Neutrosophic Wald Distribution with Applications to Reliability Analysis

Afrah Al Bossly[1] and Adnan Amin[2]*

[1] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj, 11942, Saudi Arabia

[2] School of Economics, Beijing technology and business university, Beijing, China

[1] a.basli@psau.edu.sa
[2] amin.adnan12@yahoo.com

* Correspondence: amin.adnan12@yahoo.com

Abstract: In this article, we develop neutrosophic extension of the Wald (Inverse Gaussian) distribution to present more realistic modelling for real data by introducing uncertainty in its parameters. We derive fundamental statistical properties such as the probability density function (PDF), cumulative distribution function (CDF) and quantile function, and compare it with the classical model. This comparison shows the versatility and great robustness of the neutrosophic model against the imprecise data. Considering that the Wald distribution plays a significant role in the theory of reliability, we extend some key reliability functions into a neutrosophic framework. Under neutrosophic uncertainty, we derive and study the survival function, the reliability function and the hazard function which results in a more generalized and pragmatic approach for modeling reliability. These functions provide an improved decision-making process for situations in which classical models are unable to capture the inbuilt uncertainties of systems. To make it even more applicable, we propose an approach to generate random samples from neutrosophic Wald distribution using quantile function so that neutrosophic Wald distribution can be simulated and empirically validated. In addition, we also develop an estimation procedure through the method of moments (mom), which shows a simple way of estimating the parameters.

Keywords: Neutrosophic distribution; neutrosophic probability; estimation; simulation

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1. Introduction

Statistical distributions are commonly used tools for modeling, analyzing, and predicting real-world phenomena across various domains, but especially in reliability engineering and survival analysis [1]. Reliability theory describes the time-to-failure behavior of systems, components, and machinery through statistical distributions, allowing engineers to assess product lifespan, failure rates, maintenance schedules, and product reliability (i.e. estimates of the probability of survival within a certain time) [2]. The exponential, Weibull, gamma, and inverse Gaussian (Wald) distributions are commonly used distributions in reliability analysis, each suitable for different failure patterns [3]. The exponential distribution, for example, represents fixed failure rates, and the Weibull, with its parameters, is more flexible to capture failure rates that rise or fall as time progresses [4]. Likewise, the inverse Gaussian (Wald) distribution has been applied extensively to a variety of first-passage time problems, so it is useful for modeling system lifetimes when they are exposed to random shocks [5]. Statistical distributions are pivotal in survival analysis for modeling lifespan data in the fields of medical and biological studies [6]. The Kaplan-Meier estimator, Cox proportional hazards model, and parametric distributions (i.e. log-normal, gamma and inverse Gaussian distributions) are well established methods to estimate survival probabilities and hazard rates [7]. These distributions are used to derive the hazard function, which describes the probability of an event occurring at a specific point in time, playing a key role in medical decision-making, risk assessment, and insurance studies [8]. Additionally, statistical distributions are utilized for quality control, warranty predictions, and risk analysis, all of which are necessary to ensure the product is designed optimally and is inexpensive to maintain over its lifetime in industrial applications. Statistical distributions are widely used in reliability and survival analysis due to their significance in providing some quantitative insights, predicting future outcomes, and helping in optimization of some of the decision-making processes [9]. The Wald distribution is one of the very importance distributions in statistical applications [10]. The Wald distribution (also known as the Inverse Gaussian distribution) is a basic probability model in statistics particularly in first-passage time problems and in reliability problems and provides better model results for stochastic processes [11]. It refers to the amount of time until a certain Brownian motion with positive drift hits a given threshold level and hence is well-suited for use in sequential analysis, financial growth approximations, and similar fields [12]. For similar reasons, the Wald distribution is of interest to reliability engineering, when we wish to model the lifetime of systems and components that are subject to random wear and degradation with time. Commonly used in biostatistics and in the survival analysis, it is used to model waiting or survival time, namely a situation when early failures as well as long-term survival probabilities vary greatly. Two of the most used characteristics of the survival functions are the location and shape parameters. In econometrics and finance, it is used to model log stock prices, default times in credit risk analysis, time-to-event processes, etc. In decision processes, the Wald distribution is instrumental; specifically, in sequential probability ratio tests (SPRT), it aids in optimal stopping rules based on cumulative evidence over time.

