Sombor Index in Neutrosophic Graphs

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Abstract: One of the most fundamental ideas in chemical graphs is topological indices. Numerous applications of the topological index, a wholly pictorial notion, may be found in the domains of nanotechnology, chemistry, materials science, medicines, and other things. Neutrosophic graphs are three-valued logic-based graphs. Although in some circumstances it is challenging to optimize and model such utilizing fuzzy graphs, they may be regarded as a fuzzy graph. In this study, the kinds of Sombor indices on neutrosophic graphs are first defined. A degree-based indicator called the Sombor index was developed by Gutman. The formula for this index is $SO(G) = \sum_{uv\in E(G)} \sqrt{deg(u)^2 + deg(v)^2}$. The following triad is the definition of the Sombor index for neutrosophic graphs.

$$So(G) = \left(So_T(G), So_I(G), So_F(G)\right), \qquad |So(G)| = \frac{4 + 2So_T(G) - So_I(G) - 2So_F(G)}{6}.$$

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1. Introduction

According to graph theory, the graph is an ordered pair $G = (V, E)$, such that V is the set of vertices and E is a relation on V when called edges. Here V and E both accept values belonging to the set of natural numbers $(\mathbb{N})$, that is, $|V| = n$, and $|E| = m$ such that $n, m \in \mathbb{N}$. In 1965, Zadeh [1] revolutionized modern mathematics by defining fuzzy sets. This development was able to create a new and practical branch of graph theory. Accordingly, m and n do not necessarily belong to natural numbers $(\mathbb{N})$, they can derive their values from real numbers $(\mathbb{R})$. Thus fuzzy graphs were born and human beings have witnessed many applications of this theory in recent years. Finally, in 1995, new collections called neutrosophic collections were identified and introduced by Smarandach [16]. These collections quickly spread to other areas of mathematical research. Those interested in graph theory, like other researchers, generalized the concept of neutrosophy in graph theory and thus led to the birth of a new subset of graphs called neutrosophic graphs. Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational. It is the base of neutrosophic logic, a multiple-value logic that generalizes fuzzy logic and deals with paradoxes, contradictions, antitheses, and antinomies [3, 21-23].

The neutrosophic graph is an ordered pair $G = (V, E)$, such that V is the set of vertices with membership function N, when N is a neutrosophic function, and E is edges set, that defined with neutrosophic function M as triad (T, I, F). For further study on the neutrosophic and neutrosophic sets of graphs, see References [2,3,24-27].
Topological indices are quantities that, by attributing a numerical value to a graph or a network according to its properties, help us greatly in analyzing, inferring, and comparing different graphic structures. These indices are divided into two general types indices based on degree and indices based on distance. Each of these types has its own applications. The Sombor index is a degree-based index introduced by Gutman in 2020 [4]. After the introduction of this index, many researchers tried to introduce and use this index and in a short time, several articles were presented about this index. Here are some articles published in this field for further reading by those who are interested [5-15].
In short, it can be said that a topological index is a real number that is attributed to a molecular graph and is stable with respect to the uniformity of the graph. They are used as a tool to determine the chemical and physical properties of molecules. Also, currently, these indicators are widely used in networks. The Wiener index, abbreviated as W, is the first topological index used in chemistry. This index was introduced by Harold Wiener in 1947 to show the relationship between the physical and chemical properties of organic compounds and their topological structure.
After the introduction of fuzzy graphs, the generalization of topological indices on this type of graph and the investigation of its applications were very interesting for mathematical and computer science researchers. In recent years, very beautiful and practical results have been obtained from this research, so readers can refer to references [28-31] to learn more about a corner of this valuable research.
In previous articles, we have thoroughly examined the connectivity index [17-19], and the Wiener index [20] in neutrosophic graphs, and in this paper, we define the Sombor index on neutrosophic graphs.

2. Preliminaries

In the section provides some definitions and theorems.

Definition 1. [3] Let $G = (N, M)$ is an single-valued Neutrosophic graph, where N is a Neutrosophic set on V and, M is a Neutrosophic set on E, which satisfy the following

$$T_M(u,v) \leq min\big(T_N(u), T_N(v)\big),$$
$$I_M(u,v) \geq max\big(I_N(u), I_N(v)\big),$$
$$F_M(u,v) \geq max\big(F_N(u), F_N(v)\big),$$

Where u and v are two vertices of G, and $(u,v) \in E$ is an edge of G.

