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A New SDIMSIM Methods with Optimized Region of Stability and Sustainable Development with Applications On Neutrosophic Data

A. Y. J. Almasoodi [1], Mohammed Kadhim Mohsin [2], Sarah A.AL-Ameedee [3], Mohammed Amer Shanyoor [4]

[1,2,3,4] College of Basic Education. Department of Mathematics. University of Babylon. Iraq.
bas397.abdullah.yehaa@uobabylon.edu.iq ; bas926.mohammed.kadhum@uobabylon.edu.iq ; bas387.sarah.abdradia@uobabylon.edu.iq ; bas955.mohammed.amaar@uobabylon.edu.iq ; mkmam1985@gmail.com .
*Correspondence: bas397.abdullah.yehaa@uobabylon.edu.iq

Abstract: General Linear Methods (GLMs) were introduced as logical extensions of the conventional Runge–Kutta and linear multistep methods. In the scenario when second derivatives, in addition to first derivatives, can be computed, a GLM modification known as SGLMs (GLM with second derivative) was created. The development of SDIMSIMs for the second type is presented in this study. For SDIMSIMs construction, we give examples with parameters (p, q, r, s), namely p = q = r = s = 1..., 4, where r is the number of external stages, s is the number of internal stages, p is the order, and q is the stage order. Furthermore, we will prove that those approaches are the largest area contained by the A-Stability and L-Stability properties; in addition to their mathematical importance, SDIMSIMs contribute widely to sustainable development by unlocking complex systems in climate modeling, renewable energy, ecological monitoring, epidemiological studies, etc. SDIMSIMs address this challenge by enhancing the numerical performance of these simulations and thus contributing to decreased resource consumption, lower energy demand and better computational sustainability. In this paper, we conduct construction and stability certificate for SDIMSIMs, and the supporting capacity of SDIMSIM for supporting sustainable solutions to global issues is certified. Also, we apply our study on some neutrosophic data with indeterminacy element (I), and compare the results.

Keywords: SDIMSIMs; L-stability; SGLM, neutrosophic data.

1. Introduction

The second derivative Various variants of the classical Runge–Kutta and linear multistep methods were suggested to be naturally generalized by the General Linear Methods (SGLMs) ([1], [3], [6], and [7]), e.g., integro-differential terms, differential terms within the main formula, and so forth, alternative methods were also suggested in terms of solving differential algebraic systems ([9],[11],[12],[13]). In medical applications such as ([4],[5]), a different approach was used. They are the cutting edge with wide applications in diverse fields important for sustainable development. In climate modeling, it improves the precision and efficacy of the simulations, allowing for the better predictions of long-term climate trends, and aiding climate adaptation strategies (Allen & Ingram, 2002) in [2]. In Ecological research, SDIMSIM offer advanced simulation of complex interactions within ecosystems, facilitating biodiversity conservation. Neutrosophic sets are very useful in the study of real life problems [15], Smarandache presented those sets in [14], and then, they were applied in many different areas of science and theoretical studies [16-19].

The SGLMs extend the framework of the GLMs for numerically solving autonomous ordinary differential equations:

$$y' = f\big(y(x)\big), \quad y: \mathbb{R} \to \mathbb{R}^m, \quad f: \mathbb{R}^m \to \mathbb{R}^m$$

An SGLM for the system is expressed as:

$$\begin{aligned} Y^{[n]} &= h(A \otimes I_m) f\big(Y^{[n]}\big) + h^2(A' \otimes I_m) g\big(Y^{[n]}\big) + (U \otimes I_m) y^{[n-1]}, \\ y^{[n]} &= h(B \otimes I_m) f\big(Y^{[n]}\big) + h^2(B' \otimes I_m) g\big(Y^{[n]}\big) + (V \otimes I_m) y^{[n-1]}, \end{aligned}$$

with coefficient matrices represented as:

$$\begin{bmatrix} A & A' & U \\ B & B' & V \end{bmatrix}$$

For stability analysis, we consider the linear test problem:

$$y' = \xi y$$

where $\xi \in \mathbb{C}$, and stability is governed by the matrix $M(z)$:

$$M(z) = V + z(B + zB')(I_s - zA - z^2 A')^{-1} U$$

The $M(z)$ characteristic polynomial is given by:

$$p(w, z) = \det\big(wI_r - M(z)\big) \qquad (1)$$

2. Order Condition Structure

This section revisits the order conditions for the methods, as presented in [5]. The key approach involves an input vector in the following form:

$$y_i^{[n-1]} = \sum_{k=0}^{p} h^k \alpha_{ik} y^{(k)}(x_{n-1}) + O\,(h^{p+1}), \quad i = 1, 2, \dots, r \tag{2}$$

where α_{ik} are real parameters, and $y_i^{[n]}$ represents the approximation at stage i for the integration point n. The stage values $Y_i^{[n]}$ at step size h are needed to be of order q approximations of the solution at $x_{n-1} + c_i h$:

$$Y_i^{[n]} = \sum_{k=0}^{p} \frac{c_i^k}{k!} h^k y^{(k)}(x_{n-1}) + O\,(h^{q+1}), \quad i = 1,2, \dots, s \tag{3}$$

