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Introduction and Application of the Neutrosophic Ideal in Gamma-Near-Ring

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Abstract. In this paper, we present the notion of neutrosophic ideals in gamma-near-rings. Subsequently, an examination of basic properties related to these ideals is conducted. We also provide characterizations of neutrosophic ideals. Furthermore, we establish a neutrosophic ideal by utilizing a collection of ideals. Finally, we introduce an application.

Keywords: Neutrosophic ideal, near-ring, gamma-near-ring.

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1. **Introduction**

The notion of Γ-near-rings was initially introduced by Satyanarayana [8], and the investigation of ideal theory in Γ-near-rings was further explored by Satyanarayana [8] and Booth [2]. Jun et al. [6] delved into the fuzzification of ideals in Γ-near-rings. The concept of intuitionistic fuzzy set proposed by Atanassov [1], served as a broader extension of fuzzy sets [15]. Also, Cho et al. [3] employed Atanassov's idea to establish the intuitionistic fuzzification of ideals in Γ-near-rings and scrutinize their properties. establish the intuitionistic fuzzification of ideals in Γ-near-rings and scrutinize their properties.

Smarandache [10] declared the neutrosophy as an emerging field of philosophy. Neutrosophy forms the foundation of neutrosophic logic, a novel extension of fuzzy logic that encompasses the concept of indeterminacy. The operation utilized within the framework of neutrosophic sets involves approximations, that differ from precise outcomes. These operations manage partial truths or memberships, in contrast to classical fuzzy sets. The selection of an appropriate tool within fuzzy logic relies on the specific context and the user's expertise, as various tools can yield varying degrees of precision. Therefore, making the optimal choice necessitates experience. Subsequently, researchers examined the core algebraic operations of neutrosophic sets from three distinct perspectives (e.g., [10,14,16]). Consequently, various topics on algebraic structures have been studied, such as neutrosophic rings [4], and neutrosophic modules [5, 9].

Recently, the notion of neutrosophic fuzzy sub-near-ring was proposed by Solairaju et al. [13]. They also dealt with the study of neutrosophic fuzzy ideals of near-rings and examined specific algebraic properties associated with them.

The use of a neutrosophic set (single value neutrosophic set, for short SVNS) streamlines the handling of erroneous, unforeseen, susceptible, flawed, vulnerable, and complex information. This arises from the reality that such forms of data are highly responsive to inaccuracies. This susceptibility is attributed to the particular nature of this information, which is more predisposed to encountering inaccuracies. The notions of fuzzy sets and intuitionistic fuzzy sets have both progressed significantly due to the emergence of this innovative framework. Within SVNS, uncertainty is quantified in a manner that is transparent and unequivocal, with truth membership, uncertainty membership, and falsity membership being entirely dissociated. In the field of algebraic structures, specific binary operations can be envisioned as interacting with Γ-near-ring, intricate and pervasive structures. Γ-near-rings have diverse applications such as quantum computing, graph theory, cryptography and design of experiments, serving as indispensable tools.

The motivation of this paper is that neither the neutrosophic ideals nor their properties and applications were ever studied in Γ-near rings. This new concept is important since it has many algebraic implications.

This article's primary goal is to present and comprehend the concept of neutrosophic ideals in Γ-near- rings. Moreover, the basic properties of these ideals are carried out. We also provide characterizations of neutrosophic ideals.

Following is the outline of this article: Section 2 covers the ideas and findings that we will use in the next section. Section 3 is offered to introduce neutrosophic left (resp. right) ideal in a Γ-near-ring $\mathcal{E}$ and a few idealistic results. In Section 4, we present an application in voting game. In the last section, we give some conclusion.

Table 1 indicates specific symbols that will be used everywhere in this article.

Table 1. Description and symbols used in this article.

Symbol	Description	Symbol	Description
NS	Neutrosophic set	NSs	Neutrosophic sets
SVNS	Single-valued neutrosophic set	NR	Near-ring
Γ-NR	Γ-Near-ring	SNR	Sub-near-ring

2. Preliminaries

The concepts and results we utilize in the following part are covered in this section.

Definition 2.1. [7] Let E be a non-empty set, then $(E,+,.)$ is called a right (resp., left) NR if the following are satisfied:
1. $(E,+)$ is a group.
2. $(E,.)$ is a semi-group.
3. For all $u,v,w \in N$, $(u+v).w = u.w + v.w$. (resp., left $w.(u+v) = w.u + w.v$).

