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New approach towards different ideals using bipolar valued intuitionistic neutrosophic set of an ordered semigroups

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Abstract. This paper introduces the notion of bipolar valued intuitionistic neutrosophic subsemigroup (BIntNS), bipolar valued intuitionistic neutrosophic left ideal (BIntNLI), bipolar valued intuitionistic neutrosophic right ideal (BIntNRI), bipolar valued intuitionistic neutrosophic ideal (NI), and bipolar valued intuitionistic neutrosophic bi-ideal (BIntNBI) of an ordered semigroups, along with some of its properties. A new extension over ordered semigroups of the bipolar valued intuitionistic neutrosophic ideal. The lower level set is a subsemigroup ($BVLI, BVRI, BVBI$) of an ordered semigroup. Examples are provided to exemplify our results, however the contrary may not be true. Every BIntNS is a BIntNS of an ordered semigroup.

Keywords: Left ideal, right ideal, ideal, bi-ideal.

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1. **Introduction**

Uncertainty has resulted in the development of several uncertain theories, including FS [1], intuitionistic FS (IFS) [2], Pythagorean FS (PFS) [3], and spherical FS (SFS) [4]. An FS consists of sets of membership values, known as MV, ranging from 0 to 1. Although Atanassov's claim that non-membership values (NMV) may only have a value of 1, IFS is classified as MV. The total of MVs and NMVs may occasionally exceed 1 during the decision-making process. Yager [3] developed the generalized MV and NMV logic using PFS logic, with a value of no more than 1 and based on the square of the MVs and NMVs. These notions cannot describe the neutral situation, which is neither positive nor negative. He explored their features in a manner similar to that of set theory. Rosenfeld [5] defined fuzzy subgroups and their features in 1971. Kuroki [6] introduced fuzzy semigroups as an extension of classical semigroups. Mordeson [7] proposed a fuzzy semigroup characterization. Sen et al. [8, 9] described semigroups and their properties. Kehayopula [10] examined ordered semigroups. Somsak Lekkoksung [11] discusses Q-FIs in ordered semigroups.

Kehayopula et al. [12] began research on fuzzy ordered semigroups. Faiz Muhammad Khan et al. [13] introduced the (τ_1, τ_2)-fuzzy bi-ideal and (τ_1, τ_2)-fuzzy subsemigroup. Jun et al. [14] provided results for ordered semigroups with $(\hbar, \hbar \veebar q)$-fuzzy bi-ideals. Bhakat et al. [15] developed $(\hbar, \hbar \veebar q)$-fuzzy subgroups based on the concept of membership. Kazanci et al. [16] defined FBI characteristics using $(\hbar, \hbar \veebar q)$ and proposed a generalized fuzzy bi-ideal in semigroups. Smarandache [17] invented the neutrosophic set (NS) to handle ambiguous and contradictory data. Huseyin et al. [18] explored the idea of intuitionistic neutrosophic subgroups. Palanikumar et al. [19] introduced an intuitionistic fuzzy normal subbisemiring of bisemiring. Palanikumar et al. [20] introduced bisemiring using bipolar-valued neutrosophic normal sets. The neutrosophic ideals of the ordered semigroup (τ, υ) are explored, and numerous properties are demonstrated using examples. Hila [21] et al. explored the concept of bi-ideals on ordered semigroups. Recently, Udhayakumar et al. discussed the many fuzzy applications and its generalization [22–25].

2. Preliminaries

Definition 2.1. An ordered semigroup $\mathscr{O}$ together with an order relation $\leq$ such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in \mathscr{O}$.

Definition 2.2. Let $\gimel$ and $\top$ be two non empty subsets of $\mathscr{O}$. We denote

(1) $(\gimel] = \{t \in \mathscr{O} \,|\, t \leq h \, for \, some \, h \in \gimel\}$,
(2) $\gimel\top = \{a_1 a_2 : a_1 \in \gimel, \, a_2 \in \top\}$,
(3) $\gimel_a = \{(b, c) \in \mathscr{O} \times S \,|\, a \leq bc\}$.

Definition 2.3. Let $\flat$ be a fuzzy subset of $\mathscr{O}$. A mapping $\flat : \mathscr{O} \to [0, 1]$ is called a fuzzy subsemigroup (FSS) of $\mathscr{O}$ if $\flat(xy) \geq \min\{\flat(x), \flat(y)\}$ for all $x, y \in \mathscr{O}$.

Definition 2.4. A fuzzy $\flat$ subset of $\mathscr{O}$ is called a FBI of $\mathscr{O}$ if

(1) $a_1 \leq a_2 \implies \flat(a_1) \geq \flat(a_2)$ and
(2) $\flat(xyz) \geq \min\{\flat(x), \flat(z)\}$ for all $x, y, z \in \mathscr{O}$.

Definition 2.5. A fuzzy subset υ of an ordered semigroup $\mathscr{O}$ is called a FRI(FLI) of $\mathscr{O}$ if

(1) $x \leq y \Rightarrow \upsilon(x) \geq \upsilon(y)$ for all $x, y \in \mathscr{O}$,
(2) $\upsilon(xy) \geq \upsilon(x)$ (resp. $\upsilon(xy) \geq \upsilon(y)$) for all $x, y \in \mathscr{O}$,

(3) A fuzzy subset υ of an ordered semigroup $\mathscr{O}$ is called a FI of $\mathscr{O}$, if it is both FRI and FLI.

Definition 2.6. Let $\flat$ be a fuzzy subset of $\mathscr{O}$ and $t \in [0,1]$. The set $\flat_t = \{x \in \mathscr{O} | \flat(x) \geq t\}$ is called the level subset of $\flat$. Clearly $\flat_t \subseteq \flat_s$ whenever $t \geq s$.

Definition 2.7. Let A be a fuzzy set,if Λ_A is the characteristic function of A, then $(\Lambda_A)^{\ell}_{\xi}$ is defined as

$$(\Lambda_A)^{\ell}_{\xi}(x) := \begin{cases} \ell \, if \, x \in A, \\ \xi \, if \, x \notin A. \end{cases}$$

Definition 2.8. An intuitionistic neutrosophic set $\tilde{N}$ on the set X is defined as follows:
$\tilde{N} = \{(x, \langle \tilde{\Re}(x), \tilde{\Im}(x), \tilde{\mho}(x) \rangle) : x \in X\}$ where for all $x \in X$,
$\min\{\tilde{\Re}(x), \tilde{\Im}(x)\} \leq 0.5, \min\{\tilde{\Re}(x), \tilde{\mho}(x)\} \leq 0.5, \min\{\tilde{\Im}(x), \tilde{\mho}(x)\} \leq 0.5$ with the condition
$0 = \tilde{\Re}(x) + \tilde{\Im}(x) + \tilde{\mho}(x) \leq 2$.

3. (τ, υ) bipolar valued intuitionistic neutrosophic ideals

The ordered semigroup is indicated in this section by $\mathscr{O}$. If $(\tau, \upsilon) \in [0,1]$ is such that $0 \geq \tau^- > \upsilon^- \geq -1$ and $0 \leq \tau^+ < \upsilon^+ \leq 1$, then both (τ, υ) are arbitrarily fixed.

Definition 3.1. A bipolar valued intuitionistic neutrosophic subset
$\flat = [(\tilde{\Re}^-_{\mathsf{T}}, \tilde{\Re}^+_{\mathsf{T}}), (\tilde{\Im}^-_{\mathsf{T}}, \tilde{\Im}^+_{\mathsf{T}}), (\tilde{\mho}^-_{\mathsf{T}}, \tilde{\mho}^+_{\mathsf{T}})]$ of $\mathscr{O}$ is called a (τ, υ) BIntNS of $\mathscr{O}$ if

(1) $\epsilon \leq \varsigma \Rightarrow \tilde{\Re}^-(\epsilon) \leq \tilde{\Re}^-(\varsigma)$, $\tilde{\Im}^-(\epsilon) \leq \tilde{\Im}^-(\varsigma)$ and $\tilde{\mho}^-(\epsilon) \geq \tilde{\mho}^-(\varsigma)$, $\tilde{\Re}^+(\epsilon) \geq \tilde{\Re}^+(\varsigma)$, $\tilde{\Im}^+(\epsilon) \geq \tilde{\Im}^+(\varsigma)$ and $\tilde{\mho}^+(\epsilon) \leq \tilde{\mho}^+(\varsigma)$,

(2) $\min\{\tilde{\Re}^-(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Re}^-(\epsilon), \tilde{\Re}^-(\varsigma), \upsilon^-\}$,
$\min\{\tilde{\Im}^-(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Im}^-(\epsilon), \tilde{\Im}^-(\varsigma), \upsilon^-\}$,
$\max\{\tilde{\mho}^-(\epsilon\varsigma), \tau^-\} \geq \min\{\tilde{\mho}^-(\epsilon), \tilde{\mho}^-(\varsigma), \upsilon^-\}$,

(3) $\max\{\tilde{\Re}^+(\epsilon\varsigma), \tau^+\} \geq \min\{\tilde{\Re}^+(\epsilon), \tilde{\Re}^+(\varsigma), \upsilon^+\}$,
$\max\{\tilde{\Im}^+(\epsilon\varsigma), \tau^+\} \geq \min\{\tilde{\Im}^+(\epsilon), \tilde{\Im}^+(\varsigma), \upsilon^+\}$,
$\min\{\tilde{\mho}^+(\epsilon\varsigma), \tau^+\} \leq \max\{\tilde{\mho}^+(\epsilon), \tilde{\mho}^+(\varsigma), \upsilon^+\}$. for all $\epsilon, \varsigma \in \mathscr{O}$.

Example 3.2. Let $\mathscr{O} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$ is defined on $\mathscr{O}$ with the following Cayley table:

$\cdot$	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
$\hbar_1$	$\hbar_1$	$\hbar_1$	$\hbar_1$	$\hbar_1$
$\hbar_2$	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$
$\hbar_3$	$\hbar_1$	$\hbar_3$	$\hbar_3$	$\hbar_3$
$\hbar_4$	$\hbar_1$	$\hbar_3$	$\hbar_3$	$\hbar_3$

$\leq := \{(\hbar_1,\hbar_1),(\hbar_1,\hbar_2),(\hbar_1,\hbar_3),(\hbar_1,\hbar_4),(\hbar_2,\hbar_2),(\hbar_2,\hbar_3),(\hbar_2,\hbar_4),(\hbar_3,\hbar_3),(\hbar_4,\hbar_3),(\hbar_4,\hbar_4)\}$.

The mapping $\mathsf{T} = [(\tilde{\Re}^-_{\mathsf{T}},\tilde{\Re}^+_{\mathsf{T}}),(\tilde{\Im}^-_{\mathsf{T}},\tilde{\Im}^+_{\mathsf{T}}),(\tilde{\mho}^-_{\mathsf{T}},\tilde{\mho}^+_{\mathsf{T}})] : \mathscr{O}\times\mathscr{O}\to[-1,0]\times[0,1]$.

$$(\tilde{\Re}^-,\tilde{\Re}^+)(\hbar) = \begin{cases}(-0.47,44) & if\ \hbar=\hbar_1\\(-0.32,37) & if\ \hbar=\hbar_2\\(-0.02,07) & if\ \hbar=\hbar_3\\(-0.12,17) & if\ \hbar=\hbar_4\end{cases}\qquad (\tilde{\Im}^-,\tilde{\Im}^+)(\hbar) = \begin{cases}(-0.27,23) & if\ \hbar=\hbar_1\\(-0.17,12) & if\ \hbar=\hbar_2\\(-0.02,07) & if\ \hbar=\hbar_3\\(-0.07,10) & if\ \hbar=\hbar_4\end{cases}$$

$$(\tilde{\mho}^-,\tilde{\mho}^+)(\hbar) = \begin{cases}(-0.22,27) & if\ \hbar=\hbar_1\\(-0.27,32) & if\ \hbar=\hbar_2\\(-0.37,42) & if\ \hbar=\hbar_3\\(-0.32,37) & if\ \hbar=\hbar_4\end{cases}$$

Then T is a $(0.42, 0.48)$ BIntNS of $\mathscr{O}$.