Definition 2. [3] Given $G = (N, M)$ be a single-valued neutrosophic graph and P is a path in G. P is a collection of different vertices, $v_0, v_1, v_2, \dots, v_n$ such that $\big(T_M(v_{i-1}, v_i), I_M(v_{i-1}, v_i), F_M(v_{i-1}, v_i)\big) > 0$ for $0 \leq i \leq n$. P is a neutrosophic cycle if $v_0 = v_n$ and $n \geq 3$.

Definition 3. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, and $v \in V$ is vertex of G. the degree of the vertex v is the sum of the truth membership values, the sum of the indeterminacy membership values, and the sum of the falsity membership values of all the edges that are adjacent to vertex v, and is denoted by $d(v)$, that

$$d(v) = \big(d_T(v), d_I(v), d_F(v)\big) = \left(\sum_{\substack{v\in V\\ v\neq u}} T_M(v,u), \sum_{\substack{v\in V\\ v\neq u}} I_M(v,u), \sum_{\substack{v\in V\\ v\neq u}} F_M(v,u)\right).$$

$$Td(v) = \left(Td_T(v), Td_I(v), Td_F(v)\right) = \left(\sum_{\substack{v\in V\\ v\neq u}} T_M(v,u) + T_N(v), \sum_{\substack{v\in V\\ v\neq u}} I_M(v,u) + I_N(v), \sum_{\substack{v\in V\\ v\neq u}} F_M(v,u) + F_N(v)\right).$$

Definition 4. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, and the d_m–degree of any vertex v in G is denoted as $d_m(v)$ where

$$d_m(v) = \left(\sum_{u\neq v\in V} T_M^m(u,v), \sum_{u\neq v\in V} I_M^m(u,v), \sum_{u\neq v\in V} F_M^m(u,v)\right)$$

Here, the path $v = v_0, v_1, v_2, \dots, v_n = u$ is the shortest path between the vertices v and u, when the length of this path is m.

Definition 5. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, G is called a regular neutrosophic graph if any vertex has the same degree, that is, for all u in $V(G)$ we have $d_G(u) = (d_1, d_2, d_3)$.

Definition 6. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, G is called a totally regular neutrosophic graph if any vertex has the same degree, that is, for all u in $V(G)$ we have $Td_G(u) = (d_1, d_2, d_3)$.

Definition 7. [3] Given $G = (N, M)$ is a single-valued neutrosophic graph, G is a complement regular neutrosophic graph if it satisfies the following,

$$\sum_{v\neq u} T_M(v,u) = c, \qquad \sum_{v\neq u} I_M(v,u) = c, \qquad \sum_{v\neq u} F_M(v,u) = c,$$

Where c is a constant value.

Definition 8. [3] A neutrosophic graph $G = (N, M)$ is called a *complete neutrosophic graph* if the following conditions are satisfied:

$$T_M(uv) = \min\{T_N(u), T_N(v)\},$$
$$I_M(uv) = \min\{I_N(u), I_N(v)\},$$
$$F_M(uv) = \max\{F_N(u), F_N(v)\},$$

For all $u, v \in V$.

3. Mane results

This section provides some definitions and theorems.

Definition 9. let $G = (N, M)$ be a neutrosophic graph. the Sombor index (SO) of the graph G is defined by,

$$So_T(G) = \frac{1}{2} \sum_{uv\in E(G)} (T_N^2(u) d_{2_T}(u) + T_N^2(v) d_{2_T}(v))^{\frac{1}{2}},$$
$$So_I(G) = \frac{1}{2} \sum_{uv\in E(G)} (I_N^2(u) d_{2_I}(u) + I_N^2(v) d_{2_I}(v))^{\frac{1}{2}},$$
$$So_F(G) = \frac{1}{2} \sum_{uv\in E(G)} (F_N^2(u) d_{2_F}(u) + F_N^2(v) d_{2_F}(v))^{\frac{1}{2}}.$$

$$So(G) = \big(So_T(G), So_I(G), So_F(G)\big), \qquad |So(G)| = \frac{4 + 2So_T(G) - So_I(G) - 2So_F(G)}{6}.$$

Too, we have:

$$d_2(u) = \left(d_{2_T}(u), d_{2_I}(u), d_{2_F}(u)\right) = \left(\sum_{u\neq v\in V} T_M^2(u,v)\,, \sum_{u\neq v\in V} I_M^2(u,v)\,, \sum_{u\neq v\in V} F_M^2(u,v)\right).$$