Definition 2.2. [8] Let Γ be a non-empty set of binary operations, then $(\mathcal{E},+,\Gamma)$ is called a Γ-NR if the following are satisfied:
1. $(\mathcal{E},+)$ is a group.
2. For any $o \in \Gamma$ we get $(\mathcal{E},+,o)$ is an NR.
3. $u \bullet (v * w) = (u \bullet v) * w \ \forall u,v,w \in \mathcal{E}$ and $\bullet, * \in \Gamma$.

Definition 2.3. [8] $(I,+,\Gamma)$ is said to be a right (resp. left) ideal of $\mathcal{E}$ if
(i) $(I,+)$ is a normal sub-group of $(\mathcal{E},+)$.
(ii) $u*(v+w) - v*u \in I$ (resp. $u*w \in I$) $\forall v \in I, * \in \Gamma$ and $u,w \in \mathcal{E}$,
where I is a subset of Γ-NR $\mathcal{E}$.

Definition 2.4. [10,14] On a universe set R, we define an NS $\mathcal{N}$ as follows:

$$\mathcal{N} = \{< i, \varphi(i), \chi(i), \psi(i) >: i \in R\},$$

with $\varphi,\chi,\psi : R \to]0,1[$. Moreover, an NS $\mathcal{N}$ on R is said to be an SVNS if $\varphi,\chi,\psi : R \to [0,1]$.

Definition 2.5. [11, 12] Let $\mathcal{N}_1 = \{< s, \varphi_1(s), \chi_1(s), \psi_1(s) >: s \in R\}$ and $\mathcal{N}_2 = \{< s, \varphi_2(s), \chi_2(s), \psi_2(s) >: s \in R\}$ be two NSs on R. Then:
1. $\mathcal{N}_1 \subset_1 \mathcal{N}_2 = \{< s, \varphi_1(s) \leq \varphi_2(s), \chi_1(s) \geq \chi_2(s), \psi_1(s) \geq \psi_2(s) >: s \in R\}$,
2. $\mathcal{N}_1 \cup_1 \mathcal{N}_2 = \{< s, \varphi_1(s) \vee \varphi_2(s), \chi_1(s) \wedge \chi_2(s), \psi_1(s) \wedge \psi_2(s) >: s \in R\}$,
3. $\mathcal{N}_1 \cap_1 \mathcal{N}_2 = \{< s, \varphi_1(s) \wedge \varphi_2(s), \chi_1(s) \vee \chi_2(s), \psi_1(s) \vee \psi_2(s) >: s \in R\}$.

Definition 2.6. [13] A neutrosophic SNR $\mathcal{S} = \{< s, \varphi(s), \chi(s), \psi(s) >: s \in \mathrm{N}\}$ of E is an NS in NR E with the following conditions:
1. $\varphi(x-y) \geq \min(\varphi(x),\varphi(y)), \chi(x-y) \leq \max(\chi(x),\chi(y)), \psi(x-y) \leq \max(\psi(x),\psi(y))$.
2. $\varphi(xy) \geq \min(\varphi(x),\varphi(y)), \chi(xy) \leq \max(\chi(x),\chi(y)), \psi(xy) \leq \max(\psi(x),\psi(y))$.

Definition 2.7. [13] A neutrosophic ideal I of E is an NS in an NR E with the following conditions:
(i) $\varphi(x-y) \geq \min(\varphi(x),\varphi(y)), \chi(x-y) \leq \max(\chi(x),\chi(y)), \psi(x-y) \leq \max(\psi(x),\psi(y))$.
(ii) $\varphi(y+x-y) \geq \varphi(x), \chi(y+x-y) \leq \chi(x), \psi(y+x-y) \leq \psi(x)$.

(iii) $\varphi(xy) \geq \varphi(y), \chi(xy) \leq \chi(y), \psi(xy) \leq \psi(y)$.
(iv) $\varphi((x+z)y - xy) \geq \varphi(z), \chi((x+z)y - xy) \leq \chi(z), \psi((x+z)y - xy) \leq \psi(z)$.
A neutrosophic ideal I is called neutrosophic left ideal when the conditions (i)-(iii) is hold, while is called neutrosophic right ideal when the conditions (i), (ii) and (iv) are satisfied.

Definition 2.8. [4] Assume that $\mathcal{I}$ is a neutrosophic ideal of a ring $\mathcal{R}$ with $0 \leq \alpha \leq \varphi(0)$, and $0 \leq \chi(0), \psi(0) \leq \alpha$. The ideal $\mathcal{I}_\alpha$ is said to be a level ideal of $\mathcal{R}$.

