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An Overview of Fermatean Neutrosophic Graphs

Prasanta Kumar Raut[1], Surapati Pramanik [2*] and Bhabani S. Mohanty [3]

[1] Department of Mathematics, Trident Academy of Technology, Bhubaneswar, Odisha, India, email: prasantaraut95@gmail.com

[2] Nandalal Ghosh B.T. College, Panpur, Narayanpur, Dist-North 24 Parganas, West Bengal, India, PIN-743126, email:surapati.math@gmail.com

[3] Department of Statistics and Applied Mathematics, Central University of Tamil Nadu, India, email: bhabanishankar@cutn.ac.in

* Correspondence: surapati.math@gmail.com; Tel.:+919477035544

Abstract: Fermatean Neutrosophic graph theory plays a significant role in the modeling and organization of optimization problems. In real-world scenarios, uncertainty is often present due to undetermined or incomplete information. Consequently, experts face challenges in designing optimization problems using fuzzy graphs. To address concerns about inconsistency, unpredictability, and vagueness in graphical optimization problems, various expansions of graph mathematical concepts have been developed. One such expansion is the concept of Fermatean neutrosophic graphs, which effectively manage uncertainties associated with unspecified and inadequate data in optimization problems. This article explores different types of Fermatean neutrosophic graphs and their operations. Additionally, it provides a numerical example showcasing the application of Fermatean Neutrosophic Sets in decision-making. These operations serve as valuable tools for analyzing and manipulating neutrosophic graphs in optimization problems.

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Keywords: Fuzzy set, Fermatean, fuzzy set, neutrosophic set, single valued neutrosophic set, Fermatean neutrosophic set, Fermatean neutrosophic graph

1. Introduction

Graph theory possesses numerous applications including computer applications, networking, transportation, and system analysis. It acts as a relational model, representing relationships between real-world objects. A graph's vertices and edges represent the objects and their associations in a given problem. Factors such as insufficient data, lack of evidence, or insufficient information frequently lead to inaccurate information in optimization problems. Zadeh [1] initially introduced the idea of the Fuzzy Set (FS) to address uncertainty.

A FS assigns membership degrees ranging from 0 to 1 to objects. Sitara et al. [2] conducted a comprehensive study on fuzzy graphs, showcasing some of their essential characteristics. Fuzzy graphs are significant because they can better handle ambiguity. However, Atanassov [3] noted that FSs are only capable of managing uncertainties by introducing degree of membership. Following this, Atanassov proposed intuitionistic FSs, a more generalized form of FSs. Every element in the intuitionistic FSs has a membership and non-membership function. Smarandache [4] introduced the novel concept known as Neutrosophic Set (NS), considered an effective method for handling vague, indistinguishable, and indetermined information in practical scenarios. Wang et al. [5] suggested the idea of single-valued NS (SVNS) to make NSs more usable in real-life applications. Real-life applications along with theoretical development of NSs were depicted in the studies [6, 7, 8, 9, 10, 11, 12, 13]. Broumi et al. [14] presented the single-valued neutrosophic graphs. Overview of neutrosophic graphs was presented in [15]. NSs have been used for dealing with shortest-path problems [16, 17, 18, 19, 20, 21].

Graph theories have widely studied in fuzzy and bipolar FS environment [22, 23, 24, 25, 26] . Fermatean FS (FFS) [27] was developed to effectively handle indeterminate evidence by expanding the domain of influence for membership and non-membership evaluations. Thamizhendhi et al. [28] introduced the Fermatean Fuzzy Hyper Graphs (FFHGs) and provided definitions and properties for this concept. Jaikum et al. [29] and Anirudh et al. [30] investigated vulnerability parameters in the NS environment. Broumi et al. [31] recently proposed the complex FNS and its application to decision-making. There are numerous applications of Fermatean Neutrosophic Graph (FNG)s in various fields, including shortest-path problems [32, 33].

The following are the primary contributions to the study:

- This paper presents several types of operations on FNGs, including lexicographic product, Cartesian product, union, composition, and join, as well as their properties.
- This study evaluates the significance of a novel class of graphs and looks into their potential applications in decision-making problems.

Fermatean neutrosophic graphs make significant contributions to real-world applications by capturing and representing uncertainty. Neutrosophic graphs can handle uncertain data, while traditional graphs can only represent clear and precise information. This capability is especially useful in fields like social networks, decision-making, and pattern recognition, where uncertainty and ambiguity are common.

Organization of the manuscripts

The subsequent sections of the research are titled as follows: Section-2 highlights some definitions. Section-3 highlights different types of FNGs. Section-4 highlights the different operations of FNGs. Section-5 highlights a numerical example, i.e., the application of Fermatean Neutrosophic Sets (FNS) in decision-making. Section-6 summarizes the conclusions of this article.