Definition 3.3. A subset $\flat$ of $\mathscr{O}$ is called a (τ,υ)-BIntNBI of $\mathscr{O}$ if

(1) If $\epsilon\leq\varsigma$, then $\tilde{\Re}^-(\epsilon)\leq\tilde{\Re}^-(\varsigma)$, $\tilde{\Im}^-(\epsilon)\leq\tilde{\Im}^-(\varsigma)$ and $\tilde{\mho}^-(\epsilon)\geq\tilde{\mho}^-(\varsigma)$, $\tilde{\Re}^+(\epsilon)\geq\tilde{\Re}^+(\varsigma)$, $\tilde{\Im}^+(\epsilon)\geq\tilde{\Im}^+(\varsigma)$ and $\tilde{\mho}^+(\epsilon)\leq\tilde{\mho}^+(\varsigma)$,

(2) $\min\{\tilde{\Re}^-(\epsilon_1\varsigma),\tau^-\}\leq\max\{\tilde{\Re}^-(\epsilon),\tilde{\Re}^-(\varsigma),\upsilon^-\}$,
$\min\{\tilde{\Im}^-(\epsilon_1\varsigma),\tau^-\}\leq\max\{\tilde{\Im}^-(\epsilon),\tilde{\Im}^-(\varsigma),\upsilon^-\}$,
$\max\{\tilde{\mho}^-(\epsilon_1\varsigma),\tau^-\}\geq\min\{\tilde{\mho}^-(\epsilon),\tilde{\mho}^-(\varsigma),\upsilon^-\}$,
$\min\{\tilde{\Re}^-(\epsilon_1\varsigma_2\kappa),\tau^-\}\leq\max\{\tilde{\Re}^-(\epsilon),\tilde{\Re}^-(\kappa),\upsilon^-\}$,
$\min\{\tilde{\Im}^-(\epsilon_1\varsigma_2\kappa),\tau^-\}\leq\max\{\tilde{\Im}^-(\epsilon),\tilde{\Im}^-(\kappa),\upsilon^-\}$,
$\max\{\tilde{\mho}^-(\epsilon_1\varsigma_2\kappa),\tau^-\}\geq\min\{\tilde{\mho}^-(\epsilon),\tilde{\mho}^-(\kappa),\upsilon^-\}$,

(3) $\max\{\tilde{\Re}^+(\epsilon_1\varsigma),\tau^+\}\geq\min\{\tilde{\Re}^+(\epsilon),\tilde{\Re}^+(\varsigma),\upsilon^+\}$,
$\max\{\tilde{\Im}^+(\epsilon_1\varsigma),\tau^+\}\geq\min\{\tilde{\Im}^+(\epsilon),\tilde{\Im}^+(\varsigma),\upsilon^+\}$,
$\min\{\tilde{\mho}^+(\epsilon_1\varsigma),\tau^+\}\leq\max\{\tilde{\mho}^+(\epsilon),\tilde{\mho}^+(\varsigma),\upsilon^+\}$,
$\max\{\tilde{\Re}^+(\epsilon_1\varsigma_2\kappa),\tau^+\}\geq\min\{\tilde{\Re}^+(\epsilon),\tilde{\Re}^+(\kappa),\upsilon^+\}$,
$\max\{\tilde{\Im}^+(\epsilon_1\varsigma_2\kappa),\tau^+\}\geq\min\{\tilde{\Im}^+(\epsilon),\tilde{\Im}^+(\kappa),\upsilon^+\}$,
$\min\{\tilde{\mho}^+(\epsilon_1\varsigma_2\kappa),\tau^+\}\leq\max\{\tilde{\mho}^+(\epsilon),\tilde{\mho}^+(\kappa),\upsilon^+\}$, for $\epsilon,\varsigma,\kappa\in\mathscr{O}$

Theorem 3.4. *A non-empty subset $\flat_\tau$ is a $\tilde{\Re}_\tau$ is a (τ,υ) BIntNS (BIntNLI, BIntNRI, BIntNBI) of $\mathscr{O}$. Then the lower level set is an BSS $(BVLI, BVRI, BVBI)$ of $\mathscr{O}$, where $\tilde{\Re}_{\tau^-}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)<\tau^-\}$, $\tilde{\Im}_{\tau^-}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)<\tau^-\}$, $\tilde{\mho}_{\tau^-}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)>\tau^-\}$, $\tilde{\Re}_{\tau^+}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)>\tau^+\}$, $\tilde{\Im}_{\tau^+}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)>\tau^+\}$ and $\tilde{\mho}_{\tau^+}=\{\epsilon\in\mathscr{O}|\tilde{\Re}(\epsilon)<\tau^+\}$*

Proof. Suppose that $\flat_{\tau^-}$ is a (τ,υ) BIntNS of $\mathscr{O}$. Let $\epsilon,\varsigma\in\mathscr{O}$ such that $\epsilon,\varsigma\in\tilde{\Re}^-_{\tau^-}$. Then $\tilde{\Re}^-(\epsilon)<\tau^-,\tilde{\Re}^-(\varsigma)<\tau^-$. Therefore $\min\{\tilde{\Re}^-(\epsilon\varsigma),\tau^-\}\leq\max\{\tilde{\Re}^-(\epsilon),\tilde{\Re}^-(\varsigma),\upsilon^-\}<\max\{\tau^-,\tau^-,\upsilon^-\}=\tau^-$. Hence $\tilde{\Re}^-(\epsilon\varsigma)<\tau^-$. It shows that $\epsilon\varsigma\in\tilde{\Re}^-_{\tau^-}$. Therefore $\tilde{\Re}^-_{\tau^-}$ is a

BSS of $\mathscr{O}$. Let $\epsilon,\varsigma \in \mathscr{O}$ such that $\epsilon,\varsigma \in \tilde{\Im}^{-}_{\tau^-}$. Then $\tilde{\Im}^{-}(\epsilon) < \tau^-, \tilde{\Im}^{-}(\varsigma) < \tau^-$. Therefore $\min\{\tilde{\Im}^{-}(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Im}^{-}(\epsilon), \tilde{\Im}^{-}(\varsigma), \upsilon^-\} < \max\{\tau^-, \tau^-, \upsilon^-\} = \tau^-$. Hence $\tilde{\Im}^{-}(\epsilon\varsigma) < \tau^-$. It shows that $\epsilon\varsigma \in \tilde{\Im}^{-}_{\tau^-}$. Therefore $\tilde{\Im}^{-}_{\tau^-}$ is a BSS of $\mathscr{O}$. Let $\epsilon,\varsigma \in \mathscr{O}$ such that $\epsilon,\varsigma \in \tilde{\mho}^{-}_{\tau^-}$. Then $\tilde{\mho}^{-}(\epsilon) > \tau^-, \tilde{\mho}^{-}(\varsigma) > \tau^-$. Therefore $\max\{\tilde{\mho}^{-}(\epsilon\varsigma), \tau^-\} \geq \min\{\tilde{\mho}^{-}(\epsilon), \tilde{\mho}^{-}(\varsigma), \upsilon^-\} > \min\{\tau^-, \tau^-, \upsilon^-\} = \upsilon^-$. Hence $\tilde{\mho}^{-}(\epsilon\varsigma) > \tau^-$. It shows that $\epsilon\varsigma \in \tilde{\mho}^{-}_{\tau^-}$. Therefore $\tilde{\mho}^{-}_{\tau^-}$ is a BSS of $\mathscr{O}$.

Suppose that $\flat_{\tau^+}$ is a (τ,υ) BIntNS of $\mathscr{O}$. Let $\epsilon,\varsigma \in \mathscr{O}$ such that $\epsilon,\varsigma \in \tilde{\Re}^{+}_{\tau^+}$. Then $\tilde{\Re}^{+}(\epsilon) > \tau^+, \tilde{\Re}^{+}(\varsigma) > \tau^+$. Therefore $\max\{\tilde{\Re}^{+}(\epsilon\varsigma), \tau^+\} \geq \min\{\tilde{\Re}^{+}(\epsilon), \tilde{\Re}^{+}(\varsigma), \upsilon^+\} > \min\{\tau^+, \tau^+, \upsilon^+\} = \tau^+$. Hence $\tilde{\Re}^{+}(\epsilon\varsigma) > \tau^+$. It shows that $\epsilon\varsigma \in \tilde{\Re}^{+}_{\tau^+}$. Therefore $\tilde{\Re}^{+}_{\tau^+}$ is a BSS of $\mathscr{O}$. Let $\epsilon,\varsigma \in \mathscr{O}$ such that $\epsilon,\varsigma \in \tilde{\Im}^{+}_{\tau^+}$. Then $\tilde{\Im}^{+}(\epsilon) > \tau^+, \tilde{\Im}^{+}(\varsigma) > \tau^+$. Therefore $\max\{\tilde{\Im}^{+}(\epsilon\varsigma), \tau^+\} \geq \min\{\tilde{\Im}^{+}(\epsilon), \tilde{\Im}^{+}(\varsigma), \upsilon^+\} > \min\{\tau^+, \tau^+, \upsilon^+\} = \tau^+$. Hence $\tilde{\Im}^{+}(\epsilon\varsigma) > \tau^+$. It shows that $\epsilon\varsigma \in \tilde{\Im}^{+}_{\tau^+}$. Therefore $\tilde{\Im}^{+}_{\tau^+}$ is a BSS of $\mathscr{O}$. Let $\epsilon,\varsigma \in \mathscr{O}$ such that $\epsilon,\varsigma \in \tilde{\mho}^{+}_{\tau^+}$. Then $\tilde{\mho}^{+}(\epsilon) < \tau^+, \tilde{\mho}^{+}(\varsigma) < \tau^+$. Therefore $\min\{\tilde{\mho}^{+}(\epsilon\varsigma), \tau^+\} \leq \max\{\tilde{\mho}^{+}(\epsilon), \tilde{\mho}^{+}(\varsigma), \upsilon^+\} < \max\{\tau^+, \tau^+, \upsilon^+\} = \upsilon^+$. Hence $\tilde{\mho}^{+}(\epsilon\varsigma) < \tau^+$. It shows that $\epsilon\varsigma \in \tilde{\mho}^{+}_{\tau^+}$. Therefore $\tilde{\mho}^{+}_{\tau^+}$ is a BSS of $\mathscr{O}$. Therefore $\flat_{\tau^+}$ is a BSS of $\mathscr{O}$. Therefore $\flat_{\tau}$ is a BSS of $\mathscr{O}$.

Theorem 3.5. *A non-empty subset $\daleth$ of $\mathscr{O}$ is a BSS $[BVLI, BVRI, BVBI]$ of $\mathscr{O}$ if and only if the intuitionistic neutrosophic subset $\flat = [(\tilde{\Re}^{-}_{\daleth}, \tilde{\Re}^{+}_{\daleth}), (\tilde{\Im}^{-}_{\daleth}, \tilde{\Im}^{+}_{\daleth}), (\tilde{\mho}^{-}_{\daleth}, \tilde{\mho}^{+}_{\daleth})]$ of $\mathscr{O}$ is defined as*

$$\tilde{\Re}^{-}(\epsilon) = \begin{cases} \leq \upsilon^- \ for\,all\ \ \epsilon \in (\daleth] \\ \tau^- \ for\,all\ \epsilon \notin (\daleth] \end{cases} \quad \tilde{\Im}^{-}(\epsilon) = \begin{cases} \leq \upsilon^- \ for\,all\ \ \epsilon \in (\daleth] \\ \tau^- \ for\,all\ \epsilon \notin (\daleth] \end{cases} \quad \tilde{\mho}^{-}(\epsilon) = \begin{cases} \geq \upsilon^- \ for\,all\ \ \epsilon \in (\daleth] \\ \tau^- \ for\,all\ \epsilon \notin (\daleth] \end{cases}$$

$$\tilde{\Re}^{+}(\epsilon) = \begin{cases} \geq \upsilon^+ \ for\,all\ \ \epsilon \in (\daleth] \\ \tau^+ \ for\,all\ \epsilon \notin (\daleth] \end{cases} \quad \tilde{\Im}^{+}(\epsilon) = \begin{cases} \geq \upsilon^+ \ for\,all\ \ \epsilon \in (\daleth] \\ \tau^+ \ for\,all\ \epsilon \notin (\daleth] \end{cases} \quad \tilde{\mho}^{+}(\epsilon) = \begin{cases} \leq \upsilon^+ \ for\,all\ \ \epsilon \in (\daleth] \\ \tau^+ \ for\,all\ \epsilon \notin (\daleth] \end{cases}$$