In what follow we refer to the set of all level ideals by $\mathcal{M}_\alpha$.

3. **Results**

In this section, we present and investigate a neutrosophic left (resp. right) ideal in a Γ-NR $\mathcal{E}$.

Definition 3.1. An NS $\mathcal{S}$ in a Γ-NR $\mathcal{E}$ is said to be a neutrosophic left (resp. right) ideal if the following are hold:
(i) $\varphi(x-y) \geq \min\left(\varphi(x), \varphi(y)\right), \chi(x-y) \leq \max\left(\chi(x), \chi(y)\right), \psi(x-y) \leq \max\left(\psi(x), \psi(y)\right)$.
(ii) $\varphi(y+x-y) \geq \varphi(x), \chi(y+x-y) \leq \chi(x), \psi(y+x-y) \leq \psi(x)$.
(iii) $\varphi(y * (x+z) - y * z) \geq \varphi(x), \chi(y * (x+z) - y * z) \leq \chi(x), \psi(y * (x+z) - y * z) \leq \psi(x)$
(resp. $\varphi(x * y) \geq \varphi(x),\ \chi(x * y) \leq \chi(x),\ \psi(x * y) \leq \psi(x), \forall x, y, z \in N$ and $* \in \Gamma$.

Example 3.2. Take the classical group $Z_3 = \{0, 1, 2\}$ under a binary operation $\oplus_3$ and $\Gamma = \{*, \circledast\}$ defined as follows: $a * b = a$ and $a \circledast b = b$ for all $a, b \in Z_3$. Clearly, $(Z_3, \oplus_3, \Gamma)$ is Γ-NR. Now, we define a neutrosophic subset $\mathcal{S} = \{< i, \varphi(i), \chi(i), \psi(i) >: s \in Z_3\}$ defined as follows:

$$\varphi(i) = \begin{cases} 0.7 & \text{if } i = 0, \\ 0.5 & \text{otherwise} \end{cases},$$

$$\chi(i) = \begin{cases} 0.4 & \text{if } i = 0, \\ 0.7 & \text{otherwise}, \end{cases}$$

$$\psi(i) = \begin{cases} 0.6 & \text{if } i = 0, \\ 0.8 & \text{otherwise}. \end{cases}$$

Thus, $\mathcal{S}$ is a neutrosophic left (resp. right) ideal in $(Z_3, \oplus_3, \Gamma)$.

Remark 3.3. In a neutrosophic left (resp. right) ideal of a Γ-NR $\mathcal{E}$ the following are true

$$\varphi(0) \geq \varphi(x), \chi(0) \leq \chi(x), \text{ and } \psi(0) \leq \psi(x), \forall\ 0, x \in \mathcal{E}.$$

Theorem 3.4. *Presume $\mathcal{S}$ is a neutrosophic left (resp. right) ideal of a $\mathit{\Gamma}$-NR $\mathcal{E}$, then the set*

$$\mathcal{N}_{\mathcal{S}} = \{< s, \varphi(s), \chi(s), \psi(s) >: s \in \mathcal{E}, \varphi(s) = \varphi(0), \chi(s) = \chi(0), \psi(s) = \psi(0)\},$$

is a left (resp. right) ideal of $\mathcal{E}$.

Proof. Since $\mathcal{S}$ is a neutrosophic left (resp. right) ideal of a $\mathit{\Gamma}$-NR $\mathcal{E}$, for any $x, y \in \mathcal{N}_{\mathcal{S}}$

$$\varphi(x - y) \geq \min(\varphi(x), \varphi(y)) = \varphi(0),$$

$$\chi(x - y) \leq \max(\chi(x), \chi(y)) = \chi(0),$$

$$\psi(x - y) \leq \max(\psi(x), \psi(y)) = \chi(0),$$

therefore $x - y \in \mathcal{N}_{\mathcal{S}}$. Again, for any $y \in \mathcal{E}$ and $x \in \mathcal{N}_{\mathcal{S}}$ we get

$$\varphi(y + x - y) \geq \varphi(x) = \varphi(0),$$

$$\chi(y + x - y) \leq \chi(x) = \chi(0),$$

$$\psi(y + x - y) \leq \psi(x) = \chi(0),$$

therefore $y + x - y \in \mathcal{N}_{\mathcal{S}}$. Finally, suppose that $x \in \mathcal{N}_{\mathcal{S}}$, $y, z \in \mathcal{E}$ and $* \in \mathit{\Gamma}$, we have

$$\varphi(y * (x + z) - y * z) \geq \varphi(x) = \varphi(0),$$

$$\chi(y * (x + z) - y * z) \leq \chi(x) = \chi(0),$$

$$\psi(y * (x + z) - y * z) \leq \psi(x) = \chi(0),$$

therefore $y * (x + z) - y * z \in \mathcal{N}_{\mathcal{S}}$. Thus, $\mathcal{N}_{\mathcal{S}}$ is a left ideal of $\mathcal{E}$. The right case is similarly.