2.. **Preliminaries:**

2.1 FNS [27]

A FNS $\hat{A}$ in $\hat{X}$ is defined as: $\widehat{\mathrm{A}} = \{\langle \hat{\mathrm{x}}, \mathrm{T}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}), \mathrm{I}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}), \mathrm{F}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}})\rangle \mid \hat{\mathrm{x}} \in \hat{X}\}$

where $\mathrm{T}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}), \mathrm{I}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}), \mathrm{F}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}): \widehat{\mathrm{X}} \to [0,1]$ represents respectively the membership, indeterminacy, and falsity degrees such that

$0 \le (\mathrm{T}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 + (\mathrm{F}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 \le 1$

$0 \le (\mathrm{I}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 \le 1$

then $0 \le (\mathrm{T}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 + (\mathrm{F}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 + (\mathrm{I}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 \le 2 \ \forall \hat{\mathrm{x}} \in \widehat{\mathrm{X}}$.

2.2 Intervalued FNS [32]

An IFNS $\hat{A}$ in $\hat{X}$ is defined as: $\widehat{\mathrm{A}} = \{\langle \hat{\mathrm{x}}, \mathrm{T}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}), \mathrm{I}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}), \mathrm{F}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}})\rangle \mid \hat{\mathrm{x}} \in \hat{X}\}$

where $\mathrm{T}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}) = [\mathrm{T}_{\widehat{\mathrm{A}}}^{-}(\hat{\mathrm{x}}), \mathrm{T}_{\widehat{\mathrm{A}}}^{+}(\hat{\mathrm{x}})]$, $\mathrm{I}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}) = [\mathrm{I}_{\widehat{\mathrm{A}}}^{-}(\hat{\mathrm{x}}), \mathrm{I}_{\widehat{\mathrm{A}}}^{+}(\hat{\mathrm{x}})]$, and $\mathrm{F}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}} = [\mathrm{F}_{\widehat{\mathrm{A}}}^{-}(\hat{\mathrm{x}}), \mathrm{F}_{\widehat{\mathrm{A}}}^{+}(\hat{\mathrm{x}})]$ indicates the truth, indeterminacy and falsity membership degree.

$0 \le (\mathrm{T}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 + (\mathrm{F}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 \le 1$ and $0 \le (\mathrm{I}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 \le 1$

$$0 \le (\mathrm{T}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 + (\mathrm{F}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 + (\mathrm{I}_{\widehat{\mathrm{A}}}(\hat{\mathrm{x}}))^3 \le 2, \forall \hat{\mathrm{x}} \in \widehat{\mathrm{X}}$$

Means

$$0 \le (\mathrm{T}_{\widehat{\mathrm{A}}}^{+}(\hat{\mathrm{x}}))^3 + \left(\mathrm{F}_{\widehat{\mathrm{A}}}^{+}(\hat{\mathrm{x}})\right)^3 + \left(\mathrm{I}_{\widehat{\mathrm{A}}}^{+}(\hat{\mathrm{x}})\right)^3 \le 2, \forall \hat{\mathrm{x}} \in \widehat{\mathrm{X}}$$

2.3 Fermatean neutrosophic graphs [28]

FNGs combine the ideas of Fermatean graphs and neutrosophic graphs. The vertices and edges are assigned with Fermatean neutrosophic values, which can have true, false, and indeterminate components

A graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ is called FNG, if the following condition holds

Let $\mathrm{G} = (\widehat{\mathrm{P}}, \widehat{\mathrm{R}})$ where $\widehat{\mathrm{P}}$ is set on $\hat{X}$ and $\widehat{\mathrm{R}}$ is a relation on $\hat{X}$ then:

$$\mathrm{T}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}) \le \min\{\mathrm{T}_{\widehat{\mathrm{P}}}(\hat{\mathrm{u}}), \mathrm{T}_{\widehat{\mathrm{P}}}(\hat{\mathrm{v}})\}$$

$$\mathrm{I}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}) \ge \max\{\mathrm{I}_{\widehat{\mathrm{P}}}(\hat{\mathrm{u}}), \mathrm{I}_{\widehat{\mathrm{P}}}(\hat{\mathrm{v}})\}$$

$$\mathrm{F}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}) \ge \max\{\mathrm{F}_{\widehat{\mathrm{P}}}(\hat{\mathrm{u}}), \mathrm{F}_{\widehat{\mathrm{P}}}(\hat{\mathrm{v}})\}$$

and $0 \le (\mathrm{T}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}))^3 + \left(\mathrm{F}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}})\right)^3 + \left(\mathrm{I}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}})\right)^3 \le 2, \forall \hat{\mathrm{u}}, \hat{\mathrm{v}} \in \hat{X}$

Where, $\mathrm{T}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}): \hat{X} \times \hat{X} \to [0,1], \mathrm{I}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}): \hat{X} \times \hat{X} \to [0,1], \mathrm{F}_{\widehat{\mathrm{R}}}(\hat{\mathrm{u}}, \hat{\mathrm{v}}): \hat{X} \times \hat{X} \to [0,1]$ represents three membership degree such as truth, indeterminacy and falsity of $\hat{R}$.

3. Different Types of FNGs

We provide regular FNGs, strong FNGs, and uniform FNGs in this section.

3.1 Regular FNGs

A regular FNG is a specialized type of graph in the domain of FNSs, characterized by nodes and edges that carry degrees of truth, indeterminacy, and falsity membership, all defined within the Fermatean neutrosophic framework.