Definition 10. Sombor index whit the totally degree of the graph G is defined as:

$$So_T(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)Td_{2_T}(u) + T_N^2(v)Td_{2_T}(v))^{\frac{1}{2}},$$
$$So_I(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (I_N^2(u)Td_{2_I}(u) + I_N^2(v)Td_{2_I}(v))^{\frac{1}{2}},$$
$$So_F(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (F_N^2(u)Td_{2_F}(u) + F_N^2(v)Td_{2_F}(v))^{\frac{1}{2}}.$$
$$So(G_T) = \big(So_T(G_T), So_I(G_T), So_F(G_T)\big), \qquad |So(G_T)| = \frac{4 + 2So_T(G_T) - So_I(G_T) - 2So_F(G_T)}{6}.$$

Too, we have:

$$Td_2(u) = \left(Td_{2_T}(u), Td_{2_I}(u), Td_{2_F}(u)\right) = \left(\sum_{u\neq v\in V} T_M^2(u,v) + T_N^2(u)\,, \sum_{u\neq v\in V} I_M^2(u,v) + I_N^2(u)\,, \sum_{u\neq v\in V} F_M^2(u,v) + F_N^2(u)\right).$$

Definition 11. The Reduced Sombor index of the graph G is defined as:

$$So_{RedT}(G) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)d_{2Red_T}(u) + T_N^2(v)d_{2Red_T}(v))^{\frac{1}{2}},$$
$$So_{RedI}(G) = \frac{1}{2}\sum_{uv\in E(G)} (I_N^2(u)d_{2Red_I}(u) + I_N^2(v)d_{2Red_I}(v))^{\frac{1}{2}},$$
$$So_{RedF}(G) = \frac{1}{2}\sum_{uv\in E(G)} (F_N^2(u)d_{2Red_F}(u) + F_N^2(v)d_{2Red_F}(v))^{\frac{1}{2}}.$$

$$So_{Red}(G) = \big(So_{RedT}(G), So_{RedI}(G), So_{RedF}(G)\big),$$

$$|So_{Red}(G)| = \frac{4 + 2So_{RedT}(G) - So_{RedI}(G) - 2So_{RedF}(G)}{6}.$$

Too, we have:

$$d_{2Red}(u) = \left(d_{2Red_T}(u), d_{2Red_I}(u), d_{2Red_F}(u)\right) = \left(\left|\sum_{u\neq v\in V} T_M^2(u,v) - T_N^2(u)\right|, \left|\sum_{u\neq v\in V} I_M^2(u,v) - I_N^2(u)\right|, \left|\sum_{u\neq v\in V} F_M^2(u,v) - F_N^2(u)\right|\right).$$

Definition 12. The Average Sombor index of the graph G is defined as:

$$So_{AvgT}(G) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)d_{2Avg_T}(u) + T_N^2(v)d_{2Avg_T}(v))^{\frac{1}{2}},$$
$$So_{AvgI}(G) = \frac{1}{2}\sum_{uv\in E(G)} (I_N^2(u)d_{2Avg_I}(u) + I_N^2(v)d_{2Avg_I}(v))^{\frac{1}{2}},$$

$$So_{AvgF}(G) = \frac{1}{2}\sum_{uv\in E(G)} (F_N^2(u)d_{2Avg_F}(u) + F_N^2(v)d_{2Avg_F}(v))^{\frac{1}{2}}.$$

$$So_{Avg}(G) = \left(So_{AvgT}(G), So_{AvgI}(G), So_{AvgF}(G)\right),$$

$$\left|So_{Avg}(G)\right| = \frac{4 + 2So_{AvgT}(G) - So_{AvgI}(G) - 2So_{AvgF}(G)}{6}.$$

Too, we have:

$$d_{2Avg}(u) = \left(d_{2Avg_T}(u), d_{2Avg_I}(u), d_{2Avg_F}(u)\right)$$
$$= \left(\left|\sum_{u\neq v\in V} T_M^2(u,v) - \frac{2\sum_{uv\epsilon E(G)} T_M(uv)}{\sum_{u\epsilon V(G)} T_N(u)}\right|, \left|\sum_{u\neq v\in V} I_M^2(u,v) - \frac{2\sum_{uv\epsilon E(G)} I_M(uv)}{\sum_{u\epsilon V(G)} I_N(u)}\right|, \left|\sum_{u\neq v\in V} F_M^2(u,v) - \frac{2\sum_{uv\epsilon E(G)} F_M(uv)}{\sum_{u\epsilon V(G)} F_N(u)}\right|\right).$$