Although classical probability theory is the most widely used method for modeling uncertainty and is based on well-defined probability measures, it requires crisp, well-defined and complete information about events. But in a large class of real-word problems, the uncertainties are not purely stochastic but also linguistic, ambiguous, or imprecise, which makes classical probability models inadequate [13]. This also highlights one of the main limitations of classical probability theory: that it does not deal well with subjectivity, uncertainty or vagueness at the level of decision-making, medical diagnosis, financial forecasting, and risk assessment [14]. In contrast, traditional probability models require an rigorously defined sample space and velvet probabilities that are often impossible to have in complex systems where uncertainty arises from human perception, incomplete data, or simply lack of precision measuring devices [15]. For instance, in medical prognosis, determining a patient's recovery period may not always be expressed in terms of crisp probabilities since it relies on many uncertain factors such as lifestyle choices, genetic compositions, treatment efficacy, etc. To overcome these limitations, the use of fuzzy set theory, fuzzy probability theory, and fuzzy distributions have been proposed to deal with situations with uncertain, vague, or natural language type uncertainties [16]. In fuzzy probability theory, classical probability is generalized to where membership functions specify the likelihood or degree of belief concerning an event. They basically represent probability values as fuzzy numbers rather than precise values; therefore, they are appropriate for problems in which uncertainty is not purely random but also imprecise or subjective. This paradigm is central to artificial intelligence, machine learning, decision support systems, climate modeling and human-centered processes where data is often incomplete, unclear or inconsistent [17]. Through fuzzy possibility decision making, prediction, delay, and uncertainty analysis, fuzzy probability can expand the application of fuzzy mathematics and help solve problems in many fields.

Although fuzzy set theory is ineffective in reconstructing higher levels of vagueness, conflict, and undetermined data [18]. Fuzzy logic attaches a membership in the range of 0 to 1, yet it does not distinguish between lack of information and contradictory data. Fuzzy set theory and fuzzy logic cannot completely model uncertainty that results in the real- world problems, especially in the decision, in the medical diagnosis, and artificial intelligence where uncertainty originates from partial, inconsistent, and contradictory information [19]. Aiming to solve these problems, neutrosophic logic was proposed by Smarandache, where each of the entities inside an uncertain statement, truth (T), indeterminacy (I) and falsity (F), is treated as a query that works independently from the others, and each of these entities takes values in the [0,1] domain [20-23]. Neutrosophic logic, as compared to fuzzy logic which can only accommodate partial truth, explicitly models uncertainty and contradiction [24]. This is particularly helpful in the areas of decision making in very complex situations, machine learning, medical diagnosis, and disparate uncertain data analysis, where uncertainty and counter-intuitive data pair [25-27]. Since classical probability fails to take into consideration real-world uncertainty, neutrosophic probability and neutrosophic distributions have emerged as important and applicable areas in many engineering and scientific fields. Neutrosophic theory is also helpful to model uncertainty because it adds indeterminacy [28].

In this work, we present the neutrosophic version of the Wald model for better modelling the real-life problems. Neutrosophic extension of Wald distribution provides additional power in dealing with ambiguous and incomplete information, rendering the models more suitable for practical scenarios.

The work is structured as follows. In section 2 the statistical model of the proposed model with uncertainty parameters are presented. Section 3 data generating model using the quantile function is discussed. Section 4 provides estimation procedure of the proposed model. Finally, Section 5 concludes the major findings of the work.

2. Proposed Model

In this section, we derive and present the statistical properties of the Wald distribution with essential functions used in the reliability analysis. The proposed model, as the classical one, is a continuous probability distribution on the positive real axis, whose parameters, through the use of the neutrosophic logic, can take the values of indeterminate and uncertain events. By assuming the imprecision in the distributional parameters, proposed model can be defined as:

$$f_N(t;\mu_N,\lambda_N)=\sqrt{\frac{\lambda_N}{2\pi t^3}}\exp\left(-\frac{\lambda_N(t-\mu_N)^2}{2\mu_N^2 t}\right),\ t,\lambda_N,\mu_N>0 \quad (1)$$

where $\mu_N=[\mu_l,\mu_u]$ and $\lambda_N=[\lambda_l,\lambda_u]$ are vague parameters of the proposed model.

The proposed model generalizes the traditional Wald distribution, as the mean and shape parameters are crisp values in the traditional Wald distribution, while in the neutrosophic Wald distribution, they can be imprecise, vague, indeterminate, or interval-based, leading to a broader application in real-world where exact values are not available. It finds application especially in reliability analysis, survival studies, and stochastic modeling when uncertainty originates from measurement errors, human judgment, or random environmental factors. For certain values of μ_N and λ_N the PDF of the proposed distribution is shown in Figure 1.

