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The double integrals of 2- refined neutrosophic functions and its applications

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Abstract: The importance of this paper comes from the fact that it presented a study of the double integrals of 2- refined neutrosophic functions and its applications, where we presented several theories for the concept of the double integrals of 2- refined neutrosophic functions over a general region and over a rectangle, including the 2- refined neutrosophic Fubini's theorem. In addition to discussing applications of double integrals of 2- refined neutrosophic functions to calculate the area of region. Also, We took into account the definition of the positive 2- refined neutrosophic number, and the n-refined AH-Isometry.

Keywords: integrals, double, Fubini's theorem, neutrosophic functions, areas, 2- refined neutrosophic.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]. Agboola introduced the concept of refined neutrosophic algebraic structures [2]. In addition, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. Abobala presented the papers on some special substructures of refined neutrosophic rings and a study of ah-substructures in n-refined neutrosophic vector spaces [6-7].

Alhasan.Y and Abdulfatah. R also presented a study on the division of refined neutrosophic number [8].

There are papers presented in n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Neutrosophic Rings I [4-5] and studying the integral calculus according to the logic of neutrosophic by presenting a set of papers on that [9-10].

In addition, the AH-Isometry was extended to n-Refined AH-Isometry by Smarandache & Abobala in 2024 [11].

2. The double integrals of 2- refined neutrosophic functions

2.1 The double integrals of 2- refined neutrosophic functions over a rectangle

Theorem 1 (The 2- refined neutrosophic Fubini's theorem)

Let $f(x,y,I_1,I_2)$ integrable over the neutrosophic rectangle

$R \cup I_1 \cup I_2 = \{(x,y,I_1,I_2): a + a_1I_1 + a_2I_2 \le x \le b + b_1I_1 + b_2I_2 \text{ and } c + c_1I_1 + c_2I_2 \le y \le d + d_1I_1 + d_2I_2\}$, where: $a, a_1, a_2, b, b_1, b_2, c, c_1, c_2, d, d_1, d_2$ are real numbers, while $I_1, I_2 =$ indeterminacy.

Then the double integrals of 2- refined neutrosophic function $f(x,y,I_1,I_2)$ over a rectangle $R \cup I_1 \cup I_2$ given by the following formula:

$$\iint_{R\cup I_1\cup I_2} f(x,y,I_1,I_2)dA = \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} \int_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2} f(x,y,I_1,I_2)dydx$$

$$= \int_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2} \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x,y,I_1,I_2)dxdy$$

where:

- using the horizontal slice:

$$\int_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2} \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x,y,I_1,I_2)dxdy$$

- using the vertical slice:

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} \int_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2} f(x,y,I_1,I_2)dydx$$

Example 1

Let $R \cup I_1 \cup I_2 = \{(x,y,I_1,I_2): 0 \le x \le 2 + 2I_1 + 2I_2 \text{ and } 1 + I_1 + I_2 \le y \le 4 + 4I_1 + 4I_2\}$, find:

$$\iint_{R\cup I_1\cup I_2} (x + 4I_1xy + 2I_2y)\, dA$$

Solution:

- <u>using the vertical slice:</u>

$$\int_0^{2+2I_1+2I_2} \int_{1+I_1+I_2}^{4+4I_1+4I_2} (x + 4I_1xy + 2I_2y)dydx = \int_0^{2+2I_1+2I_2} (xy + 2I_1xy^2 + I_2y^2)\Big|_{1+I_1+I_2}^{4+4I_1+4I_2} dx$$

$$= \int_0^{2+2I_1+2I_2} [(4 + 4I_1 + 4I_2)x + 2I_1(4 + 4I_1 + 4I_2)^2x + I_2(4 + 4I_1 + 4I_2)^2] - [(1 + I_1 + I_2)x + 2I_1(1 + I_1 + I_2)^2x + I_2(1 + I_1 + I_2)^2]dx$$

$$= \int_{0}^{2+2I_1+2I_2} [(4+4I_1+4I_2)x + 288I_1x + I_2(16+80I_1+48I_2)] - [(1+19I_1+I_2)x + I_2(1+I_1+I_2)^2]dx$$