is a (τ,υ) BIntNS[BIntNLI, BIntNRI, BIntNBI] of $\mathscr{O}$

Proof. Suppose that $\daleth$ is an BSS of $\mathscr{O}$. Let $\epsilon,\varsigma \in \mathscr{O}$ be such that $\epsilon,\varsigma \in (\daleth]$ then $\epsilon\varsigma \in (\daleth]$. Hence $\tilde{\Re}^{-}(\epsilon\varsigma) \leq \upsilon^-$, $\tilde{\Im}^{-}(\epsilon\varsigma) \leq \upsilon^-$ and $\tilde{\mho}^{-}(\epsilon\varsigma) \geq \upsilon^-$. Thus, $\min\{\tilde{\Re}^{-}(\epsilon\varsigma), \tau^-\} \leq \upsilon^- = \max\{\tilde{\Re}^{-}(\epsilon), \tilde{\Re}^{-}(\varsigma), \upsilon^-\}$, $\min\{\tilde{\Im}^{-}(\epsilon\varsigma), \tau^-\} \leq \upsilon^- = \max\{\tilde{\Im}^{-}(\epsilon), \tilde{\Im}^{-}(\varsigma), \upsilon^-\}$ and $\max\{\tilde{\mho}^{-}(\epsilon\varsigma), \tau^-\} \geq \upsilon^- = \min\{\tilde{\mho}^{-}(\epsilon), \tilde{\mho}^{-}(\varsigma), \upsilon^-\}$.
If $\epsilon \notin (\daleth]$ or $\varsigma \notin (\daleth]$, then $\max\{\tilde{\Re}^{-}(\epsilon), \tilde{\Re}^{-}(\varsigma), \upsilon^-\} = \tau^-$, $\max\{\tilde{\Im}^{-}(\epsilon), \tilde{\Im}^{-}(\varsigma), \upsilon^-\} = \tau^-$ and $\min\{\tilde{\mho}^{-}(\epsilon), \tilde{\mho}^{-}(\varsigma), \upsilon^-\} = \upsilon^-$. That is $\min\{\tilde{\Re}^{-}(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Re}^{-}(\epsilon), \tilde{\Re}^{-}(\varsigma), \upsilon^-\}$, $\min\{\tilde{\Im}^{-}(\epsilon\varsigma), \tau^-\} \leq \max\{\tilde{\Im}^{-}(\epsilon), \tilde{\Im}^{-}(\varsigma), \upsilon^-\}$ and $\max\{\tilde{\mho}^{-}(\epsilon\varsigma), \tau^-\} \geq \min\{\tilde{\mho}^{-}(\epsilon), \tilde{\mho}^{-}(\varsigma), \upsilon^-\}$.

Let $\epsilon,\varsigma \in \mathscr{O}$ be such that $\epsilon,\varsigma \in (\daleth]$ then $\epsilon\varsigma \in (\daleth]$. Hence $\tilde{\Re}^{+}(\epsilon\varsigma) \geq \upsilon^+$, $\tilde{\Im}^{+}(\epsilon\varsigma) \geq \upsilon^+$ and $\tilde{\mho}^{+}(\epsilon\varsigma) \leq \upsilon^+$. Thus, $\max\{\tilde{\Re}^{+}(\epsilon\varsigma), \tau^+\} \geq \upsilon^+ = \min\{\tilde{\Re}^{+}(\epsilon), \tilde{\Re}^{+}(\varsigma), \upsilon^+\}$, $\max\{\tilde{\Im}^{+}(\epsilon\varsigma), \tau^+\} \geq \upsilon^+ = \min\{\tilde{\Im}^{+}(\epsilon), \tilde{\Im}^{+}(\varsigma), \upsilon^+\}$ and $\min\{\tilde{\mho}^{+}(\epsilon\varsigma), \tau^+\} \leq \upsilon^+ = \max\{\tilde{\mho}^{+}(\epsilon), \tilde{\mho}^{+}(\varsigma), \upsilon^+\}$.
If $\epsilon \notin (\daleth]$ or $\varsigma \notin (\daleth]$, then $\min\{\tilde{\Re}^{+}(\epsilon), \tilde{\Re}^{+}(\varsigma), \upsilon^+\} = \tau^+$, $\min\{\tilde{\Im}^{+}(\epsilon), \tilde{\Im}^{+}(\varsigma), \upsilon^+\} = \tau^+$ and $\max\{\tilde{\mho}^{+}(\epsilon), \tilde{\mho}^{+}(\varsigma), \upsilon^+\} = \upsilon^+$.

That is $\max\{\tilde{\Re}^+(\epsilon\varsigma),\tau^+\} \geq \min\{\tilde{\Re}^+(\epsilon),\tilde{\Re}^+(\varsigma),\upsilon^+\}$, $\max\{\tilde{\Im}^+(\epsilon\varsigma),\tau^+\} \geq \min\{\tilde{\Im}^+(\epsilon),\tilde{\Im}^+(\varsigma),\upsilon^+\}$ and $\min\{\tilde{\mho}^+(\epsilon\varsigma),\tau^+\} \leq \max\{\tilde{\mho}^+(\epsilon),\tilde{\mho}^+(\varsigma),\upsilon^+\}$. Therefore $\flat$ is a (τ,υ) BIntNS of $\mathscr{O}$.

Conversely, assume that $\flat = [\tilde{\Re}^-,\tilde{\Im}^-,\tilde{\mho}^-]$ is a (τ,υ) BIntNS of $\mathscr{O}$. Let $\epsilon,\varsigma \in (\gimel]$. Then $\tilde{\Re}^-(\epsilon) \leq \upsilon^-, \tilde{\Re}^-(\varsigma) \leq \upsilon^-$, $\tilde{\Im}^-(\epsilon) \leq \upsilon^-, \tilde{\Im}^-(\varsigma) \leq \upsilon^-$ and $\tilde{\mho}^-(\epsilon) \geq \upsilon^-, \tilde{\mho}^-(\varsigma) \geq \upsilon^-$. Now $\flat = [\tilde{\Re}^-,\tilde{\Im}^-,\tilde{\mho}^-]$ is a (τ,υ) BIntNS of $\mathscr{O}$. Therefore $\min\{\tilde{\Re}^-(\epsilon\varsigma),\tau^-\} \leq \max\{\tilde{\Re}^-(\epsilon),\tilde{\Re}^-(\varsigma),\upsilon^-\} \leq \max\{\upsilon^-,\upsilon^-,\upsilon^-\} = \upsilon^-$, $\min\{\tilde{\Im}^-(\epsilon\varsigma),\tau^-\} \leq \max\{\tilde{\Im}^-(\epsilon),\tilde{\Im}^-(\varsigma),\upsilon^-\} \leq \max\{\upsilon^-,\upsilon^-,\upsilon^-\} = \upsilon^-$ and $\max\{\tilde{\mho}^-(\epsilon\varsigma),\tau^-\} \geq \min\{\tilde{\mho}^-(\epsilon),\tilde{\mho}^-(\varsigma),\upsilon^-\} \geq \min\{\upsilon^-,\upsilon^-,\upsilon^-\} = \upsilon^-$. It follows that $\epsilon\varsigma \in (\gimel]$. Let $\epsilon,\varsigma \in (\gimel]$. Then $\tilde{\Re}^+(\epsilon) \geq \upsilon^+, \tilde{\Re}^+(\varsigma) \geq \upsilon^+$, $\tilde{\Im}^+(\epsilon) \geq \upsilon^+, \tilde{\Im}^+(\varsigma) \geq \upsilon^+$ and $\tilde{\mho}^+(\epsilon) \leq \upsilon^+, \tilde{\mho}^+(\varsigma) \leq \upsilon^+$. Now $\flat = [\tilde{\Re}^+,\tilde{\Im}^+,\tilde{\mho}^+]$ is a (τ,υ) BIntNS of $\mathscr{O}$.
Therefore, $\max\{\tilde{\Re}^+(\epsilon\varsigma),\tau^+\} \geq \min\{\tilde{\Re}^+(\epsilon),\tilde{\Re}^+(\varsigma),\upsilon^+\} \geq \min\{\upsilon^+,\upsilon^+,\upsilon^+\} = \upsilon^+$, $\max\{\tilde{\Im}^+(\epsilon\varsigma),\tau^+\} \geq \min\{\tilde{\Im}^+(\epsilon),\tilde{\Im}^+(\varsigma),\upsilon^+\} \geq \min\{\upsilon^+,\upsilon^+,\upsilon^+\} = \upsilon^+$ and $\min\{\tilde{\mho}^+(\epsilon\varsigma),\tau^+\} \leq \max\{\tilde{\mho}^+(\epsilon),\tilde{\mho}^+(\varsigma),\upsilon^+\} \leq \max\{\upsilon^+,\upsilon^+,\upsilon^+\} = \upsilon^+$. It follows that $\epsilon\varsigma \in (\gimel]$. Therefore $\gimel$ is a BSS of $\mathscr{O}$.

Theorem 3.6. *A subset* $\flat = [(\tilde{\Re}^-_{\mathsf{T}},\tilde{\Re}^+_{\mathsf{T}}),(\tilde{\Im}^-_{\mathsf{T}},\tilde{\Im}^+_{\mathsf{T}}),(\tilde{\mho}^-_{\mathsf{T}},\tilde{\mho}^+_{\mathsf{T}})]$
is a (τ,υ) *BIntNS[BIntNLI,BIntNRI,BIntNBI] of* $\mathscr{O}$ *if and only if each non-empty level subset* $\flat^{(t_1,t_2)}$ *is a BSS* $[BVLI, BVRI, BVBI]$ *of* $\mathscr{O}$ *for all* $t_1 \in (\tau^-,\upsilon^-]$ *and* $t_2 \in (\tau^+,\upsilon^+]$.

Proof. Assume that $\flat^{(t_1,t_2)}$ is a BSS of $\mathscr{O}$ for each $t_1 \in [-1,0]$ and $t_2 \in [0,1]$.
Let $t_1 = \max\{\tilde{\Re}^-(\epsilon_1),\tilde{\Re}^-(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\Re}^-_t$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\min\{\tilde{\Re}^-(\epsilon\varsigma),\tau^-\} \leq t_1 = \max\{\tilde{\Re}^-(\epsilon_1),\tilde{\Re}^-(\epsilon_2),\upsilon^-\}$. Let $t_1 = \max\{\tilde{\Im}^-(\epsilon_1),\tilde{\Im}^-(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\Im}^-_t$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\min\{\tilde{\Im}^-(\epsilon\varsigma),\tau^-\} \leq t_1 = \max\{\tilde{\Im}^-(\epsilon_1),\tilde{\Im}^-(\epsilon_2),\upsilon^-\}$. Let $t_1 = \min\{\tilde{\mho}^-(\epsilon_1),\tilde{\mho}^-(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\mho}^-_t$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\max\{\tilde{\mho}^-(\epsilon\varsigma),\tau^-\} \geq t_1 = \min\{\tilde{\mho}^-(\epsilon_1),\tilde{\mho}^-(\epsilon_2),\upsilon^-\}$. Let $t_2 = \min\{\tilde{\Re}^+(\epsilon_1),\tilde{\Re}^+(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\Re}^+_s$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\max\{\tilde{\Re}^+(\epsilon\varsigma),\tau^+\} \geq t_2 = \min\{\tilde{\Re}^+(\epsilon_1),\tilde{\Re}^+(\epsilon_2),\upsilon^+\}$. Let $t_2 = \min\{\tilde{\Im}^+(\epsilon_1),\tilde{\Im}^+(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\Im}^+_t$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\max\{\tilde{\Im}^+(\epsilon\varsigma),\tau^+\} \geq t_2 = \min\{\tilde{\Im}^+(\epsilon_1),\tilde{\Im}^+(\epsilon_2),\upsilon^+\}$. Let $t_2 = \max\{\tilde{\mho}^+(\epsilon_1),\tilde{\mho}^+(\epsilon_2)\}$. Then $\epsilon_1,\epsilon_2 \in \tilde{\mho}^+_{t_2}$ for each $\epsilon_1,\epsilon_2 \in \mathscr{O}$. Thus $\min\{\tilde{\mho}^+(\epsilon\varsigma),\tau^+\} \leq t_2 = \max\{\tilde{\mho}^+(\epsilon_1),\tilde{\mho}^+(\epsilon_2),\upsilon^+\}$. This shows that $\flat^{(t_1,t_2)}$ is NS of $\mathscr{O}$.

