NSS
Neutrosophic Sets and Systems, Vol. 82, 2025
University of New Mexico

Fixed point results in complete neutrosophic fuzzy metric spaces for NF-L contractions

Ayman. A Hazaymeh[1,*]
[1]Department of mathematics, Faculty of Science, Jadara University, Irbid 21110, Jordan;
aymanha@jadara.edu.jo; ayman_hazaymeh@yahoo.com
*Correspondence: aymanha@jadara.edu.jo

Abstract. Fixed point theory occupies a significant position in both applied and pure mathematics due to its wide-ranging applications. The Banach contraction principle serves as a foundational theorem in metric spaces, which has been subject to numerous generalizations and applications. In this study, we explore the concept of neutrosophic fuzzy metric spaces and introduce the notion of NF-L contractions through a novel class of functions. We demonstrate that these contractions possess a unique fixed point. Initially, we present a crucial lemma that will be utilized in subsequent sections, followed by the proof of our primary result. Ultimately, we derive several fixed point results based on these types of contractions, which are derived from the main theorem.

Keywords: Fixed point; Neutrosophic set; Neutrosophic fuzzy metric space; NF-L contraction; Non linear contraction.

1. Introduction

The concept of Fuzzy Sets (FSs), initially proposed by Zadeh [1], has had a profound impact across various scientific disciplines since its inception. Although this framework holds significant relevance for practical applications, it has not always provided adequate solutions to numerous challenges over time. Consequently, there has been a resurgence of research aimed at addressing these issues. In this regard, Atanassov [2] introduced Intuitionistic Fuzzy Sets (IFSs) as a means to confront such challenges. In addition, Smarandache [3] introduced Neutrosophic Set (NS), represents a complex extension of conventional set theory.

Neutrosophic sets demonstrate a wide array of applications across various fields. For example, Barbosa and Smarandache [4] introduced the Neutrosophic One-Round Zero-Knowledge Proof protocol (N-1-R) ZKP, which expands upon the One-Round (1-R) ZKP framework by incorporating Neutrosophic numbers. Additionally, the authors in [5] offer a detailed characterization of effective and optimally suitable solutions pertaining to scalar optimization challenges. They also delineate the Kuhn-Tucker conditions that are pertinent to both efficiency and proper efficiency. We cant make better decisions when temporal value is not considered, which leads to less accurate decision-making. This is made possible by including historical knowledge into the state representation. In the event that we wish to consider the opinions of many time periods, we should do so for both the current and prior eras (forecasting information). For instance, some researchers would want to think about using the influence of temporal variables in their calculations. The concept of time fuzzy soft sets and its use in design-making were presented by [6]. The effect of time on fuzzy soft expert sets was also discussed by [9]. [13] also introduced the concepts and applications of time-shadow soft sets, For a more in-depth investigation into the applications of neutrosophic sets and their extensive uses with time factor , it is advisable to refer to the literature cited in [12]. [14]Bipolar Complex Fuzzy Soft Sets and Their Application. We can consult the following for a more thorough examination of the applications of neutrosophic sets and their wide range of uses, we can refer to following, it is advisable to refer to the literature as [7] introduced Neutrosophic Logic to Navigate Uncertainty of Security Events in Mexico, [8] introduced Set and Machine Learning Approach for Breast Cancer Prediction, [10] introduced Multi Attribute Neutrosophic Optimization Technique for Optimal Crop Selection in Ariyalur District, [11] introduced Tele-Medical Realization via Integrating Vague T2NSs with OWCM-RAM Toward Intelligent Medical 4.0 Evaluator Framework.

[15] presented Utilizing a single-valued neutrosophic fuzzy soft expert setting, a strong framework for decision-making Applications of a novel extension of interval-valued Q-neutrosophic soft matrix Decision-making methods based on similarity metrics of possible neutrosophic soft expert sets were introduced by [17]. An approach for determining decisions based on similarity measures of possibility interval-valued neutrosophic soft setting settings was presented by [18]. The Banach fixed-point theorem [19], commonly known as the contraction mapping theorem, is a fundamental principle in the theory of metric spaces. It asserts that within any complete metric space, a contraction mapping—defined as a function that reduces the distance between points—will possess a unique fixed point. This theorem is significant because it provides a systematic approach to identifying fixed points, which are defined as points that remain invariant under a specific function. Moreover, it guarantees both the existence and uniqueness of such points under well-defined conditions.

The Banach fixed-point theorem is widely recognized for its significant applications in numerous fields, such as differential equations and numerical analysis. Its primary function lies in demonstrating the existence and uniqueness of solutions to equations, as well as facilitating iterative methods for solution approximation. Initially proposed by Banach in 1922, this theorem has spurred a multitude of mathematicians to investigate various extensions and generalizations across diverse mathematical areas. For instance, the authors in [20, 21] have derived fixed-point results by introducing novel types of distance spaces associated with metric spaces. This line of inquiry was further advanced in [22], where the author established fixed-point results for contractions of Geraghty type. Additionally, Bataihah [23] contributed new fixed-point theorems through a distinct type of distance based on b-metric spaces.

Moreover, the theorem has inspired a wide array of generalizations and extensions. Aydi et al. [24] introduced triple coincidence point theorems for weak Φ-contractions within the framework of partially ordered metric spaces. The research referenced in [25, 26] presents various fixed-point results in b-metric spaces, while [27,28] discusses several fixed-point results in extended b-metric spaces. Furthermore, the authors in [29–31] explored fixed-point results in generalized metric spaces. For additional insights into fixed-point theory, we direct the reader to [32–34] and references therin.

Triangular norms (TN), introduced by Menger [35], generalize the triangle inequality using probability distributions instead of numerical distances. Their duals, triangular conorms (CN), are essential in fuzzy operations, particularly for intersections and unions.

In this document, we define the sets as follows: $\mathbb{R}^+ = [0, \infty)$ and $I_0 = [0, 1]$.

Definition 1.1. Let $\bullet : I_0 \times I_0 \to I_0$ be an operation. Then, $\bullet$ is said to be as continuous TN (CN) if it satisfies the following: for any $\varsigma, \varsigma', s, s' \in I_0$.

(1) $\varsigma \bullet 1 = \varsigma$,
(2) If $\varsigma \leq \varsigma'$ and $s \leq s'$, than $\varsigma \bullet s \leq \varsigma' \bullet s'$,
(3) $\bullet$ is continuous,
(4) $\bullet$ is associate and commutative.

Definition 1.2. Let $\diamond : I_0 \times I_0 \to I_0$ be an operation. Then, $\diamond$ is designated as continuous TN (TN) if it satisfies the following: for all $\varsigma, \varsigma', s, s' \in I_0$:

(1) $\varsigma \diamond 0 = \varsigma$,
(2) If $\varsigma \leq \varsigma'$ and $s \leq s'$, than $\varsigma \diamond s \leq \varsigma' \diamond s'$,
(3) $\diamond$ is continuous,
(4) $\diamond$ is associate and commutative.

The notion of neutrosophic metric space (NMS) was initially presented by Kirisci and Simsek. This setting has been utilized to investigate several fixed point theorems. The definition of neutrosophic metric spaces is outlined as follows.

Definition 1.3. [36] A 6-tuple $(\mathcal{W}, \Theta, \Phi, \Psi, \bullet, \diamond)$ is defined as a NMS when the set $\mathcal{W}$ constitutes a non-empty arbitrary collection. In this context, $\bullet$ represents a continuous t-norm, while $\diamond$ denotes a continuous t-conorm. Additionally, the elements Θ, Φ, and Ψ are three fuzzy sets defined on the Cartesian product $\mathcal{W}^2 \times (0, \infty)$. These components are required to fulfill specific conditions applicable to all elements $\zeta, \varrho, c \in \mathcal{W}$ and for all positive real numbers γ, ϖ.