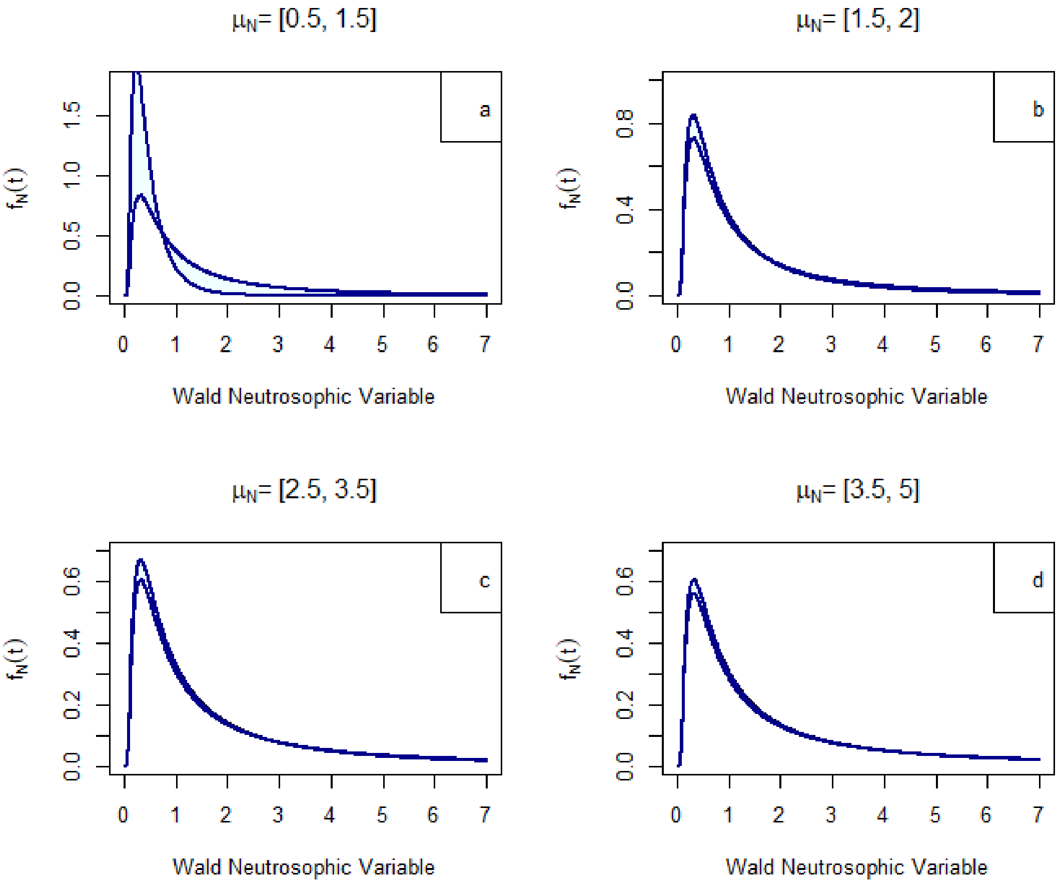

Figure 1 The PDF of the proposed distribution with imprecise location parameter values

The PDF of neutrosophic Wald distribution for different values of μ_N is displayed in Figure 1, and explains the behaviour of the distribution under indeterminacy. Each panel in the Figure 1 corresponds to a different interval of the parameter μ_N, and encompasses a range of density curves rather than a single deterministic function. The shaded areas between the two curves in both panels show the uncertainty in the PDF as a consequence of the neutrosophic representation for a, meaning that the actual value of the parameter is unknown (it may take any value within a defined interval).The distribution in the first panel μ_N=[0.5, 1.5] is highly right-skewed, it has a higher peak and a slower decay, which indicates that higher probability density points towards to further left towards of the Wald neutrosophic variable. The initial left panel is consistent with a very exaggerated deformation of periodicity as the mean interval shifts upward, and it serves as a representation of complete degradation towards random noise with a steadily increasing scandals in the mean interval as μ_N =[1.5,2], μ_N=[2.5,3.5], μ_N=[3.5,5] panels, as the peak moves to the right and the peak appears flat, showing an increasing dispersion and lower skewness.

Similarly, the CDF of the proposed model is given by:

$$F_N(t; \mu_N, \lambda_N) = \Phi\left(\sqrt{\frac{\lambda_N}{t}}\left(\frac{t}{\mu_N} - 1\right)\right) + \exp\left(\frac{2\lambda_N}{\mu_N}\right)\Phi\left(-\sqrt{\frac{\lambda_N}{t}}\left(\frac{t}{\mu_N} + 1\right)\right) \quad (2)$$

By assuming different values of μ_N and λ_N the CDF curves of the proposed model is shown in Figure 2.