Conversely, assume that $\flat^{(t_1,t_2)}$ is a BIntNS of $\mathscr{O}$. For each $t_1 \in [-1,0]$ and $t_2 \in [0,1]$ and $\epsilon_1,\epsilon_2 \in \tilde{\Re}^-_t$. We have $\tilde{\Re}^-(\epsilon_1) \leq t_1, \tilde{\Re}^-(\epsilon_2) \leq t_1$.
Since $\tilde{\Re}^-$ is a BSS of $\mathscr{O}$, $\min\{\tilde{\Re}^-(\epsilon_1\epsilon_2),\tau^-\} \leq \max\{\tilde{\Re}^-(\epsilon_1),\tilde{\Re}^-(\epsilon_2),\upsilon^-\} \leq t_1$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\Re}^-_t$. We have $\tilde{\Im}^-(\epsilon_1) \leq t_1, \tilde{\Im}^-(\epsilon_2) \leq t_1$.
Since $\tilde{\Im}^-$ is a BSS of $\mathscr{O}$, $\min\{\tilde{\Im}^-(\epsilon_1\epsilon_2),\tau^-\} \leq \max\{\tilde{\Im}^-(\epsilon_1),\tilde{\Im}^-(\epsilon_2),\upsilon^-\} \leq t_1$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\Im}^-_t$. We have $\tilde{\mho}^-(\epsilon_1) \geq t_1, \tilde{\mho}^-(\epsilon_2) \geq t_1$.
Since $\tilde{\mho}^-$ is a BSS of $\mathscr{O}$, $\max\{\tilde{\mho}^-(\epsilon_1\epsilon_2),\tau^-\} \geq \min\{\tilde{\mho}^-(\epsilon_1),\tilde{\mho}^-(\epsilon_2),\upsilon^-\} \geq t_1$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\mho}^-_t$. We have $\tilde{\Re}^+(\epsilon_1) \geq t_1, \tilde{\Re}^+(\epsilon_2) \geq t_1$.
Since $\tilde{\Re}^+$ is a BSS of $\mathscr{O}$, $\max\{\tilde{\Re}^+(\epsilon_1\epsilon_2),\tau^+\} \geq \min\{\tilde{\Re}^+(\epsilon_1),\tilde{\Re}^+(\epsilon_2),\upsilon^+\} \geq t_2$. This implies

that $\epsilon_1\epsilon_2 \in \tilde{\Re}^+_t$. We have $\tilde{\Im}^+(\epsilon_1) \geq t_2, \tilde{\Im}^+(\epsilon_2) \geq t_2$.
Since $\tilde{\Im}^+$ is a BSS of $\mathscr{O}$, $\max\{\tilde{\Im}^+(\epsilon_1\epsilon_2), \tau^+\} \geq \min\{\tilde{\Im}^+(\epsilon_1), \tilde{\Im}^+(\epsilon_2), \upsilon^+\} \geq t_2$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\Im}^+_s$. We have $\tilde{\mho}^+(\epsilon_1) \leq t_2, \tilde{\mho}^+(\epsilon_2) \leq t_2$.
Since $\tilde{\mho}^+$ is a BSS of $\mathscr{O}$, $\min\{\tilde{\mho}^+(\epsilon_1\epsilon_2), \tau^+\} \leq \max\{\tilde{\mho}^+(\epsilon_1), \tilde{\mho}^+(\epsilon_2), \upsilon^+\} \leq t_2$. This implies that $\epsilon_1\epsilon_2 \in \tilde{\mho}^+_s$. Therefore $\flat_{(t_1,t_2)}$ is a BSS of $\mathscr{O}$. Similar proofs holds.

Example 3.7. Every BIntNS $\flat$ of $\mathscr{O}$ is a (τ, υ) BIntNS of $\mathscr{O}$ and reverse implication may not be true.

For the Example 3.2, we define subset $\flat$ as

$$(\tilde{\Re}^-, \tilde{\Re}^+)(\hbar) = \begin{cases} (-0.41, 44) \;\; if \;\; \hbar = \hbar_1 \\ (-0.34, 37) \;\; if \;\; \hbar = \hbar_2 \\ (-0.24, 27) \;\; if \;\; \hbar = \hbar_3 \\ (-0.29, 32) \;\; if \;\; \hbar = \hbar_4 \end{cases} \quad (\tilde{\Im}^-, \tilde{\Im}^+)(\hbar) = \begin{cases} (-0.21, 24) \;\; if \;\; \hbar = \hbar_1 \\ (-0.14, 17) \;\; if \;\; \hbar = \hbar_2 \\ (-0.04, 07) \;\; if \;\; \hbar = \hbar_3 \\ (-0.09, 12) \;\; if \;\; \hbar = \hbar_4 \end{cases}$$

$$(\tilde{\mho}^-, \tilde{\mho}^+)(\hbar) = \begin{cases} (-0.26, 29) \;\; if \;\; \hbar = \hbar_1 \\ (-0.31, 34) \;\; if \;\; \hbar = \hbar_2 \\ (-0.41, 44) \;\; if \;\; \hbar = \hbar_3 \\ (-0.36, 39) \;\; if \;\; \hbar = \hbar_4 \end{cases}$$

Then $\flat$ is a $(0.35, 0.49)$ BIntNS of $\mathscr{O}$, but not a BIntNS.

Definition 3.8. If $\Lambda_{\daleth}$ is the characteristic function of $\daleth$, then $(\Lambda_{\daleth})^{\upsilon}_{\tau}$ is defined as

$$(\Lambda^{t-}_{\daleth})^{\upsilon^-}_{\tau^-}(\epsilon) = \begin{cases} \upsilon^- \; if \, \epsilon \in (\daleth] \\ \tau^- \; if \, \epsilon \notin (\daleth] \end{cases} \quad (\Lambda^{i-}_{\daleth})^{\upsilon^-}_{\tau^-}(\epsilon) = \begin{cases} \upsilon^- \; if \, \epsilon \in (\daleth] \\ \tau^- \; if \, \epsilon \notin (\daleth] \end{cases} \quad (\Lambda^{f-}_{\daleth})^{\upsilon^-}_{\tau^-}(\epsilon) = \begin{cases} \tau^- \; if \, \epsilon \in (\daleth] \\ \upsilon^- \; if \, \epsilon \notin (\daleth] \end{cases}$$

$$(\Lambda^{t+}_{\daleth})^{\upsilon^+}_{\tau^+}(\epsilon) = \begin{cases} \upsilon^+ \; if \, \epsilon \in (\daleth] \\ \tau^+ \; if \, \epsilon \notin (\daleth] \end{cases} \quad (\Lambda^{i+}_{\daleth})^{\upsilon^+}_{\tau^+}(\epsilon) = \begin{cases} \upsilon^+ \; if \, \epsilon \in (\daleth] \\ \tau^+ \; if \, \epsilon \notin (\daleth] \end{cases} \quad (\Lambda^{f+}_{\daleth})^{\upsilon^+}_{\tau^+}(\epsilon) = \begin{cases} \tau^+ \; if \, \epsilon \in (\daleth] \\ \upsilon^+ \; if \, \epsilon \notin (\daleth] \end{cases}$$

Theorem 3.9. *A non empty subset $\daleth$ of $\mathscr{O}$ is a BSS $[BVLI, BVRI, BVBI]$ of $\mathscr{O}$ if and only if subset $\Lambda_{(\daleth]}$ is a (τ, υ) BIntNS[BIntNLI, BIntNRI, BIntNBI] of $\mathscr{O}$.*

Proof. Assume that $\daleth$ is a BSS of $\mathscr{O}$. Then $\Lambda_{(\daleth]}$ is a BIntNS of $\mathscr{O}$ and hence $\Lambda_{(\daleth]}$ is an (τ, υ) BIntNS of $\mathscr{O}$.
Conversely, Let $\Lambda_{(\daleth]}$ is an (τ, υ) BIntNS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\daleth]$. Then $\Lambda^{t-}_{(\daleth]}(\epsilon) = \upsilon^- = \Lambda^{t-}_{(\daleth]}(\varsigma) = \upsilon^-$ Since $\Lambda^{t-}_{(\daleth]}$ is a (τ, υ) BIntNS. Consider

$$\begin{aligned} \min\{\Lambda^{t-}_{(\daleth]}(\epsilon\varsigma), \tau^-\} &\leq \max\{\Lambda^{t-}_{(\daleth]}(\epsilon), \Lambda^{t-}_{(\daleth]}(\varsigma), \upsilon^-\} \\ &= \max\{\upsilon^-, \upsilon^-, \upsilon^-\} \\ &= \upsilon^- \end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{t-}_{(\gimel]}(\epsilon\varsigma) \leq \upsilon^-$. Thus $\epsilon\varsigma \in (\gimel]$. Thus $\epsilon\varsigma \in (\gimel]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\gimel]$. Then $\Lambda^{i-}_{(\gimel]}(\epsilon) = \upsilon^- = \Lambda^{i-}_{(\gimel]}(\varsigma) = \upsilon^-$. Since $\Lambda^{i-}_{(\gimel]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned} \min\{\Lambda^{i-}_{(\gimel]}(\epsilon\varsigma), \tau^-\} &\leq \max\{\Lambda^{i-}_{(\gimel]}(\epsilon), \Lambda^{i-}_{(\gimel]}(\varsigma), \upsilon^-\} \\ &= \max\{\upsilon^-, \upsilon^-, \upsilon^-\} \\ &= \upsilon^- \end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{i-}_{(\gimel]}(\epsilon\varsigma) \leq \upsilon^-$. Thus $\epsilon\varsigma \in (\gimel]$. Thus $\epsilon\varsigma \in (\gimel]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\gimel]$. Then $\Lambda^{f-}_{(\gimel]}(\epsilon) = \tau^- = \Lambda^{f-}_{(\gimel]}(\varsigma) = \tau^-$. Since $\Lambda^{f-}_{(\gimel]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned} \max\{\Lambda^{f-}_{(\gimel]}(\epsilon\varsigma), \tau^-\} &\geq \min\{\Lambda^{f-}_{(\gimel]}(\epsilon), \Lambda^{f-}_{(\gimel]}(\varsigma), \upsilon^-\} \\ &= \min\{\tau^-, \tau^-, \upsilon^-\} \\ &= \upsilon^- \end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{f-}_{(\gimel]}(\epsilon\varsigma) \geq \tau^-$. Thus $\epsilon\varsigma \in (\gimel]$. Thus $\epsilon\varsigma \in (\gimel]$.

Therefore $\gimel$ is a BSS of $\mathscr{O}$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\gimel]$. Then $\Lambda^{t-}_{(\gimel]}(\epsilon) = \tau^- = \Lambda^{t-}_{(\gimel]}(\varsigma) = \tau^-$. Since $\Lambda^{t-}_{(\gimel]}$ is a (τ, υ)BIntNS.

$$\begin{aligned} \min\{\Lambda^{t-}_{(\gimel]}(\epsilon\varsigma), \tau^-\} &\leq \max\{\Lambda^{t-}_{(\gimel]}(\epsilon), \Lambda^{t-}_{(\gimel]}(\varsigma), \upsilon^-\} \\ &= \max\{\tau^-, \tau^-, \upsilon^-\} \\ &= \tau^- \end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{t-}_{(\gimel]}(\epsilon\varsigma) \leq \tau^-$. Thus $\epsilon\varsigma \notin (\gimel]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\gimel]$. Then $\Lambda^{i-}_{(\gimel]}(\epsilon) = \tau^- = \Lambda^{i-}_{(\gimel]}(\varsigma) = \tau^-$. Since $\Lambda^{i-}{}_{(\gimel]}$ is a (τ, υ)BIntNS.

$$\begin{aligned} \min\{\Lambda^{i-}_{(\gimel]}(\epsilon\varsigma), \tau^-\} &\leq \max\{\Lambda^{i-}_{(\gimel]}(\epsilon), \Lambda^{i-}_{(\gimel]}(\varsigma), \upsilon^-\} \\ &= \max\{\tau^-, \tau^-, \upsilon^-\} \\ &= \tau^- \end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{i-}_{(\gimel]}(\epsilon\varsigma) \leq \tau^-$. Thus $\epsilon\varsigma \notin (\gimel]$.

Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\gimel]$. Then $\Lambda^{f-}_{(\gimel]}(\epsilon) = \upsilon^- = \Lambda^{f-}_{(\gimel]}(\varsigma) = \upsilon^-$. Since $\Lambda^{f-}{}_{(\gimel]}$ is a (τ, υ)BIntNS.

$$\begin{aligned} \max\{\Lambda^{f-}_{(\gimel]}(\epsilon\varsigma), \tau^-\} &\geq \min\{\Lambda^{f-}_{(\gimel]}(\epsilon), \Lambda^{f-}_{(\gimel]}(\varsigma), \upsilon^-\} \\ &= \min\{\upsilon^-, \upsilon^-, \upsilon^-\} \\ &= \upsilon^- \end{aligned}$$

as $\tau^- > \upsilon^-$, this implies that $\Lambda^{f-}_{(\gimel]}(\epsilon\varsigma) \geq \upsilon^-$. Thus $\epsilon\varsigma \notin (\gimel]$.