Example 1. Let $G = (N, M)$ be the neutrosophic graph with $V = \{a, b, c, d, e\}$ where $(T_N, I_N, F_N)(u_1) = (0.1, 0.2, 0.5)$, $(T_N, I_N, F_N)(u_2) = (0.2, 0.7, 0.3)$, $(T_N, I_N, F_N)(u_3) = (0.4, 0.3, 0.7)$, $(T_N, I_N, F_N)(u_4) = (0.5, 0.2, 0.4)$, and $(T_N, I_N, F_N)(u_5) = (0.4, 0.5, 0.3)$, The edge set contains $(T_M, I_M, F_M)(u_1, u_2) = (0.1, 0.2, 0.5)$, $(T_M, I_M, F_M)(u_3, u_2) = (0.1, 0.2, 0.6)$, $(T_M, I_M, F_M)(u_3, u_4) = (02, 0.3, 0.8)$, and $(T_M, I_M, F_M)(u_5, u_2) = (02, 0.5, 0.3)$. (Figure 1)

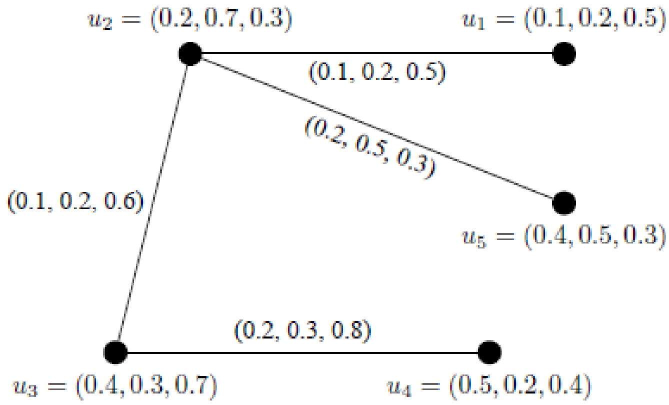

Fig1. $G = (N, M)$ be the neutrosophic graph with $V = \{a, b, c, d, e\}$.

By direct calculations,

$$d(u_1) = (0.1, 0.2, 0.5),$$
$$d(u_2) = (0.1 + 0.2 + 0.1, 0.2 + 0.2 + 0.5, 0.5 + 0.3 + 0.6) = (0.4, 0.9, 1.4),$$
$$d(u_3) = (0.1 + 0.2, 0.2 + 0.3, 0.6 + 0.8) = (0.3, 0.5, 1.4),$$
$$d(u_4) = (0.2, 0.3, 0.8),$$
$$d(u_5) = (0.2, 0.5, 0.3).$$

By definition of $d_m(v)$, we have

$$d_2(v) = \left(\sum_{u\neq v\in V} T_M^2(u,v), \sum_{u\neq v\in V} I_M^2(u,v), \sum_{u\neq v\in V} F_M^2(u,v)\right)$$

Therefore

$$d_2(u_1) = (0.01, 0.04, 0.05), \quad d_2(u_2) = (0.06, 0.33, 0.5), \quad d_2(u_3) = (0.05, 0.13, 1),$$
$$d_2(u_4) = (0.04, 0.09, 0.64), \quad d_2(u_5) = (0.04, 0.25, 0.09).$$

Also, for $u_i u_j \in E(G)$,

$$So_T(G) = \frac{1}{2} \sum_{uv \in E(G)} (T_N^2(u) d_{2_T}(u) + T_N^2(v) d_{2_T}(v))^{\frac{1}{2}} = 0.148,$$
$$So_I(G) = \frac{1}{2} \sum_{uv \in E(G)} (I_N^2(u) d_{2_I}(u) + I_N^2(v) d_{2_I}(v))^{\frac{1}{2}} = 0.771,$$
$$So_F(G) = \frac{1}{2} \sum_{uv \in E(G)} (F_N^2(u) d_{2_F}(u) + F_N^2(v) d_{2_F}(v))^{\frac{1}{2}} = 0.987,$$
$$So(G) = \big(So_T(G), So_I(G), So_F(G)\big) = (0.148, 0.771, 0.987).$$

$$|So(G)| = \frac{4 + 2So_T(G) - So_I(G) - 2So_F(G)}{6} = \frac{4 + 2(0.148) - 0.771 - 2(0.987)}{6} = 0.2585.$$

Theorem 1: let G be the Neutrosophic Graph, and H is the Neutrosophic subgraph of G such that $H = G - u$ then

$$SO(H) < SO(G).$$

Proof: Given that by omitting a vertex of G, a positive value, the sum is lost, so the proof is obvious. □

Theorem 2. Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$. Then all of the following equations are established.

$$So_T(G_1) = So_T(G_2),$$
$$So_I(G_1) = So_I(G_2),$$
$$So_F(G_1) = So_F(G_2),$$

Also, we have $So(G_1) = So(G_2)$.

