficients, a represents the determinant part and bI represents the undetermined part of the undetermined number N and it can be $[\lambda_1 , \; \lambda_2]$or $\{\lambda_1 , \; \lambda_1 \}$ or … else is any set close to the true value a.

2. Discussion:

A mathematical model is a neutrosophic model if the variables in the objective function, constraints, or both are neutrosophic values. Suppose we want to determine the optimal values of a neutrosophic nonlinear function $\boldsymbol{f_N(x)}$ subject to a set of constraints in the form of equalities or inequalities. This type of problem is addressed through the Lagrange factorial method, a method that has received great attention from students and researchers in the field of operations research, and appropriate developments have been introduced to it that have enabled it to deal with most situations with high efficiency.

- **Previous study:**

1- Neutrosophic nonlinear programming problems constrained by equality constraints: [12]

Constrained nonlinear programming problem text:

For nonlinear programming problems constrained by the equation constraints defined as follows:
Find:

$$\boldsymbol{Z_N = f_N(X) \longrightarrow (\,Max \; or \; Min)}$$

Subject to the constraints:

$$\boldsymbol{g_{Ni}(X) = b_{Ni} \quad ; i = 1, 2, \dots , m}$$

$$\boldsymbol{X = (x_1, x_2, \dots , x_n) \in R^n}$$

Since the functions $\boldsymbol{f_N(x) \; and \; g_{Ni}(X)}$ are continuous functions whose partial derivatives of the first degree can be computed, the Lagrange factorial method requires us to form a function of the type:

$$\boldsymbol{L_N(X, \lambda_i) = f_N(X) - \lambda_i(g_{Ni}(X) - b_{Ni})}$$

Where we call $\boldsymbol{L_N(X, \lambda_i)}$ the Lagrange neutrosophic function and the coefficients $\boldsymbol{\lambda_i}$ refer to the Lagrange factorials and these factorials measure the rate of change in the optimal value of $\boldsymbol{f_N(X)}$ as a result of Making small modifications to $\boldsymbol{g_{Ni}(X)}$, by solving the set of equations:

$$\boldsymbol{\frac{\partial L_N}{\partial X} = 0 \quad , \quad \frac{\partial L_N}{\partial \lambda_i} = 0}$$

We get the optimal solution vector, which is a maximum neutrosophic value of the function, which can be a minimum or a maximum. To determine its type, we apply the following test:

Maximum and minimum values of a constrained nonlinear function:

We can determine whether the value of an objective function in a nonlinear model is a minimum or a maximum as follows:

The value is minimum if the objective function is convex and the set of constraints forms a convex region

The value is maximum if the objective function is concave and the set of constraints forms a convex region

To determine the type of function $\boldsymbol{f_N(x_1 , x_2, \dots , x_n)}$ (convex or concave), we use the Hessian matrix for this function:

Hessian matrix of the neutrosophic function $f_N(x_1, x_2, \dots, x_n)$:

It is a square and symmetric matrix of order $n \times n$, denoted by $H_N(x_1, x_2, \dots, x_n)$ and is defined by the following relation:

$$H_N(x_1, x_2, \dots, x_n) = \left[\frac{\partial^2 f_N}{\partial x_i \partial x_j}\right]$$

The matrix is symmetric and must satisfy the following conditions:

- The elements of the main diagonal are positive.
- The main minor determinants are positive.

2- Nonlinear programming problems with inequalities: [15]

The problem of nonlinear programming of constrained by inequalities is defined as follows:

$$MaxZ = \mathbf{f}(x_1, x_2, \dots, x_n)$$

Subject to constraints in the form of inequalities:

$$g_i(x_1, x_2, \dots, x_n) \le b_i$$

$$(x_1, x_2, \dots, x_n) \ge 0$$

In such problems, there is a direct relationship between the size of the problem under study and the size of the calculations required to determine the maximum point (maximum or minimum) of this problem, so the Lagrange method mentioned above is considered an impractical method to solve it. In order to solve these problems, researchers in the field of operations research have presented the augmented Lagrange multipliers method., which can be summarized in the following steps:

Step 1: We solve the problem without constraints, i.e.:

$$MaxZ = \mathbf{f}(x_1, x_2, \dots, x_n)$$

- ❖ If the resulting optimal value satisfies all constraints, then this solution is an optimal solution to the constrained problem. We conclude from this that the constraints of the problem are unnecessary.
- ❖ If the resulting optimal value does not satisfy all constraints, we move to the second step.