Let $\Lambda_{(\daleth]}$ is an (τ, υ) BIntNS of $\mathscr{O}$. Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\daleth]$. Then $\Lambda^{t+}_{(\daleth]}(\epsilon) = \upsilon^+ = \Lambda^{t+}_{(\daleth]}(\varsigma) = \upsilon^+$. Since $\Lambda^{t+}_{(\daleth]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned}\max\{\Lambda^{t+}_{(\daleth]}(\epsilon\varsigma), \tau^+\} &\geq \min\{\Lambda^{t+}_{(\daleth]}(\epsilon), \Lambda^{t+}_{(\daleth]}(\varsigma), \upsilon^+\}\\ &= \min\{\upsilon^+, \upsilon^+, \upsilon^+\}\\ &= \upsilon^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{t+}_{(\daleth]}(\epsilon\varsigma) \geq \upsilon^+$. Thus $\epsilon\varsigma \in (\daleth]$. Thus $\epsilon\varsigma \in (\daleth]$.
Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\daleth]$. Then $\Lambda^{i+}_{(\daleth]}(\epsilon) = \upsilon^+ = \Lambda^{i+}_{(\daleth]}(\varsigma) = \upsilon^+$. Since $\Lambda^{i+}_{(\daleth]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned}\max\{\Lambda^{i+}_{(\daleth]}(\epsilon\varsigma), \tau^+\} &\geq \min\{\Lambda^{i+}_{(\daleth]}(\epsilon), \Lambda^{i+}_{(\daleth]}(\varsigma), \upsilon^+\}\\ &= \min\{\upsilon^+, \upsilon^+, \upsilon^+\}\\ &= \upsilon^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{i+}_{(\daleth]}(\epsilon\varsigma) \geq \upsilon^+$. Thus $\epsilon\varsigma \in (\daleth]$. Thus $\epsilon\varsigma \in (\daleth]$.
Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \in (\daleth]$. Then $\Lambda^{f+}_{(\daleth]}(\epsilon) = \tau^+ = \Lambda^{f+}_{(\daleth]}(\varsigma) = \tau^+$. Since $\Lambda^{f+}_{(\daleth]}$ is a (τ, υ)BIntNS. Consider

$$\begin{aligned}\min\{\Lambda^{f+}_{(\daleth]}(\epsilon\varsigma), \tau^+\} &\leq \max\{\Lambda^{f+}_{(\daleth]}(\epsilon), \Lambda^{f+}_{(\daleth]}(\varsigma), \upsilon^+\}\\ &= \max\{\tau^+, \tau^+, \upsilon^+\}\\ &= \upsilon^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{f+}_{(\daleth]}(\epsilon\varsigma) \leq \tau^+$. Thus $\epsilon\varsigma \in (\daleth]$. Thus $\epsilon\varsigma \in (\daleth]$.
Therefore $\daleth$ is a BSS of $\mathscr{O}$.
Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\daleth]$. Then $\Lambda^{t+}_{(\daleth]}(\epsilon) = \tau^+ = \Lambda^{t+}_{(\daleth]}(\varsigma) = \tau^+$. Since $\Lambda^{t+}_{(\daleth]}$ is a (τ, υ)BIntNS.

$$\begin{aligned}\max\{\Lambda^{t+}_{(\daleth]}(\epsilon\varsigma), \tau^+\} &\geq \min\{\Lambda^{t+}_{(\daleth]}(\epsilon), \Lambda^{t+}_{(\daleth]}(\varsigma), \upsilon^+\}\\ &= \min\{\tau^+, \tau^+, \upsilon^+\}\\ &= \tau^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{t+}_{(\daleth]}(\epsilon\varsigma) \geq \tau^+$. Thus $\epsilon\varsigma \notin (\daleth]$.
Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\daleth]$. Then $\Lambda^{i+}_{(\daleth]}(\epsilon) = \tau^+ = \Lambda^{i+}_{(\daleth]}(\varsigma) = \tau^+$. Since $\Lambda^{i+}_{(\daleth]}$ is a (τ, υ)BIntNS.

$$\begin{aligned}\max\{\Lambda^{i+}_{(\daleth]}(\epsilon\varsigma), \tau^+\} &\geq \min\{\Lambda^{i+}_{(\daleth]}(\epsilon), \Lambda^{i+}_{(\daleth]}(\varsigma), \upsilon^+\}\\ &= \min\{\tau^+, \tau^+, \upsilon^+\}\\ &= \tau^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{i+}_{(\daleth]}(\epsilon\varsigma) \geq \tau^+$. Thus $\epsilon\varsigma \notin (\daleth]$.
Let $\epsilon, \varsigma \in \mathscr{O}$ be such that $\epsilon, \varsigma \notin (\daleth]$. Then $\Lambda^{f+}_{(\daleth]}(\epsilon) = \upsilon^+ = \Lambda^{f+}_{(\daleth]}(\varsigma) = \upsilon^+$. Since $\Lambda^{f+}_{(\daleth]}$ is a

(τ, υ)BIntNS.

$$\begin{aligned}\min\{\Lambda^{f+}_{(\daleth]}(\epsilon\varsigma), \tau^+\} &\leq \max\{\Lambda^{f+}_{(\daleth]}(\epsilon), \Lambda^{f+}_{(\daleth]}(\varsigma), \upsilon^+\} \\ &= \max\{\upsilon^+, \upsilon^+, \upsilon^+\} \\ &= \upsilon^+\end{aligned}$$

as $\tau^+ < \upsilon^+$, this implies that $\Lambda^{f+}_{(\daleth]}(\epsilon\varsigma) \leq \upsilon^+$. Thus $\epsilon\varsigma \notin (\daleth]$. Therefore $\daleth$ is a BSS of $\mathscr{O}$. Similar to proof holds.

Definition 3.10. For two intuitionistic neutrosophic subsets χ and ℓ of $\mathscr{O}$, their product $\chi \cdot \ell$ is defined as

$$(\chi^{t-} \cdot \ell^{t-})(\epsilon) = \begin{cases} \inf\limits_{(s,t)\in\daleth_\epsilon} \{\chi^{t-}(s) \veebar \ell^{t-}(t)\} \; if \; \daleth_\epsilon \neq 0 \\ 0 \; otherwise \end{cases}$$

$$(\chi^{i-} \cdot \ell^{i-})(\epsilon) = \begin{cases} \inf\limits_{(s,t)\in\daleth_\epsilon} \{\chi^{i-}(s) \veebar \ell^{i-}(t)\} \; if \; \daleth_\epsilon \neq 0 \\ 0 \; otherwise \end{cases}$$

$$(\chi^{f-} \cdot \ell^{f-})(\epsilon) = \begin{cases} \sup\limits_{(s,t)\in\daleth_\epsilon} \{\chi^{f-}(s) \barwedge \ell^{f-}(t)\} \; if \; \daleth_\epsilon \neq 0 \\ 1 \; otherwise \end{cases}$$

$$(\chi^{t+} \cdot \ell^{t+})(\epsilon) = \begin{cases} \sup\limits_{(s,t)\in\daleth_\epsilon} \{\chi^{t+}(s) \barwedge \ell^{t+}(t)\} \; if \; \daleth_\epsilon \neq 0 \\ 0 \; otherwise \end{cases}$$

$$(\chi^{i+} \cdot \ell^{i+})(\epsilon) = \begin{cases} \sup\limits_{(s,t)\in\daleth_\epsilon} \{\chi^{i+}(s) \barwedge \ell^{i+}(t)\} \; if \; \daleth_\epsilon \neq 0 \\ 0 \; otherwise \end{cases}$$

$$(\chi^{f+} \cdot \ell^{f+})(\epsilon) = \begin{cases} \inf\limits_{(s,t)\in\daleth_\epsilon} \{\chi^{f+}(s) \veebar \ell^{f+}(t)\} \; if \; \daleth_\epsilon \neq 0 \\ -1 \; otherwise \end{cases}$$

Definition 3.11. Let $\flat$ be subset of $\mathscr{O}$, we define the subset $(\tilde{\Re}^-)^{\upsilon^-}_{\tau^-}(\epsilon) = \{\tilde{\Re}^-(\epsilon) \veebar \upsilon^-\} \barwedge \tau^-$, $(\tilde{\Im}^-)^{\upsilon^-}_{\tau^-}(\epsilon) = \{\tilde{\Im}^-(\epsilon) \veebar \upsilon^-\} \barwedge \tau^-$ and $(\tilde{\mho}^-)^{\upsilon^-}_{\tau^-}(\epsilon) = \{\tilde{\mho}^-(\epsilon) \barwedge \upsilon^-\} \veebar \tau^-$, $(\tilde{\Re}^+)^{\upsilon^+}_{\tau^+}(\epsilon) = \{\tilde{\Re}^+(\epsilon) \barwedge \upsilon^+\} \veebar \tau^+$, $(\tilde{\Im}^+)^{\upsilon^+}_{\tau^+}(\epsilon) = \{\tilde{\Im}^+(\epsilon) \barwedge \upsilon^+\} \veebar \tau^+$ and $(\tilde{\mho}^+)^{\upsilon^+}_{\tau^+}(\epsilon) = \{\tilde{\mho}^+(\epsilon) \veebar \upsilon^+\} \barwedge \tau^+$, for all $\epsilon \in \mathscr{O}$.

Lemma 3.12. *Let $\daleth$ and $\top$ be non-empty subsets of $\mathscr{O}$. Then the following hold:*

(1) $(\Lambda_{(\daleth]} \veebar^{\upsilon}_{\tau} \Lambda_{(\top]}) = (\Lambda_{(\daleth \barwedge \top]})^{\upsilon}_{\tau}$,
(2) $(\Lambda_{(\daleth]} \barwedge^{\upsilon}_{\tau} \Lambda_{(\top]}) = (\Lambda_{(\daleth \veebar \top]})^{\upsilon}_{\tau}$,
(3) $(\Lambda_{(\daleth]} \cdot^{\upsilon}_{\tau} \Lambda_{(\top]}) = (\Lambda_{(\daleth\top]})^{\upsilon}_{\tau}$.

Proof. (3) Let $\epsilon \in \mathscr{O}$. If $\epsilon \in (\gimel\top]$, then $(\Lambda_{(\gimel\top]})(\epsilon) = \upsilon^-$.
Since $\epsilon \leq a_1a_2$ for some $a_1 \in (\gimel]$,$a_2 \in (\top]$,we have $(a_1, a_2) \in \gimel_\epsilon$ and $\gimel_\epsilon \neq 0$.

$$\begin{aligned}(\Lambda^{t-}_{(\gimel]} \cdot \Lambda^{t-}_{(\top]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{t-}_{(\gimel]}(y), \Lambda^{t-}_{(\top]}(z)\}\\ &\leq \max\{\Lambda^{t-}_{(\gimel]}(a_1), \Lambda^{t-}_{(\top]}(a_2)\}\\ &= \upsilon^-\end{aligned}$$

$$\begin{aligned}(\Lambda^{i-}_{(\gimel]} \cdot \Lambda^{i-}_{(\top]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{i-}_{(\gimel]}(y), \Lambda^{i-}_{(\top]}(z)\}\\ &\leq \max\{\Lambda^{i-}_{(\gimel]}(a_1), \Lambda^{i-}_{(\top]}(a_2)\}\\ &= \upsilon^-\end{aligned}$$

$$\begin{aligned}(\Lambda^{f-}_{(\gimel]} \cdot \Lambda^{f-}_{(\top]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{f-}_{(\gimel]}(y), \Lambda^{f-}_{(\top]}(z)\}\\ &\geq \min\{\Lambda^{f-}_{(\gimel]}(a_1), \Lambda^{f-}_{(\top]}(a_2)\}\\ &= \tau^-\end{aligned}$$

Therefore $(\Lambda_{(\gimel]} \cdot \Lambda_{(\top]})(\epsilon) = (\Lambda_{(\gimel\top]})(\epsilon)$.