Proof. Let $G_1 = (N_1, M_1)$ be isomorphic with $G_2 = (N_2, M_2)$, and $f: V_1 \to V_2$ be the bijection from V_1 to V_2 such that

$$T_{N_1}(u) = T_{N_2}\big(f(u)\big), \quad I_{N_1}(u) = I_{N_2}\big(f(u)\big), \quad F_{N_1}(u) = F_{N_2}\big(f(u)\big),$$

For all $u \in V_1$, and

$$T_{M_1}(uv) = T_{M_2}\big(f(u)f(v)\big), \quad I_{M_1}(uv) = I_{M_2}\big(f(u)f(v)\big), \quad F_{M_1}(uv) = F_{M_2}\big(f(u)f(v)\big),$$

For all $uv \in E_1$. Since G_1 isomorphic with G_2, any edge between two vertices, for example, u and v in G_1 are equal to that between $f(u)$ and $f(v)$ in G_2. Hence

$$d_{2T_{G_1}}(u) = d_{2T_{G_2}}\big(f(u)\big), \quad d_{2I_{G_1}}(u) = d_{2I_{G_2}}\big(f(u)\big), \quad d_{2F_{G_1}}(u) = d_{2F_{G_2}}(f(u)),$$

For $u, v \in N_1^*$. Therefore

$$So_T(G_1) = So_T(G_2), \quad So_I(G_1) = So_I(G_2), \quad So_F(G_1) = So_F(G_2),$$

And

$$So(G_1) = \frac{4 + 2So_T(G_1) - 2So_F(G_1) - So_I(G_1)}{6} = \frac{4 + 2So_T(G_2) - 2So_F(G_2) - So_I(G_2)}{6} = So(G_2).$$

□

Theorem 3. Let $G=(N,M)$ be a complete neutrosophic graph whit $V=\{v_1,v_2,\dots,v_n\}$ such that $t_1\leq t_2\leq\cdots\leq t_n$, $i_1\leq i_2\leq\cdots\leq i_n$ and $f_1\geq f_2\geq\cdots\geq f_n$ where $t_j=T_N(v_j)$, $i_j=I_N(v_j)$ and $f_j=F_N(v_j)$ for $j=1,2,\dots,n$. Also, G is the regular neutrosophic graph of the second rank (that is, for each vertex u in $V(G), d_{2G}(v_j)=(d_1,d_2,d_3)$). Then

$$So_T(G)=\frac{\sqrt{d_1}}{2}\sum_{\substack{1\leq j<k\\ 2\leq k\leq n}}(t_j^2+t_k^2)^{\frac{1}{2}},$$

$$So_I(G)=\frac{\sqrt{d_2}}{2}\sum_{\substack{1\leq j<k\\ 2\leq k\leq n}}(i_j^2+i_k^2)^{\frac{1}{2}},$$

$$So_F(G)=\frac{\sqrt{d_3}}{2}\sum_{\substack{1\leq j<k\\ 2\leq k\leq n}}(f_j^2+f_k^2)^{\frac{1}{2}}.$$

Too, we have for u in $V(G)$:

$$d_2(v_j)=\left(d_{2_T}(v_j),d_{2_I}(v_j),d_{2_F}(v_j)\right)=(d_1,d_2,d_3).$$

Proof: Suppose G is a complete neutrosophic graph, we have

$$So_T(G)=\frac{1}{2}\sum_{uv\in E(G)}(T_N^2(u)d_{2_T}(u)+T_N^2(v)d_{2_T}(v))^{\frac{1}{2}}=\frac{1}{2}\sum_{uv\in E(G)}(T_N^2(u)d_1+T_N^2(v)d_1)^{\frac{1}{2}}$$
$$=\frac{1}{2}\sum_{uv\in E(G)}\sqrt{d_1}\big(T_N^2(u)+T_N^2(v)\big)^{\frac{1}{2}}=\frac{\sqrt{d_1}}{2}\sum_{uv\in E(G)}\big(T_N^2(u)+T_N^2(v)\big)^{\frac{1}{2}}$$
$$=\frac{\sqrt{d_1}}{2}\Bigg(\sqrt{t_1^2+t_2^2}+\sqrt{t_1^2+t_3^2}+\cdots+\sqrt{t_1^2+t_n^2}+\sqrt{t_2^2+t_3^2}+\cdots+\sqrt{t_2^2+t_n^2}+\cdots$$
$$+\sqrt{t_{n-2}^2+t_{n-1}^2}+\sqrt{t_{n-2}^2+t_n^2}+\sqrt{t_{n-1}^2+t_n^2}\Bigg)=\frac{\sqrt{d_1}}{2}\sum_{\substack{1\leq j<k\\ 2\leq k\leq n}}(t_j^2+t_k^2)^{\frac{1}{2}}.$$

Using the same argument, we can prove the other two cases.