(1) $0 \le \Theta(\zeta, \varrho, \gamma) \le 1$, $0 \le \Phi(\zeta, \varrho, \gamma) \le 1$, $0 \le \Psi(\zeta, \varrho, \gamma) \le 1$,
(2) $0 \le \Theta(\zeta, \varrho, \gamma) + \Phi(\zeta, \varrho, \gamma) + \Psi(\zeta, \varrho, \gamma) \le 3$,
(3) $\Theta(\zeta, \varrho, \gamma) = 1$, iff $\zeta = \varrho$
(4) $\Theta(\zeta, \varrho, \gamma) = H(\varrho, \zeta, \gamma)$, for $\gamma > 0$
(5) $\Theta(\zeta, \varrho, \gamma) \bullet \Theta(\varrho, c, \varpi) \le \Theta(\zeta, c, \gamma + \varpi)$
(6) $\Theta(\zeta, \varrho, \cdot) : \mathbb{R}^+ \to I_0$ is continuous
(7) $\lim_{\gamma \to \infty} \Theta(\zeta, \varrho, \gamma) = 1$
(8) $\Phi(\zeta, \varrho, \gamma) = 0$ iff $\zeta = \varrho$
(9) $\Phi(\zeta, \varrho, \gamma) = \Phi(\varrho, \zeta, \gamma)$,
(10) $\Phi(\zeta, \varrho, \gamma) \diamond \Phi(\varrho, c, \varpi) \ge \Phi(\zeta, c, \gamma + \varpi)$,
(11) $\Phi(\zeta, \varrho, \cdot) : \mathbb{R}^+ \to I_0$ is continuous
(12) $\lim_{\gamma \to \infty} \Phi(\zeta, \varrho, \gamma) = 0$
(13) $\Psi(\zeta, \varrho, \gamma) = 0$, iff $\zeta = \varrho$
(14) $\Psi(\zeta, \varrho, \gamma) = \Psi(\varrho, \zeta, \gamma)$,
(15) $\Psi(\zeta, \varrho, \gamma) \diamond \Psi(\varrho, c, \varpi) \ge S(\zeta, c, \gamma + \varpi)$,
(16) $\Psi(\zeta, \varrho, \cdot) : \mathbb{R} \to I_0$ is continuous
(17) $\lim_{\gamma \to \infty} \Psi(\zeta, \varrho, \gamma) = 0$
(18) If $\gamma \le 0$, then $\Theta(\zeta, \varrho, \gamma) = 0$, $\Phi(\zeta, \varrho, \gamma) = \Psi(\zeta, \varrho, \gamma) = 1$

The functions $\Theta(\zeta, \varrho, \gamma)$, $\Phi(\zeta, \varrho, \gamma)$, and $\Psi(\zeta, \varrho, \gamma)$ represent the degrees of nearness, neutralness, and non-nearness between the elements ζ and ϱ in relation to the parameter γ, respectively.

The convergence, Cauchy-ness, completeness in NMS are given as follows.

Definition 1.4. [36] Let (ζ_n) be a sequence in a NMS $(\mathcal{W}, \Theta, \Phi, \Psi, \bullet, \diamond)$. Then

(1) (ζ_n) converges to $\zeta \in \mathcal{W}$ if for any $\epsilon \in (0,1)$, $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$

$$\Theta(\zeta_n, \zeta, \gamma) > 1 - \epsilon, \ \Phi(\zeta_n, \zeta, \gamma) < \epsilon, \ \Psi(\zeta_n, \zeta, \gamma) < \epsilon$$

i.e.,

$$\lim_{n \to \infty} \Theta(\zeta_n, \zeta, \gamma) = 1, \ \lim_{n \to \infty} \Phi(\zeta_n, \zeta, \gamma) = 0, \ \lim_{n \to \infty} \Psi(\zeta_n, \zeta, \gamma) = 0$$

(2) (ζ_n) is Cauchy if for any $\epsilon \in (0,1)$, $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n, m \ge n_0$

$$\Theta(\zeta_n, \zeta_m, \gamma) > 1 - \epsilon, \ \Phi(\zeta_n, \zeta_m, \gamma) < \epsilon, \ \Psi(\zeta_n, \zeta_m, \gamma) < \epsilon$$

i.e.,

$$\lim_{n.m\to\infty}\Theta(\zeta_n,\zeta_m,\gamma)=1,\ \lim_{n,m\to\infty}\Phi(\zeta_n,\zeta_m,\gamma)=0,\ \lim_{n,m\to\infty}\Psi(\zeta_n,\zeta_m,\gamma)=0$$

(3) $(\mathcal{W},\Theta,\Phi,\Psi,\bullet,\diamond)$ is complete if every Cauchy sequence in $\mathcal{W}$ is convergent to an element in $\mathcal{W}$.

In the work of Simsek and Kirisci [37], NC-contractions were introduced within the context of neutrosophic metric spaces, demonstrating that each NC-contraction possesses a unique fixed point under specific conditions.

Definition 1.5. [37] Let $(\mathcal{W},\Theta,\Phi,\Psi,\bullet,\diamond)$ be a NMS. A mapping $f:\mathcal{W}\to\mathcal{W}$ is called neutrosophic contraction (NC) if there is $k\in(0,1)$ such that for each $\zeta,\varrho\in\mathcal{W}$ and $\gamma>0$, we have

$$\frac{1}{\Theta(f\zeta,f\varrho,\gamma)}-1\le k\left(\frac{1}{\Theta(\zeta,\varrho,\gamma)}-1\right),$$
$$\frac{1}{\Phi(f\zeta,f\varrho,\gamma)}-1\ge k\left(\frac{1}{\Phi(\zeta,\varrho,\gamma)}-1\right),$$

and

$$\frac{1}{\Psi(f\zeta,f\varrho,\gamma)}-1\ge k\left(\frac{1}{\Psi(\zeta,\varrho,\gamma)}-1\right).$$

Recently, Ghosh et al. [38] introduced the concept of neutrosophic fuzzy metric spaces and presented various topological characteristics regarding to this concept.

Definition 1.6. [38] A 7-tuple $(\mathcal{W},\Theta,\Pi,\Phi,\Psi,\bullet,\diamond)$ is known as a Neutrophic Fuzzy Metric Space (NFMS) if $\mathcal{W}$ is an arbitary set, $\bullet$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and Θ,Π,Φ, and Ψ are fuzzy sets on $\mathcal{W}^2\times(0,\infty)$ satisfying the following conditions for all $\zeta,\varrho,c,\in\mathcal{W}$ and $\gamma,\varpi>0$.