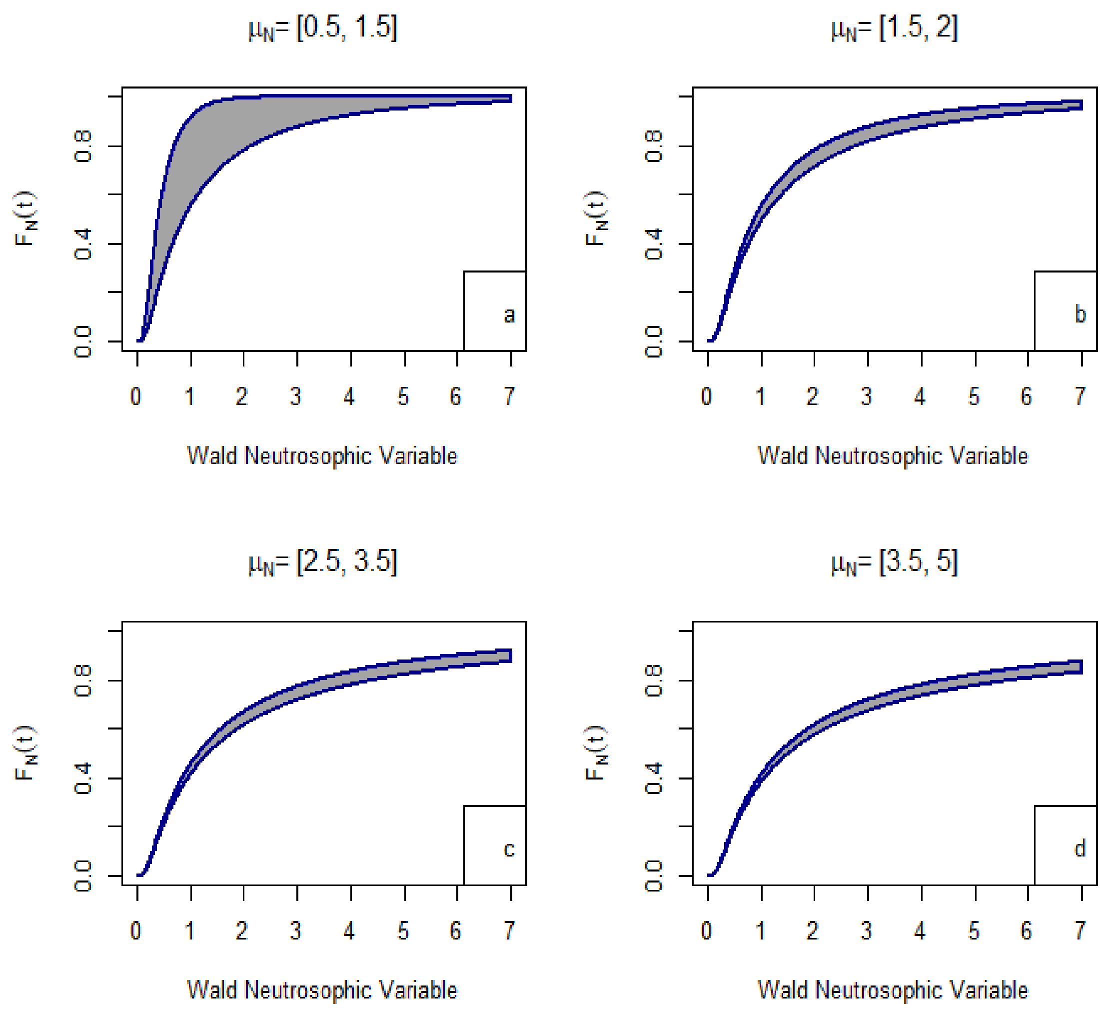

Figure 2 The CDF plot of the proposed model with different values of location parameter

The effect of uncertainty in shape parameter on the cumulative probability of the Wald neutrosophic variable is shown in Figure 2, which depicts the cumulative distribution function (CDF) of Neutrosophic Wald distribution. This means that each subpanel shows its own intervals of the parameter μ_N , describing the possible CDF curves instead of a single deterministic function. In each panel, the shaded areas representing the space between the two curves illustrate the area of uncertainty or indeterminacy associated with the neutrosophic parameter, for which the true value of μ_N is not known exactly, but for which the value belongs to the interval of the range. As we can see in the first panel (μ_N =[0.5,1.5]), the CDF increases more steeply than the previous case, indicating that the probability of finding smaller values of the Wald variable is higher since it is strongly right-skewed. In the other rows, which represent the same CDFs but for larger values of a in the mean parameter (μ_N =[1.5,2], μ_N =[2.5,3.5], and μ_N = [3.5,5]), we notice that as the value of a increases across a, the CDF curves shift to the rightmost parts of the plot, meaning that the distribution becomes wider, and larger values of the Wald variable become more likely. As μ_N becomes μ_N larger, the shaded regions become the wider, representing greater uncertainty regarding cumulative probabilities. Figure 2 illustrates how neutrosophic uncertainty on the shape parameter modifies the cumulative probability shape, allowing the Neutrosophic Wald distribution to be very flexible to different types of uncertain, imprecise, and incomplete information.

A program in R is written that use the structure of "*neutrostat* " package [29] to draw the PDF and CDF values of the proposed model. The PDF and CDF values at specified values with μ_N =[1, 2] and λ_N =[0.5,1] are given in Table 1.