$$= \int_{0}^{2+2I_1+2I_2} [(4+292I_1+4I_2)x + 80I_1 + 64I_2] - [(1+19I_1+I_2)x + 5I_1 + 4I_2]dx$$

$$= \int_{0}^{2+2I_1+2I_2} \big((3+273I_1+3I_2)x + 75I_1 + 60I_2\big)dx$$

$$= \left((3+273I_1+3I_2)\frac{x^2}{2} + (75I_1+60I_2)x\right)\Bigg|_{0}^{2+2I_1+2I_2}$$

$$= (3+273I_1+3I_2)\frac{(2+2I_1+2I_2)^2}{2} + (75I_1+60I_2)(2+2I_1+2I_2)$$

$$= (3+273I_1+3I_2)(2+10I_1+6I_2) + 570I_1 + 240I_2$$

$$= 6 + 4974I_1 + 42I_2 + 570I_1 + 240I_2$$

$$\boxed{= 6 + 5544I_1 + 282I_2}$$

- using the horizontal slice:

$$\int_{1+I_1+I_2}^{4+4I_1+4I_2} \int_{0}^{2+2I_1+2I_2} (x + 4I_1xy + 2I_2y)dxdy = \int_{1+I_1+I_2}^{4+4I_1+4I_2} \left(\frac{x^2}{2} + 2I_1x^2y + 2I_2xy\right)\Bigg|_{0}^{2+2I_1+2I_2} dy$$

$$= \int_{1+I_1+I_2}^{4+4I_1+4I_2} \left(\frac{(2+2I_1+2I_2)^2}{2} + 2I_1(2+2I_1+2I_2)^2y + 2I_2(2+2I_1+2I_2)y\right)dy$$

$$= \int_{1+I_1+I_2}^{4+4I_1+4I_2} (2+10I_1+6I_2+72I_1y+(4I_1+8I_2)y)dy$$

$$= \int_{1+I_1+I_2}^{4+4I_1+4I_2} [2+10I_1+6I_2+(76I_1+8I_2)y]dy$$

$$= \big((2+10I_1+6I_2)y + (38I_1+4I_2)y^2\big)\Big|_{1+I_1+I_2}^{4+4I_1+4I_2}$$

$$= [(2+10I_1+6I_2)(4+4I_1+4I_2) + (38I_1+4I_2)(4+4I_1+4I_2)^2] - [(2+10I_1+6I_2)(1+I_1+I_2) + (38I_1+4I_2)(1+I_1+I_2)^2]$$

$$= [(2+10I_1+6I_2)(4+4I_1+4I_2) + (38I_1+4I_2)(16+80I_1+48I_2)] - [(2+10I_1+6I_2)(1+I_1+I_2) + (38I_1+4I_2)(1+5I_1+3I_2)]$$

$$= [8+152I_1+56I_2+5792I_1+256I_2] - [2+38I_1+14I_2+362I_1+16I_2]$$

$$= 6 + 5544I_1 + 282I_2$$

Take note that the result was the same.

2.2 The double integrals of 2- refined neutrosophic functions over a general region

Theorem 2

Let $f(x, y, I_1, I_2)$ is continuous on the region $R \cup I_1 \cup I_2 = \{(x, y, I_1, I_2): a + a_1I_1 + a_2I_2 \le x \le b + b_1I_1 + b_2I_2 \text{ and } g_1(x, I_1, I_2) \le y \le g_2(x, I_1, I_2)\}$, where: a, a_1, a_2, b, b_1, b_2 are real numbers, while $I_1, I_2 =$ indeterminacy, and $g_1(x, I_1, I_2)$, $g_2(x, I_1, I_2)$ continuous 2- refined neutrosophic functions, where $g_1(x, I_1, I_2) \le g_2(x, I_1, I_2)$, for all $x \in [a + a_1I_1 + a_2I_2 , b + b_1I_1 + b_2I_2]$.

