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New Approach on Statistical Convergence of Triple Sequences in Neutrosophic Normed Spaces

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Abstract. In this project the triple sequence's statistical convergence within Neutrosophic normed space is proposed. Also, algebra of statistical limits and statistical cauchy sequences are discussed in this article. Furthermore, the article provides instances to back up a few definitions and theorems. We also looked at the special sequence space's evidence of completeness.

Keywords: Triple sequence; Statistical convergence; Neutrosophic normed spaces; Statistical Cauchy sequences; Completeness.

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1. **Introduction**

Zadeh [10] was first put forward the fuzzy sets concept. It is highly important for real-world circumstances and has a significant impact on all scientific domains. The notions of intuitionistic fuzzy sets were expanded upon by Atanassov [9]. Saadati and Park [17] presented the notion of intuitionistic fuzzy Normed space. Subsequently, Smarandache [6] presented the concept of neutrosophic sets (NS), which is an alternative form of the classical set theory concept by the addition of a indeterminant membership feature. This set is an attempt to quantify truth,indeterminacy and falsity. The distinctions between intuitionistic fuzzy sets and neutrosophic sets, as well as the related relationships between these two sets, were further examined by Smarandache.

The notion among the space of neutrosophic norm and statistical convergence was provided relatively recently by Kirisci and Simsek [14]. Statistical convergence was independently developed in 1951 by Fast [7] and Steinhaus [8] based on the idea of density of positive natural numbers. In 2003, Mursaleen, Mohuiuddine [16] and Mursaleen,Edely [15] researched the concept among statistical convergence in double sequence space. Tripathy and associates [1] [2] [3] [4] [5] formulated the notion of statistical convergence from many angles. Jeyaraman and Jenifer [12] [13] have lately created various notions related to statistical convergence in various domains.

The suggested work is wherein various topological features of statistical limits as well as statistical convergence of single and double sequences have been examined, correspondingly. We have expanded the research to encompass triple sequences within neutrosophic normed space in addition examine various topological characteristics of limits, such as the completeness of certain convergent sequence spaces that have not been covered in previous research.

2. Preliminaries

Definition 2.1. Let $X = [0,1]$. A continuous triangular norm, or t-norm, is described as given for a binary operation $\circ : X \times X \to X$: if

(i) $\circ$ can be commutative

(ii) $\circ$ can be associative

(iii) $\circ$ can be continuous

(iv) $\alpha \circ 1 = \alpha$ for every $\alpha \in X$

(v) $\alpha_1 \le \alpha_3$ and $\alpha_2 \le \alpha_4 \Rightarrow \alpha_1 \circ \alpha_2 \le \alpha_3 \circ \alpha_4$, for all $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4 \in X$

Definition 2.2. Let $X = [0,1]$. A triangular norm with continuous, or t-norm, is described as given for a binary operation $\Diamond : X \times X \to X$: if

(i) $\Diamond$ be commutative

(ii) $\Diamond$ be associative

(iii) $\Diamond$ be continuous

(iv) $\alpha \Diamond 0 = \alpha$ for all every $\alpha \in X$

(v) $\alpha_1 \le \alpha_3$ and $\alpha_2 \le \alpha_4 \Rightarrow \alpha_1 \diamond \alpha_2 \le \alpha_3 \diamond \alpha_4$, for all $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4 \in X$

Definition 2.3. [12] A 7-tuple $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$, where $\mathbb{M}$ gives linear space, $\circ$ denote a t-norm with continuous, $\Diamond$ and $*$ are t-conorms with continuous, in addition μ, ν as well as ω denote fuzzy sets upon $\mathbb{M} \times (0, \infty)$, is known as a Neutrosophic normed space $\mathfrak{NNS}$ when it satisfies the given subsequent requirements for all $\mathfrak{x}, \mathfrak{y} \in \mathbb{M}$ as well as $\lambda, s \in (0, \infty)$.

(i) $0 \le \mu(\mathfrak{x}, \lambda) + \nu(\mathfrak{x}, \lambda) + \omega(\mathfrak{x}, \lambda) \le 3$.

(ii) $\mu(\mathfrak{x}, \lambda) > 0$.

(iii) $\mu(c\mathfrak{x}, \lambda) = \mu\left(\mathfrak{x}, \frac{t}{|c|}\right)$ for every $c \neq 0$.

(iv) $\mu(\mathfrak{x}, \lambda) = 1$ if and only if $\mathfrak{x} = 0$.

(v) $\mu(\mathfrak{x} + \mathfrak{y}, \lambda + s) \geq \mu(\mathfrak{x}, t) \circ \mu(\mathfrak{y}, s)$.

(vi) $\mu(\mathfrak{x}, \lambda) : (0, \infty) \to [0, 1]$ gives a continuous function of λ.

(vii) $\lim_{\lambda \to \infty} \mu(\mathfrak{x}, t) = 1$ as well as $\lim_{\lambda \to 0} \mu(\mathfrak{x}, \lambda) = 0$.

(viii) $\nu(\mathfrak{x}, \lambda) < 1$.

(ix) $\nu(c\mathfrak{x}, \lambda) = \nu\left(\mathfrak{x}, \frac{t}{|c|}\right)$ for all $c \neq 0$.

(x) $\nu(\mathfrak{x}, \lambda) = 0$ iff $\mathfrak{x} = 0$.

(xi) $\nu(\mathfrak{x} + \mathfrak{y}, \lambda + s) \leq \nu(\mathfrak{x}, \lambda) \Diamond \nu(\mathfrak{y}, s)$.

(xii) $\nu(\mathfrak{x}, \lambda) : (0, \infty) \to [0, 1]$ gives a continuous function of t.

(xiii) $\lim_{\lambda \to \infty} \nu(\mathfrak{x}, \lambda) = 0$ as well as $\lim_{\lambda \to 0} \nu(\mathfrak{x}, \lambda) = 1$.

(xiv) $\omega(\mathfrak{x}, \lambda) < 1$.

(xv) $\omega(c\mathfrak{x}, \lambda) = \omega\left(\mathfrak{x}, \frac{\lambda}{|c|}\right)$ for all $c \neq 0$.

(xvi) $\omega(\mathfrak{x}, \lambda) = 0$ iff $\mathfrak{x} = 0$.

(xvii) $\omega(\mathfrak{x} + \mathfrak{y}, \lambda + s) \leq \omega(\mathfrak{x}, \lambda) * \omega(\mathfrak{y}, s)$.

(xviii) $\omega(\mathfrak{x}, \lambda) : (0, \infty) \to [0, 1]$ gives a continuous function of t.

(xix) $\lim_{\lambda \to \infty} \omega(\mathfrak{x}, \lambda) = 0$ and $\lim_{\lambda \to 0} \omega(\mathfrak{x}, \lambda) = 1$.