Theorem 3.5. *Presume S is a non-empty subset of Γ-NR $\mathcal{E}$ and $\mathcal{S} = \{< s, \varphi(s), \chi(s), \psi(s) >: s \in S\}$ is defined by*

$$\varphi(s) := \begin{cases} \sigma & \text{if } s \in S, \\ \beta, & \text{otherwise}, \end{cases}$$

$$\chi(s) := \begin{cases} \sigma & \text{if } s \in S, \\ \beta, & \text{otherwise}, \end{cases}$$

$$\psi(s) := \begin{cases} \sigma & \text{if } s \in S, \\ \beta, & \text{otherwise}, \end{cases}$$

for all $s \in \mathcal{E}$ and $\sigma, \beta \in [0, 1]$ with $\sigma > \beta$. Then $\mathcal{S}$ is a neutrosophic left (resp. right) ideal of if and only if $\mathcal{S}$ is a left (resp. right) ideal of $\mathcal{E}$. Moreover, $\mathcal{N}_{\mathcal{S}} = \mathcal{S}$.

Proof. Suppose that $\mathcal{S}$ is a neutrosophic left (resp. right) ideal of $\mathcal{E}$ and $x, y \in \mathcal{S}$, then

$$\varphi(x-y) \geq \min\left(\varphi(x), \varphi(y)\right) = \sigma,$$

$$\chi(x-y) \leq \max\left(\chi(x), \chi(y)\right) = \sigma,$$

$$\psi(x-y) \leq \max\left(\psi(x), \psi(y)\right) = \sigma,$$

and this leads to $x - y \in \mathcal{S}$. Again, for any $y \in \mathcal{E}$ and $x \in \mathcal{S}$ we get

$$\varphi(y+x-y) \geq \varphi(x) = \sigma,$$

$$\chi(y+x-y) \leq \chi(x) = \sigma,$$

$$\psi(y+x-y) \leq \psi(x) = \sigma,$$

therefore $y + x - y \in \mathcal{S}$. Finally, suppose that $x \in \mathcal{S}, * \in \Gamma$ and $y, z \in \mathcal{E}$.

$$\varphi(y*(x+z) - y*z) \geq \varphi(x) = \sigma,$$

$$\chi(y*(x+z) - y*z) \leq \chi(x) = \sigma,$$

$$\psi(y*(x+z) - y*z) \leq \psi(x) = \sigma,$$

therefore $y*(x+z) - y*z \in \mathcal{S}$ (resp. $x*y \in \mathcal{S}$). Thus, $\mathcal{S}$ is a left (resp. right) ideal of $\mathcal{E}$. In opposition, suppose that $\mathcal{S}$ is a neutrosophic left (resp. right) ideal of $\mathcal{E}$ and $x, y \in \mathcal{E}$, know we can have some cases:

Case 1: when at least one of x or y is not in $\mathcal{S}$, then we have

$$\varphi(x-y) \geq \beta = \min\left(\varphi(x), \varphi(y)\right),$$

$$\chi(x-y) \leq \sigma = \max\left(\chi(x), \chi(y)\right),$$

$$\psi(x-y) \leq \sigma = \max\left(\psi(x), \psi(y)\right).$$

Case 2: if $x, y \in \mathcal{S}$, then we get $x - y \in \mathcal{S}$ and

$$\varphi(x-y) = \sigma = \min\left(\varphi(x), \varphi(y)\right),$$

$$\chi(x-y) = \sigma = \max\left(\chi(x), \chi(y)\right),$$

$$\psi(x-y) = \sigma = \max\left(\psi(x), \psi(y)\right).$$

Again, for any $y \in \mathcal{E}$ and $x \in \mathcal{S}$,

$$\varphi(y+x-y) \geq \varphi(x) = \sigma,$$

$$\chi(y+x-y) \leq \chi(x) = \sigma,$$

$$\psi(y+x-y) \leq \psi(x) = \sigma,$$

therefore $y + x - y \in \mathcal{S}$. Also, we have

$$\varphi(y+x-y) \geq \varphi(x) = \beta,$$

$$\chi(y+x-y) \le \chi(x) = \beta,$$

$$\psi(y+x-y) \le \psi(x) = \beta,$$

when $x \notin \mathcal{S}$. Finally, suppose that $x \in \mathcal{S}$, $y, z \in \mathcal{E}$ and $* \in \mathit{\Gamma}$, then