Then the double integrals of 2- refined neutrosophic function $f(x, y, I_1, I_2)$ over a region $R \cup I_1 \cup I_2$ given by the following formula:

$$\iint\limits_{R\cup I_1\cup I_2} f(x, y, I_1, I_2)\, dA = \int\limits_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} \int\limits_{g_1(x,I_1,I_2)}^{g_2(x,I_1,I_2)} f(x, y, I_1, I_2)dydx$$

Example 2

Let $R \cup I_1 \cup I_2 = \{(x, y, I): 0 \le x \le 3 + 3I_1 + 3I_2 \text{ and } 0 \le y \le (1 + I_1 + I_2)x\}$, then:

$$\iint\limits_{R\cup I_1\cup I_2} \left(2e^{x^2} - siny\right) dA = \int\limits_{0}^{3+3I_1+3I_2} \int\limits_{0}^{(1+I_1+I_2)x} \left(2I_1e^{x^2} - cos\, y\right)dydx$$

$$= \int\limits_{0}^{3+3I_1+3I_2} \left(2I_1ye^{x^2} - sin\, y\right)\Big|_0^{(1+I_1+I_2)x} dx$$

$$= \int\limits_{0}^{3+3I_1+3I_2} \left(2I_1(1 + I_1 + I_2)xe^{x^2} - sin(1 + I_1 + I_2)x\right)dx$$

$$= \int\limits_{0}^{3+3I_1+3I_2} \left(6I_1xe^{x^2} - sin(1 + I_1 + I_2)x\right)dx$$

$$= \left(3I_1e^{x^2} + \frac{1}{1 + I_1 + I_2} cos(1 + I_1 + I_2)x\right)\Big|_0^{3+3I_1+3I_2}$$

$$= \left[3I_1e^{9+45I_1+27I_2} + \frac{1}{1 + I_1 + I_2} cos\big((1 + I_1 + I_2)(3 + 3I_1 + 3I_2)\big)\right] - \left[3I_1 + \frac{1}{1 + I_1 + I_2}\right]$$

$$= \left[3I_1e^{9+45I_1+27I_2} + \frac{1}{1 + I_1 + I_2} cos(3 + 15I_1 + 9I_2)\right] - \left[3I_1 + \frac{1}{1 + I_1 + I_2}\right]$$

$$= 3I_1[e^9 + I_1(e^{81} - e^{36}) + I_2(e^{36} - e^9)] + \left(1 + \frac{1}{6}I_1 + \frac{1}{2}I_2\right)[cos\, 3 + I_1(cos\, 27 - cos\, 9) + I_2(cos\, 9 - cos\, 3)] - 1 - \frac{19}{6}I_1 - \frac{1}{2}I_2$$

Example 3

Let $R \cup I_1 \cup I_2 = \{(x, y, I_1, I_2): 0 \le x \le 1 + I_1 + I_2 \ and \ 0 \le y \le x\}$, then:

$$\iint_{R\cup I_1\cup I_2} \frac{cos\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)x}{x}\, dA = \int_0^{1+I_1+I_2}\int_0^{x} \frac{cos\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)x}{x} dydx$$

$$= \int_0^{1+I_1+I_2} \frac{cos\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)x}{x}\, y\Big|_0^x \, dx$$

$$= \int_0^{1+I_1+I_2} \left(\left[\frac{cos\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)x}{x}\, x\right] - \left[\frac{cos\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)x}{x}\, (0)\right]\right) dx$$

$$= \int_0^{1+I_1+I_2} cos\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)x\, dx$$

$$= \frac{1}{\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2} sin\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)x\Big|_0^{1+I_1+I_2}$$

$$= \left[\frac{1}{\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2} sin\left(\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)(1 + I_1 + I_2)\right)\right] - [0]$$

$$= \frac{1}{\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2} sin\left(\left(\pi + \frac{\pi}{2}I_1 + \frac{\pi}{3}I_2\right)(1 + I_1 + I_2)\right)$$

$$= \left(\frac{1}{\pi} - \frac{9}{44\pi}I_1 - \frac{1}{4\pi}I_2\right) sin\left(\pi + \frac{11\pi}{6}I_1 + \frac{2\pi}{3}I_2\right)$$

$$= \left(\frac{1}{\pi} - \frac{9}{44\pi}I_1 - \frac{1}{4\pi}I_2\right)\left[sin(\pi) + I_1\left[sin\left(\frac{21\pi}{6}\right) - sin\left(\frac{5\pi}{3}\right)\right] + I_2\left[sin\left(\frac{5\pi}{3}\right) - sin(\pi)\right]\right]$$