Formally, a FNG $G = (V, E, T, I, F)$ satisfies the following:

1. Fermatean Neutrosophic Degrees:
   For each vertex $v \in V$ and edge $e \in E$, there are functions:
   - $T(v), T(e): V \cup E \to [0,1]$
   - $I(v), I(e): V \cup E \to [0,1]$

- $F(v), F(e): V \cup E \rightarrow [0,1]$ These degrees satisfy the Fermatean neutrosophic condition:

$(T(x))^3 + (F(x))^3 \leq 1 and (T(x))^3 + (F(x))^3 + (I(x))^3 \leq 2, \forall x \in \mathrm{V} \cup \mathrm{E}$.

A Regular Fermatean Neutrosophic Graph (RFNG) is a type of graph in which every vertex has the same degree under the Fermatean neutrosophic environment, meaning that each vertex has an equal number of adjacent vertices, while considering membership, non-membership, and indeterminacy in a Fermatean neutrosophic setting.

Example of a Regular Fermatean Neutrosophic Graph:

Consider a 3-regular Fermatean neutrosophic graph with 4 vertices $V = \{v_1, v_2, v_3, v_4\}$ and 4 edges $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$ forming a cycle.Each vertex has exactly 2 adjacent vertices, making it 2-regular.

Fermatean Neutrosophic Representation:

For each edge (v_i, v_j), assign Fermatean neutrosophic values:

- Membership M_{ij}: The degree of connectivity between v_i *and* v_j
- Non-membership N_{ij}: The degree of non-connection.
- Indeterminacy I_{ij}: The uncertainty in the connection.

  Consider three membership value
- $M_{ij} = 0.8$, meaning the confidence in edge existence is high.
- $N_{ij} = 0.2$, meaning the confidence in non-existence is low.
- $I_{ij} = 0.1$, representing a small amount of uncertainty.

These values satisfy the Fermatean neutrosophic condition:

$$(M_{ij})^2 + (N_{ij})^2 + (I_{ij})^2 \leq 1$$

$$(0.8)^2 + (0.2)^2 + (0.1)^2 \leq 1$$

$$0.64 + 0.04 + 0.01 \leq 1$$

$$0.69 \leq 1$$

Thus, this is a valid 2-regular Fermatean neutrosophic graph**.**

3.2 Strong FNGs

A regular FNG is a specialized type of graph in the domain of FNSs, characterized by nodes and edges that carry degrees of truth, indeterminacy, and falsity membership, all defined within the Fermatean neutrosophic framework.

Formally, a FNG $G = (V, E, T, I, F)$ satisfies the following:

1. Strong Relationship Condition:

   For any edge $e = (v_i, v_j)$

- The truth-membership degree $T(e)$ dominates, meaning it is significantly higher than the indeterminacy $I(e)$ and falsity $F(e)$ degrees: $T(e) > I(e)$and $T(e) > F(e)$.

Example of a Strong Fermatean Neutrosophic Graph:

Consider a 3-regular Strong Fermatean neutrosophic graph with 4 vertices $V = \{v_1, v_2, v_3, v_4\}$ and 5 edges $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3)\}$

Fermatean Neutrosophic Representation:

For each edge (v_i, v_j), assign Fermatean neutrosophic values:

- High Membership Degree $M_{ij} \geq 0.9$

- Low membership Degree $N_{ij} \leq 0.1$

- Low Indeterminacy Degree $I_{ij} \ \leq 0.1$

Edge (v_i, v_j)	Membership(M_{ij})	Non-Membership(N_{ij})	Indeterminacy(I_{ij})
(v_1, v_2)	0.95	0.05	0.05
(v_2, v_3)	0.92	0.07	0.06
(v_3, v_4)	0.93	0.06	0.05
(v_4, v_1)	0.94	0.05	0.06
(v_1, v_3)	0.91	0.08	0.07

These values satisfy the Fermatean neutrosophic condition:

$$(M_{ij})^2 + (N_{ij})^2 + (I_{ij})^2 \leq 1$$

$$(0.95)^2 + (0.5)^2 + (0.5)^2 \leq 1$$

$$0.9025 + 0.0025 + 0.0025 \leq 1$$

$$0.9075 \leq 1$$

Thus, this is a valid example of strong neutrosophic graph.

3.3 Uniform FNGs

A uniform FNG (RFNG) is a FNG where the Fermatean neutrosophic degrees (truth, indeterminacy, and falsity) are consistent across all vertices and edges, ensuring uniformity in the representation of uncertainty and vagueness throughout the graph.

Formally, a FNG $G = (V, E, T, I, F)$ satisfies the following:

1. uniformity conditions

- All vertices $v \in V$ have identical Fermatean neutrosophic degrees $T(v), I(v), F(v)$.

- All edges $e \in E$ have identical Fermatean neutrosophic degrees $T(e), I(e), F(e)$.

  i.e. $T(v_1) = T(v_2) = \cdots = T(v_n)$, $I(v_1) = \ I(v_2) = \cdots \ = \ I(v_n)$, $F(v_1) = F(v_2) = \cdots = F(v_n)$, for all $v \in V$, and similarly for $e \in E$.

