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m-Polar Quadripartitioned Neutrosophic Graphs with Applications in Decision-Making for Mobile Network Selection

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Abstract. In today's rapidly evolving world, the shift towards multi-polarity is increasingly evident, significantly influencing various scientific and technological domains, particularly in information and data sciences. This study introduces m-polar Quadripartitioned neutrosophic sets and graphs, along with key results and operations, such as the composition and Cartesian product of m-polar quadripartitoned neutrosophic graphs, supported by illustrative examples to clarify the proposed concepts. Additionally, recognizing the complexity of selecting a mobile network shaped by factors like coverage, data speed, pricing, and customer service-we propose a novel decision-making approach using a score function within the framework of m-polar quadripartitioned neutrosophic sets. This methodology enables users to evaluate and compare multiple mobile networks effectively across various criteria.

Keywords: m-Polar; Quadripartitioned Neutrosophic Graphs; Decision-Making; Mobile Network;

1. Introduction

An m-polar neutrosophic model is highly effective, offering greater precision, adaptability, and comparability than classical, fuzzy, or neutrosophic graph models. In neurobiology, multi-polar brain neurons demonstrate significant learning interactions with other neurons, showcasing the relevance of multi-polar systems. Likewise, multi-polar technology finds applications in information technology for managing large-scale systems. However, challenges often arise when dealing with unclear or ambiguous evaluations in the selection of reference sets.

For example, in voting scenarios, individuals may vote in favor, vote against, or abstain [33]. The mathematical properties of m-polar fuzzy graphs are studied in [10], while the concepts of m-polar valued graphs have been extended to various domains of fuzzy set theory [4,11,12]. In electrical engineering, we work with conductors and non-conductors, but some materials act as insulators. Neutrosophic set (NS) theory offers an effective framework for handling such scenarios. In many real-world applications, we encounter multipolarity, along with rankings of truth, falsity, and indeterminacy for various alternatives. To address these uncertainties and hesitations, we introduce the concept of an m-polar NS.

From a practical perspective, the applications of this model span a wide range of fields, including decision-making systems, pattern recognition, image processing, medical diagnosis, and risk management. For example, in decision-making systems, the m-polar quadripartitioned neutrosophic model captures the complex nature of real-world choices, enabling more robust and comprehensive analysis [30, 31]. Similarly, in image processing and pattern recognition, where data may involve ambiguity or uncertainty, this model supports more accurate classification and interpretation [14, 28].

In medical diagnosis, where uncertainty arises from the complexity of biological systems and variability among patients, the m-polar quadripartitioned neutrosophic model offers a systematic framework for integrating diverse information sources, enabling informed decision-making.

Similarly, in risk management, particularly in domains such as finance and insurance, where accurate risk assessment is critical, this model enhances evaluation by addressing various dimensions of risk, including probability, severity, and uncertainty [22]. Consequently, the m-polar quadripartitioned neutrosophic approach not only strengthens the theoretical foundation of neutrosophic set theory but also broadens its practical applicability to a wide array of problems characterized by uncertainty and complexity [31].

While m-polar fuzzy sets (MPFS) define membership values within $[0,1]^m$ to represent an object's m criteria, they fail to capture the indeterminacy and falsity components of the object. Neutrosophic sets (NS), on the other hand, consider truth, falsity, and ambiguity for a single criterion but lack the capability to integrate information from multiple sources with varying polarities and multiple criteria. To address this limitation, we propose a novel model -the m-polar neutrosophic set (MPNS) which combines the strengths of MPFS and NS. This model accommodates multiple criteria while handling truth, indeterminacy, and falsity for each alternative.

The MPNS builds upon earlier works such as the bipolar neutrosophic set by Deli et al. [9] [18], representing an extension of these concepts. We establish various attributes and operations of MPNS and introduce scoring functions for comparing m-polar neutrosophic numbers (MPNNs). Throughout the manuscript, we use Δ as an indexing set. In the following paper, [43] studies the concept of a novel neutrosophic multi-criteria decision-making (MCDM) approach to improve the supplier selection process under uncertainty and ambiguity

The concept of single-valued neutrosophic sets was first introduced by H. Wang et al. [27]. By dividing the indeterminacy component of NS into two parts-Unknown' (neither true nor false) and Contradiction' (both true and false)-we derive four components: $\mathcal{A}$ (truth), $\mathcal{C}$ (unknown), $\mathcal{B}$ (contradiction), and $\mathcal{D}$ (falsity). This leads to the quadripartitioned single-valued neutrosophic set, as proposed by Chatterjee et al. [8], rooted in Smarandache's "Four Numerical-Valued Neutrosophic Logic" [24] and Belnap's "Four-Valued Logic" [5]. Neutrosophic graphs were concisely explained in F. Smarandache's book [25], while the degree, order, and size of single-valued neutrosophic graphs (SVNG) were developed by S. Broumi et al. [6]. Recent advancements include bipolar quadripartitioned neutrosophic graphs (BQNG) and quadripartitioned single-valued neutrosophic graphs (QSVNG) with real-world applications [16, 17, 34]. M-polar quadripartitioned neutrosophic sets, graphs, and their topological properties were introduced by S.S. Hussain [29]

Interval neutrosophic graphs (ING), which represent membership, non-membership, and indeterminacy using intervals, offer greater flexibility and ease of use compared to single-valued neutrosophic graphs [19]. Recent studies on m-polar neutrosophic graphs (MPNG) have explored metrics [1, 2] and isomorphism properties [15]. New operations such as join, union, ring sum, and composition, along with six novel products for MPNG, have been introduced in [20].

In this paper, we present the concept of MPQNS and its associated graphs, along with key results. We introduce the composition and Cartesian product of MPQNG and provide examples to illustrate the theoretical concepts. The following highlights the significance and challenges addressed in this work.

Novelty of this paper:

- This paper provides a novel conceptual framework (m-polar quadripartitioned neutrosophic sets) to address uncertainty in decision-making, particularly in scientific and technological domains.
- It introduces innovative operations (e.g., strong product, direct product) on neutrosophic graphs, enhancing analytical capabilities and expanding the toolkit for complex system analysis.
- It explores fundamental concepts like Cartesian product, composition within the context of m-polar neutrosophic graphs, contributing to theoretical understanding and application.

- It gives the properties and theorems of m-polar quadripartitioned neutrosophic graphs, providing a solid theoretical foundation for further research and application.
- Finally, these graphs comprehensively capture decision criteria such as coverage, data speed, pricing, and customer service, enabling a holistic view of mobile network options which is crucial for the dynamic nature of mobile networks.
- The abstract nature of m-polar quadripartitioned neutrosophic sets and associated operations have present challenges in interpretation and understanding, requiring specialized knowledge and expertise for effective utilization.
- In the application viewpoint, using m-polar quadripartitioned neutrosophic sets, utilizing the score function within this framework facilitates quantitative analysis, enabling objective evaluation and comparison of mobile network options, leading to data-driven decision-making.

