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The definite integrals of 2- refined neutrosophic functions and its properties

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Abstract: This article aims to study the definite integrals of 2- refined neutrosophic functions and its properties. The definite integrals of 2- refined neutrosophic functions are defined in this study, and a set of theories are presented, including the Fundamental theorem of 2- refined neutrosophic integral calculus, the mean-value theorem of 2- refined neutrosophic integral calculus with its two parts, in addition to the properties of the definite integrals of 2- refined neutrosophic functions. Also, we used the n-Refined AH-Isometry [11].

Keywords: definite integrals; 2- refined neutrosophic functions; properties; mean-value theorem.

1. Introduction and Preliminaries

As an alternative to the existing logics, Smarandache proposed the neutrosophic logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \dots, b_nI_n)$ where $a, b_1, b_2, \dots, b_n \in R \text{ or } C$ [1]. Agboola introduced the concept of refined neutrosophic algebraic structures [2]. In addition, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. Abobala presented the papers on some special substructures of refined neutrosophic rings and a study of ah-substructures in n-refined neutrosophic vector spaces [6-7].

Alhasan.Y and Abdulfatah. R also presented a study on the division of refined neutrosophic number [8].

There are papers presented in n-Valued Refined Neutrosophic Logic and Its Applications in Physics, Neutrosophic Rings I [4-5] and studying the integral calculus according to the logic of neutrosophic by presenting a set of papers on that [9-10].

In addition, the AH-Isometry was extended to n-Refined AH-Isometry by Smarandache & Abobala in 2024 [11]

2. The definite integrals of 2- refined neutrosophic functions

Theorem 1 (Fundamental theorem of 2- refined neutrosophic integral calculus)

Let be $f(x, I_1, I_2)$ a continuous 2- refined neutrosophic function defined in $[a + a_1I_1 + a_2I_2, b + b_1I_1 + b_2I_2]$, and let $F(x, I_1, I_2)$ be the anti-derivative of $f(x, I_1, I_2)$, that is

$$\int f(x, I_1, I_2)dx = F(x, I_1, I_2)$$

then:

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = F(a + a_1I_1 + a_2I_2) - F(b + b_1I_1 + b_2I_2)$$

Where a, a_1, a_2, b, b_1, b_2 are real number, I_1, I_2 represent indeterminacy.

Example 1

(1) $$\int_{-2+I_1-I_2}^{5-I_1+I_2} (-3I_1 + 2x)dx = [-3I_1x + x^2]_{-2+I_1-I_2}^{5-I_1+I_2}$$

$$= [-3I_1(5 - I_1 + I_2) + (5 - I_1 + I_2)^2] - [3I_1(-2 + I_1 - I_2) + (-2 + I_1 - I_2)^2]$$

$$= -15I_1 + 25 - 11I_1 + 12I_2 - 6I_1 - 4 + 5I_1 - 5I_2$$

$$= 21 - 27I_1 + 7I_2$$

(2) $$\int_0^{\frac{\pi}{2}+\pi I_1+\pi I_2} cos\left(x - \pi I_1 + \frac{\pi}{4}I_2\right)dx = \left[sin\left(x - \pi I_1 + \frac{\pi}{4}I_2\right)\right]_0^{\frac{\pi}{2}+\pi I_1+\pi I_2}$$

$$= \left[sin\left(\frac{\pi}{2} + \pi I_1 + \pi I_2 - \pi I_1 + \frac{\pi}{4}I_2\right)\right] - \left[sin\left(-\pi I_1 + \frac{\pi}{4}I_2\right)\right]$$

$$= \left[sin\left(\frac{\pi}{2} + \frac{5\pi}{4}I_2\right)\right] - \left[sin\left(-\pi I_1 + \frac{\pi}{4}I_2\right)\right]$$

$$= sin\left(\frac{\pi}{2}\right) + I_1\left[sin\left(\frac{7\pi}{4}\right) - sin\left(\frac{7\pi}{4}\right)\right] + I_2\left[sin\left(\frac{7\pi}{4}\right) - sin\left(\frac{\pi}{2}\right)\right] - I_1\left[sin\left(-\frac{3\pi}{4}\right) - sin\left(\frac{\pi}{4}\right)\right] - I_2 sin\left(\frac{\pi}{4}\right)$$

$$= 1 + \sqrt{2}I_1 + \left(-\sqrt{2} - 1\right)I_2$$

(3) $$\int_{2I_2}^{-2+I_1-I_2} -2x(4 - 2I_1 + 5I_2 - x^2)^2dx = \left[\frac{(4 - 2I_1 + 5I_2 - x^2)^3}{3}\right]_{2I_2}^{-2+I_1-I_2}$$