$$\varphi(y*(x+z)-y*z) \ge \varphi(x) = \sigma,$$

$$\chi(y*(x+z)-y*z) \le \chi(x) = \sigma,$$

$$\psi(y*(x+z)-y*z) \le \psi(x) = \sigma,$$

and when $x \notin \mathcal{S}$,

$$\varphi(y*(x+z)-y*z) \ge \varphi(x) = \beta,$$

$$\chi(y*(x+z)-y*z) \le \chi(x) = \beta,$$

$$\psi(y*(x+z)-y*z) \le \psi(x) = \beta,$$

Thus, $\mathcal{S}$ is a left ideal of $\mathcal{E}$. Similarly, S is a right ideal of $\mathcal{E}$. Now

$$\begin{aligned}\mathcal{N}_{\mathcal{S}} &= \{x \in \mathcal{E} : \varphi(x) = \varphi(0), \chi(x) = \chi(0), \psi(x) = \psi(0)\}\\ &= \{x \in \mathcal{E} : \varphi(x) = \sigma, \chi(x) = \sigma, \psi(x) = \sigma\}\\ &= \{x \in \mathcal{E} : x \in \mathcal{S}\}\\ &= \mathcal{S}.\end{aligned}$$

Before, we show the next result, we define the set

$$\mathcal{J}(\mathcal{S}) = \{\alpha_i : \varphi(s) = \chi(s) = \psi(s) = \alpha_i \text{ for some } s \in \mathcal{E}\}.$$

Theorem 3.6. *Assume that $\mathcal{S}$ is an NS in a Γ-NR $\mathcal{E}$, then $\mathcal{S}$ is a neutrosophic left (resp. right) ideal of $\mathcal{E}$ if and only if every level subset $\mathcal{S}_\alpha$, $\alpha \in \mathcal{J}(\mathcal{S})$ is a left (resp. right) ideal of $\mathcal{E}$.*

Proof. Suppose that $\mathcal{S}$ is a neutrosophic left (resp. right) ideal of $\mathcal{E}$ and $\alpha \in \mathcal{J}(\mathcal{S})$, then, for any $x, y \in \mathcal{S}_\alpha$, we verify the axioms in Definition 3.1 to establish that $\mathcal{S}_\alpha$ is a left (resp. right) ideal of $\mathcal{E}$.

(i) $\varphi(x-y) \ge \min(\varphi(x), \varphi(y)) \ge \alpha$,
$\chi(x-y) \le \max(\chi(x), \chi(y)) \le \alpha$,
$\psi(x-y) \le \max(\psi(x), \psi(y)) \le \alpha$,
thus $x - y \in \mathcal{S}_\alpha$. For any $y \in \mathcal{E}$ and $x \in \mathcal{S}_\alpha$ we get

(ii) $\varphi(y+x-y) \ge \varphi(x) \ge \alpha$,
$\chi(y+x-y) \le \chi(x) \le \alpha$,
$\psi(y+x-y) \le \psi(x) \le \alpha$,
therefore $y + x - y \in \mathcal{S}_\alpha$. Now, suppose that $x \in \mathcal{S}_\alpha$, $y, z \in \mathcal{E}$ and $* \in \mathit{\Gamma}$, then

(iii) $\varphi(y*(x+z)-y*z) \geq \varphi(x) \geq \alpha$,
$\chi(y*(x+z)-y*z) \leq \chi(x) \leq \alpha$,
$\psi(y*(x+z)-y*z) \leq \psi(x) \leq \alpha$,
therefore $y*(x+z)-y*z \in \mathcal{S}_\alpha$. Thus, $\mathcal{S}_\alpha$ is a left ideal of $\mathcal{E}$. The right case is similarly.

