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Reciprocal fraction function tools used for ♮-rung Quadripartitioned neutrosophic sets

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Abstract. This paper presents a new method for creating a ♮-rung reciprocal fraction function. We introduce quadripartitioned neutrosophic sets (QNSS). The reciprocal fraction function applied to quadripartitioned neutrosophic sets are the neutrosophic sets and ♮-rung neutrosophic sets. This article will examine weighted geometric, quadripartitioned neutrosophic set weighted averaging, generalized weighted averaging, and generalized weighted geometric operators. Boundedness, idempotency, monotonicity and commutativity are also discussed.

Keywords: weighted averaging, geometric, generalized averaging, and generalized geometric operators.

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1. Introduction

Numerous uncertain theories, such as fuzzy set (FS) [19], intuitionistic FS (IFS) [3], Pythagorean FS (PFS) [18], and spherical FS (SFS) [1], have been created to address the ambiguities. Subsequently, Atanassov introduced the idea of an IFS that is restricted to one group according to non-membership degree (NMG) [3]. An FS is a group of objects in a given set that have a membership degree (MG) ranging from 0 to 1. The decision-making (DM) process could only be aware of one issue if the MG and NMG scores are both higher than one. Yager [18] defines PFS as an IFS with a value more than one and a square sum of MG and NMG less than one. According to Cuong et al. [4], the image FS concept is composed of three characteristics: positive MG, neutral MG, and negative MG. It therefore has certain advantages over PFS and IFS. Liu et al. investigated an image FS extension employing AOs [7]. Liu et al. have described an AO-based generalized PFS and its applications [8]. AO characteristics based on interval values and PFS [12]. Liu et al. [7] were the first to present the AO-based image FS. The total of the positive, neutral, and negative MG scores in the DM approach challenge never surpasses 1. Ashraf et al. [1] established the idea of SFS, which ensures that the square sum of the positive, neutral, and negative degrees does not exceed 1. Fatmaa et al. investigated the idea of SFS [5]. Using AOs and distance measurements, Zeng et al. [20] explained how to calculate ordered weighted distances. Yager [18] produced several averaging and geometric AOs under PFS weighted, ordered weighted, and weighted power conditions. Peng et al. examined a basic PFS based on the features of AOs [13]. Ashraf et al. [2] state that fuzzy spherical Dombi AOs were developed. The concepts of SFSs and T-SFSs [15, 16]. Temel et al. [14] discussed the use of Muirhead power normal SFS to MADM. MADM is used by Peng et al. [11] to investigate neutrosophic sets using TOPSIS and MABAC techniques. The TOPSIS-based extension of PFS was discussed by Zhang and associates [21]. A range of algebraic structures and aggregation techniques with possible uses were recently covered by Palanikumar et al. [9] and [10]. Gulistan [6] et al. discussed the concept of Einstein AOs under q-rung hypersoft sets. Voskoglou [17] et al. introduced the notion of q-rung NSSs and topological spaces.

2. Operations for $\natural$-rung QNSN

The fractional part of η, where η is a real number, may be expressed as follows, assuming that f is a fractional part function: Additionally, $\mathcal{H}[\eta] = \langle\eta\rangle = \eta - \{\eta\}$. The difference between a real number and its largest integer value, which is determined by the biggest integer function, may also be expressed as a fractional part function. If η is an integer, then the fractional component of $\eta = 0$. $\mathcal{H}[\eta] = \frac{1}{\eta}$ is a reciprocal fractional part function, assuming its existence. It is well known that whenever η is an integer, its fractional part equals 0. Consequently, l cannot be an integer in order for $\mathcal{H}[\eta] = \frac{1}{\eta}$ to be defined. $\mathcal{H}[\eta] = \frac{1}{\eta}$ is its domain, and it encompasses all real numbers except integers. The $\natural$-rung QNSN and its operations were developed after we discussed the idea of the $\natural$-rung quadripartitioned neutrosophic number ($\natural$-rung QNSN). The following characteristics make it easy to identify the reciprocal functions. (i) The form of reciprocal functions is a fraction. Real numbers make up the numerator, and numbers, variables, or polynomials make up the denominator. (ii) A reciprocal of x is equal to $1/x$. The domain and range of the reciprocal function $1/x$ are the sets of all real numbers other than zero and zero, respectively.

Definition 2.1. The ♮-rung QNSS $\tau = \left\{\epsilon, \left[\left\langle \langle\Xi_\tau\rangle\langle\epsilon\rangle, \langle\daleth_\tau\rangle\langle\epsilon\rangle, \langle\Lambda_\tau\rangle\langle\epsilon\rangle, \langle\Omega_\tau\rangle\langle\epsilon\rangle \right\rangle\right] \middle| \epsilon \in \mathcal{X}\right\}$, put $\langle\Xi_\tau\rangle, \langle\daleth_\tau\rangle, \langle\Lambda_\tau\rangle, \langle\Omega_\tau\rangle : \mathcal{X} \to [0,1]$ denote the TMG, contradiction MG, unknown MG and false MG of $\epsilon \in \mathcal{X}$ to τ, respectively and $0 \leq \langle\Xi_\tau\rangle\langle\epsilon\rangle^\natural + \langle\daleth_\tau\rangle\langle\epsilon\rangle^\natural + \langle\Lambda_\tau\rangle\langle\epsilon\rangle^\natural + \langle\Omega_\tau\rangle\langle\epsilon\rangle^\natural \leq 1$. For convenience, $\tau = \left[\left\langle \Xi_\tau\rangle, \langle\daleth_\tau\rangle, \langle\Lambda_\tau\rangle, \langle\Omega_\tau \right\rangle\right]$ is represent a ♮-rung QNSN.