4. Operations on FNGs

This section introduces several significant graph-theoretic operations applied to FNGs, accompanied by key results and illustrative examples

Assume that $G_1 = (P_1, R_1)$ and $G_2 = (P_2, R_2)$ are two FNGs, which relate to the classical graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Here, P_1 & P_2 represent the Fermatean neutrosophic vertex sets corresponding to V_1 & V_2 , while R_1 & R_2 denote the Fermatean neutrosophic edge sets corresponding to E_1 & E_2 .

The subsequent section explores some of these operations within the framework of FNS theory and examines their properties in detail.

4.1 Cartesian Product of FNGs

The Cartesian product of two FNGs $G_1$and G_2 is denoted by $G_1 \times G_2$ and defined as : $G_1 \times G_2 = (P_1 \times P_2, R_1 \times R_2)$, where

$$T_{P_1\times P_2}(u_1, u_2) = min\{T_{P_1}(u_1), T_{P_2}(u_2)\}$$
$$I_{P_1\times P_2}(u_1, u_2) = max\{I_{P_1}(u_1), I_{P_2}(u_2)\}$$
$$F_{P_1\times P_2}(u_1, u_2) = max\{F_{P_1}(u_1), F_{P_2}(u_2)\}$$
$$\forall\ (u_1, u_2) \in (V_1, V_2)$$

The membership value of the edges in $G_1 \times G_2$ can be computed as

$$T_{R_1\times R_2}((u, u_2), (u, v_2)) = min\{T_{P_1}(u), T_{R_2}(u_2, v_2)\}$$
$$I_{R_1\times R_2}((u, u_2), (u, v_2)) = max\{I_{P_1}(u), I_{R_2}(u_2, v_2)\}$$
$$F_{R_1\times R_2}((u, u_2), (u, v_2)) = max\{F_{P_1}(u), F_{R_2}(u_2, v_2)\}$$
$$\forall\ u \in V_1, (u_2, v_2) \in E_2$$
$$T_{R_1\times R_2}((u_1, \beta), (v_1, \beta)) = min\{T_{R_1}(u_1, v_1), T_{P_2}(\beta)\}$$
$$I_{R_1\times R_2}((u_1, \beta), (v_1, \beta)) = max\{I_{R_1}(u_1, v_1), I_{P_2}(\beta)\}$$
$$F_{R_1\times R_2}((u_1, \beta), (v_1, \beta)) = max\{F_{R_1}(u_1, v_1), F_{P_2}(\beta)\}$$
$$\forall\ \beta \in V_2, (u_1, v_1) \in E_1$$

4.2 Composition of FNGs

The composition of two FNGs G_1 and G_2, denoted by $G_1 \circ G_2$, and defined as: $G_1 \circ G_2 = (P_1 \circ P_2, R_1 \circ R_2)$ where,

$$T_{P_1\circ P_2}(u_1, u_2) = min\{T_{P_1}(u_1), T_{P_2}(u_2)\}$$
$$I_{P_1\circ P_2}(u_1, u_2) = max\{I_{P_1}(u_1), I_{P_2}(u_2)\}$$
$$F_{P_1\circ P_2}(u_1, u_2) = min\{F_{P_1}(u_1), F_{P_2}(u_2)\}$$
$$\forall\ (u_1, u_2) \in (V_1, V_2)$$
$$T_{R_1\circ R_2}((\beta, u_2), (\beta, v_2)) = min\{T_{P_1}(\beta), T_{R_2}(u_2, v_2)\}$$
$$I_{R_1\circ R_2}((\beta, u_2), (\beta, v_2)) = max\{I_{P_1}(\beta), I_{R_2}(u_2, v_2)\}$$
$$F_{R_1\circ R_2}((\beta, u_2), (\beta, v_2)) = max\ \{F_{P_1}(\beta), F_{R_2}(u_2, v_2)\}$$
$$\forall\ \beta \in V_1, (u_2, v_2) \in E_2$$
$$T_{R_1\circ R_2}((u_1, \gamma), (v_1, \gamma)) = min\{\ T_{R_1}(u_1, v_1), T_{P_2}(\gamma)\}$$
$$I_{R_1\circ R_2}((u_1, \gamma), (v_1, \gamma)) = min\{\ I_{R_1}(u_1, v_1), I_{P_2}(\gamma)\}$$

$$F_{R_1 \circ R_2}((u_1, \gamma), (v_1, \gamma)) = min\{ F_{R_1}(u_1, v_1), F_{P_2}(\gamma)\}$$
$$\forall \gamma \in V_2, (u_1, v_1) \in E_1$$
$$T_{R_1 \circ R_2}((u_1, u_2), (v_1, v_2)) = min\{T_{P_2}(u_2), T_{P_2}(v_2), T_{R_1}(u_1, v_1)\}$$
$$I_{R_1 \circ R_2}((u_1, u_2), (v_1, v_2)) = max\{I_{P_2}(u_2), I_{P_2}(v_2), I_{R_1}(u_1, v_1)\}$$
$$F_{R_1 \circ R_2}((u_1, u_2), (v_1, v_2)) = max\{F_{P_2}(u_2), F_{P_2}(v_2), F_{R_1}(u_1, v_1)\}$$
$$\forall (u_1, u_2), (v_1, v_2) \in E_1$$

Example 4.2.1

Consider two FNGs G_1 , and G_2, as presented below. Then the composition of two graph $G_1 \circ G_2$, are shown graphically in Figure below.