The other direction, we prove it by contradiction. Assume that $\mathcal{S}_\alpha$ is a left ideal of $\mathcal{E}$ (resp. right), then, for any $\alpha \in \mathcal{J}(\mathcal{S})$, we have the following axioms

(i) Assume that

$$\varphi(s_0-t_0) < \min\{\varphi(s_0), \varphi(t_0)\}$$

$$\chi(s_0-t_0) > \max\{\chi(s_0), \chi(t_0)\}$$

$$\psi(s_0-t_0) > \max\{\psi(s_0), \psi(t_0)\}$$

for some $s_0-t_0 \in \mathcal{E}$, then by putting

$$\begin{aligned}\alpha_0 &= \frac{1}{2}(\varphi(s_0-t_0)+\min\{\varphi(s_0), \varphi(t_0)\}) \\ &= \frac{1}{2}(\chi(s_0-t_0)+\max\{\chi(s_0), \chi(t_0)\}) \\ &= \frac{1}{2}(\psi(s_0-t_0)+\max\{\psi(s_0), \psi(t_0)\})\end{aligned}$$

we get

$$\varphi(s_0-t_0) < \alpha_0,\ \varphi(s_0) \quad > \alpha_0,\ \varphi(t_0) > \alpha_0$$

$$\chi(s_0-t_0) > \alpha_0,\ \chi(s_0) \quad < \alpha_0,\ \chi(t_0) < \alpha_0$$

$$\psi(s_0-t_0) > \alpha_0,\ \psi(s_0) \quad < \alpha_0,\ \psi(t_0) < \alpha_0$$

which gives a contradiction. Therefore $s_0-t_0 \notin \mathcal{S}_{\alpha_0}$ and then $\varphi(s-t) \geq \min\{\varphi(s), \varphi(t)\}, \chi(s-t) \leq \max\{\chi(s), \chi(t)\},$ and $\psi(s-t) \leq \max\{\psi(s), \psi(t)\}$.

(ii) Assume that

$$\varphi(s_0+t_0-s_0) < \varphi(s_0)$$

$$\chi(s_0+t_0-s_0) > \chi(s_0)$$

$$\psi(s_0+t_0-s_0) > \psi(s_0)$$

for some $s_0-t_0 \in \mathcal{E}$, then by putting

$$\begin{aligned}\alpha_0 &= \frac{1}{2}(\varphi(s_0+t_0-s_0)+\varphi(s_0)) \\ &= \frac{1}{2}(\chi(s_0+t_0-s_0)+\chi(s_0)) \\ &= \frac{1}{2}(\psi(s_0+t_0-s_0)+\psi(s_0))\end{aligned}$$

so

$$\varphi(s_0+t_0-s_0) < \alpha_0 \quad < \varphi(s_0)$$

$$\chi(s_0+t_0-s_0) > \alpha_0 \quad > \chi(s_0)$$

$$\psi(s_0+t_0-s_0) > \alpha_0 \quad > \psi(s_0)$$

and this leads to $s_0+t_0-s_0 \notin \mathcal{S}_{\alpha_0}$, so $\varphi(s+t-s) \geq \varphi(s), \chi(s+t-s) \leq \chi(s),$ and $\psi(s+t-s) \leq \psi(s)$.

(iii) Suppose that the axiom (iii) in Definition 3.1 is not hold, for any $* \in \Gamma$ there exist $x, y, z \in \mathcal{E}$ with

$$\varphi(y*(x+z)-y*z) < \varphi(x),$$

$$\chi(y*(x+z)-y*z) > \chi(x),$$

$$\psi(y*(x+z)-y*z) > \psi(x).$$

Now, assume that

$$\begin{aligned}\alpha_0 &= \frac{1}{2}(\varphi(y*(x+z)-y*z)+\varphi(x))\\ &= \frac{1}{2}(\chi(y*(x+z)-y*z)+\chi(x))\\ &= \frac{1}{2}(\psi(y*(x+z)-y*z)+\psi(x)),\end{aligned}$$

which leads to $(y*(x+z)-y*z \in \mathcal{S}_{\alpha_0}$. Therefore, this leads to a contradiction, supporting our claim.

Theorem 3.7. *Presume I is a left ideal (resp. right) of a Γ-NR $\mathcal{E}$, then we get a neutrosophic a left ideal (resp. right) $\mathcal{I}$ of $\mathcal{E}$ with $\mathcal{I}_\alpha = I$, for any $0 < \alpha \leq 1$.*

Proof. Suppose that $\mathcal{I}_\alpha = \{< i, \varphi(i), \chi(i), \psi(i) >: i \in \mathcal{E}\}$ is an NS defined as follows

$$\varphi(i) := \begin{cases}\alpha & \text{if } i \in I,\\ 0, & \text{otherwise,}\end{cases}$$

$$\chi(i) := \begin{cases}\alpha & \text{if } i \in I,\\ 0, & \text{otherwise,}\end{cases}$$

$$\psi(i) := \begin{cases}\alpha & \text{if } i \in I,\\ 0, & \text{otherwise,}\end{cases}$$

obviously, we find $\mathcal{I}_\alpha = I$. Now, we examine all the axioms of Definition 3.1 as follows:

(i) From the definition, it follows that $\mathcal{I}_\alpha = I$.