$$= \left(\frac{1}{\pi} - \frac{9}{44\pi}I_1 - \frac{1}{4\pi}I_2\right)\left(\left(-1 + \frac{\sqrt{3}}{2}\right)I_1 + \frac{\sqrt{3}}{2}I_2\right)$$

$$= -\frac{1}{\pi}I_1 + \frac{\sqrt{3}}{2\pi}I_1 + \frac{\sqrt{3}}{2\pi}I_2 + \frac{9}{44\pi}I_1 - \frac{9\sqrt{3}}{44\pi}I_1 + \frac{1}{4\pi}I_1 - \frac{\sqrt{3}}{8\pi}I_1 - \frac{\sqrt{3}}{8\pi}I_2$$

$$= \left(-\frac{6}{11\pi} + \frac{15\sqrt{3}}{44\pi}\right)I_1 + \frac{3\sqrt{3}}{8\pi}I_2$$

Example 4

$$\int_0^{\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2}\int_0^{x} 3I_1 x \sin y\, dydx = \int_0^{\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2} -3I_1 x \cos y|_0^x dx$$

$$= \int_0^{\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2} -3I_1 x \cos x\, dx$$

derivation	integration
$(+) - 3I_1 x$	$\cos x$
$(-)\ - 3I_1$	$\sin x$
0	$-\cos x$

$$= [-3I_1 x \sin x - 3I_1 \cos x]|_0^{\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2}$$

$$= \left[-3I_1\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right)\sin\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right) - 3I_1\cos\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right)\right] - [-3I_1]$$

$$= \left[-3\pi I_1 \sin\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right) - 3I_1\cos\left(\frac{\pi}{3}+\pi I_1-\frac{\pi}{3}I_2\right)\right] + 3I_1$$

$$= \left[-3\pi I_1\left(\sin\left(\frac{\pi}{3}\right) + I_1[\sin(\pi)-\sin(0)] + I_2\left[\sin(0)-\sin\left(\frac{\pi}{3}\right)\right]\right) - 3I_1\left(\cos\left(\frac{\pi}{3}\right) + I_1[\cos(\pi)-\cos(0)] + I_2\left[\cos(0)-\cos\left(\frac{\pi}{3}\right)\right]\right)\right] + 3I_1$$

$$= \left[-3\pi I_1\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}I_2\right) - 3I_1\left(\frac{1}{2}-2I_1+\frac{1}{2}I_2\right)\right] + 3I_1 = 6I_1$$

Theorem 3

Let $f(x,y,I_1,I_2)$ is continuous on the region $R \cup I_1 \cup I_2 = \{(x,y,I_1,I_2): c + c_1I_1 + c_2I_2 \le y \le d + d_1I_1 + d_2I_2 \text{ and } h_1(y,I_1,I_2) \le x \le h_2(y,I_1,I_2)\}$, where: c, c_1, c_2, d, d_1, d_2 are real numbers, while $I_1, I_2 =$ indeterminacy, and $h_1(y,I_1,I_2)$, $h_2(y,I_1,I_2)$ continuous 2- refined neutrosophic functions, where $h_1(y,I_1,I_2) \le h_2(y,I_1,I_2)$, for all $y \in [c + c_1I_1 + c_2I_2\,, d + d_1I_1 + d_2I_2]$.

Then the double integrals of 2- refined neutrosophic function $f(x,y,I_1,I_2)$ over a region $R \cup I_1 \cup I_2$ given by the following formula:

$$\iint_{R\cup I_1\cup I_2} f(x,y,I_1,I_2)\, dA = \int_{c+c_1I_1+c_2I_2}^{d+d_1I_1+d_2I_2}\int_{h_1(y,I_1,I_2)}^{h_2(y,I_1,I_2)} f(x,y,I_1,I_2)\, dxdy$$