(1) $0\le\Theta(\zeta,\varrho,\gamma)\le1$, $0\le\Pi(\zeta,\varrho,\gamma)\le1$, $0\le\Phi(\zeta,\varrho,\gamma)\le1$, $0\le\Psi(\zeta,\varrho,\gamma)\le1$,

(2) $0\le\Theta(\zeta,\varrho,\gamma)+\Pi(\zeta,\varrho,\gamma)+\Phi(\zeta,\varrho,\gamma)+\Psi(\zeta,\varrho,\gamma)\le4$,

(3) $\Theta(\zeta,\varrho,\gamma)=1$, iff $\zeta=\varrho$

(4) $\Theta(\zeta,\varrho,\gamma)=H(\varrho,\zeta,\gamma)$,

(5) $\Theta(\zeta,\varrho,\gamma)\bullet\Theta(\varrho,c,\varpi)\le\Theta(\zeta,c,\gamma+\varpi)$,

(6) $\Theta(\zeta,\varrho,\cdot):\mathbb{R}^+\to I_0$ is continuous

(7) $\lim_{\gamma\to\infty}\Theta(\zeta,\varrho,\gamma)=1$

(8) $\Pi(\zeta,\varrho,\gamma)=1$, iff $\zeta=\varrho$

(9) $\Pi(\zeta,\varrho,\gamma)=\Pi(\varrho,\zeta,\gamma)$, for $\gamma>0$

(10) $\Pi(\zeta,\varrho,\gamma)\bullet\Pi(\varrho,c,\varpi)\le\Pi(\zeta,c,\gamma+\varpi)$,

(11) $\Pi(\zeta,\varrho,\cdot):\mathbb{R}^+\to I_0$ is continuous

(12) $\lim_{\gamma\to\infty}\Pi(\zeta,\varrho,\gamma)=1$

(13) $\Phi(\zeta,\varrho,\gamma)=0$, iff $\zeta=\varrho$

(14) $\Phi(\zeta,\varrho,\gamma)=\Phi(\varrho,\zeta,\gamma)$,

(15) $\Phi(\zeta,\varrho,\gamma)\diamond\Phi(\varrho,c,\varpi)\geq\Phi(\zeta,c,\gamma+\varpi)$,

(16) $\Phi(\zeta,\varrho,\cdot):\mathbb{R}^+\to I_0$ is continuous

(17) $\lim_{\gamma\to\infty}\Phi(\zeta,\varrho,\gamma)=0$

(18) $\Psi(\zeta,\varrho,\gamma)=0$, iff $\zeta=\varrho$

(19) $\Psi(\zeta,\varrho,\gamma)=\Psi(\varrho,\zeta,\gamma)$,

(20) $\Psi(\zeta,\varrho,\gamma)\diamond\Psi(\varrho,c,\varpi)\geq S(\zeta,c,\gamma+\varpi)$,

(21) $\Psi(\zeta,\varrho,\cdot):\mathbb{R}^+\to I_0$ is continuous

(22) $\lim_{\gamma\to\infty}\Psi(\zeta,\varrho,\gamma)=0$

(23) If $\gamma\leq 0$, then $\Theta(\zeta,\varrho,\gamma)=\Pi(\zeta,\varrho,\gamma)=0$, $\Phi(\zeta,\varrho,\gamma)=\Psi(\zeta,\varrho,\gamma)=1$

In this context, $\Theta(\zeta,\varrho,\gamma)$ represents the certainity that distance between ζ and ϱ is less than γ, $\Pi(\zeta,\varrho,\gamma)$ represents the degree of nearness, $\Phi(\zeta,\varrho,\gamma)$ stadns for the degree of neutralness, and $\Psi(\zeta,\varrho,\gamma)$ denotes the degree of non nearness between ζ and ϱ with respect to γ, respectively.

The convergence, Cauchyness, completeness are given as follows.

Definition 1.7. [38] Let (ζ_n) be a sequence in a NFMS $(\mathcal{W},\Theta,\Pi,\Phi,\Psi,\bullet,\diamond)$. Then

(1) (ζ_n) converges to $\zeta\in\mathcal{W}$ if for any $\epsilon\in(0,1)$, $\gamma>0$ there is $n_0\in\mathbb{N}$ such that for each $n\geq n_0$

$$\Theta(\zeta_n,\zeta,\gamma)>1-\epsilon,\ \Pi(\zeta_n,\zeta,\gamma)>1-\epsilon,\ \Phi(\zeta_n,\zeta,\gamma)<\epsilon,\ \Psi(\zeta_n,\zeta,\gamma)<\epsilon$$

i.e.,

$$\lim_{n\to\infty}\Theta(\zeta_n,\zeta,\gamma)=1,\ \lim_{n\to\infty}\Pi(\zeta_n,\zeta,\gamma)=1,\ \lim_{n\to\infty}\Phi(\zeta_n,\zeta,\gamma)=0,\ \lim_{n\to\infty}\Psi(\zeta_n,\zeta,\gamma)=0$$

(2) (ζ_n) is called Cauchy if any $\epsilon\in(0,1)$, $\gamma>0$ there is $n_0\in\mathbb{N}$ such that for each $n,m\geq n_0$

$$\Theta(\zeta_n,\zeta_m,\gamma)>1-\epsilon,\ \Pi(\zeta_n,\zeta_m,\gamma)>1-\epsilon,\ \Phi(\zeta_n,\zeta_m,\gamma)<\epsilon,\ \Psi(\zeta_n,\zeta_m,\gamma)<\epsilon$$

i.e.,

$$\lim_{n.m\to\infty}\Theta(\zeta_n,\zeta_m,\gamma)=1,\ \lim_{n.m\to\infty}\Pi(\zeta_n,\zeta_m,\gamma)=1,\ \lim_{n,m\to\infty}\Phi(\zeta_n,\zeta_m,\gamma)=0,\ \lim_{n,m\to\infty}\Psi(\zeta_n,\zeta_m,\gamma)=0$$

(3) $(\mathcal{W},\Theta,\Pi,\Phi,\Psi,\bullet,\diamond)$ is called complete if every Cauchy sequence in $\mathcal{W}$ is convergent to an element in $\mathcal{W}$.

In 2008, Popescu [39] introduced the innovative concept of P-contraction mappings in the realm of metric spaces. He demonstrated that any P-contraction mapping defined on complete metric spaces has a unique fixed point.

The definition of P-contractions is articulated as follows:

Definition 1.8. [39] Let $(\mathcal{W}, d)$ represent a metric space, and let f be a self-mapping on $\mathcal{W}$. The mapping f is classified as a P-contraction if

$$d(f\zeta, f\xi) \leq k\ \{d(\zeta, \xi) + |d(\zeta, f\zeta) - d(\xi, f\xi)|\ \},$$

holds for all $\zeta, \xi \in \mathcal{W}$.

It is noteworthy that when $k = 1$, the mapping f is referred to as p-contractive, as indicated in [40].

The theorem established by Popescu is articulated as follows.

Theorem 1.9. *[39] If $(\mathcal{W}, d)$ is a complete metric space and $f : \mathcal{W} \to \mathcal{W}$ is a P-contraction, then f possesses a unique fixed point.*

This study investigates fixed point theory within the context of neutrosophic metric spaces, drawing inspiration from previous works in the field, such as those by [41–43]. This paper draws inspiration from the P-contractions concept introduced by Popescu to establish a novel category of contractions, referred to as N-$P\mathcal{L}$-contractions. This new classification employs an auxiliary function within the neutrosophic metric space framework. We prove that these contractions possess a unique fixed point under specific conditions. The main findings of this study enable the formulation of various fixed point theorems.

2. **Main Result**

We commence with the subsequent essential lemmas that are required to derive our principal result.