(ii) Since $|Image(\varphi)| = |Image(\chi)| = |Image(\psi)| = 2$, if either $\varphi(y+x-y), \chi(y+x-y) = 0$ or $\psi(y+x-y) = 0$, and $\varphi(x) = \chi(x) = \psi(x) = \alpha$, then we have $y+x-y \notin I$ and $x \in I$ which leads to a contradiction. Thus, the item (ii) of Definition 3.1 is hold.

(*iii*) Again, since $|Image(\varphi)| = |Image(\chi)| = |Image(\psi)| = 2$. If either $\varphi(y * (x + z) - y * z), \chi(y * (x + z) - y * z) = 0$ or $\psi(y * (x + z) - y * z) = 0$ and $\varphi(x) = \chi(x) = \psi(x) = \alpha$ (resp., if any one $\varphi(x*y), \chi(x*y)$ and $\psi(x*y)$ equal zero and $\varphi(x) = \chi(x) = \psi(x) = \alpha$). Thus, $y * (x + z) - y * z \in I$ and $x \in I$ (resp., $x * x \in I$ and $x \in I$). Thus, the theory is established.

Theorem 3.8. *Presume $\mathcal{I}$ is a neutrosophic left ideal of (resp. right) of a Γ-NR $\mathcal{E}$, and $0 \leqslant \alpha \leqslant 1$, then we have*

$$\varphi(i) = \sup\{i : i \in \varphi_\alpha, \ \forall i \in \mathcal{E}\},$$

$$\chi(i) = \inf\{i : i \in \chi_\alpha, \ \forall i \in \mathcal{E}\},$$

$$\psi(i) = \inf\{i : i \in \varphi_\alpha, \ \forall i \in \mathcal{E}\}.$$

Proof. Suppose that

$$u_1 = \sup\{i : i \in \varphi_\alpha, \ \forall i \in \mathcal{E}\},$$

$$u_2 = \inf\{i : i \in \chi_\alpha, \ \forall i \in \mathcal{E}\},$$

$$u_3 = \inf\{i : i \in \varphi_\alpha, \ \forall i \in \mathcal{E}\},$$

and $v_1, \ v_2, \ v_3 > 0$ are given, then we obtain

$$u_1 - v_1 < \alpha,$$

$$u_2 - v_2 > \alpha,$$

$$u_3 - v_3 > \alpha,$$

such that $i \in \varphi_\alpha, \ i \in \chi_\alpha, \ i \in \varphi_\alpha$ and thus

$$u_1 - v_1 < \varphi(i),$$

$$u_2 - v_2 > \chi(i),$$

$$u_3 - v_3 > \psi(i).$$

Assuming that $v_1, \ v_2, \ v_3$ are arbitrary, it follows that $u_1 \leqslant \varphi(i), \ u_2 \leqslant \chi(i)$, and $u_3 \leqslant \psi(i)$. The other hand, suppose that $\varphi(i) = w_1, \ \chi(i) = w_2$ and $\psi(i) = w_3$, then we have $i \in \varphi_{w_1}, \ i \in \chi_{w_2}$ and $i \in \psi_{w_3}$, so

$$w_1 \in \{i : i \in \varphi_\alpha, \ \forall i \in \mathcal{E}\},$$

$$w_2 \in \{i : i \in \chi_\alpha, \ \forall i \in \mathcal{E}\},$$

$$w_3 \in \{i : i \in \varphi_\alpha, \ \forall i \in \mathcal{E}\}.$$

Thus, $w_1 \leqslant u_1$, $w_2 \leqslant u_2$ and $w_3 \leqslant u_3$. Therefore, $w_1 = u_1$, $w_2 = u_2$ and $w_3 = u_3$.

Let us now examine the converse of Theorem 3.8. Suppose we have a non-empty subset Z of $[0, 1]$. It is possible, without sacrificing the general nature of the argument, to employ Z as an indexing set in the subsequent context. Consider that $\{\mathcal{I}_\alpha : \alpha \in Z\}$ is a set of neutrosophic ideals of $\mathcal{I}$ with

(i) $\mathcal{I} = \cup_{\alpha\in Z}\mathcal{I}_\alpha$,

(ii) For any $\alpha_1, \alpha_2 \in Z$, $\alpha_1 > \alpha_2$ iff $\mathcal{I}_{\alpha_1} \subset \mathcal{I}_{\alpha_2}$.