Example 5

Let $R \cup I_1 \cup I_2 = \{(x,y,I_1,I_2): 1 + I_1 + I_2 \le y \le 2 + 2I_1 + 2I_2 \text{ and } (1 + I_1 + I_2)y \le x \le 3I_1y^2\}$, then:

$$\iint_{R\cup I_1\cup I_2} 2\, dA = \int_{1+I_1+I_2}^{2+2I_1+2I_2}\int_{(1+I_1+I_2)y}^{3I_1y^2} 2\, dxdy = \int_{1+I_1+I_2}^{2+2I_1+2I_2} 2x|_{(1+I_1+I_2)y}^{3I_1y^2} dy$$

$$= 2\int_{1+I_1+I_2}^{2+2I_1+2I_2} (3I_1y^2 - (1+I_1+I_2)y)\, dy$$

$$= (2I_1y^3 - (1+I_1+I_2)y^2)|_{1+I_1+I_2}^{2+2I_1+2I_2}$$

$$= [2I_1(2+2I_1+2I_2)^3 - (1+I_1+I_2)(2+2I_1+2I_2)^2] - [2I_1(1+I_1+I_2)^3 - (1+I_1+I_2)(1+I_1+I_2)^2]$$

$$= (1+I_1+I_2)^3[14I_1 - 3] = (1+19I_1+7I_2)[14I_1 - 3]$$

$$= -3 + 321I_1 - 21I_2$$

Theorem 4

Let $f(x, y, I_1, I_2)$ and $h(x, y, I_1, I_2)$ be integrable over region $R \cup I_1 \cup I_2$, and let $c + c_1I_1 + c_2I_2$ any constant, then:

(1) $$\iint_{R\cup I_1\cup I_2} (c + c_1I_1 + c_2I_2) f(x, y, I_1, I_2)\, dA = (c + c_1I_1 + c_2I_2) \iint_{R\cup I_1\cup I_2} f(x, y, I_1, I_2)\, dA$$

(2) $$\iint_{R\cup I_1\cup I_2} [f(x, y, I_1, I_2) + h(x, y, I_1, I_2)]\, dA = \iint_{R\cup I_1\cup I_2} f(x, y, I_1, I_2)\, dA + \iint_{R\cup I_1\cup I_2} h(x, y, I_1, I_2)\, dA$$

(3) If $R \cup I_1 \cup I_2 = R_1 \cup R_2 \cup I_1 \cup I_2$, where R_1, R_2 are not overlap regions, then:

$$\iint_{R\cup I_1\cup I_2} f(x, y, I_1, I_2)\, dA = \iint_{R_1\cup I_1\cup I_2} f(x, y, I_1, I_2)\, dA + \iint_{R_2\cup I_1\cup I_2} f(x, y, I_1, I_2)\, dA$$

3.2 Applications of double integrals of 2- refined neutrosophic functions

We can use double integrals of 2- refined neutrosophic functions to calculate the area of region:

$$A = \iint_{R\cup I_1\cup I_2} dxdy = \iint_{R\cup I_1\cup I_2} dydx$$

Example 6

Let's find the area of the plane region bounded by the curve of $y = x^2$ and $y = (1 + I_1 + I_2)x$ by using the double integrals of 2- refined neutrosophic functions.

intersection points:

$$x^2 = (1 + I_1 + I_2)x$$

$$x(x - 1 + I_1 + I_2) = 0$$

$$\Rightarrow \begin{cases} x = 0 \\ x = 1 + I_1 + I_2 \end{cases}$$

then:

$$\begin{cases} y = 0 \\ y = 1 + 5I_1 + 3I_2 \end{cases}$$

so, the intersection points are: $(0, 0)$ and $\left(1 - \frac{1}{4}I_1, 1 + 5I_1 + 3I_2\right)$