Definition 2.2. Let $\tau = [\langle\Xi_\tau\rangle, \langle\daleth_\tau\rangle, \langle\Lambda_\tau\rangle, \langle\Omega_\tau\rangle]$, $\tau_1 = [\langle\Xi_1\rangle, \langle\daleth_1\rangle, \langle\Lambda_1\rangle, \langle\Omega_1\rangle]$ and $\tau_2 = [\langle\Xi_2\rangle, \langle\daleth_2\rangle, \langle\Lambda_2\rangle, \langle\Omega_2\rangle]$ be any three ♮-rung QNSNs, and $\rho > 0$. Then

$$(1)\ \tau_1 \Cup \tau_2 = \begin{bmatrix} \sqrt[\natural]{\langle\Xi_1\rangle^\natural + \langle\Xi_2\rangle^\natural - \langle\Xi_1\rangle^\natural \cdot \langle\Xi_2\rangle^\natural}, \\ \sqrt[\natural]{\langle\daleth_1\rangle^\natural + \langle\daleth_2\rangle^\natural - \langle\daleth_1\rangle^\natural \cdot \langle\daleth_2\rangle^\natural}, \\ \langle\Lambda_1\rangle^\natural \cdot \langle\Lambda_2\rangle^\natural, \langle\Omega_1\rangle^\natural \cdot \langle\Omega_2\rangle^\natural \end{bmatrix},$$

$$(2)\ \tau_1 \Cap \tau_2 = \begin{bmatrix} \langle\Xi_1\rangle^\natural \cdot \langle\Xi_2\rangle^\natural, \langle\daleth_1\rangle^\natural \cdot \langle\daleth_2\rangle^\natural \\ \sqrt[\natural]{\langle\Lambda_1\rangle^\natural + \langle\Lambda_2\rangle^\natural - \langle\Lambda_1\rangle^\natural \cdot \langle\Lambda_2\rangle^\natural}, \\ \sqrt[\natural]{\langle\Omega_1\rangle^\natural + \langle\Omega_2\rangle^\natural - \langle\Omega_1\rangle^\natural \cdot \langle\Omega_2\rangle^\natural} \end{bmatrix}$$

$$(3)\ \rho \cdot \tau = \begin{bmatrix} \sqrt[\natural]{\shortmid - \langle \shortmid - \langle\langle\Xi\rangle^\natural\rangle^\rho}, \sqrt[\natural]{\shortmid - \langle \shortmid - \langle\langle\daleth\rangle^\natural\rangle^\rho}, \\ \langle\langle\Lambda\rangle^\natural\rangle^\rho, \langle\langle\Omega\rangle^\natural\rangle^\rho \end{bmatrix},$$

$$(4)\ \tau^\rho = \begin{bmatrix} \langle\langle\Xi\rangle^\natural\rangle^\rho, \langle\langle\daleth\rangle^\natural\rangle^\rho, \\ \sqrt[\natural]{\shortmid - \langle \shortmid - \langle\langle\Lambda\rangle^\natural\rangle^\rho}, \sqrt[\natural]{\shortmid - \langle \shortmid - \langle\langle\Omega\rangle^\natural\rangle^\rho} \end{bmatrix}.$$

3. AOs based on ♮-rung RFFQNSN

Here we describe the AOs using ♮-rung RFFQNWA, ♮-rung RFFQNWG, G ♮-rung RFFQNWA, and G ♮-rung RFFQNWG.

3.1. $\natural - rungQNWA$

Definition 3.1. Let $\tau_i = [\langle\Xi_i\rangle, \langle\daleth_i\rangle, \langle\Lambda_i\rangle, \langle\Omega_i\rangle]$ be the ♮-rung RFFQNSNs, $W = \langle\gamma_1, \gamma_2, ..., \gamma_n\rangle$ be the weight of τ_i, $\gamma_i \geq 0$ and $\Cup_{i=1}^n \gamma_i = 1$. Then ♮-rung RFFQNWA $\langle\tau_1, \tau_2, ..., \tau_n\rangle = \Cup_{i=1}^n \gamma_i\tau_i$.

Theorem 3.2. *Let* $\tau_i = \left[\ \langle\Xi_i\rangle, \langle\daleth_i\rangle, \langle\Lambda_i\rangle, \langle\Omega_i\rangle\rangle\ \right]$ *be the ♮-rung RFFQNSNs.*
Then $\langle\natural\rangle QNWA\langle\tau_1, \tau_2, ..., \tau_n\rangle$

$$= \begin{bmatrix} \sqrt[\natural]{\shortmid - \boxdot_{i=1}^n \left\langle \shortmid - \langle\Xi_i\rangle^\natural \right\rangle^{\gamma_i}}, \sqrt[\natural]{\shortmid - \boxdot_{i=1}^n \left\langle \shortmid - \langle\daleth_i\rangle^\natural \right\rangle^{\gamma_i}}, \\ \boxdot_{i=1}^n \langle\langle\Lambda_i\rangle^\natural\rangle^{\gamma_i}, \boxdot_{i=1}^n \langle\langle\Omega_i\rangle^\natural\rangle^{\gamma_i} \end{bmatrix}.$$

Proof. If $n = 2$, then ♮-rung RFFQNWA$\langle\tau_1, \tau_2\rangle = \gamma_1\tau_1 \Cup \gamma_2\tau_2$, put

$$\gamma_1\tau_1 = \begin{bmatrix} \sqrt[\natural]{\shortmid - \left\langle \shortmid - \langle\Xi_1\rangle^\natural \right\rangle^{\gamma_1}}, \sqrt[\natural]{\shortmid - \left\langle \shortmid - \langle\daleth_1\rangle^\natural \right\rangle^{\gamma_1}} \\ \langle\langle\Lambda_1\rangle^\natural\rangle^{\gamma_1}, \langle\langle\Omega_1\rangle^\natural\rangle^{\gamma_1} \end{bmatrix}$$

$$\gamma_2\tau_2 = \begin{bmatrix} \sqrt[\natural]{\imath - \left\langle \imath - \langle \Xi_2 \rangle^{\natural} \right\rangle^{\gamma_2}}, \sqrt[\natural]{\imath - \left\langle \imath - \langle \daleth_2 \rangle^{\natural} \right\rangle^{\gamma_2}} \\ \langle \langle \Lambda_2 \rangle^{\natural} \rangle^{\gamma_2}, \langle \langle \Omega_2 \rangle^{\natural} \rangle^{\gamma_2} \end{bmatrix}.$$