The flowchart of the proposed QNG model is shown below:1

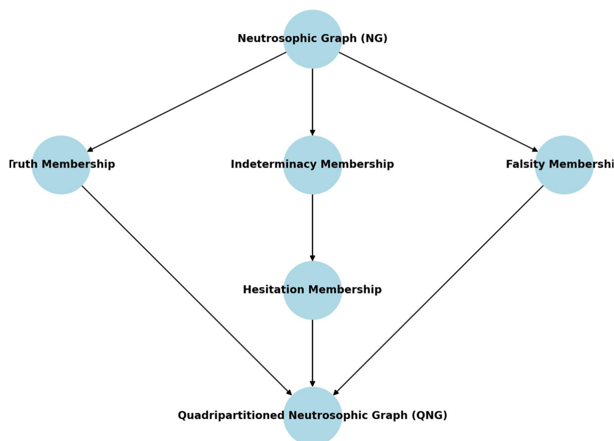

FIGURE 1. Flow Chart

2. Operation on m-polar Quadripartitioned Neutrosophic Graphs

Definition 2.1. A graph $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta, \Gamma)$ of the graph $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}}, \mathcal{E}_{\mathfrak{G}})$, where $\beta = (\mathcal{A}_\beta, \mathcal{B}_\beta, \mathcal{C}_\beta, \mathcal{D}_\beta)$, and $\Gamma = (\mathcal{A}_\Gamma, \mathcal{B}_\Gamma, \mathcal{C}_\Gamma, \mathcal{D}_\Gamma)$ is called m-polar QNG if, for each $i = 1, 2, .., m$,

(1) $0 \leq \chi_i(\mathcal{A}_\beta(r)) + \chi_i(\mathcal{B}_\beta(r)) + \chi_i(\mathcal{C}_\beta(r)) + \chi_i(\mathcal{D}_\beta(r)) \leq 4, \forall r \in \mathbb{V}_{\mathfrak{G}}$

(2) $\mathcal{A}_\Gamma : \mathcal{E}_{\mathfrak{G}} \to [0,1]^m, \mathcal{B}_\Gamma : \mathcal{E}_{\mathfrak{G}} \to [0,1]^m, \mathcal{C}_\Gamma : \mathcal{E}_{\mathfrak{G}} \to [0,1]^m, \mathcal{D}_\Gamma : \mathcal{E}_{\mathfrak{G}} \to [0,1]^m$ are the TMF, CMF, IMF and FMF of the edge rs respectively, such that

$\chi_i(\mathcal{A}_\Gamma(rs)) \leq \chi_i(\mathcal{A}_\beta(r)) \wedge \chi_i(\mathcal{A}_\beta(s)), \quad \chi_i(\mathcal{B}_\Gamma(rs)) \leq \chi_i(\mathcal{B}_\beta(r)) \wedge \chi_i(\mathcal{B}_\beta(s)) \ \chi_i(\mathcal{C}_\Gamma(rs)) \leq \chi_i(\mathcal{C}_\beta(r)) \vee \chi_i(\mathcal{C}_\beta(s)), \quad \chi_i(\mathcal{D}_\Gamma(rs)) \leq \chi_i(\mathcal{D}_\beta(r)) \vee \chi_i(\mathcal{D}_\beta(s))$ for all $rs \in \mathcal{E}_{\mathfrak{G}}$

Definition 2.2. The Cartesian products of two m-QNGs $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta_1, \Gamma_1)$ and $\mathbb{H} = (\mathbb{V}_{\mathbb{H}}, \beta_2, \Gamma_2)$ of the graph $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}}, \mathcal{E}_{\mathfrak{G}})$ and $\mathbb{H}' = (\mathbb{V}_{\mathbb{H}}, \mathcal{E}_{\mathbb{H}})$ respectively, is denoted by $\mathfrak{G} \times \mathbb{H}$

and is explained as $(\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}}, \beta_1 \times \beta_2, \Gamma_1 \times \Gamma_2)$ where $\beta_1 \times \beta_2 = (\mathcal{A}_{\beta_1} \times \mathcal{A}_{\beta_2}, \mathcal{B}_{\beta_1} \times \mathcal{B}_{\beta_2}, \mathcal{C}_{\beta_1} \times \mathcal{C}_{\beta_2}, \mathcal{D}_{\beta_1} \times \mathcal{D}_{\beta_2})$ and $\Gamma_1 \times \Gamma_2 = (\mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2}, \mathcal{B}_{\Gamma_1} \times \mathcal{B}_{\Gamma_2}, \mathcal{C}_{\Gamma_1} \times \mathcal{C}_{\Gamma_2}, \mathcal{D}_{\Gamma_1} \times \mathcal{D}_{\Gamma_2})$ s.t for each $i = 1, 2, 3, ..., m$.

$$\begin{aligned}
(1) \quad \chi_i((\mathcal{A}_{\beta_1} \times \mathcal{A}_{\beta_2})(r,s)) &= \chi_i(\mathcal{A}_{\beta_1}(r)) \wedge \chi_i(\mathcal{A}_{\beta_2}(s))\\
\chi_i((\mathcal{B}_{\beta_1} \times \mathcal{B}_{\beta_2})(r,s)) &= \chi_i(\mathcal{B}_{\beta_1}(r)) \wedge \chi_i(\mathcal{B}_{\beta_2}(s))\\
\chi_i((\mathcal{C}_{\beta_1} \times \mathcal{C}_{\beta_2})(r,s)) &= \chi_i(\mathcal{C}_{\beta_1}(r)) \vee \chi_i(\mathcal{C}_{\beta_2}(s))\\
\chi_i((\mathcal{D}_{\beta_1} \times \mathcal{D}_{\beta_2})(r,s)) &= \chi_i(\mathcal{D}_{\beta_1}(r)) \vee \chi_i(\mathcal{D}_{\beta_2}(s))\\
\text{for all} \quad (r,s) \in \mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}},
\end{aligned}$$

$$\begin{aligned}
(2) \quad \chi_i((\mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2})(\alpha,r)(\alpha,s)) &= \chi_i(\mathcal{A}_{\beta_1}(\alpha)) \wedge \chi_i(\mathcal{A}_{\Gamma_2}(rs)\\
\chi_i((\mathcal{B}_{\Gamma_1} \times \mathcal{B}_{\Gamma_2})(\alpha,r)(\alpha,s)0 &= \chi_i(\mathcal{B}_{\beta_1}(\alpha)) \wedge \chi_i(\mathcal{B}_{\Gamma_2}(rs))\\
\chi_i((\mathcal{C}_{\Gamma_1} \times \mathcal{C}_{\Gamma_2})(\alpha,r)(\alpha,s)) &= \chi_i(\mathcal{C}_{\beta_1}(\alpha)) \vee \chi_i(\mathcal{C}_{\Gamma_2}(rs))\\
\chi_i((\mathcal{D}_{\Gamma_1} \times \mathcal{D}_{\Gamma_2})(\alpha,r)(\alpha,s)) &= \chi_i(\mathcal{D}_{\beta_1}(\alpha)) \vee \chi_i(\mathcal{D}_{\Gamma_2}(rs))\\
\text{for all} \quad \alpha \in \mathbb{V}_{\mathfrak{G}} &\text{and } \forall rs \in \mathcal{E}_{\mathbb{H}}
\end{aligned}$$

$$\begin{aligned}
(iii) \quad \chi_i((\mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2})(r,\alpha)(s,\alpha)) &= \chi_i(\mathcal{A}_{\Gamma_1}(rs)) \wedge \chi_i(\mathcal{A}_{\beta_2}(\alpha))\\
\chi_i((\mathcal{B}_{\Gamma_1} \times \mathcal{B}_{\Gamma_2})(r,\alpha)(s,\alpha)) &= \chi_i(\mathcal{B}_{\Gamma_1}(rs)) \wedge \chi_i(\mathcal{B}_{\beta_2}(\alpha))\\
\chi_i((\mathcal{C}_{\Gamma_1} \times \mathcal{C}_{\Gamma_2})(r,\alpha)(s,\alpha)) &= \chi_i(\mathcal{C}_{\Gamma_1}(rs)) \vee \chi_i(\mathcal{C}_{\beta_2}(\alpha))\\
\chi_i((\mathcal{D}_{\Gamma_1} \times \mathcal{D}_{\Gamma_2})(r,\alpha)(s,\alpha)) &= \chi_i(\mathcal{D}_{\Gamma_1}(rs)) \vee \chi_i(\mathcal{D}_{\beta_2}(\alpha))\\
\text{for all} \quad \alpha \in \mathbb{V}_{\mathbb{H}} &\text{and } \forall rs \in \mathcal{E}_{\mathfrak{G}}
\end{aligned}$$