- <u>using the horizontal slice:</u>

$$\int_0^{1+I_1+I_2}\int_{x^2}^{(1+I_1+I_2)x} dydx = \int_0^{1-\frac{1}{4}I_1} y|_{x^2}^{(1+I_1+I_2)x} dx$$

$$= \int_0^{1+I_1+I_2} \left((1+I_1+I_2)x - x^2\right)dx$$

$$= \left((1+I_1+I_2)\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^{1+I_1+I_2}$$

$$= (1+I_1+I_2)\frac{(1+I_1+I_2)^2}{2} - \frac{(1+I_1+I_2)^3}{3}$$

$$= \frac{(1+I_1+I_2)^3}{2} - \frac{(1+I_1+I_2)^3}{3}$$

$$= \boxed{\frac{1}{6} + \frac{19}{6}I_1 + \frac{7}{6}I_2}$$

- using the vertical slice:

$$\int_0^{1+5I_1+3I_2}\int_{\left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)y}^{\sqrt{y}} dxdy = \int_0^{1+5I_1+3I_2} x|_{\left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)y}^{\sqrt{y}} dy$$

$$= \int_0^{1+5I_1+3I_2} \left(\sqrt{y} - \left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)y\right)dy$$

$$= \left(\frac{2}{3}\sqrt{y^3} - \left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)\frac{y^2}{2}\right)\Big|_0^{1+5I_1+3I_2}$$

$$= \left[\frac{2}{3}\sqrt{(1+5I_1+3I_2)^3} - \left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right)\frac{(1+5I_1+3I_2)^2}{2}\right] - [0]$$

$$= \frac{2}{3}\sqrt{1+665I_1+63I_2} - \left(\frac{1}{2} + \frac{19}{2}I_1 + \frac{7}{2}I_2\right)$$

$$= \frac{2}{3}(1+19I_1+7I_2) - \left(\frac{1}{2} + \frac{19}{2}I_1 + \frac{7}{2}I_2\right)$$

$$= \boxed{\frac{1}{6} + \frac{19}{6}I_1 + \frac{7}{6}I_2}$$

Take note that the result was the same.

Definition 1

Let $R(I_1, I_2) = \{a + bI_1 + cI_2;\ a, b, c \in R\}$ be the real neutrosophic field, then $a + bI_1 + cI_2 \geq 0$ if and only if $a \geq 0, a + c \geq 0, a + b + c \geq 0$

Example 7

Let's find the area of the plane region bounded by the curve of $y = 1 + 5I_1 + 3I_2 - x^2$ and $x - axis$ by using the double integrals of 2- refined neutrosophic functions.

$$1 + 5I_1 + 3I_2 - x^2 = 0$$
$$x^2 = 1 + 5I_1 + 3I_2$$

Let's find: $\sqrt{1 + 5I_1 + 3I_2}$

$$\sqrt{1 + 5I_1 + 3I_2} = x_1 + x_2I_1 + x_3I_2$$

$$1 + 5I_1 + 3I_2 = (x_1 + x_2I_1 + x_3I_2)^2$$

$$1 + 5I_1 + 3I_2 = (x_1 + x_2I_1)^2 + 2(x_1 + x_2I_1)(x_3I_2) + (x_3I_2)^2$$

$$1 + 5I_1 + 3I_2 = x_1{}^2 + 2x_1x_2I_1 + (x_2I_1)^2 + 2(x_1 + x_2I_1)(x_3I_2) + (x_3I_2)^2$$

$$1 + 5I_1 + 3I_2 = x_1{}^2 + 2x_1x_2I_1 + x_2{}^2I_1 + 2xx_3I_2 + 2x_2x_3I_1 + x_3{}^2I_2$$

$$1 + 5I_1 + 3I_2 = x_1{}^2 + (x_2{}^2 + 2x_1x_2 + 2x_2x_3)I_1 + (x_3{}^2 + 2x_1x_3)I_2$$

Whence:

$$\begin{cases} x_1{}^2 = 1 \\ x_2{}^2 + 2x_1x_2 + 2x_2x_3 = 5 \\ x_3{}^2 + 2x_1x_3 = 3 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = \pm 1 \\ x_2{}^2 + 2x_1x_2 + 2x_2x_3 = 5 \ (1) \\ x_3{}^2 + 2x_1x_3 = 3 \ (2) \end{cases}$$