Now,

$$\gamma_1\tau_1 \Cup \gamma_2\tau_2 = \begin{bmatrix} \sqrt[\natural]{\begin{matrix} \left\langle \imath - \left\langle \imath - \langle \Xi_1 \rangle^{\natural} \right\rangle^{\gamma_1} \right\rangle + \left\langle \imath - \left\langle \imath - \langle \Xi_2 \rangle^{\natural} \right\rangle^{\gamma_2} \right\rangle \\ - \left\langle \imath - \left\langle \imath - \langle \Xi_1 \rangle^{\natural} \right\rangle^{\gamma_1} \right\rangle \cdot \left\langle \imath - \left\langle \imath - \langle \Xi_2 \rangle^{\natural} \right\rangle^{\gamma_2} \right\rangle \end{matrix}}, \\ \\ \sqrt[\natural]{\begin{matrix} \left\langle \imath - \left\langle \imath - \langle \daleth_1 \rangle^{\natural} \right\rangle^{\gamma_1} \right\rangle + \left\langle \imath - \left\langle \imath - \langle \daleth_2 \rangle^{\natural} \right\rangle^{\gamma_2} \right\rangle \\ - \left\langle \imath - \left\langle \imath - \langle \daleth_1 \rangle^{\natural} \right\rangle^{\gamma_1} \right\rangle \cdot \left\langle \imath - \left\langle \imath - \langle \daleth_2 \rangle^{\natural} \right\rangle^{\gamma_2} \right\rangle \end{matrix}}, \\ \\ \langle \langle \Lambda_1 \rangle^{\natural} \rangle^{\gamma_1} \cdot \langle \langle \Lambda_2 \rangle^{\natural} \rangle^{\gamma_2} \\ \langle \langle \Omega_1 \rangle^{\natural} \rangle^{\gamma_1} \cdot \langle \langle \Omega_2 \rangle^{\natural} \rangle^{\gamma_2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\natural]{\imath - \left\langle \imath - \langle \Xi_1 \rangle^{\natural} \right\rangle^{\gamma_1} \left\langle \imath - \langle \Xi_2 \rangle^{\natural} \right\rangle^{\gamma_2}}, \\ \sqrt[\natural]{\imath - \left\langle \imath - \langle \daleth_1 \rangle^{\natural} \right\rangle^{\gamma_1} \left\langle \imath - \langle \daleth_2 \rangle^{\natural} \right\rangle^{\gamma_2}}, \\ \langle \langle \Lambda_1 \rangle^{\natural} \rangle^{\gamma_1} \cdot \langle \langle \Lambda_2 \rangle^{\natural} \rangle^{\gamma_2} \\ \langle \langle \Omega_1 \rangle^{\natural} \rangle^{\gamma_1} \cdot \langle \langle \Omega_2 \rangle^{\natural} \rangle^{\gamma_2} \end{bmatrix}$$

Hence, $\natural - rungRFFQNWA\langle \tau_1, \tau_2 \rangle$

$$= \begin{bmatrix} \sqrt[\natural]{\imath - \boxdot_{i=1}^{2} \left\langle \imath - \langle \Xi_i \rangle^{\natural} \right\rangle^{\gamma_i}}, \sqrt[\natural]{\imath - \boxdot_{i=1}^{2} \left\langle \imath - \langle \daleth_i \rangle^{\natural} \right\rangle^{\gamma_i}} \\ \boxdot_{i=1}^{2} \langle \langle \Lambda_i \rangle^{\natural} \rangle^{\gamma_i}, \boxdot_{i=1}^{2} \langle \langle \Omega_i \rangle^{\natural} \rangle^{\gamma_i} \end{bmatrix}.$$

Also, $n \geq 3$,

Thus, $\natural - rungRFFQNWA\langle \tau_1, \tau_2, ..., \tau_l \rangle$

$$= \begin{bmatrix} \sqrt[\natural]{\imath - \boxdot_{i=1}^{l} \left\langle \imath - \langle \Xi_i \rangle^{\natural} \right\rangle^{\gamma_i}}, \sqrt[\natural]{\imath - \boxdot_{i=1}^{l} \left\langle \imath - \langle \daleth_i \rangle^{\natural} \right\rangle^{\gamma_i}} \\ \boxdot_{i=1}^{l} \langle \langle \Lambda_i \rangle^{\natural} \rangle^{\gamma_i}, \boxdot_{i=1}^{l} \langle \langle \Omega_i \rangle^{\natural} \rangle^{\gamma_i} \end{bmatrix}.$$

If $n = l + 1$, then $\natural$-rung RFFQNWA $\langle \tau_1, \tau_2, ..., \tau_l, \tau_{l+1} \rangle$

$$= \begin{bmatrix} \sqrt[\natural]{\begin{matrix} \Cup_{i=1}^{l} \left\langle \imath - \left\langle \imath - \langle \Xi_i \rangle^{\natural} \right\rangle^{\gamma_i} \right\rangle + \left\langle \imath - \left\langle \imath - \langle \Xi_{l+1} \rangle^{\natural} \right\rangle^{\gamma_{l+1}} \right\rangle \\ - \boxdot_{i=1}^{l} \left\langle \imath - \left\langle \imath - \langle \Xi_i \rangle^{\natural} \right\rangle^{\gamma_i} \right\rangle \cdot \left\langle \imath - \left\langle \imath - \langle \Xi_{l+1} \rangle^{\natural} \right\rangle^{\gamma_{l+1}} \right\rangle \end{matrix}} \\ \sqrt[\natural]{\begin{matrix} \Cup_{i=1}^{l} \left\langle \imath - \left\langle \imath - \langle \daleth_i \rangle^{\natural} \right\rangle^{\gamma_i} \right\rangle + \left\langle \imath - \left\langle \imath - \langle \daleth_{l+1} \rangle^{\natural} \right\rangle^{\gamma_{l+1}} \right\rangle \\ - \boxdot_{i=1}^{l} \left\langle \imath - \left\langle \imath - \langle \daleth_i \rangle^{\natural} \right\rangle^{\gamma_i} \right\rangle \cdot \left\langle \imath - \left\langle \imath - \langle \daleth_{l+1} \rangle^{\natural} \right\rangle^{\gamma_{l+1}} \right\rangle \end{matrix}} \\ \boxdot_{i=1}^{l} \langle \langle \Lambda_i \rangle^{\natural} \rangle^{\gamma_i} \cdot \langle \Lambda_{l+1} \rangle^{\natural} \rangle^{\gamma_{l+1}} \\ \boxdot_{i=1}^{l} \langle \langle \Omega_i \rangle^{\natural} \rangle^{\gamma_i} \cdot \langle \Omega_{l+1} \rangle^{\natural} \rangle^{\gamma_{l+1}} \end{bmatrix}$$

$$= \left[\begin{array}{c} \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{l+1}\left\langle \text{ı} - \langle \Xi_i \rangle^{\natural} \right\rangle^{\gamma_i}},, \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{l+1}\left\langle \text{ı} - \langle \daleth_i \rangle^{\natural} \right\rangle^{\gamma_i}}, \\ \boxdot_{i=1}^{l+1}\langle\langle \Lambda_i \rangle^{\natural}\rangle^{\gamma_i}, \boxdot_{i=1}^{l+1}\langle\langle \Omega_i \rangle^{\natural}\rangle^{\gamma_i} \end{array}\right].$$