The triplet (μ, ν, ω) gives Neutrosophic Normed space ($\mathfrak{NNS}$) and described by
$(\mu, \nu, \omega) = \{(\alpha, \lambda) : \mu(\mathfrak{x}, \lambda), \nu(\mathfrak{x}, \lambda), \omega(\mathfrak{x}, \lambda) : \mathfrak{x} \in \mathbb{M}\}$

Definition 2.4. Let a triple sequence $\mathfrak{x} = (\mathfrak{x}_{ijk})$ among $\mathfrak{NNS}$ $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$ is indicated as a convergent towards $\mathfrak{l}$, if there esists $i_0, j_0, k_0 \in \mathbb{N}$ for every $i \geq i_0, j \geq j_0, k \geq k_0$, and $\lambda \in (0, \infty)$ we have $1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}$, $\nu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}$ and $\omega(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}$ for all $\dot{\varepsilon} \in (0, 1)$.

Lemma 2.5. *[12] For any sequence $(\mathfrak{x}_n)$ on $\mathfrak{NNS}$ $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$, $\mathfrak{l}$ is the convergence point with respect to $\mathfrak{NN}$ (μ, ν, ω) iff $\mu(\mathfrak{x}_n - \mathfrak{l}, \lambda) \to 1$, $\nu(\mathfrak{x}_n - \mathfrak{l}, \lambda) \to 0$, and $\omega(\mathfrak{x}_n - \mathfrak{l}, \lambda) \to 0$ as $n \to \infty$.*

Definition 2.6. Let $\mathfrak{A}$ be a triple natural density is described as follows:
$\delta_3(\mathfrak{A}) = \lim_{i,j,k \to \infty} \frac{|\Lambda(i,j,k)|}{ijk}$, in which vertical bars represent the set cardinality, given that described limit exists. Consider $\mathfrak{A} \subseteq \mathbb{N} \times \mathbb{N}$ along with
$\Lambda(i, j, k) = \{(p, q, r) \in \mathfrak{A} : p \leq i; q \leq j; r \leq k\}$.

Example 2.7. Let us assume $\mathfrak{A} = \{(p, q, 3r) : p, q, r \in \mathbb{N}\}$ after now
$\delta_3(\mathfrak{A}) = \lim_{i,j,k \to \infty} \frac{|\Lambda(i,j,k)|}{ijk} = \lim_{i,j,k \to \infty} \frac{ijk}{3ijk} = \frac{1}{3}$, i.e. the density of triple nature among $\mathfrak{A}$ denote $\frac{1}{3}$.
Take another set $\mathfrak{B} = \{(p^2, q^2, r^2) : p, q, r \in \mathbb{N}\}$ then the density of triple nature among $\mathfrak{B}$ is provided by $\delta_3(\mathfrak{B}) = \lim_{i,j,k \to \infty} \frac{|\Lambda(i,j,k)|}{ijk} = \lim_{i,j,k \to \infty} \frac{\sqrt{ijk}}{ijk} = 0$.

3. Statistical Convergence

Definition 3.1. If the triple natural density of the set $\{(i,j,k) \in \mathbb{N}^3 : |\mathfrak{x}_{ijk} - \mathfrak{l}| \geq \dot{\varepsilon}\} = 0$ for all $\dot{\varepsilon} > 0$, then the triple sequence $\mathfrak{x} = (\mathfrak{x}_{ijk})$ exist statistically convergent towards a number $\mathfrak{l}$. We express it symbolically as $\mathfrak{st}_3 - \lim \mathfrak{x} = \mathfrak{l}$.

Remark 3.2. Every real triple sequence $\mathfrak{x} = (\mathfrak{x}_{ijk})$ that converges to a number $\mathfrak{l}$ does so statistically towards the same limit.

Definition 3.3. An element has triple sequence $\mathfrak{x} = (\mathfrak{x}_{ijk})$ among $\mathfrak{NNS}$ $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$ is referred to be statistically convergent towards $\mathfrak{l}$ with respect to $\mathfrak{NN}$ (μ, ν, ω) when for all those $\dot{\varepsilon} \in (0,1)$ along with for certain $\lambda > 0$ the triple natural density through the set $\{(i,j,k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon},\ \ \nu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon}$ as well as $\omega(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon}\} = 0$. Equivalently,

$$\lim_{p,q,r\to\infty} \frac{1}{pqr} \left| \left\{ \begin{array}{c} (i,j,k) \in \mathbb{N}^3 : i \leq p; j \leq q; k \leq r; \\ 1 - \mu(\mathfrak{x}_{ijk} - l, \lambda) \geq \dot{\varepsilon}, \nu(\mathfrak{x}_{lkj} - l, \lambda) \geq \dot{\varepsilon} \text{ and} \\ \omega(\mathfrak{x}_{lkj} - l, \lambda) \geq \dot{\varepsilon} \end{array} \right\} \right|.$$

Example 3.4. Let $\mathbb{M} = \mathbb{R}^3$ where $(\mathbb{R}^3, ||\cdot||_2)$ denote a $\mathfrak{NLS}$ in addition $\mathbb{R}$ denotes the real numbers set. Consider $\dot{a} \circ \dot{b} = \dot{a}\dot{b}$, $\dot{a} \Diamond \dot{b} = \dot{a} + \dot{b} - \dot{a}\dot{b}$ as well as $\dot{a} * \dot{b} = \dot{a} + \dot{b} - \dot{a}\dot{b}$. Take into consideration that the $\mathfrak{NNS}$ $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$ where
$\mu(\dot{v}, \lambda) = \frac{\lambda}{\lambda + ||\dot{v}||_2}$, $\nu(\dot{v}, \lambda) = \frac{||\dot{v}||_2}{\lambda + ||\dot{v}||_2}$ as well as $\omega(\dot{v}, \lambda) = \frac{||\dot{v}||_2}{\lambda}$, in which $||\dot{v}||_2$ corresponds to the second vector norm v, where λ denote a positive real number.
Define the sequence now $\mathfrak{x} = (\mathfrak{x}_{ijk})$ as

$$\mathfrak{x} = \mathfrak{x}_{ijk} = \begin{cases} \left(i, \frac{1}{j}, \frac{1}{k}\right)^t, & i = r^2, \text{where}\, r, jk \in \mathbb{N}, \\ \left(\frac{1}{i^2}, \frac{1}{j+k}, \frac{1}{k^2}\right)^t, & \text{otherwise}, \end{cases}$$

We have, $E = \{(i,j,k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - 0, \lambda) \geq \dot{\varepsilon}, \nu(\mathfrak{x}_{ijk} - 0, \lambda) \geq \dot{\varepsilon}$ and $\omega(\mathfrak{x}_{ijk} - 0, \lambda) \geq \dot{\varepsilon}\}$ then $E = \{(i^2, j, k) \in \mathbb{N}^3 : i^2 \leq p, j \leq q, k \leq r \in \mathbb{N}\}$.
Therefore, $\delta_3(E) = \lim_{p,q,r\to\infty} \frac{\sqrt{p}qr}{pqr} = 0$.
Hence $\mathfrak{st}_3^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = 0$.