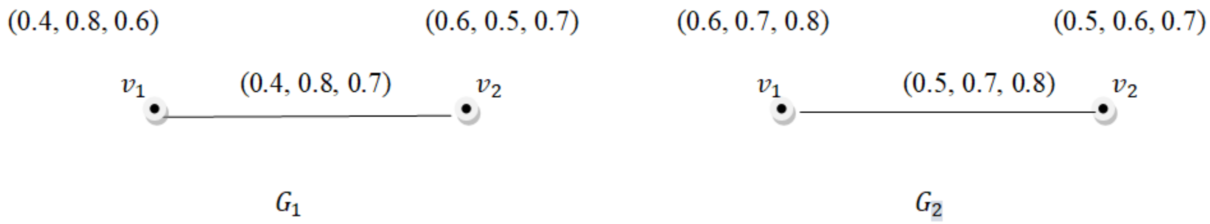

Fig. 1: Two FNGs

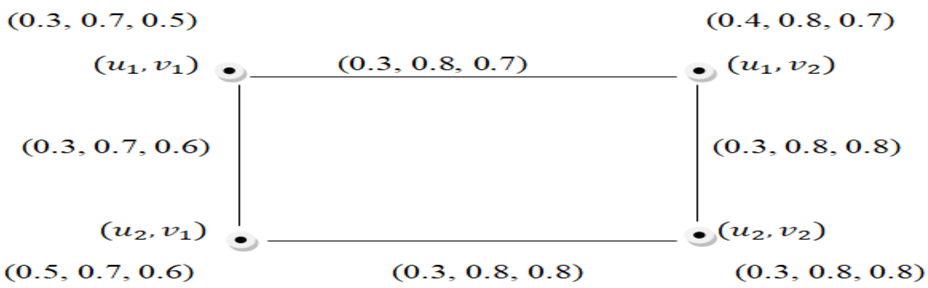

Fig. 2: Composition of two FNGs

4.3 Lexicographic product of FNGs

The lexicographic product of two FNGs G_1 and G_2 , denoted by $G_1 \cdot G_2$, and defined as: $G_1 \cdot G_2 = (P_1 \cdot P_2, R_1 \cdot R_2)$ where

$T_{P_1 \cdot P_2}(u_1, u_2) = min\{T_{P_1}(u_1), T_{P_2}(u_2)\}$

$I_{P_1 \cdot P_2}(u_1, u_2) = max\{I_{P_1}(u_1), I_{P_2}(u_2)\}$

$F_{P_1 \cdot P_2}(u_1, u_2) = max\{F_{P_1}(u_1), F_{P_2}(u_2)\}$

$\forall (u_1, u_2) \in (V_1, V_2)$

$T_{R_1 \cdot R_2}((\beta, u_2), (\beta, v_2)) = min\{T_{P_1}(\beta), T_{R_2}(u_2, v_2)\}$

$I_{R_1 \cdot R_2}((\beta, u_2), (\beta, v_2)) = max\{I_{P_1}(\beta), I_{R_2}(u_2, v_2)\}$

$F_{R_1 \cdot R_2}((\beta, u_2), (\beta, v_2)) = max\{F_{P_1}(\beta), F_{R_2}(u_2, v_2)\}$

$\forall \beta \in V_1, (u_2, v_2) \in E_2$

$T_{R_1 \cdot R_2}((u_1, u_2), (v_1, v_2)) = min\{T_{R_1}(u_1, v_1), T_{R_2}(u_2, v_2)\}$

$I_{R_1 \cdot R_2}((u_1,u_2),(v_1,v_2)) = max\{I_{R_1}(u_1,v_1), I_{R_2}(u_2,v_2)\}$

$F_{R_1 \cdot R_2}((u_1,u_2),(v_1,v_2)) = max\ \{F_{R_1}(u_1,v_1), F_{R_2}(u_2,v_2)\}$

$\forall\ (u_1,v_1) \in E_1, (u_2,v_2) \in E_2$

Example 4.3.1

Consider two FNGs G_1 G_2, as presented below. Then the lexicographic product of two graph $G_1 \cdot G_2$, are shown graphically in Figure below.