Theorem 3.3. *Let* $\tau_i = \left[\langle \Xi_i \rangle, \langle \daleth_i \rangle, \langle \Lambda_i \rangle, \langle \Omega_i \rangle\rangle\right]$ *be the* $\natural$*-rung RFFQNSNs. Then* $\natural$*-rung RFFQNWA* $\langle \tau_1, \tau_2, ..., \tau_n \rangle = \tau$

Proof. Since $\langle \Xi_i \rangle = \langle \Xi \rangle$, $\langle \daleth_i \rangle = \langle \daleth \rangle$, $\langle \Lambda_i \rangle = \langle \Lambda \rangle$ and $\langle \Omega_i \rangle = \langle \Omega \rangle$ and $\mathbb{U}_{i=1}^{n} \gamma_i = 1$. Now, $\langle \natural \rangle QNWA \langle \tau_1, \tau_2, ..., \tau_n \rangle$

$$\begin{aligned} &= \left[\begin{array}{c} \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle \Xi_i \rangle^{\natural} \right\rangle^{\gamma_i}},, \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle \daleth_i \rangle^{\natural} \right\rangle^{\gamma_i}}, \\ \boxdot_{i=1}^{n}\langle\langle \Lambda_i \rangle^{\natural}\rangle^{\gamma_i}, \boxdot_{i=1}^{n}\langle\langle \Omega_i \rangle^{\natural}\rangle^{\gamma_i} \end{array}\right] \\ &= \left[\begin{array}{c} \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\langle \Xi \rangle^{\natural} \right\rangle^{\mathbb{U}_{i=1}^{n}\gamma_i}}, \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\langle \daleth \rangle^{\natural} \right\rangle^{\mathbb{U}_{i=1}^{n}\gamma_i}}, \\ \langle\langle \Lambda \rangle^{\natural}\rangle^{\mathbb{U}_{i=1}^{n}\gamma_i}, \langle\langle \Omega \rangle^{\natural}\rangle^{\mathbb{U}_{i=1}^{n}\gamma_i}, \end{array}\right] \\ &= \left[\begin{array}{c} \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\langle \Xi \rangle^{\natural} \right\rangle}, \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\langle \daleth \rangle^{\natural} \right\rangle}, \\ \langle \Lambda \rangle^{\natural}, \langle \Omega \rangle^{\natural} \end{array}\right] \\ &= \tau. \end{aligned}$$

Theorem 3.4. *Let* $\tau_i = \left[\langle \Xi_i \rangle, \langle \daleth_i \rangle, \langle \Lambda_i \rangle, \langle \Omega_i \rangle\rangle\right]$ *be the* $\natural$*-rung RFFQNSNs. Then* $\natural$*-rung RFFQNWA*$\langle \tau_1, \tau_2, ..., \tau_n \rangle$*, put* $\underbrace{\Xi} = \min\langle \Xi_{ij} \rangle$, $\overbrace{\langle \Xi \rangle} = \max\langle \Xi_{ij} \rangle$, $\underbrace{\langle \daleth \rangle} = \min\langle \daleth_{ij} \rangle$, $\overbrace{\langle \daleth \rangle} = \max\langle \daleth_{ij} \rangle$, $\underbrace{\langle \Lambda \rangle} = \min\langle \Lambda_{ij} \rangle$, $\overbrace{\langle \Lambda \rangle} = \max\langle \Lambda_{ij} \rangle$, $\underbrace{\langle \Omega \rangle} = \min\langle \Omega_{ij} \rangle$, $\overbrace{\langle \Omega \rangle} = \max\langle \Omega_{ij} \rangle$ *and put* $1 \le i \le n,\ j = 1, 2, ..., i_j$. *Then,* $\left[\underbrace{\langle \Xi \rangle}, \underbrace{\langle \daleth \rangle}, \overbrace{\langle \Lambda \rangle}, \overbrace{\langle \Omega \rangle}\right]$

$$\begin{aligned} &\le \langle \natural \rangle QNWA \langle \tau_1, \tau_2, ..., \tau_n \rangle \\ &\le \left[\overbrace{\langle \Xi \rangle}, \overbrace{\langle \daleth \rangle}, \underbrace{\langle \Lambda \rangle}, \underbrace{\langle \Omega \rangle}\right]. \end{aligned}$$

(Boundedness property).

Proof. Since, $\underbrace{\langle \Xi \rangle} = \min\langle \Xi_{ij} \rangle$, $\overbrace{\langle \Xi \rangle} = \max\langle \Xi_{ij} \rangle$ and $\underbrace{\langle \Xi \rangle} \le \langle \Xi_{ij} \rangle \le \overbrace{\langle \Xi \rangle}$.

Now, $\underbrace{\langle \Xi \rangle} = \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle \underbrace{\langle \Xi \rangle} \rangle^{\natural} \right\rangle^{\gamma_i}} \le \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle\langle \Xi_{ij} \rangle\rangle^{\natural} \right\rangle^{\gamma_i}}$
$\le \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle \overbrace{\langle \Xi \rangle} \rangle^{\natural} \right\rangle^{\gamma_i}} = \overbrace{\langle \Xi \rangle}$.

Since, $\underbrace{\langle \daleth \rangle} = \min\langle \daleth_{ij} \rangle$, $\overbrace{\langle \daleth \rangle} = \max\langle \daleth_{ij} \rangle$ and $\underbrace{\langle \daleth \rangle} \le \langle \daleth_{ij} \rangle \le \overbrace{\langle \daleth \rangle}$.

Now, $\underbrace{\langle \daleth \rangle} = \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle \underbrace{\langle \daleth \rangle} \rangle^{\natural} \right\rangle^{\gamma_i}} \le \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle \daleth_{ij} \rangle^{\natural} \right\rangle^{\gamma_i}} \le$
$\sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle \overbrace{\langle \daleth \rangle} \rangle^{\natural} \right\rangle^{\gamma_i}} = \overbrace{\langle \daleth \rangle}$.

Since, $\underbrace{\langle\langle\Lambda\rangle^\natural} = \min\langle\Lambda_{ij}\rangle^\natural$, $\overbrace{\langle\langle\Lambda\rangle^\natural} = \max\langle\Lambda_{ij}\rangle^\natural$ and $\underbrace{\langle\langle\Lambda\rangle^\natural} \leq \langle\Lambda_{ij}\rangle^\natural \leq \overbrace{\langle\langle\Lambda\rangle^\natural}$.