Step 2: We take $k = 1$ (the number of constraints used is k, we take one of the existing constraints) and convert this constraint to an equality constraint and search for the optimal solution for $\mathbf{f}(x_1, x_2, \dots, x_n)$ that is subject to $k = 1$ constraint in an equal manner using the Lagrange multipliers method mentioned above,

- ❖ If the resulting solution satisfies all the constraints of the problem, then the solution determines a local optimal point
- ❖ If the resulting value does not satisfy all the constraints, we delete this solution because it is not possible

We repeat this step for all possible sets of constraints, each of which consists of $k = 1$ equality constraint and record all the local optimal points that we obtain, then move to the third step.

Step 3: We take $k = 2$ (the number of constraints used is k. We take two constraints from the existing constraints) and convert these two constraints into equality constraints and search for the optimal solution for $\mathbf{f}(x_1, x_2, \dots, x_n)$ that is subject to $k = 2$ constraints in an equal manner using the Lagrange multipliers method mentioned above.

- ❖ If the resulting solution satisfies all the constraints of the problem, then the solution determines a local optimal point.
- ❖ If the resulting value does not satisfy all the constraints, we delete this solution because it is not possible.

We repeat this step for all possible sets of constraints each consisting of $k = 2$ equality constraints and record all the local optimal points we obtain.

We repeat the work for $k = 3$, $k = 4$ …, $k = m$

We calculate the values of the objective function for all local optimal points and choose the optimal value that satisfies the objective of the problem under study. The corresponding point is an absolute optimal point.

Note: If we do not encounter a local optimal point in the resulting solutions, we decide that the problem has no possible solution.

We illustrate the above through the following example:

Example1:

Find the optimal solution to the following nonlinear programming problem:

$$MaxZ = \mathrm{f}(x_1, x_2) = -(2x_1 - 5)^2 - (2x_2 - 1)^2$$

Subject to constraints:

$$x_1 + 2x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Solution:

Step 1: We solve the problem without restrictions:

To find the optimal value of $f(x)$ without constraints, we find the solutions to the following equations:

$$\frac{\partial f}{\partial x_1} = -4(2x_1 - 5) = 0$$
$$\frac{\partial f}{\partial x_2} = -4(2x_2 - 1) = 0$$

The solution to the set of equations is $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, \frac{1}{2}\right)$ This solution does not satisfy the first constraint in the problem, so we resort to applying steps (2) and (3) in the solution algorithm. Applying these two steps requires solving seven problems using the Lagrange multipliers method.

We summarize these problems and their solutions as follows:

For $k = 1$ for all constraints we get Problems 1, 2, 3:

Problem 1: For the constraint $x_1 = 0$:

The Lagrange function is:

$$L_1(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda x_1$$

We calculate the partial derivatives of this function with respect to x_1 ‹ x_2 ‹ λ:

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) = 0$$
$$\frac{\partial L}{\partial \lambda} = x_1 = 0$$

The solution to the system of equations is the point $\bar{x} = (x_1, x_2, \lambda) = \left(0, \frac{1}{2}, -20\right)$ The point satisfies all the constraints and is a local maximum.

Problem 2: For the constraint $x_2 = 0$:

The Lagrange function is:

$$L_2(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda x_2$$

We calculate the partial derivatives of this function with respect to x_1 ‹ x_2 ‹ λ:

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 2x_2 = 0$$

The solution to the previous set of equations is the point $\bar{x} = (x_1, x_2, \lambda) = \left(\frac{5}{2}, 0, 0\right)$.