Additionally, the computed values at the end of the current step for output should satisfy:

$$y_i^{[n]} = \sum_{k=0}^{p} h^k \alpha_{ik} y^{(k)}(x_n) + O\,(h^{p+1}), \quad i = 1,2, \dots, r$$

Let $\alpha_k = [\alpha_{1k}\; \alpha_{2k}\; \dots\; \alpha_{rk}]^T$ for $k = 0,1, \dots, p$. Here, α_0 and α_1 are recognized as the pre-consistency and consistency vectors, respectively (refer to [4]). Let $Z := [1\; z\; \dots\; z^p]^T \in \mathbb{C}^{p+1}$, and define the matrix W as:

$$W = \left[\alpha_0\; \alpha_1\; \dots\; \alpha_p\right]$$

We now present the following theorem:

Theorem 1.

Assume that equation (1) is satisfied by $y^{[n-1]}$. Equations (2) and (3) are then satisfied by the SGLM system of order p and stage order q=p if and only if:

$$\exp(cz) = zA\exp(cz) + z^2 A' \exp(cz) + UWZ + O(z^{p+1}),$$

$$\exp(z)WZ = zB\exp(cz) + z^2 B' \exp(cz) + VWZ + O(z^{p+1}).$$

The product WZ is the same vector w(z) with a polynomial-valued shape as explained in [5], where the exponential function is applied element-wise to a vector.

When $U = I_s$, in Theorem 1 the stage order conditions will be satisfied if and only if:

$$W = C - ACK - A'CK^2,$$

Where the matrix of Vandermonde is $C = \left(C_{ij}\right) \in \mathbb{R}^{s\times(p+1)}$, with coefficients given by:

$$C_{ij} = \frac{c_i^{j-1}}{(j-1)!}, \quad 1 \le i \le s,\ 1 \le j \le p+1,$$

the matrix of shifting $K \in \mathbb{R}^{(p+1)\times(p+1)}$ is described by $K = \left[0\ e_1\ \dots\ e_p\right]$, where e_j is the j-th vector of unity (see [5]).

3. Construction of SDIMSIMs

Let $r = s$, and $U = I_s$. The coefficient matrix B is computed as:

$$B = B_0 - AB_1 - AB_2 - VB_3 - (B - VA)B_4 + VA$$

The matrices B_0, B_1, B_2, B_3, and B_4 are defined as:

$B_0 = \int_{1+c_i}^{1} \phi_j\,(x)dx, \quad B_1 = \phi_j(1 + c_i), \quad B_2 = \phi_j{}'(1 + c_i), \quad B_3 = \int_0^{c_i} \phi_j\,(x)dx,$
As in [1] and [6].

a) SDIMSIMs of Type 2 (with $p = q = r = s = 1$)

As given in Theorem 1, the SDIMSIMs order of type 2 is less than 3. The matrix of Coefficients for the method of order 2 is given by:

$$\begin{bmatrix} \lambda & \mu & 1 \\ 1 & \mu & 1 \end{bmatrix}.$$

With abscissae vector $c = [1]^T$, coefficients are:

$$\begin{bmatrix} 0.131631473311373 & 0.004702134734068 & 1 \\ 1 & 0.004702134734068 & 1 \end{bmatrix}.$$

The area of the stability region is $S = -175.7794$, where the sign is negative because the area is on the left side. Figure 1 displays it in contrast to Figures 2, 3 the works of A. Abdi and G. Hojjati etcetera in ([6], [8]).