$$\begin{aligned}
(iv)\chi_i((\mathcal{A}_{\Gamma_1} \times \mathcal{A}_{\Gamma_2})(r,s)(\alpha,\delta)) &= 0\\
\chi_i((\mathcal{B}_{\Gamma_1} \times \mathcal{B}_{\Gamma_2})(r,s)(\alpha,\delta)) &= 0\\
\chi_i((\mathcal{C}_{\Gamma_1} \times \mathcal{C}_{\Gamma_2})(r,s)(\alpha,\delta)) &= 0\\
\chi_i((\mathcal{D}_{\Gamma_1} \times \mathcal{D}_{\Gamma_2})(r,s)(\alpha,\delta)) &= 0\\
(r,s)(\alpha,\delta) \in (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}})^2 - \mathcal{E}.
\end{aligned}$$

Here $\mathcal{E} = \{(r,\alpha)(s,\alpha) : rs \in \mathcal{E}_{\mathfrak{G}}, \alpha \in \mathbb{V}_{\mathbb{H}}\} \cup \{(r,\alpha)(r,\delta) : r \in \mathbb{V}_{\mathfrak{G}}, \alpha\delta \in \mathcal{E}_{\mathbb{H}}\}$.

Theorem 2.3. *The Cartesian Product* $(\mathfrak{G} \times \mathbb{H}) = (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}}, \beta_{\mathfrak{G}} \times \beta_{\mathbb{H}}, \Gamma_{\mathfrak{G}} \times \Gamma_{\mathbb{H}})$ *where* $\beta_{\mathfrak{G}} \times \beta_{\mathbb{H}} = (\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}}, \mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}}, \mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}}, \mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})$ *and* $\Gamma_{\mathfrak{G}} \times \Gamma_{\mathbb{H}} = (\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}}, \mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}}, \mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}}, \mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})$ *of two m-polar QNG* $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta_{\mathfrak{G}}, \Gamma_{\mathfrak{G}})$ *and* $\mathbb{H} = (\mathbb{V}_{\mathbb{H}}, \beta_{\mathbb{H}}, \Gamma_{\mathbb{H}})$ *of the graphs* $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}} \ \ \mathcal{E}_{\mathfrak{G}})$ *and* $\mathbb{H}' = (\mathbb{V}_{\mathbb{H}} \ \ \mathcal{E}_{\mathbb{H}})$ *is an* $\mathfrak{m}$*-polar quadripartitioned neutrosophic graph of* $\mathfrak{G}' \times \mathbb{H}'$

Proof. Let $\alpha \in \mathbb{V}_{\mathfrak{G}}$ and $rs \in \mathcal{E}_{\mathbb{H}}$ then for each $i = 1, 2, 3, .., m$.

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &= \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})((\alpha, r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s))) \\
\therefore \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &\leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})((\alpha, r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s)))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &= \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha, r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s))) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &\leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha, r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s)))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &= \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \{\chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(r))\} \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})((\alpha, r)) \vee \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s))) \\
\therefore \quad \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &\leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})((\alpha, r)) \vee \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s)))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &= \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \{\chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(r))\} \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})((\alpha, r)) \vee \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha, s))) \\
\therefore \quad \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha, r)(\alpha, s))) &\leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})((\alpha, r)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha, s))
\end{aligned}$$

Let $\alpha \in \mathbb{V}_{\mathbb{H}}$ and $rs \in \mathcal{E}_{\mathfrak{G}}$ then for each $i = 1, 2, 3, .., m$.

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{A}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\leq \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(s))\} \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(s,\alpha))) \\
\therefore \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(s,\alpha)))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{B}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\leq \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha,r)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})(s,\alpha))) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})(s,\alpha))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{C}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\leq \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})(s,\alpha))) \\
\therefore \quad \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})(s,\alpha))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{D}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\leq \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(s,\alpha))) \\
\therefore \quad \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(s,\alpha))
\end{aligned}$$

Let $(r,s)(\alpha,\delta) \in (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}})^2 - \mathcal{E}$. Then,

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \times \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) &= 0 \leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \times \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \times \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) &= 0 \leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \times \mathcal{B}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \times \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) &= 0 \leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \times \mathcal{C}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \times \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) &= 0 \leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \times \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha,\delta))
\end{aligned}$$

for each $i = 1, 2, 3, ..., m$. □

Definition 2.4. The composition $\mathfrak{G} \circ \mathbb{H} = (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}}, \beta_{\mathfrak{G}} \times \beta_{\mathbb{H}}, \Gamma_{\mathfrak{G}} \times \Gamma_{\mathbb{H}})$ where $(\beta_{\mathfrak{G}} \circ \beta_{\mathbb{H}}) = (\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}}, \mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}}, \mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}}, \mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})$ and $(\Gamma_{\mathfrak{G}} \circ \Gamma_{\mathbb{H}}) = (\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}}, \mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}}, \mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}}, \mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})$ of two m-polar QNG $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta_{\mathfrak{G}}, \Gamma_{\mathfrak{G}})$ and $\mathbb{H} = (\mathbb{V}_{\mathbb{H}}, \beta_{\mathbb{H}}, \Gamma_{\mathbb{H}})$, where $\beta_{\mathfrak{G}} = (\mathcal{A}_{\beta_{\mathfrak{G}}}, \mathcal{B}_{\beta_{\mathfrak{G}}}, \mathcal{C}_{\beta_{\mathfrak{G}}}, \mathcal{D}_{\beta_{\mathfrak{G}}})$, $\beta_{\mathbb{H}} = (\mathcal{A}_{\beta_{\mathbb{H}}}, \mathcal{B}_{\beta_{\mathbb{H}}}, \mathcal{C}_{\beta_{\mathbb{H}}}, \mathcal{D}_{\beta_{\mathbb{H}}})$, $\Gamma_{\mathfrak{G}} = (\mathcal{A}_{\Gamma_{\mathfrak{G}}}, \mathcal{B}_{\Gamma_{\mathfrak{G}}}, \mathcal{C}_{\Gamma_{\mathfrak{G}}}, \mathcal{D}_{\Gamma_{\mathfrak{G}}})$, $\Gamma_{\mathbb{H}} = (\mathcal{A}_{\Gamma_{\mathbb{H}}}, \mathcal{B}_{\Gamma_{\mathbb{H}}}, \mathcal{C}_{\Gamma_{\mathbb{H}}}, \mathcal{D}_{\Gamma_{\mathbb{H}}})$, of the graphs $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}}, \mathcal{E}_{\mathfrak{G}})$ and $\mathbb{H}' = (\mathbb{V}_{\mathbb{H}}, \mathcal{E}_{\mathbb{H}})$ respectively, is explained as follows: for each $i = 1, 2, .., m$,

$$\begin{aligned}
(i) \quad \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(rs)) &= \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s)) \\
\chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(rs)) &= \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s)) \\
\chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(rs)) &= \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s)) \\
\chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(rs)) &= \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s)) \\
&\text{for} \quad \text{all} \quad rs \in \mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}}
\end{aligned}$$