Table 1 PDF and CDF values of the proposed model using neutrostat R package

Z values	PDF values	CDF values
0.5	[0.643,0.967]	[0.232, 0.249]
1	[0.352, 0.564]	[0.490, 0.627]
1.5	[0.212, 0.259]	[0.627, 0.824]
2	[0.120, 0.141]	[0.713, 0.915]
2.5	[0.058,0.099]	[0.773, 0.957]
3	[0.028, 0.073]	[0.815, 0.978]
3.5	[0.014, 0.056]	[0.848, 0.988]
4	[0.007, 0.044]	[0.873, 0.994]

Results in Table 1 show at each value of neutrosophic random variable Z we have interval value of PDF and CDF. This is because of indeterminacy that we have assumed in the neutrosophic distributional parameters.

The survival function is another important function of the proposed model. For uncertain parameters, survival function of the proposed model is given by:

$$S_N(t; \mu_N, \lambda_N) = 1 - F_N(t; \mu_N, \lambda_N) \quad (3)$$

The survival function indicates the probability of subject or system survival past time. Mathematically T is a random variable representing the time to occurrence of an event of interest (e.g., failure, death, or any type of endpoint of interest). In reliability study, clinical studies, and actuarial science, the survival function provides important information about the longevity or durability of a system or population. It is a non-increasing function. For some specific values of μ_N and λ_N,survival function of the proposed model is shown in Figure 3.

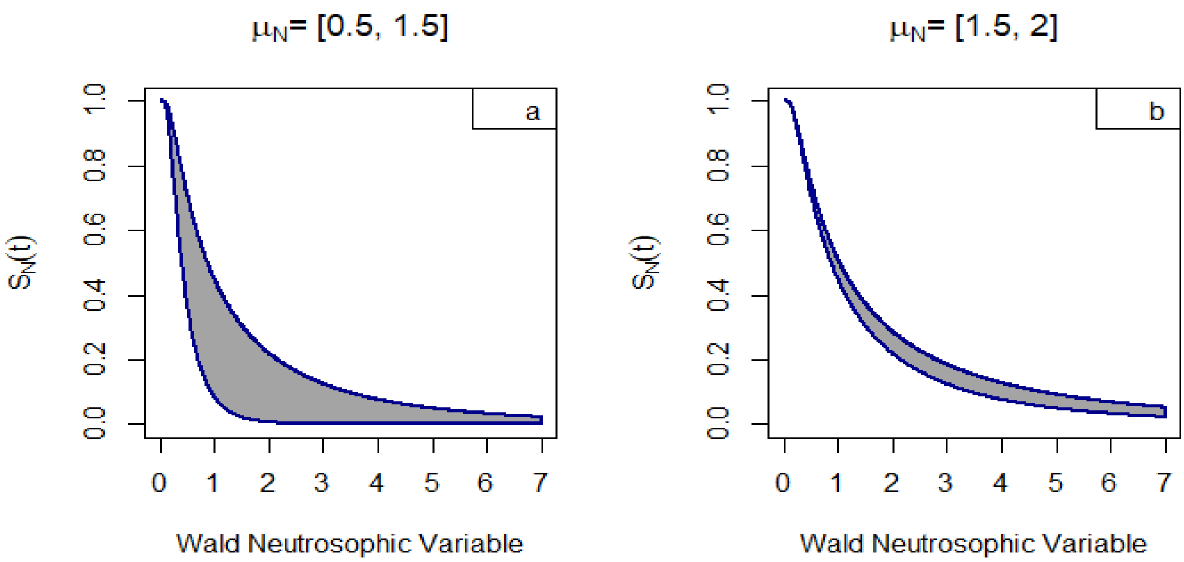

Figure 3 Survival function of the proposed model with imprecise parameter values.

Interval values of the neutrosophic mean parameters of the survival function of the Neutrosophic Wald Distribution are shown in this Figure 3. The survival function $S_N(t; \mu_N, \lambda_N)$ corresponds to the probability that the Wald neutrosophic variable T > t. It plots the survival functions corresponding to two edge values of the neutrosophic location parameter, namely μ_N=0.5 and μ_N=1.5 (left panel). The area in gray symbolizes the uncertainty in the neutrosophic extension, as each realization would result in a possible survival probability ranging from a minimum to a maximum value. In the right panel on the left is plotted survival function for the interval μ_N=1.5 and μ_N=2. Survival probabilities decrease as t increases, and the shaded area once more emphasizes the fluctuations of possible values fueled by neutrosophic uncertainty. The results show how the uncertainty of the location parameter impacts the survival probabilities, and confirming the proposed distribution's ability to model a survival problem where the values of the parameters are not exact. Neutrosophic interval effect is visually depicted by the upper and lower boundaries of the shaded region.
The other function related to survival is hazard function which is defined below:

$$H_N(t; \mu_N, \lambda_N) = \frac{f_N(t;\mu_N,\lambda_N)}{S_N(t;\mu_N,\lambda_N)} \quad (4)$$

The exact expression of Eq (4) is quite complicated, how visual depiction of hazard function is given in Figure 4.