If $\epsilon \in (\gimel\top]$, then $(\Lambda_{(\gimel\top]})(\epsilon) = \upsilon^+$.
Since $\epsilon \leq a_1a_2$ for some $a_1 \in (\gimel]$,$a_2 \in (\top]$,we have $(a_1, a_2) \in \gimel_\epsilon$ and $\gimel_\epsilon \neq 0$.

$$\begin{aligned}(\Lambda^{t+}_{(\gimel]} \cdot \Lambda^{t+}_{(\top]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{t+}_{(\gimel]}(y), \Lambda^{t+}_{(\top]}(z)\}\\ &\geq \min\{\Lambda^{t+}_{(\gimel]}(a_1), \Lambda^{t+}_{(\top]}(a_2)\}\\ &= \upsilon^+\end{aligned}$$

$$\begin{aligned}(\Lambda^{i+}_{(\gimel]} \cdot \Lambda^{i+}_{(\top]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{i+}_{(\gimel]}(y), \Lambda^{i+}_{(\top]}(z)\}\\ &\geq \min\{\Lambda^{i+}_{(\gimel]}(a_1), \Lambda^{i+}_{(\top]}(a_2)\}\\ &= \upsilon^+\end{aligned}$$

$$\begin{aligned}(\Lambda^{f+}_{(\gimel]} \cdot \Lambda^{f+}_{(\top]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{f+}_{(\gimel]}(y), \Lambda^{f+}_{(\top]}(z)\}\\ &\leq \max\{\Lambda^{f+}_{(\gimel]}(a_1), \Lambda^{f+}_{(\top]}(a_2)\}\\ &= \tau^+\end{aligned}$$

Therefore $(\Lambda_{(\gimel]} \cdot \Lambda_{(\top]})(\epsilon) = (\Lambda_{(\gimel\top]})(\epsilon)$.

If $\epsilon \notin (\gimel\top]$ then $(\Lambda^{t-}_{(\gimel\top]})(\epsilon) = \tau^-$, $(\Lambda^{i-}_{(\gimel\top]})(\epsilon) = \tau^-$ and $(\Lambda^{f-}_{(\gimel\top]})(\epsilon) = \upsilon^-$. Since $\epsilon \leq a_1a_2$ for some $a_1 \notin (\gimel]$,$a_2 \notin (\top]$. We have

$$\begin{aligned}(\Lambda^{t-}_{(\gimel]} \cdot \Lambda^{t-}_{(\top]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{t-}_{(\gimel]}(y), \Lambda^{t-}_{(\top]}(z)\}\\ &\leq \max\{\Lambda^{t-}_{(\gimel]}(a_1), \Lambda^{t-}_{(\top]}(a_2)\}\\ &= \tau^-\end{aligned}$$

$$
\begin{aligned}
(\Lambda^{i-}_{(\beth]} \cdot \Lambda^{i-}_{(\top]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{i-}_{(\beth]}(y), \Lambda^{i-}_{(\top]}(z)\} \\
&\leq \max\{\Lambda^{i-}_{(\beth]}(a_1), \Lambda^{i-}_{(\top]}(a_2)\} \\
&= \tau^-
\end{aligned}
$$

$$
\begin{aligned}
(\Lambda^{f-}_{(\beth]} \cdot \Lambda^{f-}_{(\top]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{f-}_{(\beth]}(y), \Lambda^{f-}_{(\top]}(z)\} \\
&\geq \min\{\Lambda^{f-}_{(\beth]}(a_1), \Lambda^{f-}_{(\top]}(a_2)\} \\
&= \upsilon^-
\end{aligned}
$$

If $\epsilon \notin (\beth\top]$ then $(\Lambda^{t+}_{(\beth\top]})(\epsilon) = \tau^+$, $(\Lambda^{i+}_{(\beth\top]})(\epsilon) = \tau^+$ and $(\Lambda^{f+}_{(\beth\top]})(\epsilon) = \upsilon^+$. Since $\epsilon \leq a_1a_2$ for some $a_1 \notin (\beth]$,$a_2 \notin (\top]$. We have

$$
\begin{aligned}
(\Lambda^{t+}_{(\beth]} \cdot \Lambda^{t+}_{(\top]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{t+}_{(\beth]}(y), \Lambda^{t+}_{(\top]}(z)\} \\
&\geq \min\{\Lambda^{t+}_{(\beth]}(a_1), \Lambda^{t+}_{(\top]}(a_2)\} \\
&= \tau^+
\end{aligned}
$$

$$
\begin{aligned}
(\Lambda^{i+}_{(\beth]} \cdot \Lambda^{i+}_{(\top]})(\epsilon) &= \sup_{\epsilon=yz} \min\{\Lambda^{i+}_{(\beth]}(y), \Lambda^{i+}_{(\top]}(z)\} \\
&\geq \min\{\Lambda^{i+}_{(\beth]}(a_1), \Lambda^{i+}_{(\top]}(a_2)\} \\
&= \tau^+
\end{aligned}
$$

$$
\begin{aligned}
(\Lambda^{f+}_{(\beth]} \cdot \Lambda^{f+}_{(\top]})(\epsilon) &= \inf_{\epsilon=yz} \max\{\Lambda^{f+}_{(\beth]}(y), \Lambda^{f+}_{(\top]}(z)\} \\
&\leq \max\{\Lambda^{f+}_{(\beth]}(a_1), \Lambda^{f+}_{(\top]}(a_2)\} \\
&= \upsilon^+
\end{aligned}
$$

Hence $(\Lambda_{(\beth]} \cdot \Lambda_{(\top]})(\epsilon) = (\Lambda_{(\beth\top]})(\epsilon)$.

Theorem 3.13. *For $\beth, \top \subseteq \mathscr{O}$ and $\{\beth_i | i \in I\}$ be a family of subsets of $\mathscr{O}$ then*
(i) $(\beth] \subseteq (\top]$ if and only if $(\Lambda_{(\beth]})^{\upsilon}_{\tau} \leq (\Lambda_{(\top]})^{\upsilon}_{\tau}$.
(ii) $(\overline{\wedge}_{i\in I}\Lambda_{(\beth_i]})^{\upsilon}_{\tau} = (\Lambda_{\overline{\wedge}_{i\in I}(\beth_i]})^{\upsilon}_{\tau}$.
(iii) $(\underline{\vee}_{i\in I}\Lambda_{(\beth_i]})^{\upsilon}_{\tau} = (\Lambda_{\underline{\vee}_{i\in I}(\beth_i]})^{\upsilon}_{\tau}$.

Theorem 3.14. *If $\beth$ is a (τ, υ)-BIntNLI[BIntNS, BIntNRI] of $\mathscr{O}$, then $(\beth)^{\upsilon}_{\tau}$ is a BIntNLI[BIntNS, BIntNRI] of $\mathscr{O}$.*

Proof. Assume that $\daleth$ is a (τ, υ)BIntNLI of $\mathscr{O}$. If there exist $\epsilon, \varsigma \in \mathscr{O}$. Now

$$\begin{aligned}
\min\{(\tilde{\Re}^-)^{\upsilon^-}_{\tau^-}(\epsilon\varsigma), \tau^-\} &= \min\{(\{\tilde{\Re}^-(\epsilon\varsigma) \veebar \upsilon^-\} \barwedge \tau^-), \tau^-\} \\
&= \{\tilde{\Re}^-(\epsilon\varsigma) \veebar \upsilon^-\} \barwedge \tau^- \\
&= \{\tilde{\Re}^-(\epsilon\varsigma) \barwedge \tau^-\} \veebar \{\upsilon^- \barwedge \tau^-\} \\
&= \{(\tilde{\Re}^-(\epsilon\varsigma) \barwedge \tau^-) \barwedge \tau^-\} \veebar \upsilon^- \\
&\leq \{(\tilde{\Re}^-(\varsigma) \veebar \upsilon^-) \barwedge \tau^-\} \veebar \upsilon^- \\
&= \{(\tilde{\Re}^-(\varsigma) \veebar \upsilon^-) \veebar \upsilon^-\} \barwedge (\tau^- \veebar \upsilon^-) \\
&= \{(\tilde{\Re}^-(\varsigma) \veebar \upsilon^-) \veebar \upsilon^-\} \barwedge \tau^- \\
&\leq (\tilde{\Re}^-)^{\upsilon^-}_{\tau^-}(\varsigma) \veebar \upsilon^-.
\end{aligned}$$

$$\begin{aligned}
\min\{(\tilde{\Im}^-)^{\upsilon^-}_{\tau^-}(\epsilon\varsigma), \tau^-\} &= \min\{(\{\tilde{\Im}^-(\epsilon\varsigma) \veebar \upsilon^-\} \barwedge \tau^-), \tau^-\} \\
&= \{\tilde{\Im}^-(\epsilon\varsigma) \veebar \upsilon^-\} \barwedge \tau^- \\
&= \{\tilde{\Im}^-(\epsilon\varsigma) \barwedge \tau^-\} \veebar \{\upsilon^- \barwedge \tau^-\} \\
&= \{(\tilde{\Im}^-(\epsilon\varsigma) \barwedge \tau^-) \barwedge \tau^-\} \veebar \upsilon^- \\
&\leq \{(\tilde{\Im}^-(\varsigma) \veebar \upsilon^-) \barwedge \tau^-\} \veebar \upsilon^- \\
&= \{(\tilde{\Im}^-(\varsigma) \veebar \upsilon^-) \veebar \upsilon^-\} \barwedge (\tau^- \veebar \upsilon^-) \\
&= \{(\tilde{\Im}^-(\varsigma) \veebar \upsilon^-) \veebar \upsilon^-\} \barwedge \tau^- \\
&\leq (\tilde{\Im}^-)^{\upsilon^-}_{\tau^-}(\varsigma) \veebar \upsilon^-.
\end{aligned}$$

$$\begin{aligned}
\max\{(\tilde{\mho}^-)^{\upsilon^-}_{\tau^-}(\epsilon\varsigma), \tau^-\} &= \max\{(\{\tilde{\mho}^-(\epsilon\varsigma) \barwedge \upsilon^-\} \veebar \tau^-), \tau^-\} \\
&= \{\tilde{\mho}^-(\epsilon\varsigma) \barwedge \upsilon^-\} \veebar \tau^- \\
&= \{\tilde{\mho}^-(\epsilon\varsigma) \veebar \tau^-\} \barwedge \{\upsilon^- \veebar \tau^-\} \\
&= \{(\tilde{\mho}^-(\epsilon\varsigma) \veebar \tau^-) \veebar \tau^-\} \barwedge \tau^- \\
&\geq \{(\tilde{\mho}^-(\varsigma) \barwedge \upsilon^-) \veebar \tau^-\} \barwedge \tau^- \\
&= \{(\tilde{\mho}^-(\varsigma) \barwedge \upsilon^-) \barwedge \upsilon^-\} \veebar \tau^- \\
&\geq (\tilde{\mho}^-)^{\upsilon^-}_{\tau^-}(\varsigma) \barwedge \upsilon^-
\end{aligned}$$