$$\begin{aligned}
(ii) \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(\alpha, r)(\alpha, s)) &= \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\Gamma_{\mathbb{H}}}(rs)) \\
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})(\alpha, r)(\alpha, s)) &= \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\Gamma_{\mathbb{H}}}(rs)) \\
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})(\alpha, r)(\alpha, s)) &= \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\Gamma_{\mathbb{H}}}(rs)) \\
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})(\alpha, r)(\alpha, s)) &= \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\text{for} \quad \text{all} \quad \alpha \in \mathbb{V}_{\mathfrak{G}} \text{and} \quad (rs) \in \mathcal{E}_H.
\end{aligned}$$

$$\begin{aligned}
(iii) \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(r, \alpha)(s, \alpha)) &= \chi_i(\mathcal{A}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha)) \\
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})(r, \alpha)(s, \alpha)) &= \chi_i(\mathcal{B}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha)) \\
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})(r, \alpha)(s, \alpha)) &= \chi_i(\mathcal{C}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha)) \\
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})(r, \alpha)(s, \alpha)) &= \chi_i(\mathcal{D}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\text{for} \quad \text{all} \quad \alpha \in \mathbb{V}_{\mathbb{H}} \text{and} \quad (rs) \in \mathcal{E}_G.
\end{aligned}$$

$$(iv)\quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) = \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}}(r\alpha))) \wedge \chi_i((\mathcal{A}_{\beta_{\mathbb{H}}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\delta)))$$
$$\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) = \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}}(r\alpha))) \wedge \chi_i((\mathcal{B}_{\beta_{\mathbb{H}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\delta)))$$
$$\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) = \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}}(r\alpha))) \vee \chi_i((\mathcal{C}_{\beta_{\mathbb{H}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\delta)))$$
$$\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) = \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}}(r\alpha))) \vee \chi_i((\mathcal{D}_{\beta_{\mathbb{H}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\delta)))$$
$$\text{for all} \quad (rs)(\alpha\delta) \in \mathcal{E}_0 - \mathcal{E}$$

$$(v)\quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) = 0$$
$$\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) = 0$$
$$\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) = 0$$
$$\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) = 0$$
$$\text{for} \quad \text{all} \quad (rs)(\alpha\delta) \in (\mathbb{V}_{\mathfrak{G}} \times \mathbb{V}_{\mathbb{H}})^2 - \mathcal{E}_0$$

Here $\mathcal{E} = \{((r,\alpha)(s\alpha)) : rs \in \mathcal{E}_{\mathfrak{G}}, \alpha \in \mathbb{V}_{\mathbb{H}}\} \cup ((r,\alpha)(r\delta)) : r \in \mathbb{V}_{\mathfrak{G}}, \alpha\delta \in \mathcal{E}_{\mathbb{H}}$ and $\mathcal{E}^0 = \mathcal{E} \cup \{(r,\alpha)(s,\delta) : r\alpha \in \mathcal{E}_{\mathfrak{G}}, \alpha \neq \delta \in \mathbb{V}_{\mathbb{H}}\}$

Theorem 2.5. *The composition* $(\mathfrak{G} \circ \mathbb{H}) = (\mathbb{V}_{\mathfrak{G}} \circ \mathbb{V}_{\mathbb{H}}, \beta_{\mathfrak{G}} \circ \beta_{\mathbb{H}}, \Gamma_{\mathfrak{G}} \circ \Gamma_{\mathbb{H}})$ *where* $\beta_{\mathfrak{G}} \circ \beta_{\mathbb{H}} = (\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}}, \mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}}, \mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}}, \mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})$ *and* $\Gamma_{\mathfrak{G}} \circ \Gamma_{\mathbb{H}} = (\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}}, \mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}}, \mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}}, \mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})$ *of two* $\mathfrak{m}$*-polar QNG* $\mathfrak{G} = (\mathbb{V}_{\mathfrak{G}}, \beta_{\mathfrak{G}}, \Gamma_{\mathfrak{G}})$ *and* $\mathbb{H} = (\mathbb{V}_{\mathbb{H}}, \beta_{\mathbb{H}}, \Gamma_{\mathbb{H}})$ *of the graphs* $\mathfrak{G}' = (\mathbb{V}_{\mathfrak{G}}, \mathcal{E}_{\mathfrak{G}})$ *and* $\mathbb{H}' = (\mathbb{V}_{\mathbb{H}}, \mathcal{E}_{\mathbb{H}})$ *is an* $\mathfrak{m}$*-QNG of* $\mathfrak{G}' \circ \mathbb{H}'$

Proof. Let $\alpha \in \mathbb{V}_{\mathfrak{G}}$ and $rs \in \mathcal{E}_{\mathbb{H}}$ then for each $i = 1, 2, 3, .., m$.

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &= \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})((\alpha,r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,s)))
\end{aligned}$$
$$\therefore \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) \leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})((\alpha,r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,s))$$

$$\begin{aligned}
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &= \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(r))\} \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha,r)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,s))) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &\leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})((\alpha,r)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(\alpha,s))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &= \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \{\chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(r))\} \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(\alpha,s))) \\
\therefore \quad \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &\leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(\alpha,s))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &= \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\Gamma_{\mathbb{H}}}(rs)) \\
&\leq \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \{\chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s))\} \\
&= \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(r))\} \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s))\} \\
&= \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(\alpha,s))) \\
\therefore \quad \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((\alpha,r)(\alpha,s))) &\leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha,s))
\end{aligned}$$

Let $\alpha \in \mathbb{V}_{\mathbb{H}}$ and $rs \in \mathcal{E}_{\mathfrak{G}}$ then for each $i = 1, 2, 3, .., m$.

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{A}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\leq \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \wedge \{\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(s,\alpha))) \\
\therefore \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(s,\alpha))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{B}_{\Gamma_{\mathfrak{G}}}(rs)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\leq \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \wedge \{\chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(s,\alpha))) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})((r,\alpha)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(s,\alpha))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{C}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\leq \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \vee \{\chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(s,\alpha))) \\
\therefore \quad \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(s,\alpha))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &= \chi_i(\mathcal{D}_{\Gamma_{\mathfrak{G}}}(rs)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha)) \\
&\leq \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \vee \{\chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\alpha))\} \\
&= \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})((\alpha,r)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(s,\alpha))) \\
\therefore \quad \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,\alpha)(s,\alpha))) &\leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})((r,\alpha)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(s,\alpha))
\end{aligned}$$

Let $(r,s)(\alpha,\delta) \in (\mathbb{V}_{\mathfrak{G}} \circ \mathbb{V}_{\mathbb{H}})^2 - \mathcal{E}$. Then for each $i = 1,2,3,...,m$.