$$= \left[\frac{(4 - 5I_1 + 4I_2 - (-2 + I_1 - I_2)^2)^3}{3}\right] - \left[\frac{(4 - 5I_1 + 4I_2 - (2I_2)^2)^3}{3}\right]$$

$$= \left[\frac{(4 - 5I_1 + 4I_2 - 4 + 5I_1 - 5I_2)^3}{3}\right] - \left[\frac{(4 - 5I_1 + 4I_2 - 4I_2)^3}{3}\right]$$

$$= \left[\frac{-1}{3}I_2\right] - \left[\frac{64}{3} - \frac{65}{3}I_1\right] = -\frac{64}{3} + \frac{65}{3}I_1 - \frac{1}{3}I_2$$

(4) $$\int_{5+29I_1+6I_2}^{2+26I_1+25I_2} \frac{1}{2\sqrt{x-1-2I_1-I_2}}dx = \left[\sqrt{x-1-2I_1-I_2}\right]_{5+29I_1+6I_2}^{2+26I_1+25I_2}$$

$$= \sqrt{2+26I_1+25I_2-1-2I_1-I_2} - \sqrt{5+29I_1+6I_2-1-2I_1-I_2}$$

$$= \sqrt{1+24I_1+24I_2} - \sqrt{4+27I_1+5I_2}$$

$$= 1+2I_1+4I_2-2-3I_1-2I_2$$

$$= -1-I_1+2I_2$$

Theorem 2 (The mean- value theorem of 2- refined neutrosophic integral calculus_ part I)

If $f(x,I_1,I_2)$ is continuous on an interval, then we say that $f(x,I_1,I_2)$ has an anti- derivative on that interval. In specific, if $a+a_1I_1+a_2I_2$ is any point in the interval, then $f(x,I_1,I_2)$ defined by:

(1) $$\frac{d}{dx}\left[\int_{a+a_1I_1+a_2I_2}^{x} f(t,I_1,I_2)dt\right] = f(x,I_1,I_2)$$

(2) $$\frac{d}{dx}\left[\int_{x}^{a+a_1I_1+a_2I_2} f(t,I_1,I_2)dt\right] = -f(x,I_1,I_2)$$

Example 2

(1) $$\frac{d}{dx}\left[\int_{\pi+\frac{\pi}{3}I_1+\frac{\pi}{6}I_2}^{x} (tan^2t+2-3I_1+4I_2)dt\right] = tan^2x+2-3I_1+4I_2$$

(2) $$\frac{d}{dx}\left[\int_{9-5I_1+7I_2}^{x} \sqrt{5I_1t+4I_2}dt\right] = \sqrt{5I_1x+4I_2}$$

(3) $$\frac{d}{dx}\left[\int_{x}^{-6+I_1-I_2} (5I_1t^3+I_2t^2+t-2I_1+5I_2)dt\right] = -5I_1x^3-I_2x^2-x+2I_1-5I_2$$

Remarks 1

(1) $$\frac{d}{dx}\left[\int_{a+a_1I_1+a_2I_2}^{g(x,I_1,I_2)} f(t,I_1,I_2)dt\right] = f\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

Proof:

$$\frac{d}{dx}\left[\int_{a+a_1I_1+a_2I_2}^{g(x,I_1,I_2)} f(t,I_1,I_2)dt\right] = \frac{d}{dx}\left[F\big(g(x,I_1,I_2)\big)\right]$$

$$= \acute{F}\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

$$= f\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

(2) $\frac{d}{dx}\left[\int_{g(x,I_1,I_2)}^{a+a_1I_1+a_2I_2} f(t,I_1,I_2)dt\right] = -f\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$

Proof:

$$\frac{d}{dx}\left[\int_{g(x,I_1,I_2)}^{a+a_1I_1+a_2I_2} f(t,I_1,I_2)dt\right] = \frac{d}{dx}\left[-F\big(g(x,I_1,I_2)\big)\right]$$

$$= -\acute{F}\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

$$= -f\big(g(x,I_1,I_2)\big)\acute{g}(x,I_1,I_2)$$

(3) $\frac{d}{dx}\left[\int_{g_1(x,I_1,I_2)}^{g_2(x,I_1,I_2)} f(t,I_1,I_2)dt\right] = f\big(g_2(x,I_1,I_2)\big)\,\acute{g}_2(x,I_1,I_2) - f\big(g_1(x,I_1,I_2)\big)\,\acute{g}_1(x,I_1,I_2)$