Theorem 3.5. *Let* $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$ *be a* $\mathfrak{NNS}$*, the sequence* $(\mathfrak{x}_{ijk})$ *statistically converges towards the same limit* $\mathfrak{l}$ *when the sequence* $\mathfrak{x} = (\mathfrak{x}_{ijk})$ *converges to a number* $\mathfrak{l}$*. Conversely, however, is generally not true.*

Proof. Let $(\mu, \nu, \omega) - \lim \mathfrak{x} = \mathfrak{l}$. That i_0, j_0, as well as $k_0 \in \mathbb{N}$ exists, in which for every $(i,j,k) \in \mathbb{N}^3$ along with $i \geq i_0, j \geq j_0, k \geq k_0$, $1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}$, $\nu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}$ and $\omega(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}$ which implies the set
$E = \{(i,j,k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon},\ \nu(\mathfrak{x}_{lkj} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon}$ and $\omega(\mathfrak{x}_{lkj} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon}\}$.
As a result, $\delta_3(E) = 0$, forcing $\mathfrak{st}_3^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \mathfrak{l}$.

For the converse section, we're thinking of an illustration.

Let $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$ be a $\mathfrak{NNS}$. Given a triple sequence of real numbers $\mathfrak{x} = (\mathfrak{x}_{ijk})$ over $\mathfrak{NNS}$, which is defined as

$$\mathfrak{x} = \mathfrak{x}_{ijk} = \begin{cases} i, & i, j \text{ and } k \text{ are perfect cube} \\ 2^{-(i+j+k)}, & \text{otherwise,} \end{cases}.$$

Define, $\mu(v, \lambda) = \frac{\lambda}{\lambda + |v|}$, $\nu(v, \lambda) = \frac{|v|}{\lambda + |v|}$ and $\omega(v, \lambda) = \frac{|v|}{\lambda}$.

We have $E = \{(i, j, k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon},\ \nu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon} \text{ as well as } \omega(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon}\} = \{(i, j, k) \in \mathbb{N}^3 : i, j, \text{ and } k \text{ are perfect cube}\}$.

Now, $\Lambda(p, q, r) = \{(i, j, k) : i \leq p, j \leq q; k \leq r\}, |\Lambda(p, q, r)| = p^{\frac{1}{3}} q^{\frac{1}{3}} r^{\frac{1}{3}}$, which yields

$\delta_3(E) = \lim_{p,q,r \to \infty} \frac{|\Lambda(p,q,r)|}{pqr} = (pqr)^{-\frac{2}{3}} \to 0$.

$\mathfrak{st}_3^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = 0$. But that i_0, j_0, and $k_0 \in \mathbb{N}$ does not exists in a way that

$1 - \mu(\mathfrak{x}_{ijk}, \lambda) < \dot{\varepsilon}$, $\nu(\mathfrak{x}_{ijk}, \lambda) < \dot{\varepsilon}$ and $\omega(\mathfrak{x}_{ijk}, \lambda) < \dot{\varepsilon}$ for every $i \geq i_0, j \geq j_0, k \geq k_0$.

Hence $(\mu, \nu, \omega) - \lim \mathfrak{x}$ doesn't exist. □

Lemma 3.6. *Let* $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$ *be a* $\mathfrak{NNS}$*, in addition for all those* $\dot{\varepsilon} > 0$ *as well as* $\lambda > 0$ *the given are equivalents.*

(i) $\mathfrak{st}_3^{(\mu,\nu,\omega)} - \lim(\mathfrak{x}_{ijk}) = \mathfrak{l}$.

(ii) $\delta_3\{(i, j, k) \in \mathbb{N}^3 : 1-\mu(\mathfrak{x}_{ijk}-\mathfrak{l}, \lambda) \geq \dot{\varepsilon}\} = \delta_3\{(i, j, k) \in \mathbb{N}^3 : \nu(\mathfrak{x}_{ijk}-l, \lambda) \geq \dot{\varepsilon}\} = \delta_3\{(i, j, k) \in \mathbb{N}^3 : \omega(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) \geq \dot{\varepsilon}\} = 0$.

(iii) $\delta_3\{(i, j, k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}\} = \delta_3\{(i, j, k) \in \mathbb{N}^3 : \nu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}\} = \delta_3\{(i, j, k) \in \mathbb{N}^3 : \omega(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) < \dot{\varepsilon}\} = 1$.

(iv) $\mathfrak{st}_3 - \lim \mu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) = 1$, $\mathfrak{st}_3 - \lim \nu(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) = 0$ *and* $\mathfrak{st}_3 - \lim \omega(\mathfrak{x}_{ijk} - \mathfrak{l}, \lambda) = 0$.

4. Statistical Limit Algebra

Theorem 4.1. *The triple sequence* $\mathfrak{x} = \mathfrak{x}_{ijk}$ *Statistical limit on* $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$ *with respect to* $\mathfrak{NN}$ (μ, ν, ω) *is unique.*

Proof. Assuming,if it is possible, that $\mathfrak{st}_3^{(\mu,\nu,\omega)} - \lim \mathfrak{x} \to \mathfrak{l}_1$ while $\mathfrak{st}_3^{(\mu,\nu,\omega)} - \lim \mathfrak{x} \to \mathfrak{l}_2$ after that for each $\dot{\varepsilon} > 0$ select, $\xi > 0$ in which $(1 - \xi) \circ (1 - \xi) > 1 - \dot{\varepsilon}$, $\xi \Diamond \xi < \dot{\varepsilon}$ as well as $\xi * \xi < \dot{\varepsilon}$ after that for some $\lambda > 0$ we obtain

$$U_1(\xi, \lambda) = \{(i, j, k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}_1, \lambda) \geq \xi\},$$

$$U_2(\xi, \lambda) = \{(i, j, k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}_2, \lambda) \geq \xi\},$$

$$V_1(\xi, \lambda) = \{(i, j, k) \in \mathbb{N}^3 : \nu(\mathfrak{x}_{ijk} - \mathfrak{l}_1, \lambda) \geq \xi\},$$

$$V_2(\xi, \lambda) = \{(i, j, k) \in \mathbb{N}^3 : \nu(\mathfrak{x}_{ijk} - \mathfrak{l}_2, \lambda) \geq \xi\}$$

$$W_1(\xi, \lambda) = \{(i, j, k) \in \mathbb{N}^3 : \omega(\mathfrak{x}_{ijk} - \mathfrak{l}_1, \lambda) \geq \xi\},$$

$$W_2(\xi, \lambda) = \{(i, j, k) \in \mathbb{N}^3 : \omega(\mathfrak{x}_{ijk} - \mathfrak{l}_2, \lambda) \geq \xi\}.$$