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) = 0 \leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) = 0 \leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(\alpha,\delta))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) = 0 \leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})((r,s)(\alpha,\delta))) = 0 \leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha,\delta))
\end{aligned}$$

Let $(r,s)(\alpha,\delta) \in \mathcal{E}_0 - \mathcal{E}$. Then for each $i = 1,2,3,...,m$.

$$\begin{aligned}
\chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) &= \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}}(r\alpha))) \wedge \chi_i((\mathcal{A}_{\beta_{\mathbb{H}}}(s) \wedge \mathcal{A}_{\beta_{\mathbb{H}}}(\delta))) \\
&\leq \{(\chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)))\} \wedge \{\chi_i((\mathcal{A}_{\beta_{\mathbb{H}}}(s)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\delta)))\} \\
&= \{\chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(s)))\} \wedge \{\chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{A}_{\beta_{\mathbb{H}}}(\delta)))\} \\
&= \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\therefore \quad \chi_i((\mathcal{A}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{A}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) &\leq \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{A}_{\beta_{\mathfrak{G}}} \circ \mathcal{A}_{\beta_{\mathbb{H}}})(\alpha,\delta))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) &= \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}}(r\alpha))) \wedge \chi_i((\mathcal{B}_{\beta_{\mathbb{H}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\delta))) \\
&\leq \{\chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)))\} \wedge \{\chi_i((\mathcal{B}_{\beta_{\mathbb{H}}}(s)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\delta)))\} \\
&= \{\chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}}(r)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(s)))\} \wedge \{\chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}}(\alpha)) \wedge \chi_i(\mathcal{B}_{\beta_{\mathbb{H}}}(\delta)))\} \\
&= \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{B}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) &\leq \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(r,s)) \wedge \chi_i((\mathcal{B}_{\beta_{\mathfrak{G}}} \circ \mathcal{B}_{\beta_{\mathbb{H}}})(\alpha,\delta))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) &= \chi_i((\mathcal{C}_{\Gamma_{\mathfrak{G}}}(r\alpha))) \vee \chi_i((\mathcal{C}_{\beta_{\mathbb{H}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\delta))) \\
&\leq \{\chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)))\} \vee \{\chi_i((\mathcal{C}_{\beta_{\mathbb{H}}}(s)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\delta)))\} \\
&= \{\chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(s)))\} \vee \{\chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{C}_{\beta_{\mathbb{H}}}(\delta)))\} \\
&= \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{C}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) &\leq \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{C}_{\beta_{\mathfrak{G}}} \circ \mathcal{C}_{\beta_{\mathbb{H}}})(\alpha,\delta))
\end{aligned}$$

$$\begin{aligned}
\chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) &= \chi_i((\mathcal{D}_{\Gamma_{\mathfrak{G}}}(r\alpha))) \vee \chi_i((\mathcal{D}_{\beta_{\mathbb{H}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\delta))) \\
&\leq \{\chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)))\} \vee \{\chi_i((\mathcal{D}_{\beta_{\mathbb{H}}}(s)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\delta)))\} \\
&= \{\chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}}(r)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(s)))\} \vee \{\chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}}(\alpha)) \vee \chi_i(\mathcal{D}_{\beta_{\mathbb{H}}}(\delta)))\} \\
&= \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha,\delta)) \\
\therefore \quad \chi_i((\mathcal{B}_{\Gamma_{\mathfrak{G}}} \circ \mathcal{D}_{\Gamma_{\mathbb{H}}})(r,s)(\alpha,\delta)) &\leq \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(r,s)) \vee \chi_i((\mathcal{D}_{\beta_{\mathfrak{G}}} \circ \mathcal{D}_{\beta_{\mathbb{H}}})(\alpha,\delta))
\end{aligned}$$

□

3. Application: Network Selection for Mobile Service

In this section, m-polar quadripartitioned neutrosophic decision making model for "Network Selection for Mobile Service". The criteria are:

1. Coverage and Signal Strength
2. Data Speed and Performance
3. Service Plans and Pricing
4. Customer Service and Support

Let $\mathfrak{N} = \{\mathfrak{N}_1, \mathfrak{N}_2, \ldots, \mathfrak{N}_m\}$ be the set of alternatives and $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_n\}$ be the set of criteria. We consider that ω_j be the weight corresponding to the criteria $\mathfrak{C}_j$, $(j = 1, 2, \ldots, n)$. In the decision making problem, the evaluation information of the alternative $\mathfrak{N}_i$, $(i = 1, 2, \ldots, m)$ on the criteria is represented by the form of a quadripartitioned neutrosophic number. Coverage and signal strength are fundamental criteria for mobile network selection, ensuring reliable connectivity across different geographical areas. Consumers prioritize networks with extensive coverage, minimal dead zones, and strong signal strength, enabling uninterrupted communication and data access. Data speed and performance significantly impact the user experience, especially for tasks such as streaming, downloading, and online gaming. Buyers seek networks with high-speed data connections, low latency, and consistent performance, allowing for fast and seamless browsing, streaming, and data transfer. Service plans and pricing play a crucial role in network selection, with consumers considering factors such as monthly fees, data allowances, contract terms, and additional charges. Customers look for cost-effective plans that offer value for money, competitive pricing, and flexibility to meet their usage requirements. Customer service and support are essential considerations, as users may require assistance with billing inquiries, technical support, or network-related issues. Buyers prefer networks with responsive customer service, accessible support channels, and knowledgeable representatives who can address their concerns promptly and effectively.

3.1. *Algorithm:*

`Step-1:` Define Criteria and Weights

`Step-2:` Collect Neutrosophic Evaluations and construct single-valued quadripartitioned neutrosophic decision matrix $\mathcal{D}$:

$$\mathcal{D} = (a_{i,j})_{m,n} = \begin{pmatrix} (T_{1,1}, C_{1,1}, U_{1,1}, F_{1,1}) & (T_{1,2}, C_{1,2}, U_{1,2}, F_{1,2}) & \ldots & (T_{1,n}, C_{1,n}, U_{1,n}, F_{1,n}) \\ (T_{2,1}, C_{2,1}, U_{2,1}, F_{2,1}) & (T_{2,2}, C_{2,2}, U_{2,2}, F_{2,2}) & \ldots & (T_{2,n}, C_{2,n}, U_{2,n}, F_{2,n}) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{m,1}, C_{m,1}, U_{m,1}, F_{m,1}) & (T_{m,2}, C_{m,2}, U_{m,2}, F_{m,2}) & \ldots & (T_{m,n}, C_{m,n}, U_{m,n}, F_{m,n}) \end{pmatrix}$$

`Step-3:` Aggregate Evaluations- Aggregate the evaluations using a neutrosophic arithmetic average operator

`Step-4:` Calculate Scores and Rank Networks - Compute the score and accuracy value for each networks based on the aggregated evaluations. We determine some score function for the ranking of mQNs, in order to solve the problems with multi-criteria decision-making with m-Polar quadripartitioned neutrosophic numbers (mQNNs).

The mQNS score function defined as

$$\mathfrak{S}(\mathfrak{Q}) = \frac{m}{2} + \sum_{i=0}^{m} \frac{\omega_T \cdot T_i + \omega_C \cdot C_i + \omega_U \cdot U_i + \omega_F \cdot F_i}{\omega_T + \omega_C + \omega_U + \omega_F}$$

$$\mathfrak{A}(\mathfrak{Q}) = \sum_{i=0}^{m} \sqrt{\frac{(\omega_T \cdot T_i)^2 + (\omega_C \cdot C_i)^2 + (\omega_U \cdot U_i)^2 + (\omega_F \cdot F_i)^2}{2}}$$

`Step-5:` Make Decision - Rank the Networks based on their scores and select the one with the highest rank as the best option.