Case1: $x_1 = 1$ by substitution in (2)

$$x_3{}^2 + 2x_3 - 3 = 0$$
$$(x_3 - 1)(x_3 + 3) = 0$$

Then:

$$\begin{cases} x_3 = 1 \\ x_3 = -3 \end{cases}$$

- For $x_3 = 1$ we substitute in (1)

$$x_2{}^2 + 4x_2 - 5 = 0$$
$$(x_2 + 5)(x_2 - 1) = 0$$

Then:

$$\begin{cases} x_2 = 1 \\ x_2 = -5 \end{cases}$$

Hence:

$$\sqrt{1 + 5I_1 + 3I_2} = 1 + I_1 + I_2$$

Or:

$$= 1 - 5I_1 + I_2$$

- For $x_3 = -3$ we substitute in (1)

$$x_2{}^2 - 4x_2 - 5 = 0$$
$$(x_2 - 5)(x_2 + 1) = 0$$

Then:

$$\begin{cases} x_2 = 5 \\ x_2 = -1 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2}=1+5I_1-3I_2$$

Or:

$$=1-I_1-3I_2$$

Case2: $x_1=-1$ by substitution in (2)

$$x_3{}^2-2x_3-3=0$$
$$(x_3-3)(x_3+1)=0$$

Then:

$$\begin{cases} x_3=3 \\ x_3=-1 \end{cases}$$

- For $x_3=3$ we substitute in (1)

$$x_2{}^2+4x_2-5=0$$

$$(x_2-1)(x_2+5)=0$$

Then:

$$\begin{cases} x_2=1 \\ x_2=-5 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2}=-1+I_1+3I_2$$

Or:

$$=-1-5I_1+3I_2$$

- For $x_3=-1$ we substitute in (1)

$$x_2{}^2-4x_2-5=0$$
$$(x_2-5)(x_2+1)=0$$

$$\begin{cases} x_2=5 \\ x_2=-1 \end{cases}$$

Hence:

$$\sqrt{1+5I_1+3I_2}=-1+5I_1-I_2$$

Or:

$$=-1-I_1-I_2$$

hence intersection points are:

$$(1+I_1+I_2,0),(-1-I_1-I_2,0)$$

Or:

$$(1+5I_1-3I_2,0),(-1-5I_1+3I_2,0)$$

Or:

$$(1-I_1-3I_2,0),(-1+I_1+3I_2,0)$$

Or:

$$(1-5I_1+I_2,0),(-1+5I_1-I_2,0)$$

For: $(1+I_1+I_2,0),(-1-I_1-I_2,0)$

$$A = \int_{-1-I_1-I_2}^{1+I_1+I_2} \int_{0}^{1+5I_1+3I_2-x^2} dydx = \int_{-1-I_1-I_2}^{1+I_1+I_2} y\Big|_0^{1+5I_1+3I_2-x^2} dx$$

$$= \int_{-1-I_1-I_2}^{1+I_1+I_2} (1+5I_1+3I_2-x^2)dx$$

$$= \left((1+5I_1+3I_2)x - \frac{x^3}{3}\right)\Bigg|_{-1-I_1-I_2}^{1+I_1+I_2}$$

$$= \left[(1+5I_1+3I_2)(1+I_1+I_2) - \frac{(1+I_1+I_2)^3}{3}\right] - \left[(1+5I_1+3I_2)(-1-I_1-I_2) - \frac{(-1-I_1-I_2)^3}{3}\right]$$

$$= 2\left[(1+5I_1+3I_2)(1+I_1+I_2) - \frac{(1+I_1+I_2)^3}{3}\right]$$

$$= 2\left[1+19I_1+7I_2 - \left(\frac{1+19I_1+7I_2}{3}\right)\right]$$

$$= \frac{4}{3} + \frac{76}{3}I_1 + \frac{28}{3}I_2 > 0$$

$becous: \frac{4}{3} > 0, \frac{4}{3} + \frac{28}{3} > 0 \ and \ \frac{4}{3} + \frac{76}{3} + \frac{28}{3}, see \ definition \ 1$