Based on the preceding lemma, for certain $\lambda > 0$, we obtain
$\delta_3(U_1(\xi,\lambda)) = \delta_3(V_1(\xi,\lambda)) = \delta_3(W_1(\xi,\lambda)) = 0,$
$\delta_3(U_2(\xi,\lambda)) = \delta_3(V_2(\xi,\lambda)) = \delta_3(W_2(\xi,\lambda)) = 0,$
Now we defining,
$T_{\mu,\nu,\omega}(\xi,\lambda) = \{U_1(\xi,\lambda) \cup U_2(\xi,\lambda)\} \cap \{V_1(\xi,\lambda) \cup V_2(\xi,\lambda)\} \cap \{W_1(\xi,\lambda) \cup W_2(\xi,\lambda)\}$ which implies $\delta_3(T_{\mu,\nu,\omega}(\xi,\lambda)) = 0.$
Then $\delta_3(\mathbb{N}^3 - T_{\mu,\nu,\omega}(\xi,\lambda)) = 1.$
Now if $(p,q,r) \in \mathbb{N}^3 - T_{\mu,\nu,\omega}(\xi,\lambda)$ then
$(p,q,r) \in \mathbb{N}^3 - \{U_1(\xi,\lambda)) \cup U_2(\xi,\lambda)\}$ or
$(p,q,r) \in \mathbb{N}^3 - \{V_1(\xi,\lambda)) \cup V_2(\xi,\lambda)\}$ or
$(p,q,r) \in \mathbb{N}^3 - \{W_1(\xi,\lambda)) \cup W_2(\xi,\lambda)\}.$
Case-I. If $(p,q,r) \in \mathbb{N}^3 - \{U_1(\xi,\lambda)) \cup U_2(\xi,\lambda)\}$ then

$$1 - \mu(\mathfrak{l}_1 - \mathfrak{l}_2,\lambda) \leq 1 - \mu(\mathfrak{l}_1 - \mathfrak{x}_{pqr},\lambda) \circ \mu(\mathfrak{x}_{pqr} - \mathfrak{l}_2,\lambda) < 1 - (1-\xi) \circ (1-\xi) < 1 - (1-\dot{\varepsilon}) = \dot{\varepsilon}.$$

Case-II. If $(p,q,r) \in \mathbb{N}^3 - \{V_1(\xi,\lambda)) \cup V_2(\xi,\lambda)\}$ then

$$\nu(\mathfrak{l}_1 - \mathfrak{l}_2,\lambda) \leq \nu(\mathfrak{l}_1 - \mathfrak{x}_{pqr},\lambda) \Diamond \nu(\mathfrak{x}_{pqr} - \mathfrak{l}_2,\lambda) < \xi \Diamond \xi < \dot{\varepsilon}.$$

Case-III. If $(p,q,r) \in \mathbb{N}^3 - \{W_1(\xi,\lambda)) \cup W_2(\xi,\lambda)\}$ then

$$\omega(l_1 - l_2,\lambda) \leq \omega(\mathfrak{l}_1 - \mathfrak{x}_{pqr},\lambda) * \omega(\mathfrak{x}_{pqr} - \mathfrak{l}_2,\lambda) < \xi * \xi < \dot{\varepsilon}.$$

due to the arbitrariness of $\lambda > 0$. Therefore, based on the three aforementioned situations, we deduce that $1 - \mu(\mathfrak{l}_1 - \mathfrak{l}_2,\lambda) < \dot{\varepsilon}$, $\nu(\mathfrak{l}_1 - \mathfrak{l}_2,\lambda) < \dot{\varepsilon}$ and $\omega(\mathfrak{l}_1 - \mathfrak{l}_2,\lambda) < \dot{\varepsilon}$ for all $\dot{\varepsilon} > 0$ it brings about the outcome. □

Theorem 4.2. *If the sequences $\mathfrak{x} = (\mathfrak{x}_{ijk})$ as well as $y = (y_{ijk})$ on $\mathfrak{NNS}$ $(\mathbb{M},\mu,\nu,\omega,\circ,\Diamond,*)$ statistically convergent in relate towards $\mathfrak{NN}$ (μ,ν,ω) to $\mathfrak{l}_1$ as well as $\mathfrak{l}_2$ respectively. Then in relate towards same $\mathfrak{NN}$ (μ,ν,ω), the sequence $(\mathfrak{x}_{ijk} + y_{ijk})$ converges towards $\mathfrak{l}_1 + \mathfrak{l}_2$.*

Proof. We are provide, $\mathfrak{st}_3^{(\mu,\nu,\omega)} - \lim \mathfrak{x} = \mathfrak{l}_1$ as well as $\mathfrak{st}_3^{(\mu,\nu,\omega)} - \lim y = \mathfrak{l}_2$; now
$U = \{(i,j,k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}_1,\lambda) \geq \dot{\varepsilon} \,\text{or}\, \nu(\mathfrak{x}_{ijk} - \mathfrak{l}_1,\lambda) \geq \dot{\varepsilon}, \omega(\mathfrak{x}_{ijk} - \mathfrak{l}_1,\lambda) \geq \dot{\varepsilon}\}$ and
$V = \{(i,j,k) \in \mathbb{N}^3 : 1 - \mu(y_{ijk} - \mathfrak{l}_2,\lambda) \geq \dot{\varepsilon} \,\text{or}\, \nu(y_{ijk} - \mathfrak{l}_2,\lambda) \geq \dot{\varepsilon},\ \omega(y_{ijk} - \mathfrak{l}_2,\lambda) \geq \dot{\varepsilon}\}$ along with $\delta_3(V) = \delta_3(U) = 0$, it follows $\delta_3(U^c) = \delta_3(\mathbb{N}^3 - U) = 1$ while $\delta_3(V^c) = \delta_3(\mathbb{N}^3 - V) = 1$, in which,
$U^c = \{(i,j,k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{l}_1,\lambda) < \dot{\varepsilon}, \nu(\mathfrak{x}_{ijk} - \mathfrak{l}_1,\lambda) < \dot{\varepsilon} \,\text{and}\, \omega(\mathfrak{x}_{ijk} - \mathfrak{l}_1,\lambda) < \dot{\varepsilon}\},$
$V^c = \{(i,j,k) \in \mathbb{N}^3 : 1 - \mu(y_{ijk} - \mathfrak{l}_2,\lambda) < \dot{\varepsilon}, \nu(y_{ijk} - \mathfrak{l}_2,\lambda) < \dot{\varepsilon} \,\text{and}\, \omega(y_{ijk} - \mathfrak{l}_2,\lambda) < \dot{\varepsilon}\}.$
Then, for $\dot{\varepsilon} > 0$ select, $\xi \in (0,1)$ in which $(1-\dot{\varepsilon}) \circ (1-\dot{\varepsilon}) > 1 - \xi$, $\dot{\varepsilon} \Diamond \dot{\varepsilon} < \xi$ and $\dot{\varepsilon} * \dot{\varepsilon} < \xi$ for