The point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, 0\right)$ does not satisfy all the constraints since the first constraint is not satisfied Solution Rejected.

Problem 3: For the constraint $x_1 + 2x_2 - 2 = 0$:

The Lagrange function is:

$$L_3(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda(x_1 + 2x_2 - 2)$$

We calculate the partial derivatives of this function with respect to x_1 ‹ x_2 ‹ λ:

$$\frac{\partial L_3}{\partial x_1} = -4(2x_1 - 5) - \lambda = 0$$
$$\frac{\partial L_3}{\partial x_2} = -4(2x_2 - 1) - 2\lambda = 0$$
$$\frac{\partial L_3}{\partial \lambda} = -(x_1 + 2x_2 - 2) = 0$$

The solution to the previous set of equations is $\overline{x} = (x_1, x_2, \lambda) = \left(\frac{22}{10}, \frac{-1}{10}, \frac{12}{5}\right)$The point $(x_1, x_2) = \left(\frac{22}{10}, \frac{-1}{10}\right)$does not satisfy the third constraint. The solution is rejected.

For $k = 2$ for all constraints we get Problems 4,5,6:

Problem 4: Taking the constraints $x_1 + 2x_2 - 2 = 0$, $x_1 = 0$:

The Lagrange function is

$$L_4(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1(x_1 + 2x_2 - 2) - \lambda_2 x_1$$

We calculate the partial derivatives of this function with respect to x_1 ‘ x_2 ‘ λ_1‘ λ_2:

$$\frac{\partial L_4}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L_4}{\partial x_2} = -4(2x_2 - 1) - 2\lambda_1 = 0$$
$$\frac{\partial L_4}{\partial \lambda_1} = -(x_1 + 2x_2 - 2) = 0$$
$$\frac{\partial L_4}{\partial \lambda_2} = -x_1 = 0$$

The solution to the previous set of equations is $\overline{x} = (x_1, x_2, \lambda) = (0,1,-2,22)$ The point $(x_1, x_2) = (0,1)$ satisfies all the constraints, so it is a local maximum.

Problem 5: We take the two constraints $x_1 + 2x_2 - 2 = 0$, $x_2 = 0$

The Lagrange function is:

$$L_5(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1(x_1 + 2x_2 - 2) - \lambda_2 x_2$$

We calculate the partial derivatives of this function with respect to x_1 ‘ x_2 ‘ λ_1‘ λ_2:

$$\frac{\partial L_5}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 = 0$$
$$\frac{\partial L_5}{\partial x_2} = -4(2x_2 - 1) - 2\lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L_5}{\partial \lambda_1} = -(x_1 + 2x_2 - 2) = 0$$
$$\frac{\partial L_5}{\partial \lambda_2} = -x_2 = 0$$

The solution to the previous set of equations is $\bar{x} = (x_1, x_2, \lambda) = (2,0,4,-4)$ The point $(x_1, x_2) = (2,0)$ satisfies all the constraints, so it is a local maximum.

Problem 6: We take the two constraints $x_2 = 0$, $x_1 = 0$

The Lagrange function is:

$$L_6(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1 x_1 - \lambda_2 x_2$$

We calculate the partial derivatives of this function with respect tox_1 ‘ x_2 ‘ λ_1‘ λ_2:

$$\frac{\partial L_6}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 = 0$$
$$\frac{\partial L_6}{\partial x_2} = -4(2x_2 - 1) - \lambda_2 = 0$$
$$\frac{\partial L_6}{\partial \lambda_1} = -x_1 = 0$$
$$\frac{\partial L_6}{\partial \lambda_2} = -x_2 = 0$$

The solution to the previous set of equations is $\bar{x} = (x_1, x_2, \lambda) = (0,0,-20,-4)$The point $(x_1, x_2) = (0,0)$satisfies all the constraints and is a local maximum.