□

Theorem 4. Let $G=(N,\ M)$ be a complete neutrosophic graph whit $|V|=n$. Then,

If M is a constant function, that is, $M(uv)=(t_m,\ i_m,\ f_m), for\ uv\ \in E(G)$. Now, we have:

$$So_T(G)=\frac{\sqrt{n-1}}{2}t_m\sum_{uv\in E(G)}(T_N^2(u)+T_N^2(v))^{\frac{1}{2}},$$

$$So_I(G)=\frac{\sqrt{n-1}}{2}i_m\sum_{uv\in E(G)}(I_N^2(u)+I_N^2(v))^{\frac{1}{2}},$$

$$So_F(G)=\frac{\sqrt{n-1}}{2}f_m\sum_{uv\in E(G)}(F_N^2(u)+F_N^2(v))^{\frac{1}{2}}.$$

If $M\ and\ N$ are the constant function, that is, $M(uv)=(t_m,\ i_m,\ f_m), for\ uv\in E(G),\ and\ N(u)=(t_n,i_n,f_n), for\ u\in V(G)$. Now, we have:

$$So_T(G)=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}t_m.t_n,$$

$$So_I(G)=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}i_m.i_n,$$

$$So_F(G)=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}f_m.f_n.$$

Also,

$$|So(G)|=\frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}}(2t_m.t_n-i_m.i_n-2f_m.f_n)+\frac{2}{3}.$$

If $M\,and\,N$ are the constant and the same function, that is, $M(uv)=(t,i,f), for\ uv\in E(G)$, $and\ N(u)=(t,i,f), for\ u\in V(G)$. Now, we have:

$$So_T(G)=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}t^2,$$
$$So_I(G)=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}i^2,$$
$$So_F(G)=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}f^2.$$

Also,

$$|So(G)|=\frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}}(2t^2-i^2-2f^2)+\frac{2}{3}.$$

Proof: In the first case, suppose G is a complete neutrosophic graph, with $|V|=n$. Also, $uv\in E(G)$, we have $M(uv)=(t_m,\ i_m,\ f_m)$. Therefore,

$$d_2(u)=\left(d_{2_T}(u),d_{2_I}(u),d_{2_F}(u)\right)=\left(\sum_{u\neq v\in V}T_M^2(u,v),\sum_{u\neq v\in V}I_M^2(u,v),\sum_{u\neq v\in V}F_M^2(u,v)\right)$$
$$=\left((n-1)t_m^2,(n-1)i_m^2,(n-1)f_m^2\right).$$

Now, by placing $d_2(u)$ in relation to the Sombor index we get:

$$So_T(G)=\frac{1}{2}\sum_{uv\in E(G)}(T_N^2(u)d_{2_T}(u)+T_N^2(v)d_{2_T}(v))^{\frac{1}{2}}=\frac{1}{2}\sum_{uv\in E(G)}(T_N^2(u)(n-1)t_m^2+T_N^2(v)(n-1)t_m^2)^{\frac{1}{2}}$$
$$=\frac{1}{2}\sum_{uv\in E(G)}((n-1)t_m^2)^{\frac{1}{2}}(T_N^2(u)+T_N^2(v))^{\frac{1}{2}}=\frac{1}{2}((n-1)t_m^2)^{\frac{1}{2}}\sum_{uv\in E(G)}(T_N^2(u)+T_N^2(v))^{\frac{1}{2}}$$
$$=\frac{\sqrt{n-1}}{2}t_m\sum_{uv\in E(G)}(T_N^2(u)+T_N^2(v))^{\frac{1}{2}}.$$