For our calculation, we are considering the data for 4-polar QNs are given as follows:

Network $\mathfrak{N}_1$:

1. Coverage and Signal Strength - $(0.4, 0.3, 0.2, 0.1)$
2. Data Speed and Performance - $(0.4, 0.6, 0.2, 0.1)$
3. Service Plans and Pricing - $(0.2, 0.5, 0.1, 0.2)$
4. Customer Service and Support - $(0.1, 0.2, 0.5, 0.2)$

Network $\mathfrak{N}_2$:

1. Coverage and Signal Strength - $(0.8, 0.4, 0.4, 0.1)$
2. Data Speed and Performance - $(0.6, 0.2, 0.7, 0.1)$
3. Service Plans and Pricing - $(0.7, 0.3, 0.8, 0.1)$
4. Customer Service and Support - $(0.1, 0.4, 0.7, 0.1)$

Network $\mathfrak{N}_3$:

1. Coverage and Signal Strength - $(0.9, 0.1, 0.1, 0)$
2. Data Speed and Performance - $(0.6, 0.3, 0.2, 0.1)$
3. Service Plans and Pricing - $(0.8, 0.8, 0.2, 0.2)$
4. Customer Service and Support - $(0.4, 0.7, 0.1, 0.1)$

Network $\mathfrak{N}_4$:

1. Coverage and Signal Strength - $(0.7, 0.4, 0.1, 0)$
2. Data Speed and Performance - $(0.5, 0.3, 0.7, 0)$
3. Service Plans and Pricing - $(0.8, 0.1, 0.3, 0)$
4. Customer Service and Support - $(0.8, 0.4, 0.2, 0.1)$.

Now, this Figure 2 generate the given data with each group representing a criterion and each bar within a group representing a network.

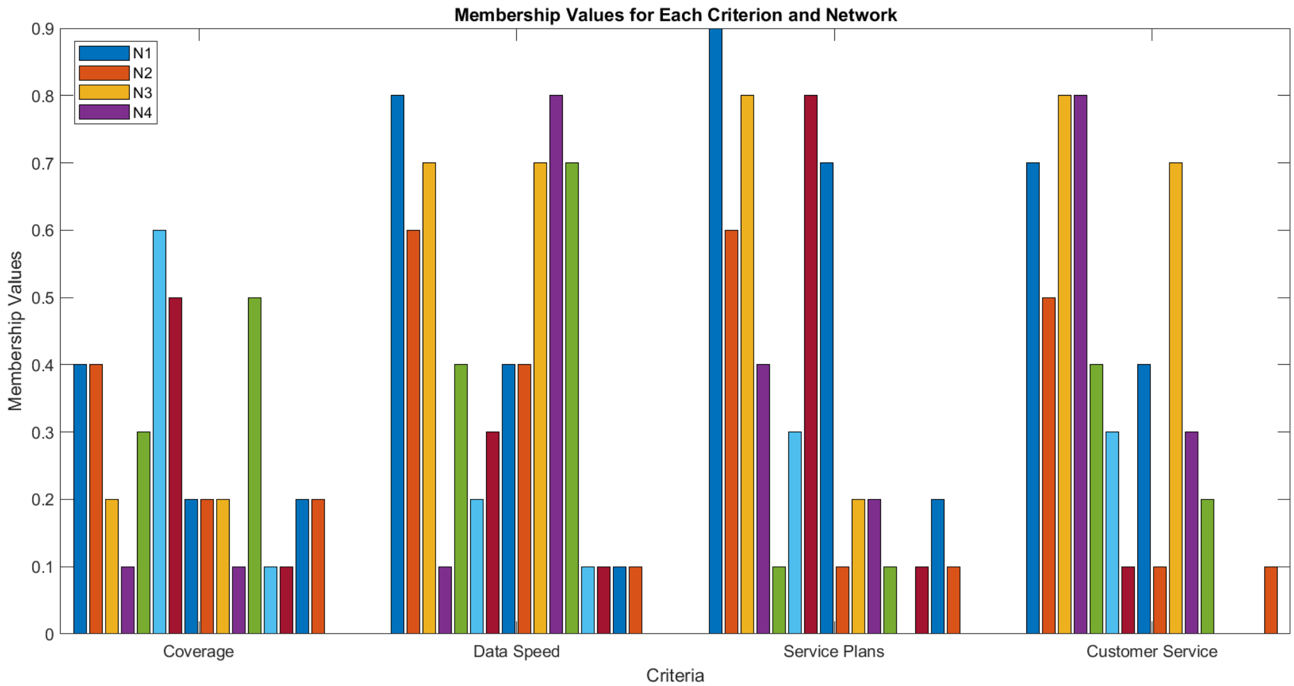

FIGURE 2. Membership values for each Criterion and Network

The weights are given as $\omega = (0.4, 0.5, 0.2, 0.1)$. Here in $\mathcal{D}$ each row represents a networks, and each column represents a criterion.

$$\mathcal{D} = \begin{pmatrix} (0.4, 0.3, 0.2, 0.1) & (0.4, 0.6, 0.2, 0.1) & (0.2, 0.5, 0.1, 0.2) & (0.1, 0.2, 0.5, 0.2) \\ (0.8, 0.4, 0.4, 0.1) & (0.6, 0.2, 0.7, 0.1) & (0.7, 0.3, 0.8, 0.1) & (0.1, 0.4, 0.7, 0.1) \\ (0.9, 0.1, 0.1, 0) & (0.6, 0.3, 0.2, 0.1) & (0.8, 0.8, 0.2, 0.2) & (0.4, 0.7, 0.1, 0.1) \\ (0.7, 0.4, 0.1, 0) & (0.5, 0.3, 0.7, 0) & (0.8, 0.1, 0.3, 0) & (0.8, 0.4, 0.2, 0.1) \end{pmatrix}$$

To aggregate the evaluations, we shall compute the quadripartitioned neutrosophic arithmetic average for each criterion across all networks. This will give a single quadripartitioned neutrosophic value for each criterion.

Let's denote the aggregated quadripartioned neutrosophic values as $\mathfrak{Q}_i$ for each criterion i.

$\mathfrak{Q}_1 = \left(\frac{0.4+0.4+0.2+0.1}{4}, \frac{0.3+0.6+0.5+0.2}{4}, \frac{0.2+0.2+0.1+0.5}{4}, \frac{0.1+0.1+0.2+0.2}{4}\right) = (0.275, 0.4, 0.25, 0.15)$.

In the similar fashion, we can also compute:

$\mathfrak{Q}_2 = (0.55, 0.325, 0.65, 0.025)$,

$\mathfrak{Q}_3 = (0.675, 0.475, 0.15, 0.1)$,

$\mathfrak{Q}_4 = (0.7, 0.3, 0.325, 0.025)$.

Now, let's calculate the scores and accuracy values for each network based on these aggregated values.

To calculate the scores and accuracy values for each networks, we shall use the aggregated quadripartitioned neutrosophic values obtained in the previous step. The score function

$\mathfrak{S}(\mathfrak{Q}_1) = 0.729$

$\mathfrak{S}(\mathfrak{Q}_2) = 0.8458$

$\mathfrak{S}(\mathfrak{Q}_3) = 0.8745$

$\mathfrak{S}(\mathfrak{Q}_4) = 0.8313.$

The accuracy function $\mathfrak{A}(\mathfrak{Q}_1) = 0.166$

$\mathfrak{A}(\mathfrak{Q}_2) = 0.219$

$\mathfrak{A}(\mathfrak{Q}_3) = 0.257$

$\mathfrak{A}(\mathfrak{Q}_4) = 0.221$

Now, let's rank the networks based on their scores and select the one with the highest rank as the best network (in Fig 3). Based on the calculated scores and accuracy values, we can rank the networks as follows: $\mathfrak{N}_3 > \mathfrak{N}_2 > \mathfrak{N}_4 > \mathfrak{N}_1$ Hence the best network is $\mathfrak{N}_3$.