For $k = 3$ for all constraints we get Problem 7:

Problem 7: We take the constraints $x_1 + 2x_2 - 2 = 0$، $x_2 = 0$, $x_1 = 0$

The Lagrange function is:

$$L_7(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3(x_1 + 2x_2 - 2)$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1 ، λ_2 ، λ_3:

$$\frac{\partial L_7}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 - \lambda_3 = 0$$

$$\frac{\partial L_7}{\partial x_2} = -4(2x_2 - 1) - \lambda_2 - 2\lambda_3 = 0$$

$$\frac{\partial L_7}{\partial \lambda_1} = -x_1 = 0$$

$$\frac{\partial L_7}{\partial \lambda_2} = -x_2 = 0$$

$$\frac{\partial L_7}{\partial \lambda_3} = -(x_1 + 2x_2 - 2) = 0$$

We note that there is no solution to the previous set of equations due to the contradiction of the last three equations. To determine the absolute optimal solution, i.e., the absolute maximum value, we take all possible points (local maximum points) and calculate the value of the objective function at each point and choose the largest value. The corresponding point is the absolute maximum solution point:

Problem 1: Point $\bar{x} = (x_1, x_2) = \left(0, \frac{1}{2}\right)$ is a local maximum point, value of the objective function:

$$f(x) = -25$$

Problem 4: Point $(x_1, x_2) = (0, 1)$ This point is a local maximum point, value of the objective function:

$$f(x) = -26$$

Problem 5: Point $(x_1, x_2) = (2, 0)$ is a local maximum point, value of the objective function:

$$f(x) = -2$$

Problem 6: Point $(x_1, x_2) = (0, 0)$ is a local maximum point, value of the objective function:

$$f(x) = -26$$

By comparing the values of the objective function in the previous problems, we find that the best possible point is point $(x_1^*, x_2^*) = (2, 0)$ and the absolute maximum value of the function is:

$$f^*(x) = -2$$

- **Current study:**

Neutrosophic Nonlinear programming problems constrained by inequalities:

The nonlinear programming problem constrained by inequalities is defined as follows:

$$(Max\ or\ Min) Z_N = f_N(x_1, x_2, \dots, x_n)$$

And subject to constraints in the form of inequalities:

$$g_{Ni}(x_1, x_2, \dots, x_n) \leq b_{Ni}$$

$$(x_1, x_2, \dots, x_n) \geq 0$$

The solution steps according to the expanded Lagrange multipliers method are as follows:

Step 1: We solve the problem without constraints, i.e.:

$$(Max\ or\ Min) Z_N = f_N(x_1, x_2, \dots, x_n)$$

If the resulting optimal neutrosophic value satisfies all constraints, then this solution is an optimal solution to the constrained neutrosophic problem, and we conclude Hence, the constraints of the problem are unnecessary if the resulting optimal value does not satisfy all the constraints.

Step 2: We take $k = 1$ (we take one of the constraints):

We convert this constraint to an equality constraint and search for the optimal solution for $f_N(x_1, x_2, \dots, x_n)$ that is subject to $k = 1$ constraint in an equal manner using the Lagrange neutrosophic multipliers method.

If the resulting solution satisfies all the constraints of the problem, then the solution determines a local neutrosophic optimal point. If the resulting neutrosophic optimal value does not satisfy all the constraints, we delete it because this solution is not possible.
We repeat the work for all possible sets of constraints, each of which consists of $k = 1$ equality constraint, and we record all the local neutrosophic optimal points that we obtain, then we move to step 3.