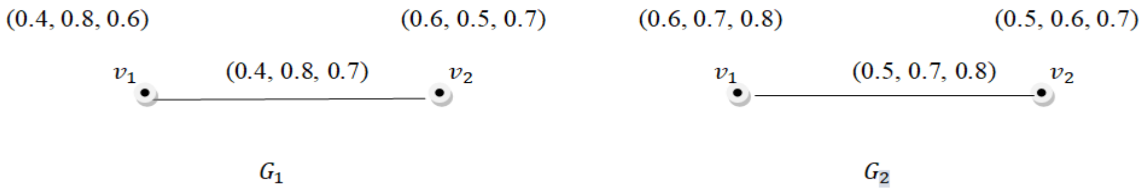

Fig. 3: Two FNGs

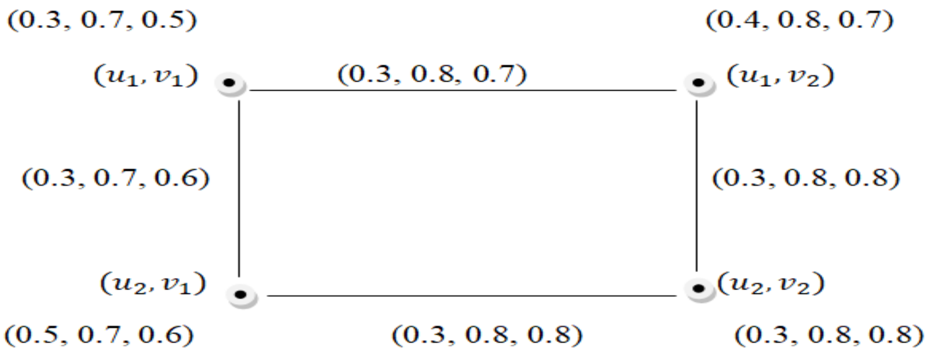

Fig. 4: Lexicographic product of two FNGs

4.4 Union of FNGs

The union of two FNGs G_1 and G_2, denoted by $G_1 \cup G_2$, and defined as: $G_1 \cup G_2 = (P_1 \cup P_2, R_1 \cup R_2)$ where

$$T_{P_1 \cup P_2}(u) = \begin{cases} T_{P_1}(u)\ if\ u \in V_1 - V_2 \\ T_{P_2}(u)\ if\ u \in V_2 - V_1 \\ max\{T_{P_1}(u), T_{P_2}(v)\}\ if\ u \in V_1 \cup V_2 \end{cases}$$

$$I_{P_1 \cup P_2}(u) = \begin{cases} I_{P_1}(u)\ if\ u \in V_1 - V_2 \\ I_{P_2}(u)\ if\ u \in V_2 - V_1 \\ min\{I_{P_1}(u), I_{P_2}(v)\}\ if\ u \in V_1 \cup V_2 \end{cases}$$

$$F_{P_1 \cup P_2}(u) = \begin{cases} F_{P_1}(u)\ if\ u \in V_1 - V_2 \\ F_{P_2}(u)\ if\ u \in V_2 - V_1 \\ min\{F_{P_1}(u), F_{P_2}(v)\}\ if\ u \in V_1 \cup V_2 \end{cases}$$

$$T_{P_1\cup P_2}(u,v)=\begin{cases} T_{R_1}(u,v)\ if\ (u,v)\ \in\ E_1-E_2 \\ T_{R_2}(u,v)\ if\ (u,v)\ \in\ E_2-E_1 \\ max\{T_{R_1}(u,v),T_{R_2}(u,v)\}\ if\ (u,v)\ \in\ E_1\cup E_2 \end{cases}$$

$$I_{P_1\cup P_2}(u,v)=\begin{cases} I_{R_1}(u,v)\ if\ (u,v)\ \in\ E_1-E_2 \\ I_{R_2}(u,v)\ if\ (u,v)\ \in\ E_2-E_1 \\ min\{I_{R_1}(u,v),I_{R_2}(u,v)\}\ if\ (u,v)\ \in\ E_1\cup E_2 \end{cases}$$

$$F_{P_1\cup P_2}(u,v)=\begin{cases} F_{R_1}(u,v)\ if\ (u,v)\ \in\ E_1-E_2 \\ F_{R_2}(u,v)\ if\ (u,v)\ \in\ E_2-E_1 \\ min\{F_{R_1}(u,v),F_{R_2}(u,v)\}\ if\ (u,v)\ \in\ E_1\cup E_2 \end{cases}$$

Example 4.4.1

Consider two FNGs G_1 *and* G_2, as presented below. Then the union of $G_1 \cup G_2$, are shown graphically in Figure 5.