Lemma 2.1. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a NFMS. Then*

(1) $\Theta(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-decreasing*
(2) $\Pi(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-decreasing*
(3) $\Phi(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-increasing*
(4) $\Psi(\zeta, \varrho, \cdot) : \mathbb{R} \to \mathbb{R}$ *is non-increasing*

Proof. (1) Let $\gamma_1, \gamma_2 > 0$, with $\gamma_1 > \gamma_2$. Then, there is $\delta > 0$ such that $\gamma_1 = \gamma_2 + \delta$.
From (5), we get

$$\begin{aligned}\Theta(\zeta, \varrho, \gamma_1) &= \Theta(\zeta, \varrho, \gamma_2 + \delta)\\ &\geq \Theta(\zeta, \varrho, \gamma_2) \bullet \Theta(\varrho, \varrho, \delta)\\ &= \Theta(\zeta, \varrho, \gamma_2).\end{aligned}$$

The demonstrations for the remaining cases are analogous to that of (1). □

Lemma 2.2. *Let $\mathcal{H} : \mathcal{D} \to \mathcal{B}$ be a bounded real valued function, where $\mathcal{D},\ \mathcal{B}$ are subsets of $\mathbb{R}$. Then*

(1) *If $\mathcal{H}$ is non increasing, then there is $c > 0$ such that for all $\lambda \in \mathcal{D}$*

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \leq c\ \mathcal{H}(\lambda)$$

(2) *If $\mathcal{H}$ is non decreasing, then there is $\tau > 0$ such that for all $\lambda \in \mathcal{D}$*

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \geq \tau\ \mathcal{H}(\lambda)$$

Proof. (1)Let $\lambda \in \mathcal{D}$ be arbitrary. Since $\frac{\lambda}{2} < \lambda$ and $\mathcal{H}$ is non increasing, then

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \geq \mathcal{H}(\lambda).$$

So, there is $\xi_\lambda \geq 0$ such that

$$\mathcal{H}\left(\frac{\lambda}{2}\right) = \mathcal{H}(\lambda) + \xi_\lambda.$$

Thus,

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \leq 2\max\{\mathcal{H}(\lambda), \xi_\lambda\}.$$

Now,

$$\mathcal{H}(\lambda).\xi_\lambda = c_\lambda \Longrightarrow c_\lambda.\mathcal{H}(\lambda) = \xi_\lambda.$$

So,

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \leq 2\max\{\mathcal{H}(\lambda), c_\lambda.\mathcal{H}(\lambda)\}$$

$$\mathcal{H}\left(\frac{\lambda}{2}\right) \leq \max\{2, 2c_\lambda\}\mathcal{H}(\lambda).$$

Let $C = \max\{2, \max_{\lambda\in\mathcal{D}}\{2c_\lambda\}\}$. Then we get the result.
The proof of (2) is identical to that of (1).□

Remark 2.3. According to Lemma 2.1, and Lemma 2.2, there are $C_\Theta, C_\Pi, C_\Phi, C_\Psi$ such that

$$\Theta(\zeta, \varrho, \frac{\lambda}{2}) \geq C_\Theta\ \Theta(\zeta, \varrho, \lambda),$$

$$\Pi(\zeta, \varrho, \frac{\lambda}{2}) \geq C_\Pi\ \Pi(\zeta, \varrho, \lambda),$$

$$\Phi(\zeta, \varrho, \frac{\lambda}{2}) \leq C_\Phi\ \Phi(\zeta, \varrho, \lambda),$$

$$\Psi(\zeta, \varrho, \frac{\lambda}{2}) \leq C_\Psi\ \Psi(\zeta, \varrho, \lambda),$$

Definition 2.4. In this context, we define a real-valued function of three variables that operates over the domain $\mathcal{W}^2 \times (0, \infty)$, where $\mathcal{W}$ is any non-empty set. We refer to this function as $\mathcal{H}$ and claim that it possesses the property (UC) if, for any sequences (ζ_n) and (ϱ_n) contained within $\mathcal{W}$, the following equality holds:

$$\lim_{\lambda \to \lambda_0} \lim_{n \to \infty} \mathcal{H}(\zeta_n, \varrho_n, \lambda) = \lim_{n \to \infty} \lim_{\lambda \to \lambda_0} \mathcal{H}(\zeta_n, \varrho_n, \lambda).$$

Whenever both limits exist.

In the following sections of this study, we will proceed with the premise that each of the fuzzy sets $\Theta,\ \Phi,\ \Psi$ exhibits the UC property.

Through this context we need the following class of functions:

Definition 2.5. Let $\mathscr{L}$ denote the collection of all functions $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the condition

$$\mathcal{L}(a, b) \leq a + b.$$

We will now present the definition of NF-L contractions.

Definition 2.6. Let $(\mathcal{W}, \Theta, \Phi, \Psi, \bullet, \diamond)$ be an NFMS. A mapping $f : \mathcal{W} \to \mathcal{W}$ is called neutrosophic fuzzy p-contraction (shortly NF-L contraction) if there is $0 < k < \frac{1}{3}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, we have

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + \mathcal{L}\left(\frac{1}{\Theta(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Theta(\varrho, f\varrho, \gamma)} - 1 \right) \right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + \mathcal{L}\left(\frac{1}{\Pi(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Pi(\varrho, f\varrho, \gamma)} - 1 \right) \right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k \left[\Phi(\zeta, \varrho, \gamma) + \mathcal{L}\left(\Phi(\zeta, f\zeta, \gamma), \Phi(\varrho, f\varrho, \gamma) \right) \right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k \left[\Psi(\zeta, \varrho, \gamma) + \mathcal{L}\left(\Psi(\zeta, f\zeta, \gamma), \Psi(\varrho, f\varrho, \gamma) \right) \right].$$

Lemma 2.7. *Let $(\mathcal{W}, \Theta, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS, Suppose that $f : \mathcal{W} \to \mathcal{W}$ is NF-L contraction. Consequently, if f has a fixed point then it is unique.*

Proof. Assume that $\mu, \nu \in \mathcal{W}$ such that $f\mu = \mu$ and $f\nu = \nu$. Then, Definition 2.6 implies

$$\frac{1}{\Theta(\mu, \nu, \gamma)} - 1 = \frac{1}{\Theta(f\mu, f\nu, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\mu, \nu, \gamma)} - 1 + \mathcal{L}\left(\frac{1}{\Theta(\mu, \mu, \gamma)} - 1, \frac{1}{\Theta(\nu, \nu, \gamma)} - 1 \right) \right],$$

$$\frac{1}{\Pi(\mu, \nu, \gamma)} - 1 = \frac{1}{\Pi(f\mu, f\nu, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\mu, \nu, \gamma)} - 1 + \mathcal{L}\left(\frac{1}{\Pi(\mu, \mu, \gamma)} - 1, \frac{1}{\Pi(\nu, \nu, \gamma)} - 1 \right) \right],$$

$$\Phi(\mu,\nu,\gamma)=\Phi(f\mu,f\nu,\gamma)\leq k\left[\Phi(\mu,\nu,\gamma)+\mathcal{L}\left(\Phi(\mu,\mu,\gamma),\Phi(\nu,\nu,\gamma)\right)\right],$$

and

$$\Psi(\mu,\nu,\gamma)=\Psi(f\mu,f\nu,\gamma)\leq k\left[\Psi(\mu,\nu,\gamma)+\mathcal{L}\left(\Psi(\mu,\mu,\gamma),\Psi(\nu,\nu,\gamma)\right)\right].$$

So, $\Theta(\mu,\nu,\gamma)=1$, $\Phi(\mu,\nu,\gamma)=0$, and $\Psi(\mu,\nu,\gamma)=0$, and hence $\mu=\nu$. □

Lemma 2.8. *Let $(\mathcal{W},\Theta,\mathcal{B},\Phi,\Psi,\diamond,\bullet)$ be a NFMS, and let (ζ_n) be a sequence such that for $\gamma>0$*

$$\begin{aligned}\Theta(\zeta_p,\zeta_q,\gamma)&\geq\Theta(\zeta_{p-1},\zeta_{q-1},\gamma)\\ \Pi(\zeta_p,\zeta_q,\gamma)&\geq\Pi(\zeta_{p-1},\zeta_{q-1},\gamma)\\ \Phi(\zeta_p,\zeta_q,\gamma)&\leq\Phi(\zeta_{p-1},\zeta_{q-1},\gamma)\\ \Psi(\zeta_p,\zeta_q,\gamma)&\leq\Psi(\zeta_{p-1},\zeta_{q-1},\gamma)\end{aligned}\tag{1}$$

and

$$\begin{aligned}\lim_{n\to\infty}\Theta(\zeta_n,\zeta_{n+1},\gamma)&=1,\\ \lim_{n\to\infty}\Pi(\zeta_n,\zeta_{n+1},\gamma)&=1,\\ \lim_{n\to\infty}\Phi(\zeta_n,\zeta_{n+1},\gamma)&=0,\\ \lim_{n\to\infty}\Psi(\zeta_n,\zeta_{n+1},\gamma)&=0.\end{aligned}\tag{2}$$