Step 3: We take $k = 2$ constraint (we take two of the constraints):

We convert this constraint to an equality constraint and search for the optimal solution for $f_N(x_1, x_2, \ldots, x_n)$ that is subject to $k = 2$ constraint in an equal manner.
If the resulting neutrosophic optimum does not satisfy all the constraints, we delete it because this solution is not possible.
We repeat the work for all possible sets of constraints, each of which consists of $k = 2$ equality constraint, and we record all the local neutrosophic optimum points that we obtain.
We continue in this way until we reach $k = m$ constraint.
We calculate the value of the objective function for all the local optimal points we have obtained and choose the largest value if the goal is to maximize and the smallest value if the goal is to minimize. These values are the absolute neutrosophic optimal value.
We illustrate the above through the following example:
We take the mathematical model given in example (1):
The neutrosophic formula for the mathematical model given in example (1) is:

$$MaxZ_N = f_N(x_1, x_2) = (c_{1N}x_1 - d_{1N})^2 - (c_{2N}x_2 - d_{2N})^2$$

Subject to constraints:

$$a_{1N}x_1 + a_{2N}x_2 \leq b_N$$
$$x_1, x_2 \geq 0$$

Where $c_{1N}, c_{2N}, d_{1N}, d_{2N}, a_{1N}, a_{2N}, b_N$ are neutrosophic values.
It is worth noting that the existence of one neutrosophic value is sufficient for the nonlinear mathematical model to be a neutrosophic model. Accordingly, since the purpose of this research is to provide a neutrosophic formulation of the extended Lagrange multipliers method, we will take the neutrosophic mathematical model as follows:

Example 2:

Find the optimal solution to the following neutrosophic nonlinear programming problem:

$$MaxZ = f(x_1, x_2) = -(2x_1 - 5)^2 - (2x_2 - 1)^2$$

Subject to the constraints:

$$x_1 + 2x_2 \leq [1,3]$$
$$x_1, x_2 \geq 0$$

In any of the constraints of the mathematical models, the second party expresses the available capabilities
(The number of raw materials - possible working hours ...) We took it as a neutrosophic value in the form of a field whose minimum limit represents the available capabilities in the worst conditions and the maximum limit represents the available capabilities in the best conditions.
Step 1: We solve the problem without restrictions:
To find the optimal value of $f(x)$ without constraints, we find the solutions to the following equations:

$$\frac{\partial f}{\partial x_1} = -4(2x_1 - 5) = 0$$
$$\frac{\partial f}{\partial x_2} = -4(2x_2 - 1) = 0$$

The solution to the set of equations is $\bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}) = \left(\frac{5}{2}, \frac{1}{2}\right)$ This solution does not satisfy the first constraint in the problem, so we resort to applying steps (2) and (3) in the solution algorithm. Applying these two steps requires solving seven problems using the Lagrange multipliers method.
We summarize these problems and their solutions as follows:

For $k = 1$ for all constraints we get Problems 1, 2, 3:

Problem 1: For the constraint $x_1 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda x_1$$

We calculate the partial derivatives of this function with respect to $\boldsymbol{x_1}$ ، $\boldsymbol{x_2}$ ، $\boldsymbol{\lambda}$.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) = 0$$
$$\frac{\partial L}{\partial \lambda} = x_1 = 0$$

The solution to the system of equations is the point $\bar{x} = (x_1, x_2, \lambda) = \left(0, \frac{1}{2}, -20\right)$ The point satisfies all the constraints and is a local maximum.

Problem 2: For the constraint $x_2 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda x_2$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 2x_2 = 0$$

The solution to the previous set of equations is the point $\bar{x} = (x_1, x_2, \lambda) = \left(\frac{5}{2}, 0, 0\right)$.

The point $\boldsymbol{\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, 0\right)}$ does not satisfy all the constraints since the first constraint is not satisfied Solution Rejected.

Problem 3: For the constraint $x_1 + 2x_2 - [1,3] = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda(x_1 + 2x_2 - [1,3])$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - 2\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = -(x_1 + 2x_2 - [1,3]) = 0$$

The solution to the previous set of equations is:

$$\bar{x}_N = (x_{1N}, x_{2N}, \lambda_N) = \left(\left[1, \frac{6}{5}\right], \left[0, \frac{2}{5}\right], \left[\frac{52}{5}, 12\right]\right)$$

The point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(\left[1, \frac{6}{5}\right], \left[0, \frac{2}{5}\right]\right)$does not satisfy the third constraint. The solution is rejected.