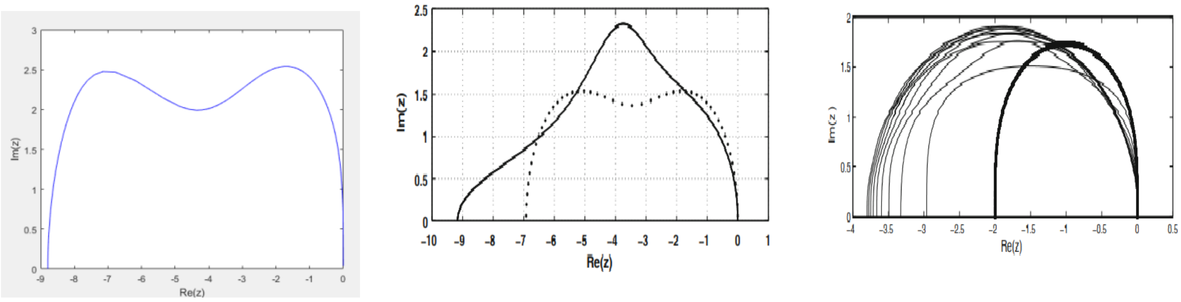

Figure 1: We compare the maximal area of the stability region for this paper's method of order 1 on the left side, the method at [6] in the middle and [8] on the right.

b) SDIMSIMs of Type 2 (with $p = q = r = s = 2$)

By selecting the abscissae vector $c = [-1,1]^T$, coefficients for the method of order 2 at least are:

$$A = \begin{pmatrix} 0.275562183474736 & 0 \\ 3.193741017519553 & 0.275562183474736 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.0124050750 & 0 \\ 0.191395813953893 & 0.0124050750 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.836540100273188 & 0.1634598997 \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} 0.04166288324 & 0.002027732315 \end{pmatrix}$$

$$B = \begin{pmatrix} 1.171861588 & -0.0124001620 \\ 0.05535242576 & 0.1285704909 \end{pmatrix}$$

The area of the stability region is $S = -164.5267$. Figure 4 displays the stability region in contrast to Figures 5 and 6 from the works ([6],[8]).

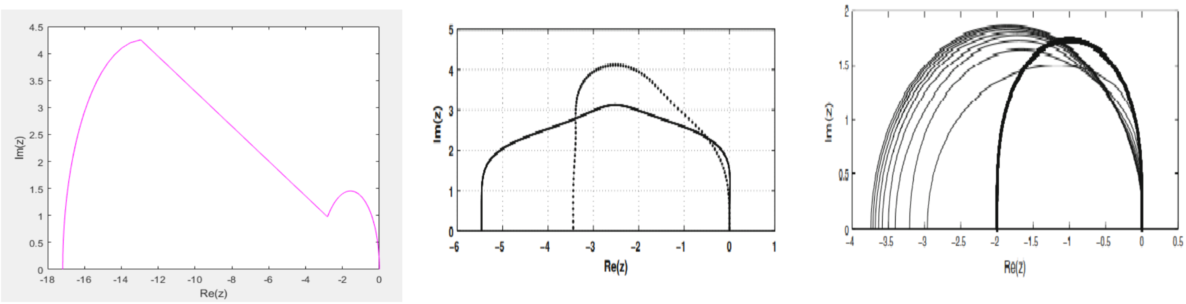

Figure 2: The method of order 2's maximum region of stability on the left side, [6] of the same order

at the middle and [8] to the right side.

c) SDIMSIMs of Type 2 (with $p = q = r = s = 3$)

By choosing abscissae vector $c = [-1, 0.222610896326282, 0.997211035225582]^T$, coefficients of the method of order 3 are given by:

$$A = \begin{pmatrix} 0.185949019058918 & 0 & 0 \\ -0.029251166425091 & 0.185949019058918 & 0 \\ 0.180266217726395 & 0.762611999618729 & 0.185949019058918 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.015294685296723 & 0 & 0 \\ -0.004313489973732 & 0.015294685296723 & 0 \\ 0.069788314022775 & 0.175636485019311 & 0.015294685296723 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.227767612412112 & 0.188598515058830 & 0.583633872529058 \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} 0.04340094012 & 0.1053919568 & 0.008926496410 \end{pmatrix}$$

$$B = \begin{pmatrix} -0.4401257211 & -0.1020157260 & -0.4736454833 \\ 0.8117363413 & 1.149846336 & 0.7782165791 \\ 0.614207097 & 0.952317092 & 0.580687335 \end{pmatrix}$$

Area: $S = -65.6410$. See Figure 7 in compare to 8,9 from ([1],[7]).