$(i,j,k) \in (\mathbb{N}^3 - U) \cap (\mathbb{N}^3 - V) = \mathbb{N}^3 - (U \cup V)$ we have

$$\begin{aligned} 1-\mu(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda) &\leq 1-\mu\left(\mathfrak{x}_{ijk}-\mathfrak{l}_1,\frac{\lambda}{2}\right)\circ\mu\left(y_{ijk}-\mathfrak{l}_2,\frac{\lambda}{2}\right) \\ &< 1-(1-\dot{\varepsilon})\circ(1-\dot{\varepsilon}) \\ &< 1-(1-\xi)=\xi. \end{aligned}$$

Also, $\nu(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda) \leq \nu\left(\mathfrak{x}_{ijk}-\mathfrak{l}_1,\frac{\lambda}{2}\right)\Diamond\nu\left(y_{ijk}-\mathfrak{l}_2,\frac{\lambda}{2}\right) < \dot{\varepsilon}\Diamond\dot{\varepsilon} < \xi$,
Also, $\omega(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda) \leq \omega\left(\mathfrak{x}_{ijk}-\mathfrak{l}_1,\frac{\lambda}{2}\right)*\omega\left(y_{ijk}-\mathfrak{l}_2,\frac{\lambda}{2}\right) < \dot{\varepsilon}*\dot{\varepsilon} < \xi$, and we also have, $\delta_3(U\cup V) \leq \delta_3(U)+\delta_3(V)$ which implies $\delta_3(U\cup V)=0$ equivalently, $\delta_3(\mathbb{N}^3-(U\cup V))=1$, which yields,
$\delta_3\{(i,j,k)\in\mathbb{N}^3 : 1-\mu(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda)<\xi \,\text{or}\, \nu(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda)<\xi$,
$\omega(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda)<\xi\}=1$ then
$\delta_3\{(i,j,k)\in\mathbb{N}^3 : 1-\mu(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda)\geq\xi \,\text{or}\, \nu(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda)\geq\xi$,
$\omega(\mathfrak{x}_{ijk}+y_{ijk}-\mathfrak{l}_1-\mathfrak{l}_2,\lambda)\geq\xi\}=0$, for all $\dot{\varepsilon}$.
Hence $\mathfrak{st}_3^{(\mu,\nu,\omega)}-\lim(\mathfrak{x}+\mathfrak{x})=\mathfrak{l}_1+\mathfrak{l}_2$. □

Theorem 4.3. *If the sequences $\mathfrak{x}=(\mathfrak{x}_{ijk})$ over $\mathfrak{NNS}$ $(\mathbb{M},\mu,\nu,\omega,\circ,\Diamond,*)$ statistically convergent in relate towards $\mathfrak{NN}$ (μ,ν,ω) to $\mathfrak{l}$. Then in relate towards same $\mathfrak{NN}$ (μ,ν,ω), the sequence $(c\mathfrak{x}_{ijk})$, where c is a scalar, converges to $c\mathfrak{l}$.*

Proof. We have, $\mathfrak{st}_3^{(\mu,\nu,\omega)}-\lim(\mathfrak{x})=\mathfrak{l}, \delta_3(U)=\delta_3\{(i,j,k)\in\mathbb{N}^3 : 1-\mu(\mathfrak{x}_{ijk}-\mathfrak{l},\lambda)\geq\dot{\varepsilon}$ or $\nu(\mathfrak{x}_{ijk}-\mathfrak{l},\lambda)\geq\dot{\varepsilon},\ \omega(\mathfrak{x}_{ijk}-\mathfrak{l},\lambda)\geq\dot{\varepsilon}\}=0$ another hand, $\delta_3(\mathbb{N}^3-U)=1$.
Case I. When $c\neq 0$; for $(i,j,k)\in U$ for all $j\neq 0$. Setting $t=\frac{\lambda}{|c|}>0$ then,
$1-\mu(c\mathfrak{x}_{ijk}-c\mathfrak{l},\lambda)=1-\mu\left(\mathfrak{x}_{ijk}-l,\frac{\lambda}{|c|}\right)=1-\mu(\mathfrak{x}_{ijk}-\mathfrak{l},\lambda)\geq\dot{\varepsilon}$,
$\nu(c\mathfrak{x}_{ijk}-c\mathfrak{l},\lambda)=\nu\left(\mathfrak{x}_{ijk}-\mathfrak{l},\frac{\lambda}{|c|}\right)=\nu(\mathfrak{x}_{ijk}-\mathfrak{l},\lambda)\geq\dot{\varepsilon}$ and
$\omega(c\mathfrak{x}_{ijk}-c\mathfrak{l},\lambda)=\omega\left(\mathfrak{x}_{ijk}-\mathfrak{l},\frac{\lambda}{|c|}\right)=\omega(\mathfrak{x}_{ijk}-\mathfrak{l},\lambda)\geq\dot{\varepsilon}$
which gives, $\delta_3\{(i,j,k)\in\mathbb{N}^3 : 1-\mu(c\mathfrak{x}_{ijk}-c\mathfrak{l},\lambda)\geq\dot{\varepsilon} \,\text{or}\, \nu(c\mathfrak{x}_{ijk}-c\mathfrak{l},\lambda)\geq\dot{\varepsilon}$,
$\omega(c\mathfrak{x}_{ijk}-c\mathfrak{l},\lambda)\geq\dot{\varepsilon}\}=0$.
Case II. When $c=0$. The theorem within above instance is clear. Consequently, the theorem is proven. □

5. Statistical Cauchy Sequences

Definition 5.1. Let $(\mathbb{M},\mu,\nu,\omega,\circ,\Diamond,*)$ be a $\mathfrak{NNS}$. Additionally, a triple sequence $(\mathfrak{x}_{ijk})$ is known to be statistically Cauchy in relate towards $\mathfrak{NN}$ (μ,ν,ω) when for all $\dot{\varepsilon}>0$ along with for $\lambda>0$ that $i,j,$ and $k\in\mathbb{N}$ exist in which for each $p\geq i, q\geq j, r\geq k$. We have
$\delta_3(\{(p,q,r)\in\mathbb{N}^3 : 1-\mu(\mathfrak{x}_{pqr}-\mathfrak{x}_{ijk},\lambda)\geq\dot{\varepsilon} \,\text{or}\, \nu(x_{pqr}-\mathfrak{x}_{ijk},\lambda)\geq\dot{\varepsilon},\ \omega(\mathfrak{x}_{pqr}-\mathfrak{x}_{ijk},\lambda)\geq\dot{\varepsilon}\})=0.$

Theorem 5.2. *A triple sequence* $\mathfrak{x} = (\mathfrak{x}_{ijk})$ *on* $\mathfrak{NNS}$ $(\mathbb{M}, \mu, \nu, \omega, \circ, \Diamond, *)$ *gives statistically convergent in relate towards* $\mathfrak{NN}$ (μ, ν, ω) *iff this is statistically Cauchy in relate towards same* $\mathfrak{NN}$ (μ, ν, ω).