For $k = 2$ all constraints we get Problems 4,5,6:

Problem 4: Taking the constraints $x_1 + 2x_2 - [1,3] = 0$, $x_1 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1(x_1 + 2x_2 - [1,3]) - \lambda_2 x_1$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1، λ_2.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - 2\lambda_1 = 0$$
$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + 2x_2 - [1,3]) = 0$$
$$\frac{\partial L}{\partial \lambda_2} = -x_1 = 0$$

The solution to the previous set of equations is:

$$\overline{x_N} = (x_{1N}, x_{2N}, \lambda_{1N}, \lambda_{2N}) = \left(0, \left[\frac{1}{2}, \frac{3}{2}\right], [0,4], [16,20]\right)$$

The point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(0, \left[\frac{1}{2}, \frac{3}{2}\right]\right)$, not all constraints are satisfied since the first constraint is not satisfied, the solution is rejected.

Problem 5: We take the two constraints $x_1 + 2x_2 - [1,3] = 0$, $x_2 = 0$.

The Lagrange function is:

$$\mathrm{L}(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1(x_1 + 2x_2 - [1,3]) - \lambda_2 x_2$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1، λ_2.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 = 0$$

$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - 2\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + 2x_2 - [1,3]) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -x_2 = 0$$

The solution to the previous set of equations is:

$$\overline{x_N} = (x_{1N}, x_{2N}, \lambda_{1N}, \lambda_{2N}) = ([1,3], 0, [-4,12], [-20,12])$$

The point $\overline{x_N} = (x_{1N}, x_{2N}) = ([1,3], 0)$ satisfies all the constraints, so it is a local maximum.

Problem 6: We take the two constraints $x_2 = 0$, $x_1 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1 x_1 - \lambda_2 x_2$$

We calculate the partial derivatives of this function with respect tox_1 ، x_2 ، λ_1، λ_2.

$$\frac{\partial L_6}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 = 0$$

$$\frac{\partial L_6}{\partial x_2} = -4(2x_2 - 1) - \lambda_2 = 0$$

$$\frac{\partial L_6}{\partial \lambda_1} = -x_1 = 0$$

$$\frac{\partial L_6}{\partial \lambda_2} = -x_2 = 0$$

The solution to the previous set of equations is $\overline{x} = (x_1, x_2, \lambda) = (0,0,-20,-4)$The point $\overline{x} = (x_1, x_2) = (0,0)$satisfies all the constraints and is a local maximum.

For $k = 3$ for all constraints we get Problem 7:

Problem 7: We take the constraints $x_1 + 2x_2 - [1,3] = 0$، $x_2 = 0$, $x_1 = 0$:

The Lagrange function is:

$$L(x, \lambda) = -(2x_1 - 5)^2 - (2x_2 - 1)^2 - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3(x_1 + 2x_2 - [1,3])$$

We calculate the partial derivatives of this function with respect to x_1 ، x_2 ، λ_1 ، λ_2 ، λ_3.

$$\frac{\partial L}{\partial x_1} = -4(2x_1 - 5) - \lambda_1 - \lambda_3 = 0$$

$$\frac{\partial L}{\partial x_2} = -4(2x_2 - 1) - \lambda_2 - 2\lambda_3 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -x_1 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -x_2 = 0$$

$$\frac{\partial L}{\partial \lambda_3} = -(x_1 + 2x_2 - [1,3]) = 0$$

To determine the absolute optimal neutrosophic solution, i.e., the absolute neutrosophic maximum, we take all possible points (local maxima) and calculate the value of the objective function at each point and choose the largest values. The corresponding point is the absolute neutrosophic maximum solution point:

Problem 1: The point $\bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}) = \left(\boldsymbol{0}, \frac{\boldsymbol{1}}{\boldsymbol{2}}\right)$ is a local maximum point, the value of the objective function:

$$\boldsymbol{f(x) = -25}$$

Problem 3: The point $\overline{\boldsymbol{x_N}} = (\boldsymbol{x_{1N}}, \boldsymbol{x_{2N}}) = \left(\left[\boldsymbol{1}, \frac{\boldsymbol{6}}{\boldsymbol{5}}\right], \left[\boldsymbol{0}, \frac{\boldsymbol{2}}{\boldsymbol{5}}\right]\right)$ is a local maximum point, the value of the objective function:

$$\boldsymbol{f_N(x) \in [-33.8, -13]}$$

Problem 5: The point $\bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}) = ([\boldsymbol{1}, \boldsymbol{3}], \boldsymbol{0})$ is a local maximum point, the value of the objective function:

$$\boldsymbol{f_N(x) \in [-9, -1]}$$

Problem 6: The point $\bar{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}) = (\boldsymbol{0}, \boldsymbol{0})$ is a local maximum point, the value of the objective function:

$$\boldsymbol{f(x) = -26}$$

By comparing the values of the objective function in the previous questions, we find that the best possible point is the point $(\boldsymbol{x_1^*}, \boldsymbol{x_2^*}) = ([\boldsymbol{1}, \boldsymbol{3}], \boldsymbol{0})$ and the absolute maximum value of the function is:

$$\boldsymbol{f_N^*(x) \in [-9, -1]}$$

Note: This method does not guarantee an absolute optimal value for the problem, however the best possible point can be considered an absolute optimal point (general).

3. Conclusion and results:

In the previous study, we presented a neutrosophic formulation of the extended Lagrange multipliers method, and we summarize what we have reached through the following table:

	Classical Problem	**Neutrosophic Problem**
The Problem without restrictions	Optimal value of $f(x)$ without constraints Solution point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, \frac{1}{2}\right)$ Does not satisfy all constraints as the first constraint	Optimal value of $f(x)$ without constraints Solution point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, \frac{1}{2}\right)$ Does not satisfy all constraints as the first constraint
Problem1	$k = 1$ Constraint $x_1 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(0, \frac{1}{2}\right)$ It satisfies all constraints, so it is a local maximum Value of objective function $f(x) = -25$	$k = 1$ Constraint $x_1 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(0, \frac{1}{2}\right)$ It satisfies all constraints, so it is a local maximum Value of objective function $f(x) = -25$
Problem2	$k = 1$ Constraint $x_2 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, 0\right)$ Not all constraints are satisfied as the first constraint is not satisfied Solution rejected	$k = 1$ Constraint $x_2 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(\frac{5}{2}, 0\right)$ Not all constraints are satisfied as the first constraint is not satisfied Solution rejected
Problem3	$k = 1$ Constraint $x_1 + 2x_2 - 2 = 0$ Solution point $\bar{x} = (x_1, x_2) = \left(\frac{22}{10}, \frac{-1}{10}\right)$ Does not satisfy all constraints as the third constraint is not satisfied Solution rejected	$k = 1$ Constraint $x_1 + 2x_2 - [1,3] = 0$ Solution Point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(\left[1, \frac{6}{5}\right], \left[0, \frac{2}{5}\right]\right)$ It satisfies all constraints, so it is a local maximum Value of objective function $f_N(x) \in [-33.8, -13]$
Problem4	$k = 2$ Constraints $x_1 + 2x_2 - 2 = 0 \quad , x_1 = 0$ The solution is the point $\bar{x} = (x_1, x_2) = (0,1)$ that satisfies all the constraints It is a local maximum	$k = 2$ Constraints $x_1 + 2x_2 - [1,3] = 0 \quad , x_1 = 0$ Solution Point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(0, \left[\frac{1}{2}, \frac{3}{2}\right]\right)$ Does not satisfy all constraints as the first constraint is not satisfied