We have, $\underbrace{\langle\langle\Lambda\rangle^\natural} = \boxdot_{i=1}^{n}\langle\underbrace{\langle\langle\Lambda\rangle^\natural}\rangle^{\gamma_i} \leq \boxdot_{i=1}^{n}\langle\langle\Lambda_{ij}\rangle^\natural\rangle^{\gamma_i} \leq \boxdot_{i=1}^{n}\langle\overbrace{\langle\langle\Lambda\rangle^\natural}\rangle^{\gamma_i} = \overbrace{\langle\langle\Lambda\rangle^\natural}$.

Since, $\underbrace{\langle\langle\Omega\rangle^\natural} = \min\langle\Omega_{ij}\rangle^\natural$, $\overbrace{\langle\langle\Omega\rangle^\natural} = \max\langle\Omega_{ij}\rangle^\natural$ and $\underbrace{\langle\langle\Omega\rangle^\natural} \leq \langle\Omega_{ij}\rangle^\natural \leq \overbrace{\langle\langle\Omega\rangle^\natural}$.

We have, $\underbrace{\langle\langle\Omega\rangle^\natural} = \boxdot_{i=1}^{n}\langle\underbrace{\langle\langle\Omega\rangle^\natural}\rangle^{\gamma_i} \leq \boxdot_{i=1}^{n}\langle\langle\Omega_{ij}\rangle^\natural\rangle^{\gamma_i} \leq \boxdot_{i=1}^{n}\langle\overbrace{\langle\langle\Omega\rangle^\natural}\rangle^{\gamma_i} = \overbrace{\langle\langle\Omega\rangle^\natural}$.

Therefore,

$$\left[\begin{array}{c}\left\langle\sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle \imath - \langle\underbrace{\langle\Xi\rangle}\rangle^\natural\right\rangle^{\gamma_i}}\right\rangle^2 + \left\langle\sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle \imath - \langle\underbrace{\langle\daleth\rangle}\rangle^\natural\right\rangle^{\gamma_i}}\right\rangle^2 \\ + \imath - \left\langle\boxdot_{i=1}^{n}\langle\overbrace{\langle\Lambda\rangle}\rangle^\natural\rangle^{\gamma_i}\right\rangle^2 - \left\langle\boxdot_{i=1}^{n}\langle\overbrace{\langle\Omega\rangle}\rangle^\natural\rangle^{\gamma_i}\right\rangle^2\end{array}\right]$$

$$\leq \left[\begin{array}{c}\left\langle\sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle \imath - \langle\Xi_{ij}\rangle^\natural\right\rangle^{\gamma_i}}\right\rangle^2 + \left\langle\sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle \imath - \langle\daleth_{ij}\rangle^\natural\right\rangle^{\gamma_i}}\right\rangle^2 + \\ +1 - \left\langle\boxdot_{i=1}^{n}\langle\langle\Lambda_{ij}\rangle^\natural\rangle^{\gamma_i}\right\rangle^2 - \left\langle\boxdot_{i=1}^{n}\langle\langle\Omega_{ij}\rangle^\natural\rangle^{\gamma_i}\right\rangle^2\end{array}\right]$$

$$\leq \left[\begin{array}{c}\left\langle\sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle \imath - \langle\overbrace{\langle\Xi\rangle}\rangle^\natural\right\rangle^{\gamma_i}}\right\rangle^2 + \left\langle\sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle \imath - \langle\overbrace{\langle\daleth\rangle}\rangle^\natural\right\rangle^{\gamma_i}}\right\rangle^2 + \\ +1 - \left\langle\boxdot_{i=1}^{n}\langle\underbrace{\langle\Lambda\rangle}\rangle^\natural\rangle^{\gamma_i}\right\rangle^2 - \left\langle\boxdot_{i=1}^{n}\langle\underbrace{\langle\Omega\rangle}\rangle^\natural\rangle^{\gamma_i}\right\rangle^2\end{array}\right].$$

Hence, $\left[\underbrace{\langle\Xi\rangle}, \underbrace{\langle\daleth\rangle}, \overbrace{\langle\Lambda\rangle}, \overbrace{\langle\Omega\rangle}\right] \leq \langle\natural\rangle QNWA\langle\tau_1, \tau_2, ..., \tau_n\rangle \leq \left[\overbrace{\langle\Xi\rangle}, \overbrace{\langle\daleth\rangle}, \underbrace{\langle\Lambda\rangle}, \underbrace{\langle\Omega\rangle}\right]$.

Theorem 3.5. *Let $\tau_i = [\langle\Xi_{\alpha_{ij}}\rangle, \langle\daleth_{\alpha_{ij}}\rangle, \langle\Lambda_{\alpha_{ij}}\rangle, \langle\Omega_{\alpha_{ij}}\rangle]$ and $\varpi_i = [\langle\Xi_{\beta_{ij}}\rangle, \langle\daleth_{\beta_{ij}}\rangle, \langle\Lambda_{\beta_{ij}}\rangle, \langle\Omega_{\beta_{ij}}\rangle]$, be the $\natural$-rung RFFQNWAs. For any i, if there is $\langle\Xi_{\alpha_{ij}}\rangle^2 \leq \langle\Xi_{\beta_{ij}}\rangle^2$ and $\langle\daleth_{\alpha_{ij}}\rangle^2 \leq \langle\daleth_{\beta_{ij}}\rangle^2$ $\langle\Lambda_{\alpha_{ij}}\rangle^2 \geq \langle\Lambda_{\beta_{ij}}\rangle^2$ and $\langle\Omega_{\alpha_{ij}}\rangle^2 \geq \langle\Omega_{\beta_{ij}}\rangle^2$ or $\tau_i \leq \varpi_i$. Prove that $\langle\natural\rangle QNWA\langle\tau_1, \tau_2, ..., \tau_n\rangle \leq \langle\natural\rangle QNWA\langle\varpi_1, \varpi_2, ..., \varpi_n\rangle$, put $\langle i = 1, 2, ..., n\rangle, \langle j = 1, 2, ..., i_j\rangle$ (monotonicity property).*