If (ζ_n) is not Cauchy, then there exist an $1>\epsilon>0$ and $\gamma>0$ along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) derived from (ζ_n), where (m_k) such that one at least of the following holds.

$$\lim_{k\to\infty}\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)=1-\epsilon,$$

$$\lim_{k\to\infty}\Pi(\zeta_{n_k},\zeta_{m_k},\gamma)=1-\epsilon,$$

$$\lim_{k\to\infty}\Phi(\zeta_{n_k},\zeta_{m_k},\gamma)=\epsilon,$$

$$\lim_{k\to\infty}\Psi(\zeta_{n_k},\zeta_{m_k},\gamma)=\epsilon.$$

Proof. If (ζ_n) is not Cauchy, then for each $\gamma>0$

$$\lim_{n,m\to\infty}\Theta(\zeta_n,\zeta_m,\gamma)\neq1,$$

$$\lim_{n,m\to\infty}\Pi(\zeta_n,\zeta_m,\gamma)\neq1,$$

$$\lim_{n,m\to\infty}\Phi(\zeta_n,\zeta_m,\gamma)\neq0,$$

or

$$\lim_{n,m\to\infty}\Psi(\zeta_n,\zeta_m,\gamma)\neq0.$$

Case 1: If $\lim_{n,m\to\infty} \Theta(\zeta_n, \zeta_m, \gamma) \neq 1$, then there are $\gamma > 0$, and $\epsilon > 0$ along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) derived from (ζ_n), where (m_k) is chosen as the smallest index which satisfying the followiing condition.

$$\Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma) \leq 1 - \epsilon, \quad m_k > n_k > k. \tag{3}$$

This implies that

$$\Theta(\zeta_{n_k}, \zeta_{m_k-1}, \gamma) > 1 - \epsilon. \tag{4}$$

chose $\delta > 0$. Then

$$\begin{aligned} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma + \delta) &\geq \Theta(\zeta_{n_k}, \zeta_{m_k-1}, \gamma) \diamond \Theta(\zeta_{m_k-1}, \zeta_{m_k}, \delta) \\ &> (1 - \epsilon) \diamond \Theta(\zeta_{m_k-1}, \zeta_{m_k}, \delta). \end{aligned}$$

Using Equation 2, we get

$$\liminf_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma + \delta) \geq (1 - \epsilon).$$

Also,

$$\begin{aligned} (1 - \epsilon) &\leq \lim_{\delta\to 0^+} \liminf_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma + \delta) \\ &= \liminf_{k\to\infty} \lim_{\delta\to 0^+} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma + \delta) \\ &= \liminf_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma). \end{aligned}$$

Also, from 3, it follows

$$\limsup_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma) \leq (1 - \epsilon).$$

So, we get

$$\lim_{k\to\infty} \Theta(\zeta_{n_k}, \zeta_{m_k}, \gamma) = (1 - \epsilon).$$

Again, we have

$$\begin{aligned} \Theta(\zeta_{n_k-1}, \zeta_{m_k-1}, \gamma + \delta) &\geq \Theta(\zeta_{n_k-1}, \zeta_{n_k}, \delta) \diamond \Theta(\zeta_{n_k}, \zeta_{m_k-1}, \gamma) \\ &> \Theta(\zeta_{n_k-1}, \zeta_{n_k}, \delta) \diamond (1 - \epsilon). \end{aligned}$$

Using Equation 2, we get $\liminf_{k\to\infty} \Theta(\zeta_{n_k-1}, \zeta_{m_k-1}, \gamma + \delta) \geq (1 - \epsilon)$.
Also,

$$\begin{aligned}(1-\epsilon) &\leq \lim_{\delta\to 0^+}\liminf_{k\to\infty}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma+\delta)\\ &= \liminf_{k\to\infty}\lim_{\delta\to 0^+}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma+\delta)\\ &= \liminf_{k\to\infty}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma).\end{aligned}$$

From Eq 3, we get

$$\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)\leq\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)\leq(1-\epsilon).$$

So,

$$\limsup_{k\to\infty}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)\leq(1-\epsilon).$$

Hence,

$$\lim_{k\to\infty}\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)=(1-\epsilon).$$

The demonstration for the remaining cases is Similar to that of Case (1). □

Theorem 2.9. *Let $(\mathcal{W},\Theta,\Pi,\Phi,\Psi,\bullet,\diamond)$ be a complete NFMS, Suppose that $f:\mathcal{W}\to\mathcal{W}$ is NF-L contraction where $k.C_\Psi<1$. Therefor, the function f has a unique fixed point.*

Proof. Let $\zeta_0\in\mathcal{W}$ denote an arbitrary point. We consider the Picard sequence (ζ_n) defined by the relation $\zeta_{n+1}=f^n(\zeta_0)$ for all $n\geq 0$. According to Definition 2.6, this holds for each $n\in\mathbb{N}$.

$$\frac{1}{\Theta(\zeta_n,\zeta_{n+1},\gamma)}-1\leq k\left[\frac{1}{\Theta(\zeta_{n-1},\zeta_n,\gamma)}-1+\mathcal{L}\left(\frac{1}{\Theta(\zeta_{n-1},\zeta_n,\gamma)}-1,\frac{1}{\Theta(\zeta_n,\zeta_{n+1},\gamma)}-1\right)\right],$$

$$\frac{1}{\Pi(\zeta_n,\zeta_{n+1},\gamma)}-1\leq k\left[\frac{1}{\Pi(\zeta_{n-1},\zeta_n,\gamma)}-1+\mathcal{L}\left(\frac{1}{\Pi(\zeta_{n-1},\zeta_n,\gamma)}-1,\frac{1}{\Pi(\zeta_n,\zeta_{n+1},\gamma)}-1\right)\right],$$

$$\Phi(\zeta_n,\zeta_{n+1},\gamma)\leq k\left[\Phi(\zeta_{n-1},\zeta_n,\gamma)+\mathcal{L}\left(\Phi(\zeta_{n-1},\zeta_n,\gamma),\Phi(\zeta_n,\zeta_{n+1},\gamma)\right)\right],$$

and

$$\Psi(\zeta_n,\zeta_{n+1},\gamma)\leq k\left[\Psi(\zeta_{n-1},\zeta_n,\gamma)+\mathcal{L}\left(\Psi(\zeta_{n-1},\zeta_n,\gamma),\Psi(\zeta_n,\zeta_{n+1},\gamma)\right)\right].$$

From Definition 2.5, we get

$$\frac{1}{\Theta(\zeta_n,\zeta_{n+1},\gamma)}-1\leq\frac{2k}{1-k}\left(\frac{1}{\Theta(\zeta_{n-1},\zeta_n,\gamma)}-1\right),$$

$$\frac{1}{\Pi(\zeta_n,\zeta_{n+1},\gamma)}-1\leq\frac{2k}{1-k}\left(\frac{1}{\Pi(\zeta_{n-1},\zeta_n,\gamma)}-1\right),$$

$$\Phi(\zeta_n,\zeta_{n+1},\gamma)\leq\frac{2k}{1-k}\left(\Phi(\zeta_{n-1},\zeta_n,\gamma)\right),$$

and

$$\Psi(\zeta_n,\zeta_{n+1},\gamma)\leq\frac{2k}{1-k}\left(\Psi(\zeta_{n-1},\zeta_n,\gamma)\right).$$