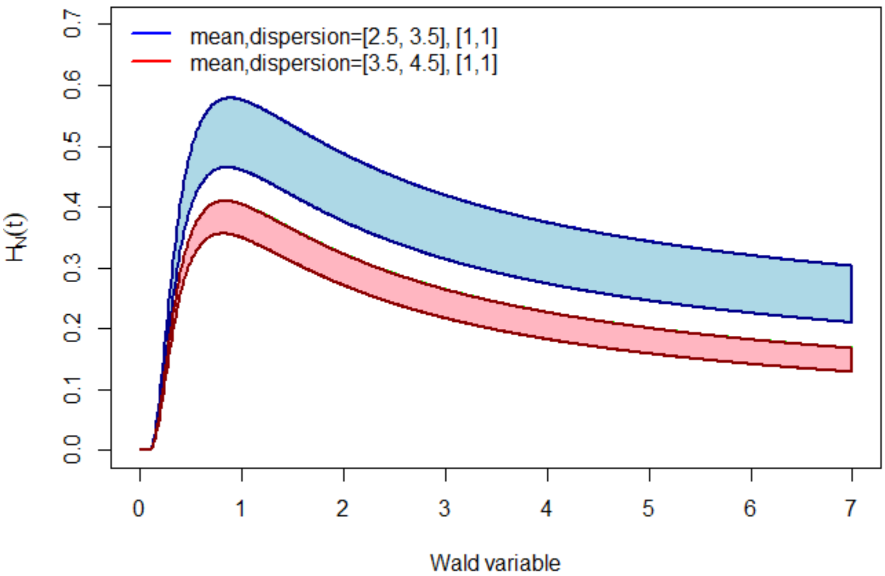

Figure 4 Hazard function of the proposed model with different inexact distribution parameters

In survival analysis and reliability studies, the hazard function describes the instantaneous failure rate of a subject or system at a given time, subject to prior survival. It gives insights on how failure of components changes over its life and is widely used in reliability modeling, reliability evaluation, medicine, engineering applications, etc.
The cumulative hazard function for the proposed model can be derived by :

$$H_N(t; \mu_N, \lambda_N) = -\ln S_N\,(t; \mu_N, \lambda_N) \quad (5)$$

The $H_N(t; \mu_N, \lambda_N)$ is has an important role in survival and reliability data modelling. It provides a measure of the expected number of failures occurring up to a specific time in survival and reliability analysis. One of the primary reliability metrics that must be computed is the Mean Time to Failure (MTTF), which represents the average failure time for a system or component that cannot be repaired. The average time to failure (MTTF) is the average time that the failure occurs or is equal to the mean of the failure time distribution, which is often broadly applied in engineering, manufacturing, and survival analysis to estimate product life, calculate optimal maintenance approaches, increase system reliability, and reduce incidence of surprises.

The MTTF for the proposed model is defined by:

$$\text{MTTF}_N = E[T] = \int_0^\infty t\, f_N(t; \mu_N, \lambda_N)\, dt$$

$E[T] = \int_0^\infty t \sqrt{\frac{\lambda_N}{2\pi t^3}} \exp\left(-\frac{\lambda_N (t-\mu_N)^2}{2\mu_N^2 t}\right) dt$ (6)

Solving (6) further yielded:

$$E[T] = \mu_N$$

The mean residual life (MRL) function can also derived as:

$\text{MRL}_N(t; \mu_N, \lambda_N) = \frac{\int_t^\infty S_N(u; \mu_N, \lambda_N) du}{S_N(t; \mu_N, \lambda_N)}$ (7)

The MRL function characterizes the expected remaining functional lifetime of an item (system) given that it has survived up to the instant t. It is used in reliability analysis, survival studies, and maintenance planning. Ultimately, MRL assists in assessing longevity, predictions on failures, and optimization of replacement strategies, both for engineering and health purposes.

The variance of the model can be derived as follows:

$\text{Var}(T) = E[T^2] - (E[T])^2$ (8)

Now

$E[T^2] = \int_0^\infty t^2 f_N(t; \mu_N, \lambda_N)\ dt$ (9)

$E[T^2] = \mu_N^2 + \frac{\mu_N^3}{\lambda_N}$ (10)

$E[T] = \mu_N$ (11)

Using Eq (10) and Eq (11) in Eq (9), we get:

$\text{Var}(T) = \frac{\mu_N^3}{\lambda_N}$ (12)

Likewise, mode of the distribution can be obtained as:

$\frac{d}{dT} f_N(T; \mu_N, \lambda_N) = 0$ (13)

$$f_N(T) = \sqrt{\frac{\lambda_N}{2\pi T^3}} \exp\left(-\frac{\lambda_N (T-\mu_N)^2}{2\mu_N^2 T}\right)$$

Solving for T, mode is given by:

$$\frac{d}{dt} f_N(t) = f_N(t) \left[\frac{3}{2t} - \frac{\lambda_N}{\mu_N^2}\left(1 - \frac{\mu_N}{t}\right)\right]$$

$$\frac{3}{2t} - \frac{\lambda_N}{\mu_N^2}\left(1 - \frac{\mu_N}{t}\right) = 0$$

Solving for T, mode is given by:

$\text{Mode}(T) = \mu_N \left(\sqrt{1 + \frac{9\mu_N}{4\lambda_N}} - \frac{3\mu_N}{2\lambda_N}\right)$ (14)

The skewness can be determined by solving the expression:

$\gamma_1 = \frac{E[(T - E[T])^3]}{\text{Var}(T)^{3/2}}$ (15)

$$\text{Var}(T) = \frac{\mu_N^3}{\lambda_N}$$

and $E[T] = \mu_N$

Using these expressions in Eq (15), yielded:

$$\gamma_{1N} = \frac{\frac{3\mu_N^5}{\lambda_N}}{\left(\frac{\mu_N^3}{\lambda_N}\right)^{3/2}}$$

$\gamma_{1N} = \frac{3\mu_N}{\sqrt{\lambda_N}}$ (16)

We can derive the expression for kurtosis coefficient as:

$$\gamma_2 = \frac{E[(T-E[T])^4]}{\mathrm{Var}(T)^2} \tag{17}$$

Eq (17) further can be simplified as:

$$E[(T-\mu_N)^4] = \frac{15\mu_N^7}{\lambda_N^3}$$

$$\gamma_{2N} = \frac{\frac{15\mu_N^7}{\lambda_N^3}}{\left(\frac{\mu_N^3}{\lambda_N}\right)^2}$$

$$\gamma_{2N} = \frac{15\mu_N^7}{\lambda_N^3} \times \frac{\lambda_N^2}{\mu_N^6}$$

$$\gamma_{2N} = \frac{15\mu_N}{\lambda_N} \tag{18}$$

Skewness and kurtosis coefficients explain probability distribution shape. Skewness tells us about the asymmetry of a distribution a positive number means the distribution is right-skewed (has a longer right tail), and a negative number means that it is left-skewed. On other hand kurtosis refers to the tails of a distribution; high kurtosis means a lot of extreme outliers, low kurtosis means a flatter distribution.

3. Random Sample Generation

The inverse cumulative function method, also referred to as the inverse transform sampling technique [30], is the basic technique for generating random samples from a given probability distribution. Uniform random numbers is then generated according to the standard uniform distribution (a simple approach is that R use runif(.), or Python use random.uniform(.) etc). Using the inverse of the cumulative distribution function (CDF) of the desired distribution, these uniform values are then transformed. The CDF gives us the probability that a random variable will be less than or equal to a value at a given point. The inverse function is then applied to map the uniform values based on values of the target distribution. This technique is popular, since, provided that appropriate sampling techniques are followed, the produced samples will adhere exactly to the theoretical distribution without any error of approximation. This is valid for several distributions. The inverse cumulative function method is especially beneficial in Monte Carlo simulations, statistical modeling, and stochastic processes, where random sample generation is critical for decision-making and predictive analysis.
The inverse cumulative function or quantile function of the proposed model is defined as:

$$F_N(t;\mu_N,\lambda_N) = \Phi\left(\sqrt{\frac{\lambda_N}{t}}\left(\frac{t}{\mu_N}-1\right)\right) + \exp\left(\frac{2\lambda_N}{\mu_N}\right)\Phi\left(-\sqrt{\frac{\lambda_N}{t}}\left(\frac{t}{\mu_N}+1\right)\right)$$

Now quantile function is obtained by solving the expression:

$$p = F_N(t;\mu_N,\lambda_N)$$

$$Q_N(p) = \frac{\mu_N}{2\lambda_N}\left(2\lambda_N\Phi^{-1}(p) + \sqrt{4\lambda_N^2\left(\Phi^{-1}(p)\right)^2 + 4\mu_N\lambda_N}\right) \tag{19}$$

To generate random samples from the neutrosophic Wald distribution, we follow the simulation procedure described below. The neutrosophic parameters (e.g., mean and shape factor) become involved by defining them as interval or fuzzy values (in other words, uncertainty is taken into account). We assume that $\mu_N = [0.5, 2]$ and $\lambda_N = [1,1]$. Then, uniform random numbers from a uniform distribution are generated to use as transformation inputs. The output is directive adjustment based on the accepted neutrosophic likelihood parameters and will return neutrosophic Wald distributed samples. In cases when the parameters are represented by intervals, lower and upper bounds are computed separately, yielding an interval-valued dataset. This gives an opportunity to model on uncertain and imprecise data and versatility in reliability analysis, survival studies and uncertain decision-making environments. A program in R is used to generate random samples from the proposed model. At specific seed setting first forty random samples values using Eq (19) are given in Table 2.