Proof. For any i, $\langle\Xi_{\alpha_{ij}}\rangle^2 \leq \langle\Xi_{\beta_{ij}}\rangle^2$.
Therefore, $\imath - \langle\Xi_{\alpha_i}\rangle^2 \geq \imath - \langle\Xi_{\beta_i}\rangle^2$.
Hence, $\boxdot_{i=1}^{n}\left\langle\imath - \langle\Xi_{\alpha_i}\rangle\rangle^2\right\rangle^{\gamma_i} \geq \boxdot_{i=1}^{n}\left\langle\imath - \langle\Xi_{\beta_i}\rangle\rangle^2\right\rangle^{\gamma_i}$
and $\sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle\imath - \langle\Xi_{\alpha_i}\rangle\rangle^\natural\right\rangle^{\gamma_i}} \leq \sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle\imath - \langle\Xi_{\beta_i}\rangle\rangle^\natural\right\rangle^{\gamma_i}}$.
For any i, $\langle\daleth_{\alpha_{ij}}\rangle^\natural \leq \langle\daleth_{\beta_{ij}}\rangle\rangle^\natural$.
Therefore, $\imath - \langle\daleth_{\alpha_i}\rangle\rangle^\natural \geq \imath - \langle\daleth_{\beta_i}\rangle\rangle^\natural$.
Hence, $\boxdot_{i=1}^{n}\left\langle\imath - \langle\daleth_{\alpha_i}\rangle\rangle^\natural\right\rangle^{\gamma_i} \geq \boxdot_{i=1}^{n}\left\langle\imath - \langle\daleth_{\beta_i}\rangle\rangle^\natural\right\rangle^{\gamma_i}$.
This implies that $\sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle\imath - \langle\daleth_{\alpha_i}\rangle\rangle^\natural\right\rangle^{\gamma_i}}$
$\leq \sqrt[\natural]{\imath - \boxdot_{i=1}^{n}\left\langle\imath - \langle\daleth_{\beta_i}\rangle\rangle^\natural\right\rangle^{\gamma_i}}$.

For any i, $\langle\langle\Lambda_{\alpha_{ij}}\rangle\rangle^2 \geq \langle\langle\Lambda_{\beta_{ij}}\rangle\rangle^2$ and $\langle\langle\Lambda_{\alpha_{ij}}\rangle\rangle^{\natural} \geq \langle\langle\Lambda_{\beta_{ij}}\rangle\rangle^{\natural}$.
Therefore, $-\langle\boxdot_{i=1}^{n}\langle\Lambda_{\alpha_{ij}}\rangle\rangle^{\natural} \leq -\langle\boxdot_{i=1}^{n}\langle\Lambda_{\beta_{ij}}\rangle\rangle^{\natural}$.

For any i, $\langle\langle\Omega_{\alpha_{ij}}\rangle\rangle^2 \geq \langle\langle\Omega_{\beta_{ij}}\rangle\rangle^2$ and $\langle\langle\Omega_{\alpha_{ij}}\rangle\rangle^{\natural} \geq \langle\langle\Omega_{\beta_{ij}}\rangle\rangle^{\natural}$.
Therefore, $\text{ı} - \langle\boxdot_{i=1}^{n}\langle\Omega_{\alpha_{ij}}\rangle\rangle^{\natural} \leq \text{ı} - \langle\boxdot_{i=1}^{n}\langle\Omega_{\beta_{ij}}\rangle\rangle^{\natural}$.

$$\begin{bmatrix} \left\langle \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle\Xi_{ti}\rangle^{\natural}\right\rangle^{\gamma_i}} \right\rangle^2 + \left\langle \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle\daleth_{\alpha_i}\rangle^{\natural}\right\rangle^{\gamma_i}} \right\rangle^2 \\ +1 - \langle\boxdot_{i=1}^{n}\langle\Lambda_{\alpha_i}\rangle^{\natural}\rangle^2 - \langle\boxdot_{i=1}^{n}\langle\Omega_{\alpha_i}\rangle^{\natural}\rangle^2 \end{bmatrix}$$
$$\leq \begin{bmatrix} \left\langle \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle\Xi_{hi}\rangle^{\natural}\right\rangle^{\gamma_i}} \right\rangle^2 + \left\langle \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle\daleth_{\beta_i}\rangle^{\natural}\right\rangle^{\gamma_i}} \right\rangle^2 \\ +1 - \langle\boxdot_{i=1}^{n}\langle\Lambda_{\beta_i}\rangle^{\natural}\rangle^2 - \langle\boxdot_{i=1}^{n}\langle\Omega_{\beta_i}\rangle^{\natural}\rangle^2 \end{bmatrix}.$$

Hence, $\langle q\rangle QNWA\,\langle\tau_1, \tau_2, ..., \tau_n\rangle \leq \langle q\rangle QNWA\,\langle\varpi_1, \varpi_2, ..., \varpi_n\rangle$.

3.2. *♮-rung RFFQNWG*

Definition 3.6. Let $\tau_i = \left[\langle\langle\Xi_i\rangle, \langle\daleth_i\rangle, \langle\Lambda_i\rangle, \langle\Omega_i\rangle\rangle\right]$ be the ♮-rung RFFQNSNs.
Then ♮-rung RFFQNWG $\langle\tau_1, \tau_2, ..., \tau_n\rangle = \boxdot_{i=1}^{n}\tau_i^{\gamma_i}$.

Corollary 3.7. *Let* $\tau_i = \left[\langle\Xi_i\rangle, \langle\daleth_i\rangle, \langle\Lambda_i\rangle, \langle\Omega_i\rangle\rangle\right]$ *be the ♮-rung RFFQNSNs. Then ♮-rung RFFQNWG* $\langle\tau_1, \tau_2, ..., \tau_n\rangle$

$$= \begin{bmatrix} \boxdot_{i=1}^{n}\langle\langle\Xi_i\rangle^{\natural}\rangle^{\gamma_i}, \boxdot_{i=1}^{n}\langle\langle\daleth_i\rangle^{\natural}\rangle^{\gamma_i}, \\ \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle\Lambda_i\rangle^{\natural}\right\rangle^{\gamma_i}}, \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \langle\Omega_i\rangle^{\natural}\right\rangle^{\gamma_i}} \end{bmatrix}.$$

Corollary 3.8. *Let* $\tau_i = \left[\langle\Xi_i\rangle, \langle\daleth_i\rangle, \langle\Lambda_i\rangle, \langle\Omega_i\rangle\rangle\right]$ *be the ♮-rung RFFQNSNs and all are equal. Then ♮-rung RFFQNWG*$\langle\tau_1, \tau_2, ..., \tau_n\rangle = \tau$.