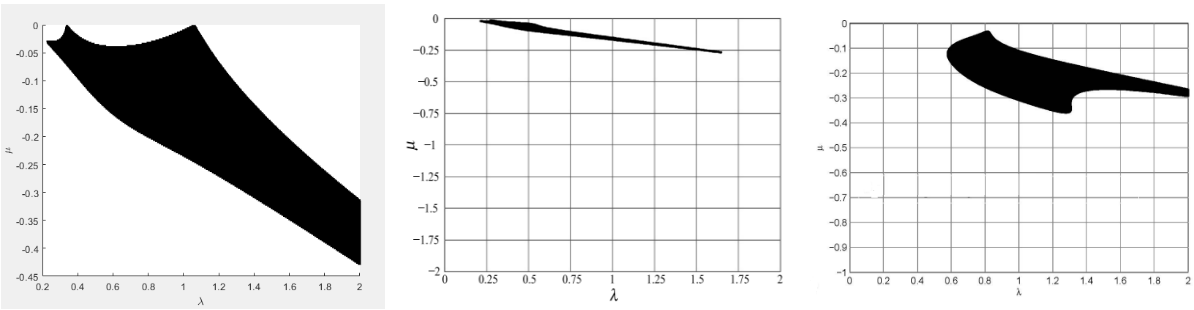

Figure 3: The area of L-stability for methods of order 3 to the left side, [1] at the middle and the work of [7] to the right side.

d) SDIMSIMs of Type 2 (with $p = q = r = s = 4$)

By choosing abscissae vector $c = [-1, 0, 0.5, 1]^T$, coefficients of the method of order 4 are given by:

$$A = \begin{pmatrix} -0.047332764332787 & 0 & 0 & 0 \\ 0.991453136844788 & -0.047332764332787 & 0 & 0 \\ 0.759226970197592 & 0.537348733117443 & -0.047332764332787 & 0 \\ 0.680653072026738 & 0.838351438070254 & 0.194437929981731 & -0.047332764332787 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.105956042213680 & 0 & 0 & 0 \\ 0.640558958225514 & 0.105956042213680 & 0 & 0 \\ 0.711684069906730 & 0.130936627137292 & 0.105956042213680 & 0 \\ 0.771001762263965 & 0.271738039373422 & 0.166677332416951 & 0.105956042213680 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.352677084796130 & 0.754516757290226 & 0.097257807025064 & -0.204451649 \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} 0.4322649863 & 0.03712292834 & -0.02377240315 & -0.02166288755 \end{pmatrix}$$

$$B = \begin{pmatrix} 1.221887526 & 0.4009782561 & 0.5114759583 & 0.5655098563 \\ 1.230434390 & 0.4095251193 & 0.5200228215 & 0.5740567195 \\ 1.425311823 & 0.6044025528 & 0.7149002550 & 0.7689341530 \\ 1.508445086 & 0.687535816 & 0.798033518 & 0.852067416 \end{pmatrix}$$

Area: $S = -37.8217$. See Figure 10 in compare to 11, 12 from ([1], [7]).

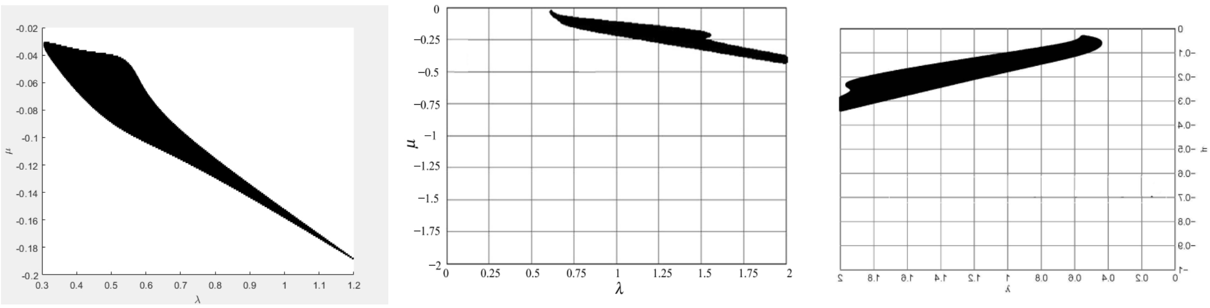

Figure 4: The area of L-stability for methods of order 4 to the left side, the work [1] at the middle and [7] to the right side.