Proof:

$$\frac{d}{dx}\left[\int_{g_1(x,I_1,I_2)}^{g_2(x,I_1,I_2)} f(t,I_1,I_2)dt\right] = \frac{d}{dx}\left[\int_{g_1(x,I_1,I_2)}^{0+0I_1+0I_2} f(t,I_1,I_2)dt + \int_{0+0I_1+0I_2}^{g_2(x,I_1,I_2)} f(t,I_1,I_2)dt\right]$$

$$= f\big(g_2(x,I_1,I_2)\big)\,\acute{g}_2(x,I_1,I_2) - f\big(g_1(x,I_1,I_2)\big)\,\acute{g}_1(x,I_1,I_2)$$

Example 3

(1) $\frac{d}{dx}\left[\int_{1+I_1+I_2}^{cos\,(x+\frac{\pi}{3}I_1+\pi I_2)} (t^3 - I_1 + 7I_2)dt\right] = -(cos^3(x + \frac{\pi}{3}I_1 + \pi I_2) - I_1 + 7I_2)sin\,(x + \frac{\pi}{3}I_1 + \pi I_2)$

(2) $\frac{d}{dx}\left[\int_{-9-I_1+6I_2}^{\sqrt{I_1+3I_2-5x}} (-8 + I_2 - t)dt\right] = (-8 + I_2 - \sqrt{I_1 + 3I_2 - 5x})\frac{-5}{2\sqrt{I_1 + 3I_2 - 5x}}$

(3) $\frac{d}{dx}\left[\int_{ln(x+4+I_1+I_2)}^{5+2I_1+I_2} tdt\right] = -\frac{1}{x + 4 + I_1 + I_2}\ ln(x + 4 + I_1 + I_2)$

(4) $\frac{d}{dx}\left[\int_{-7x+I_1+4I_2}^{6x^2-2I_1+I_2} \frac{2 - I_1 + I_2}{2t + I_1 + 3I_2}dt\right] = \frac{(2 - I_1 + I_2)x}{12x^2 - 4I_1 + 2I_2 + I_1 + 3I_2} + \frac{14 - 7I_1 + 7I_2}{-14x + 2I_1 + 8I_2 + I_1 + 3I_2}$

Theorem 3 (The mean- value theorem of 2- refined neutrosophic integral calculus_ part II)

If $f(x, I_1, I_2)$ is continuous on a closed 2- refined interval $[a + a_1I_1 + a_2I_2, b + b_1I_1 + b_2I_2]$, then there is at least one point $x^* = x_0 + x_1I_1 + x_2I_2$ in $[a + a_1I_1 + a_2I_2, b + b_1I_1 + b_2I_2]$ such that:

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = f(x^*, I_1, I_2)\big(b + b_1I_1 + b_2I_2 - (a + a_1I_1 + a_2I_2)\big)$$

Example 4

If $f(x, I_1, I_2) = 4x + 5I_1$ let's find x^* that satisfy the mean-value theorem of f on $[2 + I_1 + 3I_2, 4 + I_1 + 5I_2]$.

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = f(x^*, I_1, I_2)\big(b + b_1I_1 + b_2I_2 - (a + a_1I_1 + a_2I_2)\big)$$

$$\int_{2+I_1+3I_2}^{4+I_1+5I_2} (4x + 5I_1)dx = (4x^* + 5I_1)(2 + 2I_2)$$

$$[2x^2 + 5I_1x]_{2+I_1+3I_2}^{4+I_1+5I_2} = (4x^* + 5I_1)(2 + 2I_2)$$

$$[2(4 + I_1 + 5I_2)^2 + 5I_1(4 + I_1 + 5I_2)] - [2(2 + I_1 + 3I_2)^2 + 5I_1(2 + I_1 + 3I_2)] = (4x^* + 5I_1)(2 + 2I_2)$$

$$24 + 22I_1 - 6I_2 = (4x^* + 5I_1)(2 + 2I_2)$$

$$4x^* + 5I_1 = \frac{24 + 36I_1 + 88I_2}{2 + 2I_2}$$

$$4x^* + 5I_1 = \frac{12 + 18I_1 + 44I_2}{1 + 0 + I_2}$$

$$4x^* + 5I_1 = 12 + 9I_1 + 16I_2$$

$$4x^* = 12 + 4I_1 + 16I_2$$

$$x^* = 3 + I_1 + 4I_2 \in [2 + I_1 + 3I_2, 4 + I_1 + 5I_2]$$

Example 5

If $f(x, I_1, I_2) = 9\sqrt{x}$ let's find x^* that satisfy the mean-value theorem of f on $[0 + 0I_1 + 0I_2, 1 + 12I_1 + 3I_2]$.