It has other properties, including boundedness and monotonicity, as well as having QNWG.

3.3. *Generalized ♮-rung RFFQNWA (G ♮-rung RFFQNWA)*

Definition 3.9. Let $\tau_i = \left[\langle\Xi_i\rangle, \langle\daleth_i\rangle, \langle\Lambda_i\rangle, \langle\Omega_i\rangle\rangle\right]$ be the ♮-rung RFFQNSN. Then G ♮-rung RFFQNWA $\langle\tau_1, \tau_2, ..., \tau_n\rangle = \left\langle \Cup_{i=1}^{n} \gamma_i\tau_i^{\rho}\right\rangle^{1/\rho}$.

Theorem 3.10. *Let* $\tau_i = \Big[\langle\Xi_i\rangle, \langle\daleth_i\rangle, \langle\Lambda_i\rangle, \langle\Omega_i\rangle\Big]$ *be the* $\natural$*-rung RFFQNSNs. Then G* $\natural$*-rung RFFQNWA* $\langle\tau_1, \tau_2, ..., \tau_n\rangle$

$$= \left[\begin{array}{c} \left\langle \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \left\langle\langle\Xi_i\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_i}}\right\rangle^{1/\natural}, \\ \left\langle \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \left\langle\langle\daleth_i\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_i}}\right\rangle^{1/\natural}, \\ \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle \boxdot_{i=1}^{n}\left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\Lambda_i\rangle^{\natural}\right\rangle^{\natural}}\right\rangle^{\gamma_i}\right\rangle^{\natural}\right\rangle^{1/\natural}} \\ \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle \boxdot_{i=1}^{n}\left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\Omega_i\rangle^{\natural}\right\rangle^{\natural}}\right\rangle^{\gamma_i}\right\rangle^{\natural}\right\rangle^{1/\natural}} \end{array}\right].$$

Proof. We can prove this first by demonstrating that,

$$\Cup_{i=1}^{n}\gamma_i\tau_i^{\rho} = \left[\begin{array}{c} \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \left\langle\langle\Xi_i\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_i}}, \sqrt[\natural]{\text{ı} - \boxdot_{i=1}^{n}\left\langle \text{ı} - \left\langle\langle\daleth_i\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_i}}, \\ \boxdot_{i=1}^{n}\left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\Lambda_i\rangle^{\natural}\right\rangle^{\natural}}\right\rangle^{\gamma_i}, \boxdot_{i=1}^{n}\left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\Omega_i\rangle^{\natural}\right\rangle^{\natural}}\right\rangle^{\gamma_i}, \end{array}\right].$$

Put $n = 2$, $\gamma_1\tau_1 \Cup \gamma_2\tau_2$

$$= \left[\begin{array}{c} \sqrt[\natural]{\begin{array}{c} \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle\langle\Xi_1\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_1}}\right\rangle^{\natural} + \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle\langle\Xi_2\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_1}}\right\rangle^{\natural}, \\ -\left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle\langle\Xi_1\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_1}}\right\rangle^{\natural} \cdot \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle\langle\Xi_2\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_1}}\right\rangle^{\natural} \end{array}} \\ , \\ \sqrt[\natural]{\begin{array}{c} \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle\langle\daleth_1\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_1}}\right\rangle^{\natural} + \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle\langle\daleth_2\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_1}}\right\rangle^{\natural}, \\ -\left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle\langle\daleth_1\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_1}}\right\rangle^{\natural} \cdot \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \left\langle\langle\daleth_2\rangle^{\natural}\right\rangle^{\natural}\right\rangle^{\gamma_1}}\right\rangle^{\natural} \end{array}} \\ \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\Lambda_1\rangle^{\natural}\right\rangle^{\natural}}\right\rangle^{\gamma_1} \cdot \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\Lambda_2\rangle^{\natural}\right\rangle^{\natural}}\right\rangle^{\gamma_1}, \\ \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\Omega_1\rangle^{\natural}\right\rangle^{\natural}}\right\rangle^{\gamma_1} \cdot \left\langle \sqrt[\natural]{\text{ı} - \left\langle \text{ı} - \langle\Omega_2\rangle^{\natural}\right\rangle^{\natural}}\right\rangle^{\gamma_1}, \end{array}\right]$$

$$= \left[\begin{array}{c} \sqrt[\natural]{\imath - \boxdot_{i=1}^{2}\left\langle \imath - \left\langle \langle \Xi_1 \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}}, \sqrt[\natural]{\imath - \boxdot_{i=1}^{2}\left\langle \imath - \left\langle \langle \daleth_1 \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}}, \\ \boxdot_{i=1}^{2}\left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Lambda_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i}, \boxdot_{i=1}^{2}\left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Omega_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \end{array}\right].$$

Hence,

$$\Cup_{i=1}^{l}\gamma_i\tau_i^{\rho} = \left[\begin{array}{c} \sqrt[\natural]{\imath - \boxdot_{i=1}^{l}\left\langle \imath - \left\langle \langle \Xi_1 \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}}, \sqrt[\natural]{\imath - \boxdot_{i=1}^{l}\left\langle \imath - \left\langle \langle \daleth_1 \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \\ \boxdot_{i=1}^{l}\left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Lambda_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \boxdot_{i=1}^{l}\left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Omega_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \end{array}\right].$$

If $n = l+1$, then $\Cup_{i=1}^{l}\gamma_i\tau_i^{\rho} + \gamma_{l+1}\tau_{l+1}^{\rho} = \Cup_{i=1}^{l+1}\gamma_i\tau_i^{\rho}$.