Table 2 Random samples generated from the proposed with imprecise parameter values

Random Data				
[1.072, 1.893]	[0.954, 1.603]	[1.346, 1.939]	[1.87, 2.462]	[1.284, 1.462]
[1.186, 1.958]	[0.619, 1.377]	[1.296, 1.976]	[0.851, 0.865]	[1.563, 1.564]
[1.769, 2.199]	[1.722, 1.949]	[1.146, 1.945]	[0.957, 1.224]	[1.087, 1.762]
[0.684, 1.367]	[1.362, 1.487]	[1.125, 1.352]	[1.353, 2.287]	[1.504, 2.388]
[0.731, 1.748]	[1.135, 1.33]	[1.001, 2.233]	[1.466, 2.471]	[1.539, 1.726]
[0.741, 2.153]	[1.59, 1.706]	[1.108, 1.186]	[1.334, 1.863]	[1.521, 2.251]
[1.672, 1.839]	[1.75, 1.849]	[0.666, 2.185]	[0.987, 2.027]	[0.888, 1.645]
[0.691, 2.271]	[1.754, 1.947]	[0.532, 1.689]	[0.818, 1.614]	[0.806, 1.891]

Table 2 shows the interval random sample values generated from the proposed model with parameter setting $\mu_N = [1, 2]$ and $\lambda_N = [0.5,1]$. Each interval value indicates imprecise because of imprecise neutrosophic parameters.

4 Estimation Method

Estimation methods are statistical modeling techniques used to estimate the unknown parameters of a probability distribution based on observed neutrosophic data. Some common estimation methods in literature include the maximum likelihood estimation (MLE), method of moments (mom), and least squared method (LSM). Despite its asymptotic efficiency, which makes MLE popular, computing it involves solving complicated optimization problems (especially for non-standard distributions). However sometime, it is difficult to find the closed form likelihood expressions. In contrast, the method of moments is simpler and more intuitive, as it works by equating theoretical based moments to the sample based moments to estimate parameters. The inverse Wald distribution can be more easily fitted using the method of moments than by any of the other methods. This is due to the complex likelihood function of the inverse Wald distribution, which involves relatively computationally intensive MLE process that could even sometimes require iterative numerical methods. The method of moments, in contrast, features simple estimates where parameters can be obtained directly from sample-based properties (mean and variance). It is not subject to complex optimization routines and allows for faster estimation, making it especially useful for practical applications, especially when memory is limited or when the datasets are large.

Based on MoM approach the equating moments can be achieved as follows:

$$\widehat{\mu_N} = \bar{T} \quad (19)$$

$$\widehat{\lambda_N} = \frac{\overline{T^3}}{S^2} \quad (20)$$

If we assume the random samples generated from the proposed model with $\mu_N = [0.5, 2]$ and $\lambda_N = [1,1]$ then estimated values with other statistical characteristics are shown in Table 3.

Table 3 Estimated values of proposed model using moments method

Characteristics	Estimated values
$\widehat{\mu_N}$	[1.817, 1.187]
$\widehat{\lambda_N}$	[11.98, 42.63]
Mean	[1.187, 1.817]
Median	[1.038,1.750]
Variance	[0.139, 0.141]
Mode	[1.191,1.817]

Table 3 shows the estimated values of the proposed model using simulated data given in Table 2. Results indicate that due to imprecision in the parameters $\mu_N = [1, 2]$ and $\lambda_N = [0.5,1]$, each estimated quantity involves uncertainty. Moreover, if we assume that indeterminacy is zero then results of the classical model match with the neutrosophic version of the Wald distribution.

5 Conclusions

In this work, we proposed the neutrosophic Wald distribution which broadens the applicability of this model in practical situations by allowing uncertainty of its parameters. Utilizing this extension we also obtained key statistical properties such as the probability density function, cumulative distribution function and quantile function, which shows how flexible and robust the model is in comparison with the classical Wald distribution. The proposed addressed in our study provides a pragmatic and significant tool for fitting data with fuzzy and vague characteristics in the count of failures and analysis of reliability. Specifically, we examined some elementary reliability metrics like survival function, reliability function, and hazard function in the neutrosophic framework, thus extending definite reliability functions into a neutrosophic context. In addition, we presented a quantile-driven simulation method for the generation of random samples from the proposed model for the purposes of empirical evidence. We also proposed the moments estimation procedure that is simple and computationally inexpensive.

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Conflicts of Interest: The authors declare no conflict of interest.

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