$$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = f(x^*, I_1, I_2)\big(b + b_1I_1 + b_2I_2 - (a + a_1I_1 + a_2I_2)\big)$$

$$\int_{0+0I_1+0I_2}^{1+12I_1+3I_2} 9\sqrt{x}\, dx = 9(1 + 12I_1 + 3I_2)\sqrt{x^*}$$

$$\left[6x\sqrt{x}\right]_{0+0I_1+0I_2}^{1+12I_1+3I_2} = 9(1+12I_1+3I_2)\sqrt{x^*}$$

$$6(1+12I_1+3I_2)\sqrt{1+12I_1+3I_2} = 9(1+12I_1+3I_2)\sqrt{x^*}$$

$$\sqrt{x^*} = \frac{2\sqrt{1+12I_1+3I_2}}{3}$$

$$x^* = \frac{4(1+12I_1+3I_2)}{9} = \frac{4+48I_1+12I_2}{9}$$

$$x^* = \frac{4}{9} + \frac{16}{3}I_1 + \frac{4}{3}I_2 \ \in \ [0+0I_1+0I_2, 1+12I_1+3I_2]$$

3. Properties of the definite integrals of 2- refined neutrosophic functions

Let $f(x, I_1, I_2)$ and $g(x, I_1, I_2)$ then:

(1) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(t, I_1, I_2)dt$$

(2) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = \int_{a+a_1I_1+a_2I_2}^{c+c_1I_1+c_2I_2} f(x, I_1, I_2)dx + \int_{c+c_1I_1+c_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx$$

where: $a + a_1I_1 + a_2I_2 \leq c + c_1I_1 + c_2I_2 \leq b + b_1I_1 + b_2I_2$

(3) $$\int_{a+a_1I_1+a_2I_2}^{a+a_1I_1+a_2I_2} f(x, I_1, I_2)dx = 0$$

(4) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx = -\int_{b+b_1I_1+b_2I_2}^{a+a_1I_1+a_2I_2} f(x, I_1, I_2)dx$$

(5) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} (c+c_1I_1+c_2I_2) f(x, I_1, I_2)dx = (c+c_1I_1+c_2I_2)\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx$$

(6) $$\int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} [f(x, I_1, I_2) \pm g(x, I_1, I_2)]dx$$
$$= \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} f(x, I_1, I_2)dx \pm \int_{a+a_1I_1+a_2I_2}^{b+b_1I_1+b_2I_2} g(x, I_1, I_2)dx$$

(7) $$\int_{-(a+a_1I_1+a_2I_2)}^{a+a_1I_1+a_2I_2} f(x,I_1,I_2)dx$$

$$= \begin{cases} 2\int_{0}^{a+a_1I_1+a_2I_2} f(x,I_1,I_2)dx & ; if\ f(x,I_1,I_2)\ is\ even\ function \\ 0 & ; if\ f(x,I_1,I_2)\ is\ odd\ function \end{cases}$$

Example 6

1) $$\int_{9+4I_1-7I_2}^{9+4I_1-7I_2} (t^3+(1+I_1+3I_2)t^2-12I_1-9I_2)dx = 0$$

2) $$\int_{-\pi-\frac{\pi}{12}I_1+\frac{\pi}{3}I_2}^{\pi+\frac{\pi}{12}I_1-\frac{\pi}{3}I_2} -7I_2\ sin^7x\ cos^6x\ dx$$

Put: $f(x,I_1,I_2) = -7I_2\ sin^7x\ cos^6x$, then:

$$f(-x,I_1,I_2) = -7I_2\ sin^7(-x)cos^6(-x)$$

$$= -7I_2(sin(-x))^7(\cos(-x))^6 = 7I_2\ sin^7x\ cos^6x$$

$$= -f(x,I_1,I_2)$$

Hence $f(x,I)$ is odd function, so by property 7, we get:

$$\int_{-\pi-\frac{\pi}{12}I_1+\frac{\pi}{3}I_2}^{\pi+\frac{\pi}{12}I_1-\frac{\pi}{3}I_2} -7I_2\ sin^7x\ cos^6x\ dx = 0$$

3. Conclusions

This paper introduced the concept of the definite integrals of 2- refined neutrosophic functions and its properties, by discussing a set of theorems related to the definite integrals of 2- refined neutrosophic functions, where these theorems were proven. Also, the properties of the definite integrals of 2- refined neutrosophic functions were studied, which facilitated finding the result of the integrals of 2- refined neutrosophic function faster and directly. In addition, the importance of this paper comes from the fact that it facilitates many studies in the field of the neutrosophic integrals calculate.

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