Proof. Let $(\mathfrak{x}_{ijk})$ denote statistically convergent towards $\mathfrak{x}$ in relate towards (μ, ν, ω) it follows that for every $0 < \dot{\varepsilon} < 1$ along with $\lambda > 0$ we get $\delta_3(E) = 0$; in which,
$E = \{(i,j,k) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{ijk} - \mathfrak{x}, \lambda) \geq \dot{\varepsilon} \,\text{or}\, \nu(\mathfrak{x}_{ijk} - \mathfrak{x}, \lambda) \geq \dot{\varepsilon},\, \omega(\mathfrak{x}_{ijk} - \mathfrak{x}, \lambda) \geq \dot{\varepsilon}\}$ then $\delta_3(\mathbb{E}^c) = 1$, where
$\mathbb{E}^c = \{(p,q,r) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{pqr} - \mathfrak{x}, \lambda) < \dot{\varepsilon}, \nu(\mathfrak{x}_{pqr} - \mathfrak{x}, \lambda) < \dot{\varepsilon} \,\text{and}\, \omega(\mathfrak{x}_{pqr} - \mathfrak{x}, \lambda) < \dot{\varepsilon}\}$.
Choose (l,m,n) as well as $(u,v,w) \in \mathbb{E}^c$ in addition select certain $0 < \xi < 1$ which gives $(1-\dot{\varepsilon}) \circ (1-\dot{\varepsilon}) > 1 - \xi$, $\dot{\varepsilon} \Diamond \dot{\varepsilon} < \xi$ and $\dot{\varepsilon} * \dot{\varepsilon} < \xi$ then

$$\begin{aligned}
1 - \mu(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) &= 1 - \mu(\mathfrak{x}_{lmn} - \mathfrak{x} + \mathfrak{x} - \mathfrak{x}_{uvw}, \lambda) \\
&\leq 1 - \mu(\mathfrak{x}_{lmn} - \mathfrak{x}, \lambda) \circ \mu(\mathfrak{x}_{uvw} - \mathfrak{x}, \lambda) \\
&< 1 - (1-\dot{\varepsilon}) \circ (1-\dot{\varepsilon}) \\
&< 1 - \xi, \text{ and} \\
\nu(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) &= \nu(\mathfrak{x}_{lmn} - \mathfrak{x} + \mathfrak{x} - \mathfrak{x}_{uvw}, \lambda) \\
&\leq \nu(\mathfrak{x}_{lmn} - \mathfrak{x}, \lambda) \Diamond \nu(\mathfrak{x}_{uvw} - \mathfrak{x}, \lambda) \\
&\leq \dot{\varepsilon} \Diamond \dot{\varepsilon} < \xi \\
\omega(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) &= \omega(\mathfrak{x}_{lmn} - \mathfrak{x} + \mathfrak{x} - \mathfrak{x}_{uvw}, \lambda) \\
&\leq \omega(\mathfrak{x}_{lmn} - \mathfrak{x}, \lambda) * \omega(\mathfrak{x}_{uvw} - \mathfrak{x}, \lambda) \\
&\leq \dot{\varepsilon} * \dot{\varepsilon} < \xi.
\end{aligned}$$

Since the arbitrary are (l,m,n) as well as (u,v,w), $\mathbb{E}^c \subseteq K$, in which
$K = \left\{ \begin{array}{c} (l,m,n) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{lmn} - \mathfrak{x}_{lmn}, \lambda) < \xi, \\ \nu(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) < \xi \,\text{and}\, \omega(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) < \xi \end{array} \right\}$
then $\delta_3(\mathbb{E}^c) \leq \delta_3(K) \leq 1$, in addition, $1 \leq \delta_3(K) \leq 1$ implies $\delta_3(K) = 1$, or $\delta_3(K^c) = 0$, if
$K^c = \left\{ \begin{array}{c} (l,m,n) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) \geq \xi \,\text{or} \\ \nu(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) \geq \xi, \omega(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) \geq \xi \end{array} \right\}$.
Since $\xi > 0$, was arbitrary,
$\delta_3 \left(\left\{ \begin{array}{c} (l,m,n) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) \geq \dot{\varepsilon} \,\text{or} \\ \nu(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) \geq \dot{\varepsilon}, \omega(\mathfrak{x}_{lmn} - \mathfrak{x}_{uvw}, \lambda) \geq \dot{\varepsilon} \end{array} \right\} \right) = 0$ this gives $(\mathfrak{x}_{lmn})$ represents statistically Cauchy in relate to $\mathfrak{NN}$ (μ, ν).

In contrast, for certain l_0, m_0 as well as $n_0 \in \mathbb{N}$ we obtain,
$\delta_3 \left(\left\{ \begin{array}{c} (l,m,n) \in \mathbb{N}^3 : 1 - \mu(\mathfrak{x}_{lmn} - \mathfrak{x}_{l_0 m_0 n_0}, \lambda) \geq \dot{\varepsilon} \,\text{or} \\ \nu(\mathfrak{x}_{lmn} - \mathfrak{x}_{l_0 m_0 n_0}, \lambda) \geq \dot{\varepsilon}, \omega(\mathfrak{x}_{lmn} - \mathfrak{x}_{l_0 m_0 n_0}, \lambda) \geq \dot{\varepsilon} \end{array} \right\} \right) = 0$, it follows $\delta_3(K) = 1$, in which,

$$K=\left\{\begin{array}{c}(i,j,k)\in\mathbb{N}^3:1-\mu(\mathfrak{x}_{ijk}-\mathfrak{x}_{i_0j_0k_0},\lambda)<\dot{\varepsilon},\\ \nu(\mathfrak{x}_{ijk}-\mathfrak{x}_{i_0j_0k_0},\lambda)<\dot{\varepsilon}\,\text{and}\,\omega(\mathfrak{x}_{ijk}-\mathfrak{x}_{i_0j_0k_0},\lambda)<\dot{\varepsilon}\end{array}\right\}=0.$$

Let $\mathfrak{x}\in(\mathbb{E},\mu,\nu,\omega,\circ,\Diamond,*)$ such that $1-\mu(\mathfrak{x}_{i_0j_0k_0}-\mathfrak{x},\lambda)<\dot{\varepsilon}$, $\nu(\mathfrak{x}_{i_0j_0k_0}-\mathfrak{x},\lambda)<\dot{\varepsilon}$ as well as $\omega(\mathfrak{x}_{i_0j_0k_0}-\mathfrak{x},\lambda)<\dot{\varepsilon}$ for all $\dot{\varepsilon}>0$ for every $\dot{\varepsilon}>0$.

Assume, $\mathfrak{G}=\left\{\begin{array}{c}(u,v,w)\in\mathbb{N}^3:1-\mu(\mathfrak{x}_{uvw}-\mathfrak{x},\lambda)<\dot{\varepsilon}\,\text{and}\\ \nu(\mathfrak{x}_{uvw}-\mathfrak{x},\lambda)<\dot{\varepsilon}\ \text{and}\omega(\mathfrak{x}_{uvw}-\mathfrak{x},\lambda)<\dot{\varepsilon}\end{array}\right\}$.