In the second case, consider $M\,and\,N$ are the constant functions, such that, $M(uv)=(t_m,\ i_m,\ f_m), and\ N(u)=(t_n,i_n,f_n)$. In this state, we have for $u\in V(G)$,

$$d_2(u)=\left(d_{2_T}(u),d_{2_I}(u),d_{2_F}(u)\right)=\left(\sum_{u\neq v\in V}T_M^2(u,v),\sum_{u\neq v\in V}I_M^2(u,v),\sum_{u\neq v\in V}F_M^2(u,v)\right)$$
$$=\left((n-1)t_m^2,(n-1)i_m^2,(n-1)f_m^2\right).$$

by placing $d_2(u)$,

$$So_T(G)=\frac{1}{2}\sum_{uv\in E(G)}(T_N^2(u)d_{2_T}(u)+T_N^2(v)d_{2_T}(v))^{\frac{1}{2}}=\frac{1}{2}\sum_{uv\in E(G)}(T_N^2(u)(n-1)t_m^2+T_N^2(v)(n-1)t_m^2)^{\frac{1}{2}}$$
$$=\frac{1}{2}\sum_{uv\in E(G)}(t_n^2(n-1)t_m^2+t_n^2(n-1)t_m^2)^{\frac{1}{2}}=\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}}t_m.t_n.$$

The same can be proved for $So_I(G)$ and $So_F(G)$. Also, by direct placing in $|So(G)|$,

$$|So(G)| = \frac{4 + 2So_T(G) - So_I(G) - 2So_F(G)}{6}$$
$$= \frac{4 + 2\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} t_m.t_n - \frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} i_m.i_n - 2\frac{n(n-1)^{\frac{3}{2}}}{2\sqrt{2}} f_m.f_n}{6}$$
$$= \frac{n(n-1)^{\frac{3}{2}}}{12\sqrt{2}}(2t_m.t_n - i_m.i_n - 2f_m.f_n) + \frac{2}{3}.$$

To prove the third case, it is enough to consider $d_2(u)$ like the previous parts and then replace the value of $M(uv) = (t, i, f), and\ N(u) = (t, i, f)$. The proofs for $So_I(G)$ and $So_F(G)$ are similar to the case for $So_T(G)$.

□

Theorem 5. Let $G = (N, M)$ be a complete neutrosophic graph whit $|V| = n$. Then,

If $M\ and\ N$ are the constant functions, that is, $M(uv) = (t_m, i_m, f_m), for\ uv \in E(G), and\ N(u) = (t_n, i_n, f_n), for\ u \in V(G)$. Now, we have:

$$So_T(G_T) = \frac{n(n-1)}{2\sqrt{2}} t_n((n-1)t_m^2 + t_n^2)^{1/2},$$
$$So_I(G_T) = \frac{n(n-1)}{2\sqrt{2}} i_n((n-1)i_m^2 + i_n^2)^{1/2},$$
$$So_F(G_T) = \frac{n(n-1)}{2\sqrt{2}} f_n((n-1)f_m^2 + f_n^2)^{1/2}.$$

Also,

$$|So(G_T)| = \frac{n(n-1)}{12\sqrt{2}}\left(2t_n((n-1)t_m^2 + t_n^2)^{1/2} - i_n((n-1)i_m^2 + i_n^2)^{1/2} - 2f_n((n-1)f_m^2 + f_n^2)^{1/2}\right) + \frac{2}{3}.$$

If $M\ and\ N$ are the constant and the same functions, that is, $M(uv) = (t, i, f), for\ uv \in E(G), and\ N(u) = (t, i, f), for\ u \in V(G)$. Now, we have:

$$So_T(G_T) = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} t^2,$$
$$So_I(G_T) = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} i^2,$$
$$So_F(G_T) = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} f^2.$$

Also,

$$|So(G_T)| = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}(2t^2 - i^2 - 2f^2) + \frac{2}{3}.$$

Proof: Suppose G is a complete neutrosophic graph, with $|V| = n$. Also, $uv \in E(G)$. The first case, we have $M(uv) = (t_m, i_m, f_m), for\ uv \in E(G), and\ N(u) = (t_n, i_n, f_n), for\ u \in V(G)$. Therefore,

$$Td_2(u) = \left(Td_{2_T}(u), Td_{2_I}(u), Td_{2_F}(u)\right)$$
$$= \left(\sum_{u \neq v \in V} T_M^2(u,v) + T_N^2(u), \sum_{u \neq v \in V} I_M^2(u,v) + I_N^2(u), \sum_{u \neq v \in V} F_M^2(u,v) + F_N^2(u)\right).$$
$$= \left((n-1)t_m^2 + t_n^2, (n-1)i_m^2 + i_n^2, (n-1)f_m^2 + f_n^2\right).$$