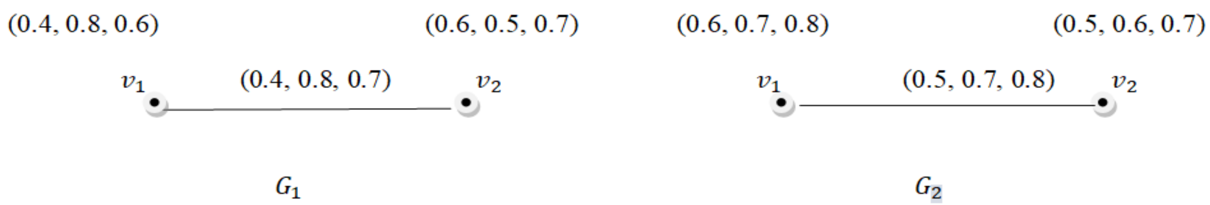

Fig. 5: Fermatean Neutrosophic Graphs

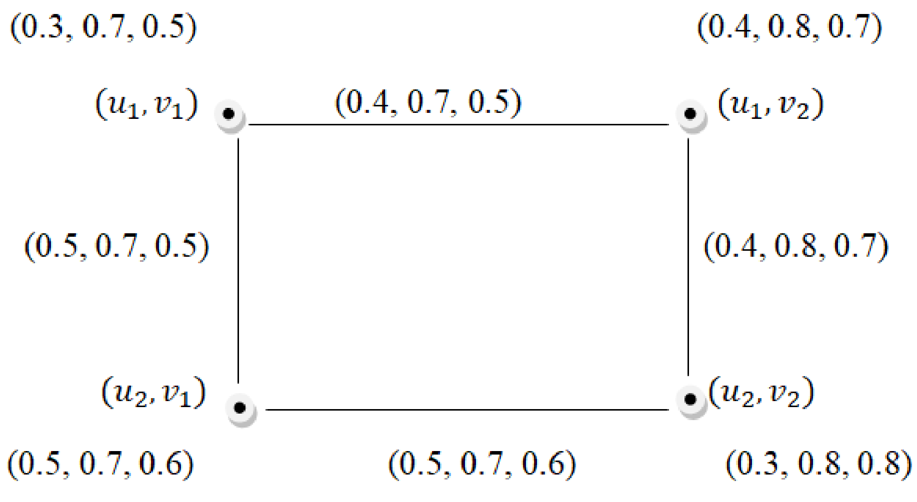

Fig. 6: Union of two FNGs

4.5 Join of FNGs

The Join of two FNGs G_1 and G_2, denoted by $G_1 + G_2$, is defined as: $G_1 + G_2 = (P_1 + P_2, R_1 + R_2)$ where,

$$T_{P_1+P_2}(u) = \begin{cases} T_{P_1}(u) \; if \; u \in V_1 - V_2 \\ T_{P_2}(u) \; if \; u \in V_2 - V_1 \\ min\{T_{P_1}(u), T_{P_2}(v)\} \; if \; u \in V_1 \cup V_2 \end{cases}$$

$$I_{P_1+P_2}(u) = \begin{cases} I_{P_1}(u) \; if \; u \in V_1 - V_2 \\ I_{P_2}(u) \; if \; u \in V_2 - V_1 \\ max\{I_{P_1}(u), I_{P_2}(v)\} \; if \; u \in V_1 \cup V_2 \end{cases}$$

$$F_{P_1+P_2}(u) = \begin{cases} F_{P_1}(u) \; if \; u \in V_1 - V_2 \\ F_{P_2}(u) \; if \; u \in V_2 - V_1 \\ max\{F_{P_1}(u), F_{P_2}(v)\} \; if \; u \in V_1 \cup V_2 \end{cases}$$

$$T_{P_1+P_2}(u,v) = \begin{cases} T_{R_1}(u,v) \; if \; (u,v) \in E_1 - E_2 \\ T_{R_2}(u,v) \; if \; (u,v) \in E_2 - E_1 \\ min\{T_{R_1}(u,v), T_{R_2}(u,v)\} \; if \; (u,v) \in E_1 \cup E_2 \end{cases}$$

$$I_{P_1+P_2}(u,v) = \begin{cases} I_{R_1}(u,v) \; if \; (u,v) \in E_1 - E_2 \\ I_{R_2}(u,v) \; if \; (u,v) \in E_2 - E_1 \\ max\{I_{R_1}(u,v), I_{R_2}(u,v)\} \; if \; (u,v) \in E_1 \cup E_2 \end{cases}$$

$$F_{P_1+P_2}(u,v) = \begin{cases} F_{R_1}(u,v) \; if \; (u,v) \in E_1 - E_2 \\ F_{R_2}(u,v) \; if \; (u,v) \in E_2 - E_1 \\ max\{F_{R_1}(u,v), F_{R_2}(u,v)\} \; if \; (u,v) \in E_1 \cup E_2 \end{cases}$$

Example 4.5.1

Consider two FNGs G_1 *and* G_2, as presented below. Then the join of two graph $G_1 + G_2$, are shown graphically in Figure below.