Now,$\Cup_{i=1}^{l}\gamma_i\tau_i^{\rho} + \gamma_{l+1}\tau_{l+1}^{\rho} = \gamma_1\tau_1^{\rho} \Cup \gamma_2\tau_2^{\rho} \Cup ... \Cup \gamma_l\tau_l^{\rho} \Cup \gamma_{l+1}\tau_{l+1}^{\rho}$

$$= \left[\begin{array}{c} \sqrt[\natural]{\begin{array}{c} \left\langle \sqrt[\natural]{\imath - \boxdot_{i=1}^{l}\left\langle \imath - \left\langle \langle \Xi_i \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \right\rangle^{\natural} + \left\langle \sqrt[\natural]{\imath - \left\langle \imath - \left\langle \Xi_{l+1} \right\rangle^{\natural} \right\rangle^{\natural\,\gamma_1}} \right\rangle^{\natural}, \\ - \left\langle \sqrt[\natural]{\imath - \boxdot_{i=1}^{l}\left\langle \imath - \left\langle \langle \Xi_i \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \right\rangle^{\natural} \cdot \left\langle \sqrt[\natural]{\imath - \left\langle \imath - \left\langle \Xi_{l+1} \right\rangle^{\natural} \right\rangle^{\natural\,\gamma_1}} \right\rangle^{\natural} \end{array}} \\ \sqrt[\natural]{\begin{array}{c} \left\langle \sqrt[\natural]{\imath - \boxdot_{i=1}^{l}\left\langle \imath - \left\langle \langle \daleth_i \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \right\rangle^{\natural} + \left\langle \sqrt[\natural]{\imath - \left\langle \imath - \left\langle \daleth_{l+1} \right\rangle^{\natural} \right\rangle^{\natural\,\gamma_1}} \right\rangle^{\natural}, \\ - \left\langle \sqrt[\natural]{\imath - \boxdot_{i=1}^{l}\left\langle \imath - \left\langle \langle \daleth_i \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \right\rangle^{\natural} \cdot \left\langle \sqrt[\natural]{\imath - \left\langle \imath - \left\langle \daleth_{l+1} \right\rangle^{\natural} \right\rangle^{\natural\,\gamma_1}} \right\rangle^{\natural} \end{array}} \\ \boxdot_{i=1}^{l}\left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Lambda_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \cdot \left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Lambda_{l+1} \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_1}, \\ \boxdot_{i=1}^{l}\left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Omega_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \cdot \left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Omega_{l+1} \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_1}, \end{array}\right]$$

$$\Cup_{i=1}^{l+1}\gamma_i\tau_i^{\natural} = \left[\begin{array}{c} \sqrt[\natural]{\imath - \boxdot_{i=1}^{l+1}\left\langle \imath - \left\langle \langle \Xi_1 \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}}, \sqrt[\natural]{\imath - \boxdot_{i=1}^{l+1}\left\langle \imath - \left\langle \langle \daleth_1 \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}}, \\ \boxdot_{i=1}^{l+1}\left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Lambda_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \boxdot_{i=1}^{l+1}\left\langle \sqrt[\natural]{\imath - \left\langle \imath - \langle \Omega_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \end{array}\right].$$

$$\left\langle \mathbb{U}_{i=1}^{l+1} \gamma_i \tau_i^{\rho} \right\rangle^{1/\rho}$$

$$= \left[\begin{array}{c} \left\langle \sqrt[\rho]{\shortmid - \boxdot_{i=1}^{l+1} \left\langle \shortmid - \left\langle \langle \Xi_i \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \right\rangle^{1/\natural} \left\langle \sqrt[\natural]{\shortmid - \boxdot_{i=1}^{l+1} \left\langle \shortmid - \left\langle \langle \daleth_i \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \right\rangle^{1/\natural} \\ \sqrt[\natural]{\shortmid - \left\langle \shortmid - \left\langle \boxdot_{i=1}^{l+1} \left\langle \sqrt[\natural]{\shortmid - \left\langle \shortmid - \langle \Lambda_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \right\rangle^{\natural} \right\rangle^{1/\natural}}, \\ \sqrt[\natural]{\shortmid - \left\langle \shortmid - \left\langle \boxdot_{i=1}^{l+1} \left\langle \sqrt[\natural]{\shortmid - \left\langle \shortmid - \langle \Omega_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \right\rangle^{\natural} \right\rangle^{1/\natural}}, \end{array} \right]$$

Theorem 3.11. *If all* $\tau_i = \left[\langle \Xi_i \rangle, \langle \daleth_i \rangle, \langle \Lambda_i \rangle, \langle \Omega_i \rangle \right]$ *and all are equal.*
Then G ♮-rung RFFQNWA $\langle \tau_1, \tau_2, ..., \tau_n \rangle = \tau$.

3.4. *Generalized ♮-rung RFFQNWG (G ♮-rung RFFQNWG)*

Definition 3.12. Let $\tau_i = \left[\langle \Xi_i \rangle, \langle \daleth_i \rangle, \langle \Lambda_i \rangle, \langle \Omega_i \rangle \right]$ be the ♮-rung RFFQNSNs.
Then G ♮-rung RFFQNWG $\langle \tau_1, \tau_2, ..., \tau_n \rangle = \frac{1}{\rho} \left\langle \boxdot_{i=1}^{n} \langle \rho \tau_i \rangle^{\gamma_i} \right\rangle$.

Corollary 3.13. *Let* $\tau_i = \left[\langle \Xi_i \rangle, \langle \daleth_i \rangle, \langle \Lambda_i \rangle, \langle \Omega_i \rangle \right]$ *be the ♮-rung RFFQNSNs.*
Then G ♮-rung RFFQNWG$\langle \tau_1, \tau_2, ..., \tau_n \rangle$

$$= \left[\begin{array}{c} \sqrt[\natural]{\shortmid - \left\langle \shortmid - \left\langle \boxdot_{i=1}^{n} \left\langle \sqrt[\natural]{\shortmid - \left\langle \shortmid - \langle \Xi_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \right\rangle^{\natural} \right\rangle^{1/\natural}}, \\ \sqrt[\natural]{\shortmid - \left\langle \shortmid - \left\langle \boxdot_{i=1}^{n} \left\langle \sqrt[\natural]{\shortmid - \left\langle \shortmid - \langle \daleth_i \rangle^{\natural} \right\rangle^{\natural}} \right\rangle^{\gamma_i} \right\rangle^{\natural} \right\rangle^{1/\natural}}, \\ \left\langle \sqrt[\natural]{\shortmid - \boxdot_{i=1}^{n} \left\langle \shortmid - \left\langle \langle \Lambda_i \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \right\rangle^{1/\natural}, \\ \left\langle \sqrt[\natural]{\shortmid - \boxdot_{i=1}^{n} \left\langle \shortmid - \left\langle \langle \Omega_i \rangle^{\natural} \right\rangle^{\natural} \right\rangle^{\gamma_i}} \right\rangle^{1/\natural} \end{array} \right].$$

Corollary 3.14. *If all* $\tau_i = \left[\langle \langle \Xi_i \rangle, \langle \daleth_i \rangle, \langle \Lambda_i \rangle, \langle \Omega_i \rangle \rangle \right]$ *are equal.*
Then G ♮-rung RFFQNWG$\langle \tau_1, \tau_2, ..., \tau_n \rangle = \tau$.

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