By proceeding this process, we get

$$\frac{1}{\Theta(\zeta_n,\zeta_{n+1},\gamma)}-1\leq\left(\frac{2k}{1-k}\right)^n\left(\frac{1}{\Theta(\zeta_0,\zeta_1,\gamma)}-1\right),$$

$$\frac{1}{\Pi(\zeta_n,\zeta_{n+1},\gamma)}-1\leq\left(\frac{2k}{1-k}\right)^n\left(\frac{1}{\Pi(\zeta_0,\zeta_1,\gamma)}-1\right),$$

$$\Phi(\zeta_n,\zeta_{n+1},\gamma)\leq\left(\frac{2k}{1-k}\right)^n\left(\Phi(\zeta_0,\zeta_1,\gamma)\right),$$

and

$$\Psi(\zeta_n,\zeta_{n+1},\gamma)\leq\left(\frac{2k}{1-k}\right)^n\left(\Psi(\zeta_0,\zeta_1,\gamma)\right).$$

Given that $k<\frac{1}{3}$, we can determine the limit as follows.

$$\begin{aligned}\lim_{n\to\infty}\Theta(\zeta_n,\zeta_{n+1},\gamma)&=1,\\ \lim_{n\to\infty}\Pi(\zeta_n,\zeta_{n+1},\gamma)&=1,\\ \lim_{n\to\infty}\Phi(\zeta_n,\zeta_{n+1},\gamma)&=0,\\ \lim_{n\to\infty}\Psi(\zeta_n,\zeta_{n+1},\gamma)&=0.\end{aligned}\tag{5}$$

We assert that the sequence (ζ_n) is a Cauchy sequence. Suppose the opposite is true. By Lemma 2.8, there exist $1>\epsilon>0$ and $\gamma>0$, along with two subsequences (ζ_{n_k}) and (ζ_{m_k}) extracted from (ζ_n), where (m_k) is such that at least one of the following conditions is satisfied.

$$\begin{aligned}\lim_{k\to\infty}\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)&=1-\epsilon,\\ \lim_{k\to\infty}\Pi(\zeta_{n_k},\zeta_{m_k},\gamma)&=1-\epsilon,\\ \lim_{k\to\infty}\Phi(\zeta_{n_k},\zeta_{m_k},\gamma)&=\epsilon,\\ \lim_{k\to\infty}\Psi(\zeta_{n_k},\zeta_{m_k},\gamma)&=\epsilon.\end{aligned}$$

Definition 2.6 implies that

$$\frac{1}{\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)}-1\leq k\left[\frac{1}{\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)}-1+\mathcal{L}\left(\frac{1}{\Theta(\zeta_{n_k-1},\zeta_{n_k},\gamma)}-1,\frac{1}{\Theta(\zeta_{m_k-1},\zeta_{m_k},\gamma)}-1\right)\right],$$

$$\frac{1}{\Pi(\zeta_{n_k},\zeta_{m_k},\gamma)}-1\leq k\left[\frac{1}{\Pi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)}-1+\mathcal{L}\left(\frac{1}{\Pi(\zeta_{n_k-1},\zeta_{n_k},\gamma)}-1,\frac{1}{\Pi(\zeta_{m_k-1},\zeta_{m_k},\gamma)}-1\right)\right],$$

$$\Phi(\zeta_{n_k},\zeta_{m_k},\gamma)\leq k\left[\Phi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)+\mathcal{L}\left(\Phi(\zeta_{n_k-1},\zeta_{n_k},\gamma),\Phi(\zeta_{m_k-1},\zeta_{m_k},\gamma)\right)\right],$$

and

$$\Psi(\zeta_{n_k},\zeta_{m_k},\gamma)\leq k\left[\Psi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)+\mathcal{L}\left(\Psi(\zeta_{n_k-1},\zeta_{n_k},\gamma),\Psi(\zeta_{m_k-1},\zeta_{m_k},\gamma)\right)\right].$$

Hence,

$$\frac{1}{\Theta(\zeta_{n_k},\zeta_{m_k},\gamma)}-1\leq k\left[\frac{1}{\Theta(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)}-1+\frac{1}{\Theta(\zeta_{n_k-1},\zeta_{n_k},\gamma)}-1+\frac{1}{\Theta(\zeta_{m_k-1},\zeta_{m_k},\gamma)}-1\right],$$

$$\frac{1}{\Pi(\zeta_{n_k},\zeta_{m_k},\gamma)}-1\leq k\left[\frac{1}{\Pi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)}-1+\frac{1}{\Pi(\zeta_{n_k-1},\zeta_{n_k},\gamma)}-1+\frac{1}{\Pi(\zeta_{m_k-1},\zeta_{m_k},\gamma)}-1\right],$$

$$\Phi(\zeta_{n_k},\zeta_{m_k},\gamma)\leq k\left[\Phi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)+\Phi(\zeta_{n_k-1},\zeta_{n_k},\gamma)+\Phi(\zeta_{m_k-1},\zeta_{m_k},\gamma)\right],$$

and

$$\Psi(\zeta_{n_k},\zeta_{m_k},\gamma)\leq k\left[\Psi(\zeta_{n_k-1},\zeta_{m_k-1},\gamma)+\Psi(\zeta_{n_k-1},\zeta_{n_k},\gamma)+\Psi(\zeta_{m_k-1},\zeta_{m_k},\gamma)\right].$$

By taking the limit when $k\to\infty$, we get

$$\frac{\epsilon}{1-\epsilon}\leq\frac{k\ \epsilon}{1-\epsilon},$$

$$\frac{\epsilon}{1-\epsilon}\leq\frac{k\ \epsilon}{1-\epsilon},$$

$$\epsilon\leq k\ \epsilon,$$

and

$$\epsilon\leq k\ \epsilon.$$

Which is a contradiction because $k<\frac{1}{3}$. Hence (ζ_n) is a Cauchy sequence, thus, there is $\mu\in\mathcal{W}$ such that $\zeta_n\to\mu$.

From Definition 2.6, we have for each $n\in\mathbb{N}$

$$\begin{aligned}\Psi(f\mu,\zeta_{n+1},\gamma)&\leq k\left[\Psi(\mu,\zeta_n,\gamma)+\mathcal{L}\left(\Psi(\mu,f\mu,\gamma),\Psi(\zeta_n,\zeta_{n+1},\gamma)\right)\right]\\&\leq k\left[\Psi(\mu,\zeta_n,\gamma)+\Psi(\mu,f\mu,\gamma)+\Psi(\zeta_n,\zeta_{n+1},\gamma)\right]\end{aligned}$$

By taking the limit, we obtain

$$\lim_{n\to\infty} \Psi(f\mu, \zeta_{n+1}, \gamma) \leq k\ \Psi(\mu, f\mu, \gamma). \tag{6}$$

On the other hand, using Remark 2.3, we get

$$\begin{aligned}\Psi(\mu, f\mu, \gamma) &\leq \Psi(\mu, \zeta_n, \tfrac{\gamma}{2}) \bullet \Psi(\zeta_n, f\mu, \tfrac{\gamma}{2})\\ &\leq \Psi(\mu, \zeta_n, \tfrac{\gamma}{2}) \bullet C_\Psi\ \Psi(\zeta_n, f\mu, \gamma).\end{aligned}$$

So,

$$\Psi(\mu, f\mu, \gamma) \leq C_\Psi \lim_{n\to\infty} \Psi(\zeta_n, f\mu, \gamma). \tag{7}$$

From Eq 6, Eq 7, we get

$$\lim_{n\to\infty} \Psi(\zeta_n, f\mu, \gamma) \leq k\ C_\Psi \lim_{n\to\infty} \Psi(\zeta_n, f\mu, \gamma).$$

So, $\lim_{n\to\infty} \Psi(\zeta_n, f\mu, \gamma) = 0$, and hence (ζ_n) converges to $f\mu$. Hence $\mu = f\mu$. The uniqueness follows from Lemma 2.7 □

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = 0$, we derive the subsequent result.