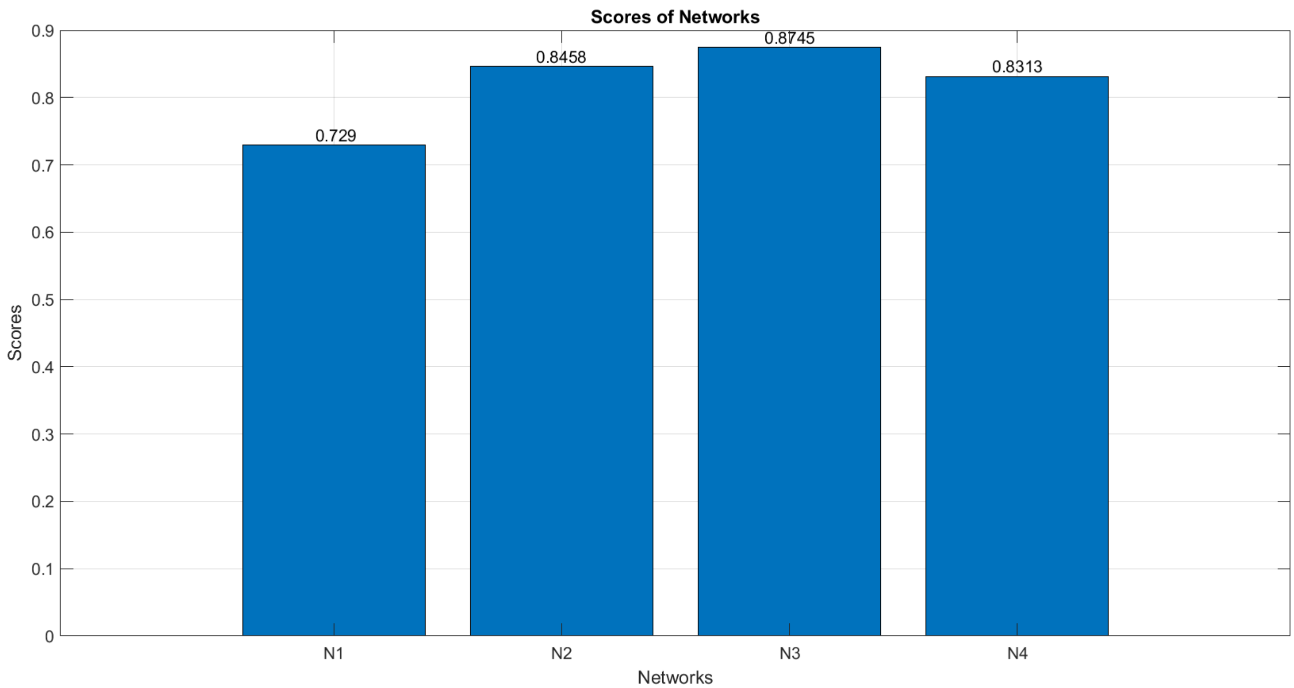

FIGURE 3. Scores of the networks

Limitations

Quadripartitioned neutrosophic graphs (QNGs) are an advanced mathematical tool with potential applications in decision-making and uncertainty modeling. However, like any emerging concept, QNGs have certain limitations, including:

Complexity in Computation:

Handling and analyzing QNGs requires significant computational effort, especially for large-scale graphs, as they involve multiple degrees of membership (truth, indeterminacy, falsity, and hesitation).

Interpretability Challenges:
The interpretation of results derived from QNGs can be complex for non-specialists, limiting their applicability in real-world scenarios where stakeholders may lack technical expertise.

Data Dependency:
QNGs rely on accurate and detailed input data for constructing membership functions. Any bias or error in the data can significantly affect the reliability of the results.

Limited Theoretical Development:
As a relatively new field, QNGs lack comprehensive theoretical frameworks, algorithms, and tools compared to traditional graph theories, which limits their widespread application.

Scalability Issues:
The practical implementation of QNGs on large datasets, such as social networks or supply chain systems, faces scalability problems due to their mathematical and computational complexity.

Integration with Existing Models:
Integrating QNGs with other models or frameworks, such as machine learning or optimization techniques, is still underexplored, which restricts their versatility.

Conclusion

In this work, we introduced and established the concept of m-polar neutrosophic structures (m-PNS) and their corresponding graphs, accompanied by several foundational results. Operations such as the Cartesian product and composition of m-polar neutrosophic graphs (m-PNGs) have been systematically defined and explored, demonstrating their applicability through illustrative examples. These developments provide a robust framework for modeling and analyzing complex systems involving uncertainty and multi-polarity.

Looking ahead, we aim to extend this study to investigate the algebraic properties of m-polar quadripartitioned neutrosophic graphs, focusing on their structural behavior and potential applications in real-world problems. The incorporation of score functions further enhances decision-making processes by providing a comprehensive assessment that balances the relative importance of multiple criteria. This ensures that decisions are not only mathematically sound but also aligned with users' preferences and priorities.

Through our proposed framework, practical applications such as mobile network selection can be significantly improved, enabling users to make informed decisions that optimize performance, affordability, and overall satisfaction. Additionally, we foresee the application of these concepts in diverse fields, such as supply chain management, social network analysis, and computational biology, paving the way for further advancements in the study of neutrosophic graphs

Compliance with ethical standards

Data availability statements Not applicable.

Conflict of interest: The authors declare that they have no conflict of interest.

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References

[1] M. Akram and N. Waseem, Certain metrics in m-polar fuzzy graphs. *New Mathematics and Natural Computation*, 12(02),2016, 135-155.

[2] M. Akram and D. Cecco, m-Polar Fuzzy Graphs. Cham:.(2019) Springer International Publishing.

[3] M. Akram and G. Shahzadi, Operations on single-valued neutrosophic graphs, Journal of Uncertain Systems, 11(1) (2017), 1-26.

[4] Talebi, A. A., Rashmanlou, H., & Sadati, S. H. (2020). New concepts on m-polar interval-valued intuitionistic fuzzy graph.

[5] N. D. Belnap, A useful four-valued logic. In Modern uses of multiple-valued logic (pp. 5-37). Springer, Dordrecht (1977).

[6] S. Broumi, F. Smarandache, Talea, M. and Bakali., "Single-valued neutrosophic graphs: Degree, Order and Size, IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2016.

[7] S. Broumi, T. Mohamed, B. Assia and F. Smarandache, "Single-valued neutrosophic graphs, The Journal of New Theory, Vol No. 10, pp. 861-101, 2016.

[8] R. Chatterjee, , P. Majumdar, & S.k. Samanta, On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets. Journal of Intelligent & Fuzzy Systems, 30(4) (2016), 2475-2485.

[9] I. Deli and S. Broumi , "Neutrosophic soft relations and some properties, Annals of Fuzzy Mathematics and Informatics, Vol 9, No. 1, pp. 169-182, 2014.

[10] G. Ghorai and M. Pal, Some properties of m-polar fuzzy graphs. Pacific Science Review A: *Natural Science and Engineering,* 18(1),2016, 38-46.

[11] Rashmanlou, H., & Borzooei, R. A. (2016). New concepts of interval-valued intuitionistic (S, T)-fuzzy graphs. Journal of Intelligent & Fuzzy Systems, 30(4), 1893-1901.

[12] Rashmanlou, H., Borzooei, R. A., Samanta, S., & Pal, M. (2016). Properties of interval valued intuitionistic (s, t) fuzzy graphs. Pacific Science Review A: Natural Science and Engineering, 18(1), 30-37.

[13] Borzooei, R. A., & Rashmanlou, H. (2016). Cayley interval-valued fuzzy graphs. UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, 78(3), 83-94.

[14] M. R. Hashmi, M. Riaz and F. Smarandache,m-Polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering analysis. International Journal of Fuzzy Systems, 22(1),2020, 273-292.