And letting $(p,q,r)\in K$

$$1-\mu(\mathfrak{x}_{pqr}-\mathfrak{x},\lambda)=1-\mu(\mathfrak{x}_{pqr}-\mathfrak{x}_{i_0j_0k_0}+\mathfrak{x}_{i_0j_0k_0}-\mathfrak{x},\lambda)<1-(1-\dot{\varepsilon})\circ(1-\dot{\varepsilon})<1-(1-\xi)<\xi.$$

Again

$$\begin{aligned}\nu(\mathfrak{x}_{pqr}-\mathfrak{x},\lambda)&=\nu(\mathfrak{x}_{pqr}-\mathfrak{x}_{i_0j_0k_0}+\mathfrak{x}_{i_0j_0k_0}-\mathfrak{x},\lambda)\\ &\le\nu\left(\mathfrak{x}_{pqr}-\mathfrak{x}_{i_0j_0k_0},\frac{\lambda}{2}\right)\Diamond\nu\left(\mathfrak{x}_{i_0j_0k_0}-\mathfrak{x},\frac{\lambda}{2}\right)<\dot{\varepsilon}\Diamond\dot{\varepsilon}<\xi,\end{aligned}$$

Also

$$\begin{aligned}\omega(\mathfrak{x}_{pqr}-\mathfrak{x},\lambda)&=\omega(\mathfrak{x}_{pqr}-\mathfrak{x}_{i_0j_0k_0}+\mathfrak{x}_{i_0j_0k_0}-\mathfrak{x},\lambda)\\ &\le\omega\left(\mathfrak{x}_{pqr}-\mathfrak{x}_{i_0j_0k_0},\frac{\lambda}{2}\right)\Diamond\omega\left(\mathfrak{x}_{i_0j_0k_0}-\mathfrak{x},\frac{\lambda}{2}\right)<\dot{\varepsilon}*\dot{\varepsilon}<\xi,\end{aligned}$$

after that $(p,q,r)\in\mathfrak{G}$ in other way, $K\subseteq\mathfrak{G}\Rightarrow\delta_3(K)\le\delta_3(\mathfrak{G}),1\le\delta_3(\mathfrak{G})\le1\Rightarrow\delta_3(\mathfrak{G})=1$ it means $\delta_3(\mathfrak{G}^c)=0$.

Hence, $\delta_3\{(u,v,w):1-\mu(\mathfrak{x}_{uvw}-\mathfrak{x},\lambda)\ge\dot{\varepsilon}\,\text{or}\,\nu(\mathfrak{x}_{uvw}-\mathfrak{x},\lambda)\ge\dot{\varepsilon},\omega(\mathfrak{x}_{uvw}-\mathfrak{x},\lambda)\ge\dot{\varepsilon}\}=0$.

Then $\mathfrak{st}_3^{(\mu,\nu,\omega)}-\lim\mathfrak{x}=\mathfrak{x}$.

Thus proof is completed. □

We are going to present some particular spaces in addition after that proved their completeness that is an immediate consequence of the preceding definitions as well as conclusion.

${}_3l_\infty=\{\mathfrak{x}=(\mathfrak{x}_{ijk}):\mathfrak{x}\,\text{represents a real number sequence of bounded triplet}\}$

${}_3l_{\infty S}=\{(\mathfrak{x}_{ijk})\in_3 l_\infty:\ \text{for few naturals}\ \ i_0,j_0,k_0\,\text{we get}\}$

$$\delta_3\left\{\begin{array}{c}(i,j,k)\in\mathbb{N}^3:1-\mu(\mathfrak{x}_{ijk}-\mathfrak{x}_{i_0j_0k_0},\lambda)\ge\dot{\varepsilon}\,\text{or}\\ \nu(\mathfrak{x}_{ijk}-\mathfrak{x}_{i_0j_0k_0},\lambda)\ge\dot{\varepsilon},\omega(\mathfrak{x}_{ijk}-\mathfrak{x}_{i_0j_0k_0},\lambda)\ge\dot{\varepsilon}\end{array}\right\}=0.$$

Theorem 5.3. *Completeness exists in the ${}_3l_{\infty S}$ space of statistically Cauchy triple bounded sequences.*

Proof. Let $\left(\mathfrak{x}_{ijk}^n\right)_{n\ge1}$ be a Cauchy sequence within ${}_3l_{\infty S}$ it gives that $n_0\in\mathbb{N}$ exist which means for each $n\ge n_0$ as well as $\dot{\varepsilon}>0$ we obtain

$1-\mu\left(\mathfrak{x}_{ijk}^n-\mathfrak{x}_{ijk}^{n_0},\lambda\right)<\dot{\varepsilon},\nu\left(\mathfrak{x}_{ijk}^n-\mathfrak{x}_{ijk}^{n_0},\lambda\right)<\dot{\varepsilon}$ and $\omega\left(\mathfrak{x}_{ijk}^n-\mathfrak{x}_{ijk}^{n_0},\lambda\right)<\dot{\varepsilon}$.

Also, $\left(\mathfrak{x}_{ijk}^n\right)$ provide a statistically Cauchy sequence for all $n\in\mathbb{N}$.

At this point, fix i,j,k in addition take the real number sequence, $\left(\mathfrak{x}_{ijk}^n\right)=\mathfrak{x}_{ijk}^1,\mathfrak{x}_{ijk}^2,\mathfrak{x}_{ijk}^3,\ldots,$

which is meant to converges towards (z_{ijk}) (owing to the completeness among $\mathbb{R}$). Since i, j, k are random, after that $\left(\mathfrak{x}_{ijk}^n\right)$ converges towards (z_{ijk}) for each of i, j and k. Thus, it occurs $n_\varepsilon \in \mathbb{N}$ which means for any $n \geq n_\varepsilon$, we have, $1-\mu(\mathfrak{x}_{ijk}^n - z_{ijk}, \lambda) < \dot{\varepsilon}$, $\nu(\mathfrak{x}_{ijk}^n - z_{ijk}, \lambda) < \dot{\varepsilon}$ and $\omega(\mathfrak{x}_{ijk}^n - z_{ijk}, \lambda) < \dot{\varepsilon}$.