Corollary 2.10. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1\right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1\right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k \left[\Phi(\zeta, \varrho, \gamma)\right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k \left[\Psi(\zeta, \varrho, \gamma)\right].$$

Therefor, the function f has a unique fixed point.

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = |a - b|$, we derive the subsequent result.

Corollary 2.11. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k\left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + \left|\frac{1}{\Theta(\zeta, f\zeta, \gamma)} - \frac{1}{\Theta(\varrho, f\varrho, \gamma)}\right|\right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k\left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + \left|\frac{1}{\Pi(\zeta, f\zeta, \gamma)} - \frac{1}{\Pi(\varrho, f\varrho, \gamma)}\right|\right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k\left[\Phi(\zeta, \varrho, \gamma) + |\Phi(\zeta, f\zeta, \gamma) - \Phi(\varrho, f\varrho, \gamma)|\right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k\left[\Psi(\zeta, \varrho, \gamma) + |\Psi(\zeta, f\zeta, \gamma) - \Psi(\varrho, f\varrho, \gamma)|\right].$$

Therefor, the function f has a unique fixed point.

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = a + b$, we derive the subsequent result.

Corollary 2.12. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k\left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + \frac{1}{\Theta(\zeta, f\zeta, \gamma)} - 1 + \frac{1}{\Theta(\varrho, f\varrho, \gamma)} - 1\right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k\left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + \frac{1}{\Pi(\zeta, f\zeta, \gamma)} - 1 + \frac{1}{\Pi(\varrho, f\varrho, \gamma)} - 1\right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k\left[\Phi(\zeta, \varrho, \gamma) + \Phi(\zeta, f\zeta, \gamma) + \Phi(\varrho, f\varrho, \gamma)\right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k\left[\Psi(\zeta, \varrho, \gamma) + \Psi(\zeta, f\zeta, \gamma) + \Psi(\varrho, f\varrho, \gamma)\right].$$

Therefor, the function f has a unique fixed point.

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = \max\{a, b\}$, we derive the subsequent result.

Corollary 2.13. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k\left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + \max\left\{\frac{1}{\Theta(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Theta(\varrho, f\varrho, \gamma)} - 1\right\}\right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k\left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + \max\left\{\frac{1}{\Pi(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Pi(\varrho, f\varrho, \gamma)} - 1\right\}\right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k \left[\Phi(\zeta, \varrho, \gamma) + \max\left\{\Phi(\zeta, f\zeta, \gamma), \Phi(\varrho, f\varrho, \gamma)\right\}\right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k \left[\Psi(\zeta, \varrho, \gamma) + \max\left\{\Psi(\zeta, f\zeta, \gamma), \Psi(\varrho, f\varrho, \gamma)\right\}\right].$$

Therefor, the function f has a unique fixed point.

By establishing the function $\mathcal{L} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ with the definition $\mathcal{L}(a, b) = 2\min\{a, b\}$, we derive the subsequent result.

Corollary 2.14. *Let $(\mathcal{W}, \Theta, \Pi, \Phi, \Psi, \bullet, \diamond)$ be a complete NFMS and $f : \mathcal{W} \to \mathcal{W}$ be a self map. Assume that there is $0 < k < \min\{\frac{1}{3}, \frac{1}{C_\Psi}\}$ such that for each $\zeta, \varrho \in \mathcal{W}$ and each $\gamma > 0$, f satisfies the following:*

$$\frac{1}{\Theta(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Theta(\zeta, \varrho, \gamma)} - 1 + 2\min\left\{\frac{1}{\Theta(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Theta(\varrho, f\varrho, \gamma)} - 1\right\}\right],$$

$$\frac{1}{\Pi(f\zeta, f\varrho, \gamma)} - 1 \leq k \left[\frac{1}{\Pi(\zeta, \varrho, \gamma)} - 1 + 2\min\left\{\frac{1}{\Pi(\zeta, f\zeta, \gamma)} - 1, \frac{1}{\Pi(\varrho, f\varrho, \gamma)} - 1\right\}\right],$$

$$\Phi(f\zeta, f\varrho, \gamma) \leq k \left[\Phi(\zeta, \varrho, \gamma) + 2\min\left\{\Phi(\zeta, f\zeta, \gamma), \Phi(\varrho, f\varrho, \gamma)\right\}\right],$$

and

$$\Psi(f\zeta, f\varrho, \gamma) \leq k \left[\Psi(\zeta, \varrho, \gamma) + 2\min\left\{\Psi(\zeta, f\zeta, \gamma), \Psi(\varrho, f\varrho, \gamma)\right\}\right].$$

Therefor, the function f has a unique fixed point.

Conclusion

Fixed-point theory broadly encompasses a range of theorems that address the behavior of transformations applied to points within a given set, guaranteeing the existence of at least one invariant point. These theorems play a pivotal role in establishing the existence of solutions to various equations and systems across multiple branches of mathematics. A notable example is Banach's Fixed Point Theorem, a cornerstone in the field of analysis, which asserts that any contraction mapping from a complete metric space to itself possesses a unique fixed point. Such theorems are essential in diverse areas, including differential equations, economics, and computer science, as they facilitate the identification of equilibria and solutions. In summary, fixed-point theorems serve as vital instruments in both theoretical and applied mathematics, offering foundational knowledge and effective strategies for addressing intricate problems by confirming the existence and, in some cases, the uniqueness of solutions.

In this study, we introduced neutrosophic P$\mathcal{L}$-contractions within the framework of neutrosophic fuzzy metric spaces, demonstrating that these contractions yield a unique fixed point.