[15] A. Hassan and M. A. Malik, m-Polar Neutrosophic Graphs, TWMS J. App. and Eng. Math. V.11, N.3, 2021, pp. 659-680,2021

[16] S.S. Hussain , N. Durga, R. Hossein , G. Ganesh , & C. Oscar , New Concepts on Quadripartitioned Single-Valued Neutrosophic Graph with Real-Life Application. International Journal of Fuzzy Systems, 24(3)(2022), 1515-1529.

[17] S. S. Hussain, G. Muhiuddin , N. Durga , & D. Al-Kadi, New Concepts on Quadripartitioned Bipolar Single Valued Neutrosophic Graph, CMES-COMPUTER MODELING IN ENGINEERING & SCIENCES, 130(1) (2022), 559-580.

[18] S.S. Hussain, J. Hussain , I. Rosyida , & S. Broumi, Quadripartitioned Neutrosophic Soft Graphs. In Handbook of Research on Advances and Applications of Fuzzy Sets and Logic (pp. 771-795) (2022). IGI Global.

[19] S. S. Hussain, I. Rosyida , H. Rashmanlou and F. Mofidnakhaei, Interval intuitionistic neutrosophic sets with its applications to interval intuitionistic neutrosophic graphs and climatic analysis. *Computational and Applied Mathematics,* 40(4), 1-20.

[20] K. Mohanta, A. Dey, A. Pal, H. V Long and L. H. Son, A study of m-polar neutrosophic graph with applications. Journal of Intelligent & Fuzzy Systems, 38(4),2020, 4809-4828.

[21] S. Roy, J. G. Lee, A. Pal, & S. K. Samanta, Similarity Measures of Quadripartitioned Single Valued Bipolar Neutrosophic Sets and Its Application in Multi-Criteria Decision Making Problems. *Symmetry*, 12(6),(2020). 1012.

[22] M. Saeed, M. Saqlain and A. Mehmood, Application of similarity measure on m-polar interval-valued neutrosophic set in decision making in sports (38), 2020, Infinite Study.

[23] F. Smarandache, “A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic, Rehoboth: American Research Press, 1999.

[24] F.Smarandache, $n-$valued Refined Neutrosophic Logic and its Applications to Physics. arXiv 2014, arXiv:1407.1041

[25] F. Smarandache, “Neutrosophic Graphs, in his book Symbolic Neutrosophic Theory, Europa, Nova, 2015

[26] H. Wang, F. Smarandache, Y. Zhang and R. Sunderraman, Single-valued neutrosophic sets, Multispace and Multistructure, Vol 4, pp. 410-413, 2010.

[27] H. Wang, F. Smarandache, Y. Zhang and R. Sunderraman, Single-valued neutrosophic sets, Multispace and Multistructure, Vol 4, pp. 410-413, 2010.

[28] J. Ye, Cosine similarity measures for intuitionistic fuzzy sets and their applications, Information and Control, 53(1)(2011), 91-97

[29] Satham Hussain, S., Nagarajan, D., Rashmanlou, H., & Mofidnakhaei, F. (2024). Novel supply chain decision making model under m-polar quadripartitioned neutrosophic environment. Journal of Applied Mathematics and Computing, 1-26.

[30] J. Ye, Generalized Dice measures for multiple attribute decision making under intuitionistic and interval-valued intuitionistic fuzzy environments, Neural Computing and Applications, 30(2018), 3623-3632.

[31] J. Ye, Multicriteria decision-making method using the Dice similarity measure based on the reduct intuitionistic fuzzy sets of interval-valued intuitionistic fuzzy sets, Applied Mathematical Modelling, 36(9)(2012), 4466-4472.

[32] Naeem, K., & Divvaz, B. (2023). Information measures for MADM under m-polar neutrosophic environment. Granular Computing, 8(3), 597-616.

[33] Deli, I., Ali, M., & Smarandache, F. (2015, August). Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. In 2015 International conference on advanced mechatronic systems (ICAMechS) (pp. 249-254). IEEE.

[34] Guo, C., Wang, Y., Yuan, S., & Li, Z. (2023). A Novel Hybrid Multiple Attribute Decision Making Framework for Enterprise Green Marketing Performance Management Evaluation Based on the Triangular Fuzzy Neutrosophic Sets. IEEE Access.

[35] X. Shi, S. Kosari, A.A. Talebi, S.H. Sadati, H. Rashmanlou, Investigation of the main energies of picture fuzzy graph and its applications, *International Journal of Computational Intelligence Systems*, vol. 15(1), P. 31, 2022.

[36] Y. Rao, Muhammad Ahsan Binyamin, Adnan Aslam, Maria Mehtab, Shazia Fazal. On the Planarity of Graphs Associated with Symmetric and Pseudo Symmetric Numerical Semigroups. *Mathematics*, 2023, 11(7):1681.

[37] Z. Shao, S. Kosari, H. Rashmanlou, M. Shoaib, New concepts in intuitionistic fuzzy graph with application in water supplier systems, *Mathematics*, vol. 8, p. 1241, 2020.

[38] KOSARI S., QIANG X., KACPRZYK J., AIN, Q. T., & RASHMANLOU, H. (2024). A Study on Topological Indices in Fuzzy Graphs with Application in Decision Making Problems. Journal of Multiple-Valued Logic & Soft Computing, 42.

[39] KOSARI, S., SHI, X., KACPRZYK, J., CHEN, Z., & RASHMANLOU, H. (2024). A Novel Description of Perfectly Regular Fuzzy Graphs with Application in Psychological Sciences. Journal of Multiple-Valued Logic & Soft Computing, 42.

[40] Khan, W. A., Arif, W., Rashmanlou, H., & Kosari, S. (2024). Interval-valued picture fuzzy hypergraphs with application towards decision making. Journal of Applied Mathematics and Computing, 70(2), 1103-1125.

[41] Kaviyarasu, M., Rashmanlou, H., Kosari, S., Broumi, S., Venitha, R., Rajeshwari, M., & Mofidnakhaei, F. (2024). , Circular Economy Strategies to Promote Sustainable Development using t-Neutrosophic Fuzzy graph. Neutrosophic Sets and Systems, 72(1), 9.

[42] Shao, Z., Kosari, S., Rao, Y., Rashmanlou, H., & Mofidnakhaei, F. (2023). New Kind of Vague Graphs With Novel Application. Journal of Multiple-Valued Logic & Soft Computing, 40.

[43] Mohamed, Z., Ismail, M. M., & Abd El-Gawad, A. (2023). Sustainable supplier selection using neutrosophic multi-criteria decision making methodology. Sustainable Machine Intelligence Journal, 3, 2-1.

[44] Rashmanlou, H., Samanta, S., Pal, M., & Borzooei, R. A. (2015). A study on bipolar fuzzy graphs. Journal of intelligent & fuzzy systems, 28(2), 571-580.

[45] Rashmanlou, H., Samanta, S., Pal, M., & Borzooei, R. A. (2015). Bipolar fuzzy graphs with categorical properties. International Journal of Computational Intelligence Systems, 8(5), 808-818.

[46] Borzooei, R. A., Rashmanlou, H., Samanta, S., & Pal, M. (2016). Regularity of vague graphs. Journal of Intelligent & Fuzzy Systems, 30(6), 3681-3689.

[47] Samanta, S., Pal, M., Rashmanlou, H., & Borzooei, R. A. (2016). Vague graphs and strengths. Journal of Intelligent & Fuzzy Systems, 30(6), 3675-3680.

[48] Romdhini, M. U., Al-Sharqi, F., Nawawi, A., Al-Quran, A., & Rashmanlou, H. (2023). Signless Laplacian energy of interval-valued fuzzy graph and its applications. Sains malaysiana, 52(7), 2127-2137.

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