Now, by placing $Td_2(u)$ in relation to the Sombor index we get:

$$So_T(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)Td_{2_T}(u) + T_N^2(v)Td_{2_T}(v))^{\frac{1}{2}}$$
$$= \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)((n-1)t_m^2 + t_n^2) + T_N^2(v)((n-1)t_m^2 + t_n^2))^{\frac{1}{2}}$$
$$= \frac{1}{2}\sum_{uv\in E(G)} (((n-1)t_m^2 + t_n^2)(T_N^2(u) + T_N^2(v)))^{\frac{1}{2}}$$
$$= \frac{1}{2}\sum_{uv\in E(G)} (((n-1)t_m^2 + t_n^2)(t_n^2 + t_n^2))^{\frac{1}{2}} = \frac{n(n-1)}{2\sqrt{2}} t_n((n-1)t_m^2 + t_n^2)^{1/2}.$$

It is similarly possible for the other two modes.

$$|So(G_T)| = \frac{4 + 2So_T(G_T) - So_I(G_T) - 2So_F(G_T)}{6}$$
$$= \frac{4 + 2\frac{n(n-1)}{2\sqrt{2}} t_n((n-1)t_m^2 + t_n^2)^{1/2} - \frac{n(n-1)}{2\sqrt{2}} i_n((n-1)i_m^2 + i_n^2)^{1/2} - 2\frac{n(n-1)}{2\sqrt{2}} f_n((n-1)f_m^2 + f_n^2)^{1/2}}{6}$$
$$= \frac{4 + \frac{n(n-1)}{2\sqrt{2}}\left(2t_n\left((n-1)t_m^2 + t_n^2\right)^{1/2} - i_n\left((n-1)i_m^2 + i_n^2\right)^{1/2} - 2f_n\left((n-1)f_m^2 + f_n^2\right)^{1/2}\right)}{6}$$
$$= \frac{n(n-1)}{12\sqrt{2}}\left(2t_n((n-1)t_m^2 + t_n^2)^{1/2} - i_n((n-1)i_m^2 + i_n^2)^{1/2} - 2f_n((n-1)f_m^2 + f_n^2)^{1/2}\right) + \frac{2}{3}.$$

For case 2: we have $M(uv) = (t,i,f), for\ uv \in E(G), and\ N(u) = (t,i,f), for\ u \in V(G),$ henc:

$$Td_2(u) = \left(Td_{2_T}(u), Td_{2_I}(u), Td_{2_F}(u)\right)$$
$$= \left(\sum_{u\neq v\in V} T_M^2(u,v) + T_N^2(u), \sum_{u\neq v\in V} I_M^2(u,v) + I_N^2(u), \sum_{u\neq v\in V} F_M^2(u,v) + F_N^2(u)\right).$$
$$= \left((n-1)t^2 + t^2, (n-1)i^2 + i^2, (n-1)f^2 + f^2\right) = (nt^2, ni^2, nf^2).$$

$$So_T(G_T) = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)Td_{2_T}(u) + T_N^2(v)Td_{2_T}(v))^{\frac{1}{2}} = \frac{1}{2}\sum_{uv\in E(G)} (T_N^2(u)(nt^2) + T_N^2(v)(nt^2))^{\frac{1}{2}}$$
$$= \frac{1}{2}\sum_{uv\in E(G)} ((nt^2)(t^2 + t^2))^{\frac{1}{2}} = \frac{1}{2}(nt^2)^{\frac{1}{2}}\sum_{uv\in E(G)} (t^2 + t^2)^{\frac{1}{2}} = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} t^2.$$

It is similarly proved for the $So_I(G_T)$ and $So_F(G_T)$. Also,

$$|So(G_T)| = \frac{4 + 2So_T(G_T) - So_I(G_T) - 2So_F(G_T)}{6}$$
$$= \frac{4 + 2\frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} t^2 - \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} i^2 - 2\frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}} f^2}{6}$$
$$= \frac{4 + \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}(2t^2 - i^2 - 2f^2)}{6} = \frac{(n-1)(n)^{\frac{3}{2}}}{2\sqrt{2}}(2t^2 - i^2 - 2f^2) + \frac{2}{3}.$$

The proof is complete.

□

Conclusion

In this paper, for the first time, Sombor indices for neutrosophic graphs are defined. Here, in addition to defining Sombor indices, we have calculated and studied these indices for complete neutrosophic graphs, which are an important category of neutrosophic graphs. In the following, we will generalize these indicators and in our next articles, we will discuss another group of neutrosophic graphs, called neutrosophic regular edge graphs, and then we will discuss its applications.

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