If there exist $\epsilon, \varsigma \in \mathscr{O}$. Now

$$\begin{aligned}
\max\{(\tilde{\Re}^+)^{\upsilon^+}_{\tau^+}(\epsilon\varsigma), \tau^+\} &= \max\{(\{\tilde{\Re}^+(\epsilon\varsigma) \barwedge \upsilon^+\} \veebar \tau^+), \tau^+\} \\
&= \{\tilde{\Re}^+(\epsilon\varsigma) \barwedge \upsilon^+\} \veebar \tau^+ \\
&= \{\tilde{\Re}^+(\epsilon\varsigma) \veebar \tau^+\} \barwedge \{\upsilon^+ \veebar \tau^+\} \\
&= \{(\tilde{\Re}^+(\epsilon\varsigma) \veebar \tau^+) \veebar \tau^+\} \barwedge \upsilon^+ \\
&\geq \{(\tilde{\Re}^+(\varsigma) \barwedge \upsilon^+) \veebar \tau^+\} \barwedge \upsilon^+ \\
&= \{(\tilde{\Re}^+(\varsigma) \barwedge \upsilon^+) \barwedge \upsilon^+\} \veebar (\tau^+ \barwedge \upsilon^+) \\
&= \{(\tilde{\Re}^+(\varsigma) \barwedge \upsilon^+) \barwedge \upsilon^+\} \veebar \tau^+ \\
&\geq (\tilde{\Re}^+)^{\upsilon^+}_{\tau^+}(\varsigma) \barwedge \upsilon^+.
\end{aligned}$$

$$\begin{aligned}
\max\{(\tilde{\Im}^+)^{\upsilon^+}_{\tau^+}(\epsilon\varsigma), \tau^+\} &= \max\{(\{\tilde{\Im}^+(\epsilon\varsigma) \barwedge \upsilon^+\} \veebar \tau^+), \tau^+\} \\
&= \{\tilde{\Im}^+(\epsilon\varsigma) \barwedge \upsilon^+\} \veebar \tau^+ \\
&= \{\tilde{\Im}^+(\epsilon\varsigma) \veebar \tau^+\} \barwedge \{\upsilon^+ \veebar \tau^+\} \\
&= \{(\tilde{\Im}^+(\epsilon\varsigma) \veebar \tau^+) \veebar \tau^+\} \barwedge \upsilon^+ \\
&\geq \{(\tilde{\Im}^+(\varsigma) \barwedge \upsilon^+) \veebar \tau^+\} \barwedge \upsilon^+ \\
&= \{(\tilde{\Im}^+(\varsigma) \barwedge \upsilon^+) \barwedge \upsilon^+\} \veebar (\tau^+ \barwedge \upsilon^+) \\
&= \{(\tilde{\Im}^+(\varsigma) \barwedge \upsilon^+) \barwedge \upsilon^+\} \veebar \tau^+ \\
&\geq (\tilde{\Im}^+)^{\upsilon^+}_{\tau^+}(\varsigma) \barwedge \upsilon^+.
\end{aligned}$$

$$\begin{aligned}
\min\{(\tilde{\mho}^+)^{\upsilon^+}_{\tau^+}(\epsilon\varsigma), \tau^+\} &= \min\{(\{\tilde{\mho}^+(\epsilon\varsigma) \veebar \upsilon^+\} \barwedge \tau^+), \tau^+\} \\
&= \{\tilde{\mho}^+(\epsilon\varsigma) \veebar \upsilon^+\} \barwedge \tau^+ \\
&= \{\tilde{\mho}^+(\epsilon\varsigma) \barwedge \tau^+\} \veebar \{\upsilon^+ \barwedge \tau^+\} \\
&= \{(\tilde{\mho}^+(\epsilon\varsigma) \barwedge \tau^+) \barwedge \tau^+\} \veebar \tau^+ \\
&\leq \{(\tilde{\mho}^+(\varsigma) \veebar \upsilon^+) \barwedge \tau^+\} \veebar \tau^+ \\
&= \{(\tilde{\mho}^+(\varsigma) \veebar \upsilon^+) \veebar \upsilon^+\} \barwedge \tau^+ \\
&\leq (\tilde{\mho}^+)^{\upsilon^+}_{\tau^+}(\varsigma) \veebar \upsilon^+
\end{aligned}$$

Hence $\daleth = [(\tilde{\Re})^{\upsilon}_{\tau}, (\tilde{\Im})^{\upsilon}_{\tau}, (\tilde{\mho})^{\upsilon}_{\tau}]$ is a BIntNLI of $\mathscr{O}$.

Theorem 3.15. *Let* $\daleth$ *be an* (τ, υ)*BIntNRI and* $\top$ *be an* (τ, υ)*BIntNLI of* $\mathscr{O}$ *then* $((\daleth \cdot \top]))^{\upsilon}_{\tau} \subseteq (\daleth \barwedge^{\upsilon}_{\tau} \top]$.

Proof. Let $\daleth = [(\tilde{\Re}^-_{\daleth}, \tilde{\Re}^+_{\daleth}), (\tilde{\Im}^-_{\daleth}, \tilde{\Im}^+_{\daleth}), (\tilde{\mho}^-_{\daleth}, \tilde{\mho}^+_{\daleth})]$ be an (τ, υ)BIntNRI and $\top = [(\tilde{\Re}^-_{\top}, \tilde{\Re}^+_{\top}), (\tilde{\Im}^-_{\top}, \tilde{\Im}^+_{\top}), (\tilde{\mho}^-_{\top}, \tilde{\mho}^+_{\top})]$ be an (τ, υ)BIntNLI of $\mathscr{O}$. Let $(\epsilon, \varsigma) \in I_\kappa$. If $I_\kappa \neq \emptyset$, then $\kappa \leq \epsilon\varsigma$. Thus $\tilde{\Re}^-_{\daleth}(\kappa) \leq \tilde{\Re}^-_{\daleth}(\epsilon\varsigma) \leq \tilde{\Re}^-_{\daleth}(\epsilon)$, $\tilde{\Im}^-_{\daleth}(\kappa) \leq \tilde{\Im}^-_{\daleth}(\epsilon\varsigma) \leq \tilde{\Im}^-_{\daleth}(\epsilon)$ and $\tilde{\mho}^-_{\daleth}(\kappa) \geq \tilde{\mho}^-_{\daleth}(\epsilon\varsigma) \geq$

$\tilde{\mho}^-_{\daleth}(\epsilon)$. Similarly $\tilde{\Re}^-_{\top}(\kappa) \leq \tilde{\Re}^-_{\top}(\epsilon\varsigma) \leq \tilde{\Re}^-_{\top}(\epsilon)$, $\tilde{\Im}^-_{\top}(\kappa) \leq \tilde{\Im}^-_{\top}(\epsilon\varsigma) \leq \tilde{\Im}^-_{\top}(\epsilon)$ and $\tilde{\mho}^-_{\top}(\kappa) \geq \tilde{\mho}^-_{\top}(\epsilon\varsigma) \geq \tilde{\mho}^-_{\top}(\epsilon)$. Let $(\epsilon, \varsigma) \in I_\kappa$. If $I_\kappa \neq \emptyset$, then $\kappa \leq \epsilon\varsigma$. Thus $\tilde{\Re}^+_{\daleth}(\kappa) \geq \tilde{\Re}^+_{\daleth}(\epsilon\varsigma) \geq \tilde{\Re}^+_{\daleth}(\epsilon)$, $\tilde{\Im}^+_{\daleth}(\kappa) \geq \tilde{\Im}^+_{\daleth}(\epsilon\varsigma) \geq \tilde{\Im}^+_{\daleth}(\epsilon)$ and $\tilde{\mho}^+_{\daleth}(\kappa) \leq \tilde{\mho}^+_{\daleth}(\epsilon\varsigma) \leq \tilde{\mho}^+_{\daleth}(\epsilon)$. Similarly $\tilde{\Re}^+_{\top}(\kappa) \geq \tilde{\Re}^+_{\top}(\epsilon\varsigma) \geq \tilde{\Re}^+_{\top}(\epsilon)$, $\tilde{\Im}^+_{\top}(\kappa) \geq \tilde{\Im}^+_{\top}(\epsilon\varsigma) \geq \tilde{\Im}^+_{\top}(\epsilon)$ and $\tilde{\mho}^+_{\top}(\kappa) \leq \tilde{\mho}^+_{\top}(\epsilon\varsigma) \leq \tilde{\mho}^+_{\top}(\epsilon)$. We have

$$
\begin{aligned}
(\tilde{\Re}^-_{(\daleth\cdot\top]})^{\upsilon^-}_{\tau^-}(\kappa) &= (\tilde{\Re}^-_{(\daleth\cdot\top]}(\kappa) \veebar \upsilon^-) \barwedge \tau^- \\
&= \Big[\big[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\Re}^-_{\daleth}(\epsilon) \veebar \tilde{\Re}^-_{\top}(\varsigma)\} \veebar \upsilon^-\big]\Big] \barwedge \tau^- \\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\Re}^-_{\daleth}(\epsilon) \veebar \tilde{\Re}^-_{\top}(\varsigma)\} \veebar \upsilon^- \veebar \upsilon^-\Big] \barwedge \tau^- \\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{(\tilde{\Re}^-_{\daleth}(\epsilon) \veebar \upsilon^-) \veebar (\tilde{\Re}^-_{\top}(\varsigma) \veebar \upsilon^-)\} \veebar \upsilon^-\Big] \barwedge \tau^- \\
&\geq (\{(\tilde{\Re}^-_{\daleth}(\kappa) \barwedge \tau^-) \veebar (\tilde{\Re}^-_{\top}(\kappa) \barwedge \tau^-)\} \veebar \upsilon^-) \barwedge \tau^- \\
&= \{((\tilde{\Re}^-_{\daleth}(\kappa) \veebar \tilde{\Re}^-_{\top}(\kappa)) \barwedge \tau^-) \veebar \upsilon^-\} \barwedge \tau^- \\
&= \{((\tilde{\Re}^-_{\daleth} \veebar \tilde{\Re}^-_{\top})(\kappa) \veebar \upsilon^-\} \barwedge \tau^- \\
&= (\tilde{\Re}^-_{\daleth \barwedge^{\upsilon^-}_{\tau^-} \top})(\kappa)
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\Im}^-_{(\daleth\cdot\top]})^{\upsilon^-}_{\tau^-}(\kappa) &= (\tilde{\Im}^-_{(\daleth\cdot\top]}(\kappa) \veebar \upsilon^-) \barwedge \tau^- \\
&= \Big[\big[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\Im}^-_{\daleth}(\epsilon) \veebar \tilde{\Im}^-_{\top}(\varsigma)\} \veebar \upsilon^-\big]\Big] \barwedge \tau^- \\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\Im}^-_{\daleth}(\epsilon) \veebar \tilde{\Im}^-_{\top}(\varsigma)\} \veebar \upsilon^- \veebar \upsilon^-\Big] \barwedge \tau^- \\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{(\tilde{\Im}^-_{\daleth}(\epsilon) \veebar \upsilon^-) \veebar (\tilde{\Im}^-_{\top}(\varsigma) \veebar \upsilon^-)\} \veebar \upsilon^-\Big] \barwedge \tau^- \\
&\geq (\{(\tilde{\Im}^-_{\daleth}(\kappa) \barwedge \tau^-) \veebar (\tilde{\Im}^-_{\top}(\kappa) \barwedge \tau^-)\} \veebar \upsilon^-) \barwedge \tau^- \\
&= \{((\tilde{\Im}^-_{\daleth}(\kappa) \veebar \tilde{\Im}^-_{\top}(\kappa)) \barwedge \tau^-) \veebar \upsilon^-\} \barwedge \tau^- \\
&= \{((\tilde{\Im}^-_{\daleth} \veebar \tilde{\Im}^-_{\top})(\kappa) \veebar \upsilon^-\} \barwedge \tau^- \\
&= (\tilde{\Im}^-_{\daleth \barwedge^{\upsilon^-}_{\tau^-} \top})(\kappa)
\end{aligned}
$$

$$\begin{aligned}
(\tilde{\mho}^{-}_{(\daleth\cdot\intercal]})^{\upsilon^{-}}_{\tau^{-}}(\kappa) &= (\tilde{\mho}^{-}_{(\daleth\cdot\intercal]}(\kappa)\barwedge\upsilon^{-})\veebar\tau^{-}\\
&= \Big[[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\mho}^{-}_{\daleth}(\epsilon)\barwedge\tilde{\mho}^{-}_{\intercal}(\varsigma)\}\barwedge\upsilon^{-}]\Big]\veebar\tau^{-}\\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\mho}^{-}_{\daleth}(\epsilon)\barwedge\tilde{\mho}^{-}_{\intercal}(\varsigma)\}\barwedge\upsilon^{-}\barwedge\upsilon^{-}\Big]\veebar\tau^{-}\\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{(\tilde{\mho}^{-}_{\daleth}(\epsilon)\barwedge\upsilon^{-})\barwedge(\tilde{\mho}^{-}_{\intercal}(\varsigma)\barwedge\upsilon^{-})\}\barwedge\upsilon^{-}\Big]\veebar\tau^{-}\\
&\le (\{(\tilde{\mho}^{-}_{\daleth}(\kappa)\veebar\tau^{-})\barwedge(\tilde{\mho}^{-}_{\intercal}(\kappa)\veebar\tau^{-})\}\barwedge\upsilon^{-})\veebar\tau^{-}\\
&= \{((\tilde{\mho}^{-}_{\daleth}(\kappa)\barwedge\tilde{\mho}^{-}_{\intercal}(\kappa))\veebar\tau^{-})\barwedge\upsilon^{-}\}\veebar\tau^{-}\\
&= \{((\tilde{\mho}^{-}_{\daleth}\barwedge\tilde{\mho}^{-}_{\intercal})(\kappa)\barwedge\upsilon^{-}\}\veebar\tau^{-}\\
&= (\tilde{\mho}^{-}_{\daleth\veebar^{\upsilon^{-}}_{\tau^{-}}\intercal})(\kappa)
\end{aligned}$$