For: $(1+5I_1-3I_2, 0), (-1-5I_1+3I_2, 0)$

$$A = \int_{-1-5I_1+3I_2}^{1+5I_1-3I_2} \int_{0}^{1+5I_1+3I_2-x^2} dydx = \int_{-1-5I_1+3I_2}^{1+5I_1-3I_2} y\Big|_0^{1+5I_1+3I_2-x^2} dx$$

$$= \int_{-1-5I_1+3I_2}^{1+5I_1-3I_2} (1+5I_1+3I_2-x^2)dx$$

$$= \left((1+5I_1+3I_2)x - \frac{x^3}{3}\right)\Bigg|_{-1-5I_1+3I_2}^{1+5I_1-3I_2}$$

$$= \left[(1+5I_1+3I_2)(1+5I_1-3I_2) - \frac{(1+5I_1-3I_2)^3}{3}\right] - \left[(1+5I_1+3I_2)(-1-5I_1+3I_2) - \frac{(-1-5I_1+3I_2)^3}{3}\right]$$

$$= 2\left[(1+5I_1+3I_2)(1+5I_1-3I_2) - \frac{(1+5I_1-3I_2)^3}{3}\right]$$

$$= 2\left[1+35I_1-9I_2 - \left(\frac{1+35I_1-9I_2}{3}\right)\right]$$

$$=\frac{4}{3}+\frac{140}{3}I_1-12I_2\ (rejected)$$

For: $(1-I_1-3I_2,0),(-1+I_1+3I_2,0)$

$$A=\int_{-1+I_1+3I_2}^{1-I_1-3I_2}\int_{0}^{1+5I_1+3I_2-x^2}dydx=\int_{-1+I_1+3I_2}^{1-I_1-3I_2}y|_0^{1+5I_1+3I_2-x^2}dx$$

$$=\int_{-1+I_1+3I_2}^{1-I_1-3I_2}(1+5I_1+3I_2-x^2)dx$$

$$=\left((1+5I_1+3I_2)x-\frac{x^3}{3}\right)\Bigg|_{-1+I_1+3I_2}^{1-I_1-3I_2}$$

$$=\left[(1+5I_1+3I_2)(1-I_1-3I_2)-\frac{(1-I_1-3I_2)^3}{3}\right]-\left[(1+5I_1+3I_2)(-1+I_1+3I_2)-\frac{(-1+I_1+3I_2)^3}{3}\right]$$

$$=2\left[(1+5I_1+3I_2)(1-I_1-3I_2)-\frac{(1-I_1-3I_2)^3}{3}\right]$$

$$=2\left[1-19I_1-9I_2-\left(\frac{1-19I_1-9I_2}{3}\right)\right]$$

$$=\frac{4}{3}-\frac{76}{3}I_1-12I_2\ (rejected)$$

For: $(1-5I_1+I_2,0),(-1+5I_1-I_2,0)$

$$A=\int_{-1+5I_1-I_2}^{1-5I_1+I_2}\int_{0}^{1+5I_1+3I_2-x^2}dydx=\int_{-1+5I_1-I_2}^{1-5I_1+I_2}y|_0^{1+5I_1+3I_2-x^2}dx$$

$$=\int_{-1+5I_1-I_2}^{1-5I_1+I_2}(1+5I_1+3I_2-x^2)dx$$

$$=\left((1+5I_1+3I_2)x-\frac{x^3}{3}\right)\Bigg|_{-1+5I_1-I_2}^{1-5I_1+I_2}$$

$$=\left[(1+5I_1+3I_2)(1-5I_1+I_2)-\frac{(1-5I_1+I_2)^3}{3}\right]-\left[(1+5I_1+3I_2)(-1+5I_1-I_2)-\frac{(-1+5I_1-I_2)^3}{3}\right]$$

$$=2\left[(1+5I_1+3I_2)(1-5I_1+I_2)-\frac{(1-5I_1+I_2)^3}{3}\right]$$

$$=2\left[1-35I_1+7I_2-\left(\frac{1-35I_1+7I_2}{3}\right)\right]$$

$$= \frac{4}{3} - \frac{140}{3} I_1 + \frac{28}{3} I_2 \ (rejected)$$

3. Conclusions

The paper presented an important study on the double integrals of 2- refined neutrosophic functions and its applications, where it presented a set of theorems in the double integrals of 2-refined neutrosophic functions, discussed several methods to calculate these integrals using the vertical slice or the horizontal slice, where we obtained the same results in both methods. Finally, we presented applications of the double integrals of 2- refined neutrosophic functions to calculate the area of region, and concluded that we take positive values by using the definition of the positive 2-refined neutrosophic number.

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