	The value of the objective function $f(x) = -26$	Solution rejected
Problem5	$k = 2$ Constraints $x_1 + 2x_2 - 2 = 0 \quad , x_2 = 0$ The solution is the point $\bar{x} = (x_1, x_2) = (2,0)$ It satisfies all the constraints, so it is a local maximum The value of the objective function $f(x) = -2$	$k = 2$ Constraints $x_1 + 2x_2 - [1,3] = 0 \quad , x_2 = 0$ Solution Point $\overline{x_N} = (x_{1N}, x_{2N}) = ([1,3], 0)$ It satisfies all constraints, so it is a local maximum Value of the objective function $f_N(x) \in [-9, -1]$
Problem6	$k = 2$ Constraints $x_2 = 0 \ , x_1 = 0$ The solution is the point $\bar{x} = (x_1, x_2) = (0,0)$ That satisfies all the constraints, so it is a local maximum The value of the objective function $f(x) = -26$	$k = 2$ Constraints $x_2 = 0 \ , x_1 = 0$ The solution is the point $\bar{x} = (x_1, x_2) = (0,0)$ That satisfies all the constraints, so it is a local maximum The value of the objective function $f(x) = -26$
Problem7	$k = 3$ Three constraints $x_1 + 2x_2 - 2 = 0 \quad , x_1 = 0 \, , x_2 = 0$ There is no solution to the problem	$k = 3$ Three constraints $x_2 = 0 \ , x_1 = 0،\ x_1 + 2x_2 - [1,3] = 0$ There is no solution to the problem

Comparing the results of the solution of the classical and neutrosophic problems, we find:
Problem 3:
In the classical problem:
$k = 1$ and $x_1 + 2x_2 - 2 = 0$
The problem has a solution at the point $\bar{x} = (x_1, x_2) = \left(\frac{22}{10}, \frac{-1}{10}\right)$ but it does not satisfy all the constraints since the third constraint is not satisfied
A rejected solution
In the neutrosophic problem:
$k = 1$ with the neutrosophic constraint $x_1 + 2x_2 - [1,3] = 0$
The problem has a solution at the point $\overline{x_N} = (x_{1N}, x_{2N}) = \left(\left[1, \frac{6}{5}\right], \left[0, \frac{2}{5}\right]\right)$ which satisfies all the constraints and is a local maximum.
The value of the objective function $\boldsymbol{f_N(x) = [-33.8, -13]}$
Problem4:
In the classical problem:
$\boldsymbol{k = 2}$ and the constraints $\boldsymbol{x_1 + 2x_2 - 2 = 0 \quad , x_1 = 0}$
The problem has a solution which is the point $\boldsymbol{\bar{x} = (x_1, x_2) = (0, 1)}$ that satisfies all the constraints, so it is a local maximum.
The value of the objective function $\boldsymbol{f(x) = -26}$
In the neutrosophic problem:
$\boldsymbol{k = 2}$ and the constraints $\boldsymbol{x_1 + 2x_2 - [1, 3] = 0 \quad , x_1 = 0}$
The problem has a solution which is the point $\boldsymbol{\overline{x_N} = (x_{1N}, x_{2N}) = \left(0, \left[\frac{1}{2}, \frac{3}{2}\right]\right)}$ that does not satisfy all the constraints, since the first constraint is not satisfied
Rejected solution
In addition to the absolute optimal value:
In the classical problem:
The point $\boldsymbol{(x_1^*, x_2^*) = (2, 0)}$ and the absolute maximum value of the function is:

$$\boldsymbol{f^*(x) = -2}$$

In the neutrosophic problem:
The point $\boldsymbol{(x_1^*, x_2^*) = ([1, 3], 0)}$ and the absolute maximum value of the function is:

$$\boldsymbol{f_N^*(x) \in [-9, -1]}$$

The absolute maximum value in the classical problem belongs to the solution domain in the neutrosophic problem, which confirms what we mentioned in the text of the research that the neutrosophic values give results that have a margin of freedom and are suitable for all conditions.

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Received: Sep 11, 2024. Accepted: Feb 20, 2025