Application on Neutrosophic Data

a) SDIMSIMs of neutrosophic Type 2 (with $p = q = r = s = 2 + I$)

By selecting the abscissae vector $c = [-1 + I, 1 + I]^T$, coefficients for the method of order 2 at least are:

$$A = \begin{pmatrix} 0.275562183474736 + I & I \\ 3.193741017519553 + I & 0.275562183474736 + I \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.0124050750 + I & I \\ 0.191395813953893 + I & 0.0124050750 + I \end{pmatrix}$$

$$V = \begin{pmatrix} 0.836540100273188 + I & 0.1634598997 + I \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} 0.04166288324 + I & 0.002027732315 + I \end{pmatrix}$$

$$B = \begin{pmatrix} 1.171861588 + I & -0.0124001620 + I \\ 0.05535242576 + I & 0.1285704909 + I \end{pmatrix}$$

The area of the stability region is $S = -164.5267 + I$.

b) SDIMSIMs of Type 2 (with $p = q = r = s = 3 + I$)

By choosing abscissae vector $c = [-1, 0.222610896326282 + I, 0.997211035225582 + I]^T$, coefficients of the method of order 3 are given by:

$$A = \begin{pmatrix} 0.185949019058918 + I & 1 + I & 2 + I \\ -0.029251166425091 + I & 0.185949019058918 + I & 1 + I \\ 0.180266217726395 + I & 0.762611999618729 + I & 0.185949019058918 + I \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.015294685296723 + I & I & I \\ -0.004313489973732 + I & 0.015294685296723 + I & I \\ 0.069788314022775 + I & 0.175636485019311 + I & 0.015294685296723 + I \end{pmatrix}$$

$$V = (0.227767612412112 + I \quad 0.188598515058830 + I \quad 0.583633872529058 + I)$$

$$\hat{B} = (0.04340094012 + I \quad 0.1053919568 + 3I \quad 0.008926496410 + 2I)$$

$$B = \begin{pmatrix} -0.4401257211 + 3I & -0.1020157260 + I & -0.4736454833 + I \\ 0.8117363413 + I & 1.149846336 + 3I & 0.7782165791 + 2I \\ 0.614207097 + 5I & 0.952317092 + I & 0.580687335 + 5I \end{pmatrix}$$

Area: $S = -65.6410 + I$.

c) SDIMSIMs of Type 2 (with $p = q = r = s = 4 + I$)

By choosing abscissae vector $c = [-1, 0, 0.5, 1]^T$, coefficients of the method of order 4 are given by:

$$A = \begin{pmatrix} -0.047332764332787 + I & 2I & 2I & 6I \\ 0.991453136844788 + I & -0.047332764332787 + I & 4I & 2I \\ 0.759226970197592 + I & 0.537348733117443 + I & -0.047332764332787 + I & 9I \\ 0.680653072026738 + I & 0.838351438070254 + I & 0.194437929981731 + I & -0.047332764332787 + I \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 0.105956042213680 & 3 + 2I & 1 + 2I & 7I \\ 0.640558958225514 & 0.105956042213680 & 5 + 2I & 2I \\ 0.711684069906730 & 0.130936627137292 & 0.105956042213680 & 7 + 2I \\ 0.771001762263965 & 0.271738039373422 & 0.166677332416951 & 0.105956042213680 \end{pmatrix}$$

$$V = (0.352677084796130 + I \quad 0.754516757290226 + I \quad 0.097257807025064 + I \quad -0.204451649 + I)$$

$$\hat{B} = (0.4322649863 + I \quad 0.03712292834 + I \quad -0.02377240315 + I \quad -0.02166288755 + I)$$

$$B = \begin{pmatrix} 1.221887526 + I & 0.4009782561 + 3I & 0.5114759583 + 4I & 0.5655098563 + I \\ 1.230434390 + 2I & 0.4095251193 + I & 0.5200228215 + I & 0.5740567195 + 5I \\ 1.425311823 + I & 0.6044025528 + 13I & 0.7149002550 + 7I & 0.7689341530 + I \\ 1.508445086 + 7I & 0.687535816 + I & 0.798033518 + I & 0.852067416 + 9I \end{pmatrix}$$

Area: $S = -37.8217 + I$

4. Conclusion

This paper introduces a new class of second derivative methods, the SDIMSIMs, within the framework of General Linear Methods (GLMs), emphasizing their potential to contribute to sustainable development. By offering enhanced stability and computational efficiency, SDIMSIMs are vital for solving complex differential equations across various fields, from climate science and renewable energy to ecology and public health. The ability to simulate large-scale systems more efficiently directly contributes to sustainability goals by reducing energy consumption and optimizing resources.

The integration of SDIMSIMs into these critical domains supports the development of sustainable solutions to pressing global challenges. Moving forward, these methods will continue to play a crucial role in advancing scientific research, particularly in the quest for sustainable solutions to environmental and health crises.

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