This primary finding paves the way for numerous subsequent results. Future research may aim to extend our findings to other types of distance spaces or enhance them by incorporating additional appropriate conditions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Comp.* **1965**, *8*, 338—353.
2. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96.
3. Smarandache, F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. *Int. J. Pure Appl. Math.* **2005**, *24*, 287–297.
4. Barbosa, R.P.; Smarandache, F. Neutrosophic One-Round Zero-Knowledge Proof. *Plithogenic Logic Comput.* **2024**, *2*, 49–54.
5. Abd Elwahed, H.; Alburaikan, A.; Smarandache, F. On characterizing efficient and properly efficient solutions for multi-objective programming problems in a complex space. *J. Optim. Ind. Eng.* **2024**, *16(2)*, 369–375.
6. Hazaymeh, A. Time Fuzzy Soft Sets and its application in design-making. *Int. J. Neutrosophic Sci.* **2025**, *3*, 37–50.
7. Wajid, M.S.; Terashima-Marin, H.; Wajid, M.A.; Smarandache, F.; Verma, S.B.; Wajid, M.K. Neutrosophic Logic to Navigate Uncertainty of Security Events in Mexico. *Neutrosophic Sets Syst.* **2024**, *73*, 120–130.
8. Ashika, T.; Grace, H.; Martin, N.; Smarandache, F. Enhanced Neutrosophic Set and Machine Learning Approach for Breast Cancer Prediction. *Neutrosophic Sets Syst.* **2024**, *73*, 206–217.
9. Hazaymeh, A. Time Factor's Impact On Fuzzy Soft Expert Sets. *Int. J. Neutrosophic Sci.* **2025**, *3*, 155–176.
10. Angammal, S.; Martin, N.; Smarandache, F. Multi Attribute Neutrosophic Optimization Technique for Optimal Crop Selection in Ariyalur District. *Neutrosophic Sets Syst.* **2024**, *73*, 153–168.
11. Salem, A.; Mohamed, M.; Smarandache, F. IM4.0EF: Tele-Medical Realization via Integrating Vague T2NSs with OWCM-RAM Toward Intelligent Medical 4.0 Evaluator Framework. *Sustainable Machine Intell. J.* **2024**, *9*, 79–88.
12. Hazaymeh, A. Time Effective Fuzzy Soft Set and Its Some Applications with and Without a Neutrosophic. *Int. J. Neutrosophic Sci.* **2024**, *2*, 129–129.
13. Hazaymeh, A. (2024). Time-Shadow Soft Set: Concepts and Applications. The International Journal of Fuzzy Logic and Intelligent Systems ,24(4),387-398
14. Alqaraleh, S.M.; Alkouri, A.U.M.; Massa'deh, M.O.; Talafha, A.G.; Bataihah, A. Bipolar Complex Fuzzy Soft Sets and Their Application. *Int. J. Fuzzy Syst. Appl.* **2022**, *11(1)*, 1–23.
15. Al-Qudah, Y. A robust framework for the decision-making based on single-valued neutrosophic fuzzy soft expert setting. *Int. J. Neutrosophic Sci.* **2024**, *23(2)*, 195–210. DOI: 10.54216/IJNS.230216.
16. Al-Qudah, Y.O.; Abdalla, A.; Al-Sharqi, F. A new generalization of interval-valued Q-neutrosophic soft matrix and its applications. *Int. J. Neutrosophic Sci.* **2025**, –, 242–257. DOI: https://doi.org/10.54216/IJNS.250322.
17. Al-Sharqi, F.; Al-Qudah, Y.; Alotaibi, N. Decision-making techniques based on similarity measures of possibility neutrosophic soft expert sets. *Neutrosophic Sets Syst.* **2023**, *55*, 358–382. DOI: 10.5281/zenodo.7832770.
18. Al-Qudah, Y.; Al-Sharqi, F. Algorithm for decision-making based on similarity measures of possibility interval-valued neutrosophic soft setting settings. *Int. J. Neutrosophic Sci.* **2023**, *22(3)*, 69–83. DOI: 10.54216/IJNS.220305.

19. Banach, S. Sur les opératios dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.* **1922**, *3*, 133-181.
20. Bataihah, A.; Qawasmeh, T.; Batiha, I. M.; Batiha, I. H.; Abdeljawad, T. Gamma Distance Mappings with Application to Fractional Boundary Differential Equation. *J. Math. Anal.* **2024**, *15*, 5, 99–106. https://doi.org/10.54379/jma-2024-5-7.
21. Bataihah, A.; Qawasmeh, T. A new type of distance spaces and fixed point results. *J. Math. Anal.* **2024**, *15*(4), 81–90.
22. Bataihah, A. Fixed point results of Geraghty type contractions with equivalent distance. *Int. J. Neutrosophic Sci.* **2025**, *25*(3), 177–186. https://doi.org/10.54216/IJNS.250316.
23. Bataihah, A. Some fixed point results with application to fractional differential equation via new type of distance spaces. *Results Nonlinear Anal.* **2024**, *7*, 202–208.
24. Aydi, H.; Karapinar, E.; Postolache, M. Tripled coincidence point theorems for weak Φ-contractions in partially ordered metric spaces. *Fixed Point Theory Appl.* **2012**, 44. https://doi.org/10.1186/1687-1812-2012-44.
25. Bataihah, A.; Qawasmeh, T.; Shatnawi, M. Discussion on b-metric spaces and related results in metric and G-metric spaces. *Nonlinear Functional Anal. Appl.* **2022**, *27*, 2, 233–247.
26. Nazam, M.; Arshad, M.; Postolache, M. Coincidence and common fixed point theorems for four mappings satisfying (α_s, F)-contraction. *Nonlinear Anal. Modelling and Control* **2018**, *23*(5), 664–690.
27. Qawasmeh, T.; Shatanawi, W.; Bataihah, A.; Tallafha, A. Fixed point results and (α, β)-triangular admissibility in the frame of complete extended b-metric spaces and application. *U.P.B. Sci. Bull., Series A* **2021**, *83*(1), 113-124.
28. Shatanawi, W.; Qawasmeh, T.; Bataihah, A.; Tallafha, A. New contractions and some fixed point results with application based on extended quasi b-metric spaces. *U.P.B. Sci. Bull., Series A* **2021**, *83*(2), 1223-7027.
29. Abu-Irwaq, I.; Nuseir, I.; Bataihah, A. Common fixed point theorems in G-metric spaces with Ω-distance. *J. Math. Anal.* **2017**, *8*(1), 120–129.
30. Abodayeh, K.; Bataihah, A.; Shatanawi, W. Generalized Ω-distance mappings and some fixed point theorems. *U.P.B. Sci. Bull. Series A* **2017**, *79*, 223–232.
31. Abu-Irwaq, I.; Shatanawi, W.; Bataihah, A.; Nuseir, I. Fixed point results for nonlinear contractions with generalized Ω-distance mappings. *U.P.B. Sci. Bull. Series A* **2019**, *81*(1), 57-64.
32. Shatanawi, W.; Bataihah, A. Remarks on G-Metric Spaces and Related Fixed Point Theorems. *Thai J. Math.* **2021**, *19*(2), 445–455. https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/1168.
33. Khan, M. S.; Singh, Y. M.; Maniu, G.; Postolache, M. On (α, p)-convex contraction and asymptotic regularity. *J. Math. Comput. Sci.* **2018**, *18*, 132–145.
34. Shatanawi, W.; Maniu, G.; Bataihah, A.; Ahmad, F. B. Common fixed points for mappings of cyclic form satisfying linear contractive conditions with Omega-distance. *UPB Sci. Bull. Series A* **2017**, *79*, 11–20.
35. Menger, K. Statistical Metrics. *Proc. Natl. Acad. Sci. USA* **1942**, *28*(12), 535–537.
36. Kirişci, M.; Şimşek, N. Neutrosophic metric spaces. *Mathematical Sciences* **2020**, *14*(3), 241–248.
37. Şimşek, N.; Kirişci, M. Fixed point theorems in Neutrosophic metric spaces. *Infinite Study* **2019**.
38. Ghosh, S.; Sonam, Bhardwaj, R.; Narayan, S. On Neutrosophic Fuzzy Metric Space and Its Topological Properties. *Symmetry* **2024**, *16*(5), 613.
39. Popescu, O. A new type of contractive mappings in complete metric spaces. *Bull. Transilv. Univ. Brasov, Ser. III, Math. Inform. Phys.* **2008**, *1*(50), 479–482.
40. Altun, I.; Durmaz, G.; Olgun, M. P-contractive mappings on metric spaces. *J. Nonlinear Functional Anal.* **2018**.

41. Bataihah, A.; Ayman, A. Neutrosophic fuzzy metric spaces and fixed points results with integral contraction type. *Int. J. Neutrosophic Sci.* **2025**, *25*(4), 561–572. https://doi.org/10.54216/IJNS.250344.
42. Hazaymeh, A.; Bataihah, A. Results on fixed points in neutrosophic metric spaces through the use of simulation functions. *J. Math. Anal.* **2024**, *15*(6).
43. Bataihah, A.; Hazaymeh, A. Quasi Contractions and Fixed Point Theorems in the Context of Neutrosophic Fuzzy Metric Spaces. *Eur. J. Pure Appl. Math.* **2025**, *18*(1), 5785–5785.

Received: Oct 7, 2024. Accepted: March 1, 2025