$$\begin{aligned}
(\tilde{\Re}^{+}_{(\daleth\cdot\intercal]})^{\upsilon^{+}}_{\tau^{+}}(\kappa) &= (\tilde{\Re}^{+}_{(\daleth\cdot\intercal]}(\kappa)\barwedge\upsilon^{+})\veebar\tau^{+}\\
&= \Big[[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\Re}^{+}_{\daleth}(\epsilon)\barwedge\tilde{\Re}^{+}_{\intercal}(\varsigma)\}\barwedge\upsilon^{+}]\Big]\veebar\tau^{+}\\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\Re}^{+}_{\daleth}(\epsilon)\barwedge\tilde{\Re}^{+}_{\intercal}(\varsigma)\}\barwedge\upsilon^{+}\barwedge\upsilon^{+}\Big]\veebar\tau^{+}\\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{(\tilde{\Re}^{+}_{\daleth}(\epsilon)\barwedge\upsilon^{+})\barwedge(\tilde{\Re}^{+}_{\intercal}(\varsigma)\barwedge\upsilon^{+})\}\barwedge\upsilon^{+}\Big]\veebar\tau^{+}\\
&\le (\{(\tilde{\Re}^{+}_{\daleth}(\kappa)\veebar\tau^{+})\barwedge(\tilde{\Re}^{+}_{\intercal}(\kappa)\veebar\tau^{+})\}\barwedge\upsilon^{+})\veebar\tau^{+}\\
&= \{((\tilde{\Re}^{+}_{\daleth}(\kappa)\barwedge\tilde{\Re}^{+}_{\intercal}(\kappa))\veebar\tau^{+})\barwedge\upsilon^{+}\}\veebar\tau^{+}\\
&= \{((\tilde{\Re}^{+}_{\daleth}\barwedge\tilde{\Re}^{+}_{\intercal})(\kappa)\barwedge\upsilon^{+}\}\veebar\tau^{+}\\
&= (\tilde{\Re}^{+}_{\daleth\barwedge^{\upsilon^{+}}_{\tau^{+}}\intercal})(\kappa)
\end{aligned}$$

$$\begin{aligned}
(\tilde{\Im}^{+}_{(\daleth\cdot\intercal]})^{\upsilon^{+}}_{\tau^{+}}(\kappa) &= (\tilde{\Im}^{+}_{(\daleth\cdot\intercal]}(\kappa)\barwedge\upsilon^{+})\veebar\tau^{+}\\
&= \Big[[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\Im}^{+}_{\daleth}(\epsilon)\barwedge\tilde{\Im}^{+}_{\intercal}(\varsigma)\}\barwedge\upsilon^{+}]\Big]\veebar\tau^{+}\\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{\tilde{\Im}^{+}_{\daleth}(\epsilon)\barwedge\tilde{\Im}^{+}_{\intercal}(\varsigma)\}\barwedge\upsilon^{+}\barwedge\upsilon^{+}\Big]\veebar\tau^{+}\\
&= \Big[\sup_{\kappa\le\epsilon\varsigma}\{(\tilde{\Im}^{+}_{\daleth}(\epsilon)\barwedge\upsilon^{+})\barwedge(\tilde{\Im}^{+}_{\intercal}(\varsigma)\barwedge\upsilon^{+})\}\barwedge\upsilon^{+}\Big]\veebar\tau^{+}\\
&\le (\{(\tilde{\Im}^{+}_{\daleth}(\kappa)\veebar\tau^{+})\barwedge(\tilde{\Im}^{+}_{\intercal}(\kappa)\veebar\tau^{+})\}\barwedge\upsilon^{+})\veebar\tau^{+}\\
&= \{((\tilde{\Im}^{+}_{\daleth}(\kappa)\barwedge\tilde{\Im}^{+}_{\intercal}(\kappa))\veebar\tau^{+})\barwedge\upsilon^{+}\}\veebar\tau^{+}\\
&= \{((\tilde{\Im}^{+}_{\daleth}\barwedge\tilde{\Im}^{+}_{\intercal})(\kappa)\barwedge\upsilon^{+}\}\veebar\tau^{+}\\
&= (\tilde{\Im}^{+}_{\daleth\barwedge^{\upsilon^{+}}_{\tau^{+}}\intercal})(\kappa)
\end{aligned}$$

$$
\begin{aligned}
(\tilde{\mho}^{+}_{(\daleth\cdot\mathsf{T}]})^{\upsilon^{+}}_{\tau^{+}}(\kappa) &= (\tilde{\mho}^{+}_{(\daleth\cdot\mathsf{T}]}(\kappa)\veebar\upsilon^{+})\barwedge\tau^{+}\\
&= \Big[[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\mho}^{+}_{\daleth}(\epsilon)\veebar\tilde{\mho}^{+}_{\mathsf{T}}(\varsigma)\}\veebar\upsilon^{+}]\Big]\barwedge\tau^{+}\\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{\tilde{\mho}^{+}_{\daleth}(\epsilon)\veebar\tilde{\mho}^{+}_{\mathsf{T}}(\varsigma)\}\veebar\upsilon^{+}\veebar\upsilon^{+}\Big]\barwedge\tau^{+}\\
&= \Big[\inf_{\kappa\leq\epsilon\varsigma}\{(\tilde{\mho}^{+}_{\daleth}(\epsilon)\veebar\upsilon^{+})\veebar(\tilde{\mho}^{+}_{\mathsf{T}}(\varsigma)\veebar\upsilon^{+})\}\veebar\upsilon^{+}\Big]\barwedge\tau^{+}\\
&\geq (\{(\tilde{\mho}^{+}_{\daleth}(\kappa)\barwedge\tau^{+})\veebar(\tilde{\mho}^{+}_{\mathsf{T}}(\kappa)\barwedge\tau^{+})\}\veebar\upsilon^{+})\barwedge\tau^{+}\\
&= \{((\tilde{\mho}^{+}_{\daleth}(\kappa)\veebar\tilde{\mho}^{+}_{\mathsf{T}}(\kappa))\barwedge\tau^{+})\veebar\upsilon^{+}\}\barwedge\tau^{+}\\
&= \{((\tilde{\mho}^{+}_{\daleth}\veebar\tilde{\mho}^{+}_{\mathsf{T}})(\kappa)\veebar\upsilon^{+}\}\barwedge\tau^{+}\\
&= (\tilde{\mho}^{+}_{\daleth\veebar^{\upsilon^{+}}_{\tau^{+}}\mathsf{T}})(\kappa)
\end{aligned}
$$

Let $\epsilon,\varsigma\notin I_{\kappa}$. If $I_{\kappa}=\emptyset$, then $(\tilde{\Re}^{-}_{\daleth}\cdot\tilde{\Re}^{-}_{\mathsf{T}})(\kappa)=0$, $(\tilde{\Im}^{-}_{\daleth}\cdot\tilde{\Im}^{-}_{\mathsf{T}})(\kappa)=0$ and $(\tilde{\mho}^{-}_{\daleth}\cdot\tilde{\mho}^{-}_{\mathsf{T}})(\kappa)=-1$ and such that $\kappa\leq\epsilon\varsigma$.

$$
\begin{aligned}
(\tilde{\Re}^{-}_{(\daleth\cdot\mathsf{T}]})^{\upsilon^{-}}_{\tau^{-}}(\kappa) &= (\tilde{\Re}^{-}_{(\daleth\cdot\mathsf{T}]}(\kappa)\veebar\upsilon^{-})\barwedge\tau^{-}\\
&= 0\barwedge\tau^{-}\\
&\geq (\tilde{\Re}^{-}_{\daleth\barwedge\mathsf{T}}(\kappa)\veebar\upsilon^{-})\barwedge\tau^{-}\\
&= (\tilde{\Re}^{-}_{\daleth\barwedge\mathsf{T}}(\kappa)\veebar\upsilon^{-})
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\Im}^{-}_{(\daleth\cdot\mathsf{T}]})^{\upsilon^{-}}_{\tau^{-}}(\kappa) &= (\tilde{\Im}^{-}_{(\daleth\cdot\mathsf{T}]}(\kappa)\veebar\upsilon^{-})\barwedge\tau^{-}\\
&= 0\barwedge\tau^{-}\\
&\geq (\tilde{\Im}^{-}_{\daleth\barwedge\mathsf{T}}(\kappa)\veebar\upsilon^{-})\barwedge\tau^{-}\\
&= (\tilde{\Im}^{-}_{\daleth\barwedge\mathsf{T}}(\kappa)\veebar\upsilon^{-})
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\mho}^{-}_{(\daleth\cdot\mathsf{T}]})^{\upsilon^{-}}_{\tau^{-}}(\kappa) &= (\tilde{\mho}^{-}_{(\daleth\cdot\mathsf{T}]}(\kappa)\barwedge\upsilon^{-})\veebar\tau^{-}\\
&= -1\veebar\tau^{-}\\
&= \tau^{-}\\
&\leq (\tilde{\mho}^{-}_{\daleth\veebar\mathsf{T}}(\kappa)\barwedge\upsilon^{-})\veebar\tau^{-}\\
&= (\tilde{\mho}^{-}_{\daleth\veebar\mathsf{T}}(\kappa)\barwedge\upsilon^{-})
\end{aligned}
$$

Let $\epsilon,\varsigma\notin I_{\kappa}$. If $I_{\kappa}=\emptyset$, then $(\tilde{\Re}^{+}_{\daleth}\cdot\tilde{\Re}^{+}_{\mathsf{T}})(\kappa)=0$, $(\tilde{\Im}^{+}_{\daleth}\cdot\tilde{\Im}^{+}_{\mathsf{T}})(\kappa)=0$ and $(\tilde{\mho}^{+}_{\daleth}\cdot\tilde{\mho}^{+}_{\mathsf{T}})(\kappa)=1$ and such that $\kappa\leq\epsilon\varsigma$.

$$
\begin{aligned}
(\tilde{\Re}^{+}_{(\daleth\cdot\mathsf{T}]})^{\upsilon^{+}}_{\tau^{+}}(\kappa) &= (\tilde{\Re}^{+}_{(\daleth\cdot\mathsf{T}]}(\kappa)\barwedge\upsilon^{+})\veebar\tau^{+}\\
&= 0\veebar\tau^{+}\\
&\leq (\tilde{\Re}^{+}_{\daleth\barwedge\mathsf{T}}(\kappa)\barwedge\upsilon^{+})\veebar\tau^{+}\\
&= (\tilde{\Re}^{+}_{\daleth\barwedge\mathsf{T}}(\kappa)\barwedge\upsilon^{+})
\end{aligned}
$$

$$
\begin{aligned}
(\tilde{\Im}^{+}_{(\beth\cdot\top]})^{\upsilon^+}_{\tau^+}(\kappa) &= (\tilde{\Im}^{+}_{(\beth\cdot\top]}(\kappa) \,\overline{\wedge}\, \upsilon^{+}) \,\underline{\vee}\, \tau^{+} \\
&= 0 \,\underline{\vee}\, \tau^{+} \\
&\leq (\tilde{\Im}^{+}_{\beth\overline{\wedge}\top}(\kappa) \,\overline{\wedge}\, \upsilon^{+}) \,\underline{\vee}\, \tau^{+} \\
&= (\tilde{\Im}^{+}_{\beth\overline{\wedge}\top}(\kappa) \,\overline{\wedge}\, \upsilon^{+}) \\
(\tilde{\mho}^{+}_{(\beth\cdot\top]})^{\upsilon^+}_{\tau^+}(\kappa) &= (\tilde{\mho}^{+}_{(\beth\cdot\top]}(\kappa) \,\underline{\vee}\, \upsilon^{+}) \,\overline{\wedge}\, \tau^{+} \\
&= 1 \,\overline{\wedge}\, \tau^{+} \\
&= \tau^{+} \\
&\geq (\tilde{\mho}^{+}_{\beth\underline{\vee}\top}(\kappa) \,\underline{\vee}\, \upsilon^{+}) \,\overline{\wedge}\, \tau^{+} \\
&= (\tilde{\mho}^{+}_{\beth\underline{\vee}\top}(\kappa) \,\underline{\vee}\, \upsilon^{+})
\end{aligned}
$$

Therefore $((\beth \cdot \top])^{\upsilon}_{\tau} \subseteq (\beth \,\overline{\wedge}^{\upsilon}_{\tau}\, \top]$.

4. Conclusion and future direction

The notions of (τ, υ)-BIntNS, BIntNLI, BIntNRI, BIntNI, and BIntNBI were introduced, along with a discussion of some of the characteristics of an ordered semigroup. Furthermore, an analysis of the properties of various transformations is conducted. We are attempting to handle novel fuzzy structures with cubic and interval values. Therefore, in the future, we should think about adopting advanced, soft settings.

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