Repeatedly, $n \geq \max\{n_0, n_\varepsilon\}$ the $\left(\mathfrak{x}_{ijk}^n\right)$ sequence represents statistically Cauchy, which implies that there are $i_0, j_0, k_0 \in \mathbb{N}$ where

$$\delta_3 \left\{ \begin{array}{c} (i,j,k) \in \mathbb{N}^3 : 1-\mu\left(\mathfrak{x}_{ijk}^n - \mathfrak{x}_{i_0j_0k_0}^n, \lambda\right) \geq \dot{\varepsilon} \text{ or} \\ \nu(\mathfrak{x}_{ijk}^n - \mathfrak{x}_{i_0j_0k_0}^n, \lambda) \geq \dot{\varepsilon},\ \omega(\mathfrak{x}_{ijk}^n - \mathfrak{x}_{i_0j_0k_0}^n, \lambda) \geq \dot{\varepsilon} \end{array} \right\} = 0.$$

Now $\delta_3(\mathbb{M}) = 0$, in which,

$$\mathbb{M} = \left\{ \begin{array}{c} (i,j,k) \in \mathbb{N}^3 : 1-\mu\left(\mathfrak{x}_{ijk}^n - \mathfrak{x}_{i_0j_0k_0}^n, \lambda\right) < \dot{\varepsilon} \text{ or} \\ \nu\left(\mathfrak{x}_{ijk}^n - \mathfrak{x}_{i_0j_0k_0}^n, \lambda\right) < \dot{\varepsilon}, \omega\left(\mathfrak{x}_{ijk}^n - \mathfrak{x}_{i_0j_0k_0}^n, \lambda\right) < \dot{\varepsilon} \end{array} \right\} = 0.$$

Now choose $\xi \in (0,1)$ which means $(1-\dot{\varepsilon}) \circ (1-\dot{\varepsilon}) > 1-\xi$, $\dot{\varepsilon} \Diamond \dot{\varepsilon} < \xi$ and $\dot{\varepsilon} * \dot{\varepsilon} < \xi$.

Letting

$$K^c = \left\{ \begin{array}{c} (e,f,g) \in \mathbb{N}^3 : 1-\mu(\mathfrak{z}_{efg} - \mathfrak{z}_{e_0f_0g_0}, \lambda) \geq \dot{\varepsilon} \text{ or} \\ \nu(\mathfrak{z}_{efg} - \mathfrak{z}_{e_0f_0g_0}, \lambda) \geq \dot{\varepsilon}, \omega(\mathfrak{z}_{efg} - \mathfrak{z}_{e_0f_0g_0}, \lambda) \geq \dot{\varepsilon} \end{array} \right\}.$$

Now we wish to prove $\delta_3(K^c) = 0$ for some naturals i_0, j_0 and k_0, for the requirement, we have

$$K = \left\{ \begin{array}{c} (e,f,g) \in \mathbb{N}^3 : 1-\mu(\mathfrak{z}_{efg} - \mathfrak{z}_{e_0f_0g_0}, \lambda) < \dot{\varepsilon} \text{ and} \\ \nu(\mathfrak{z}_{efg} - \mathfrak{z}_{e_0f_0g_0}, \lambda) < \dot{\varepsilon} \text{ and}\, \omega(\mathfrak{z}_{efg} - \mathfrak{z}_{e_0f_0g_0}, \lambda) < \dot{\varepsilon} \end{array} \right\}.$$

Put $i_0 \geq e_0; j_0 \geq f_0; k_0 \geq g_0$; in addition consider $(r,s,t) \in \mathbb{M}$. After that

$$\begin{aligned} 1-\mu(\mathfrak{z}_{rst} - \mathfrak{z}_{e_0f_0g_0}, t) &= 1-\mu\left(\mathfrak{z}_{rst} - \mathfrak{x}_{rst}^n + \mathfrak{x}_{rst}^n - \mathfrak{x}_{e_0f_0g_0}^n + \mathfrak{x}_{e_0f_0g_0}^n - \mathfrak{z}_{e_0f_0g_0}, \lambda\right) \\ &\leq 1-\mu\left(\mathfrak{z}_{rst} - \mathfrak{x}_{rst}^n, \frac{\lambda}{3}\right) \circ \mu\left(\mathfrak{x}_{rst}^n - \mathfrak{x}_{e_0f_0g_0}^n, \frac{\lambda}{3}\right) \circ \mu\left(\mathfrak{x}_{e_0f_0g_0}^n - \mathfrak{z}_{e_0f_0g_0}, \frac{\lambda}{3}\right) \\ &< 1-(1-\dot{\varepsilon}) \circ (1-\dot{\varepsilon}) \circ (1-\varepsilon) < 1-(1-\xi) < \xi. \end{aligned}$$

Furthermore, we possess

$$\begin{aligned} \nu(\mathfrak{z}_{rst} - \mathfrak{z}_{e_0f_0g_0}, \lambda) &= \nu\left(\mathfrak{z}_{rst} - \mathfrak{x}_{rst}^n + \mathfrak{x}_{rst}^n - \mathfrak{x}_{e_0f_0g_0}^n + \mathfrak{x}_{e_0f_0g_0}^n - \mathfrak{z}_{e_0f_0g_0}, \lambda\right) \\ &\leq \nu\left(\mathfrak{z}_{rst} - \mathfrak{x}_{rst}^n, \frac{\lambda}{3}\right)\ \nu\left(\mathfrak{x}_{rst}^n - \mathfrak{x}_{e_0f_0g_0}^n, \frac{\lambda}{3}\right) \Diamond \nu\left(\mathfrak{x}_{e_0f_0g_0}^n - \mathfrak{z}_{e_0f_0g_0}, \frac{\lambda}{3}\right) \\ &< \dot{\varepsilon} \Diamond \dot{\varepsilon} \Diamond \dot{\varepsilon} < \xi, \end{aligned}$$

Also,

$$\begin{aligned} \omega(\mathfrak{z}_{rst} - \mathfrak{z}_{e_0f_0g_0}, \lambda) &= \omega\left(\mathfrak{z}_{rst} - \mathfrak{x}_{rst}^n + \mathfrak{x}_{rst}^n - \mathfrak{x}_{e_0f_0g_0}^n + \mathfrak{x}_{e_0f_0g_0}^n - \mathfrak{z}_{e_0f_0g_0}, \lambda\right) \\ &\leq \omega\left(\mathfrak{z}_{rst} - \mathfrak{x}_{rst}^n, \frac{\lambda}{3}\right) * \omega\left(\mathfrak{x}_{rst}^n - \mathfrak{x}_{e_0f_0g_0}^n, \frac{\lambda}{3}\right) * \omega\left(\mathfrak{x}_{e_0f_0g_0}^n - \mathfrak{z}_{e_0f_0g_0}, \frac{\lambda}{3}\right) \\ &< \dot{\varepsilon} * \dot{\varepsilon} * \dot{\varepsilon} < \xi, \end{aligned}$$

it follows that $\mathbb{M} \subseteq K \Rightarrow K^c \subseteq \mathbb{M}^c$ thus, $\delta_3(K^c) \leq \delta_3(\mathbb{M}^c) = 0$.

Finally $\delta_3(K^c) = 0$. Hence, $(\mathfrak{z}_{ijk}) \in_3 l_{\infty S}$. This brings the proof is complete. □

6. Conclusion

This project's goal towards extend the statistical convergence of sequences to triple sequences within the $\mathfrak{NNS}$ context. We have examined the algebraic characteristics of the limits of this convergence. Along with the topological characteristics of the spaces, the relationship between statistical and typical convergence has been examined for triple sequences.

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