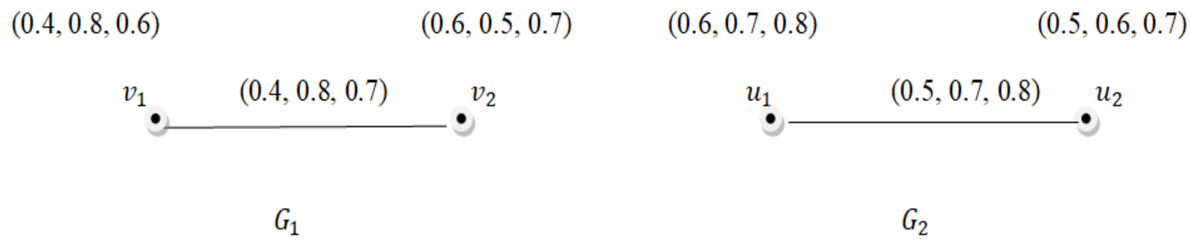

Fig. 7: Two FNGs

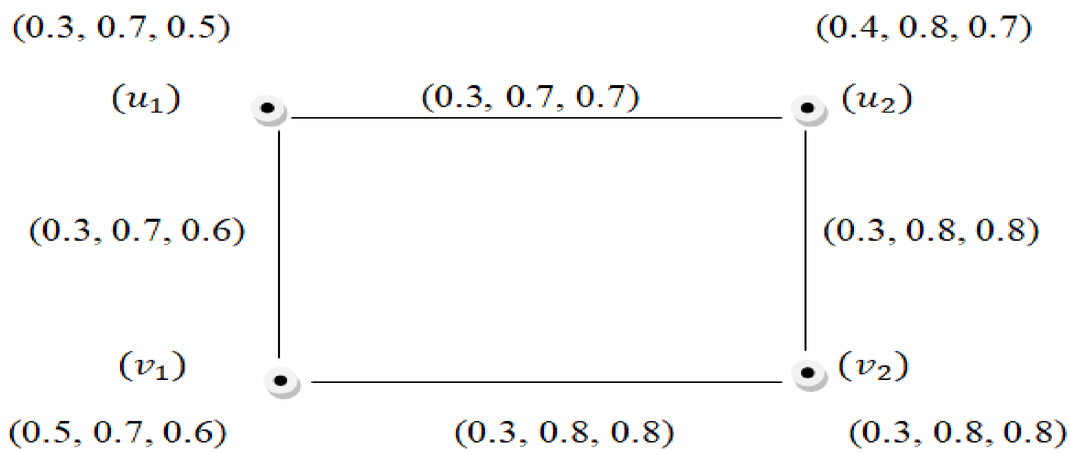

Fig. 8 Join of two FNGS

5. Numerical example (Application of FNSs) in decision-making)

NSs and their extensions have widely used in decision making such as identification of key factors affecting soil liquefaction under seismic risk [34], Choosing the most critical water quality parameter for an aquaponic system [35], green supplier selection [36], E-commerce site selection [37], career selection problem [38], vaccine selection [39], clay-brick selection [40], and so on. Different extension of NSs have been proposed in the literature such as quadripartition NS (QNS) [41], interval QNS [42], Pentapartitioned NS (PNS) [43] , Interval PNS [44], Pythagorean/ Fermatean NSs [45]. Several researchers have recently used Fermatean fuzzy sets, for multicriteria decision making such as WASPAS [46, 47], TOPSIS [48, 49], ELECTRE method [50] and so on. To demonstrate the application of Fermatean in decision making, let us consider a scenario in which a company must decide which supplier to use for a specific product. The decision is made based on three factors: price, quality, and delivery time.

We consider that there are three potential suppliers: A, B, and C. The decision-making team assigns Fermatean Neutrosophic membership values to each supplier for each criterion, ranging from 0 to 1.

Supplier A:

Table-1: Supplier A's membership values

	Belief	Indeterminacy	Unbelief
Price	0.8	0.3	0.7
Quality	0.7	0.2	0.6
Delivery Time	0.8	0.4	0.7

Supplier B:

Table-2: Supplier B's membership values

	Belief	Indeterminacy	Unbelief
Price	0.7	0.3	0.7
Quality	0.8	0.3	0.7

Delivery Time	0.9	0.4	0.6

Supplier C:

Table-3: Supplier C’s membership values

	Belief	Indeterminacy	Unbelief
Price	0.7	0.2	0.6
Quality	0.8	0.3	0.7
Delivery Time	0.7	0.4	0.6

To calculate the overall Fermatean neutrosophic score for each supplier, we can use the weighted average method. Let's assume equal weights for all criteria.

Supplier A:

Price: $(0.8 \times 1/3) + (0.3 \times 1/3) + (0.7 \times 1/3) = 0.59$

Quality: $(0.7 \times 1/3) + (0.2 \times 1/3) + (0.6 \times 1/3) = 0.49$

Delivery Time: $(0.8 \times 1/3) + (0.4 \times 1/3) + (0.7 \times 1/3) = 0.62$

Overall Score for Supplier A: $\frac{(0.59 + 0.49 + 0.62)}{3} = 0.56$

Similarly, we can compute the total scores for Suppliers B and C.

Supplier B: Overall Score is 0.59 ,Supplier C: Overall score is 0.54.

Based on the Fermatean neutrosophic scores, Supplier B has the highest overall score and should be considered the company's preferred choice.

This example demonstrates how FNSs can be used in decision making by allowing experts to consider the belief, indeterminacy, and unbelief values associated with each decision alternative and criteria.

6. Conclusions

Fermatean Neutrosophic graphs offer a unique and efficient framework for representing and examining intricate systems. These graphs utilize the Fermatean Neutrosophic set to effectively capture and assess the varying degrees of membership, non-membership, and uncertainty present in a system. This facilitates a more comprehensive comprehension and interpretation of the information, making Fermatean Neutrosophic graphs an invaluable tool in areas like decision-making, pattern recognition, and data analysis. The development and application of Fermatean Neutrosophic graphs have the potential to significantly enhance our understanding of complex systems, enabling more precise and well-informed decision-making. In the future the concept of Fermentinean neutrosophic graph can be extended to Fermentinean pentapartitioned neutrosophic graph [51].

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