Theorem 3.9. *Let* $\mathcal{A} = \{< i, \varphi(i), \chi(i), \psi(i) >: i \in \mathcal{E}\}$ *be an NS defined as*

$$\varphi(i) = \sup\{\alpha \in Z : i \in \mathcal{I}_\alpha\},$$

$$\chi(i) = \inf\{\alpha \in Z : i \in \mathcal{I}_\alpha\},$$

$$\psi(i) = \inf\{\alpha \in Z : i \in \mathcal{I}_\alpha\}.$$

Then $\mathcal{A}$ *is a neutrosophic left (resp. right) ideal of* $\mathcal{E}$.

Proof. According Theorem 3.6, to prove that $\mathcal{A}_\beta = \{< i, \varphi_{\beta_1}(i), \chi_{\beta_2}(i), \psi_{\beta_3}(i) >: i \in \mathcal{E}\}$ is a neutrosophic left (resp. right) ideal for any $\beta_1, \beta_2, \beta_3 \in [0, 1]$, we have two cases:
Case (1) $\beta_1 = \sup\{\alpha \in Z : \alpha < \beta_1\}$, $\beta_2 = \inf\{\alpha \in Z : \alpha < \beta_2\}$ and $\beta_3 = \inf\{\alpha \in Z : \alpha < \beta_3\}$. This implies that:

(i) $i \in \varphi_{\beta_1} \Leftrightarrow i \in \varphi_\alpha \forall\alpha < \beta_1 \Leftrightarrow i \in \wedge_{\alpha<\beta_1}\varphi_\alpha$, thus $\varphi_{\beta_1} = \wedge_{\alpha<\beta_1}\varphi_\alpha$.

(ii) $i \in \chi_{\beta_2} \Leftrightarrow i \in \chi_\alpha \forall\alpha < \beta_2 \Leftrightarrow i \in \vee_{\alpha<\beta_2}\chi_\alpha$, thus $\chi_{\beta_2} = \vee_{\alpha<\beta_2}\chi_\alpha$.

(iii) $i \in \psi_{\beta_3} \Leftrightarrow i \in \psi_\alpha \forall\alpha < \beta_3 \Leftrightarrow i \in \vee_{\alpha<\beta_3}\psi_\alpha$, thus $\psi_{\beta_3} = \vee_{\alpha<\beta_3}\psi_\alpha$.

Therefore, the intersection is a neutrosophic left (resp. right) ideal of $\mathcal{E}$.
Case (2) $\beta_1 \neq \sup\{\alpha \in Z : \alpha < \beta_1\}$, $\beta_2 \neq \inf\{\alpha \in Z : \alpha < \beta_2\}$ and $\beta_3 \neq \inf\{\alpha \in Z : \alpha < \beta_3\}$. Now, we explain that $\varphi_{\beta_1} = \vee_{\alpha\geqslant\beta_1}\varphi_\alpha$, $\chi_{\beta_2} = \wedge_{\alpha\geqslant\beta_2}\chi_\alpha$ and $\psi_{\beta_3} = \wedge_{\alpha\geqslant\beta_3}\psi_\alpha$. Suppose that i in $\vee_{\alpha\geqslant\beta_1}\varphi_\alpha$, $\wedge_{\alpha\geqslant\beta_2}\chi_\alpha$ and $\wedge_{\alpha\geqslant\beta_3}\psi_\alpha$. Then we get i in φ_α, χ_α and ψ_α for every $\alpha \geqslant \beta_1, \beta_2, \beta_3$. Obviously, $\varphi(i) \geqslant \alpha \geqslant \beta_1$, $\alpha \geqslant \beta_2 \geqslant \chi(i)$, and $\alpha \geqslant \beta_3 \geqslant \psi(i)$. Thus, i in φ_{β_1}, χ_{β_2} and ψ_{β_3}. On the other hand, assume that i is does not belong to any of $\vee_{\alpha\geqslant\beta_1}\varphi_\alpha$, $\wedge_{\alpha\geqslant\beta_2}\chi_\alpha$ and $\wedge_{\alpha\geqslant\beta_3}\psi_\alpha$. Then i does not belong to any φ_α, χ_α and ψ_α for every $\alpha \geqslant \beta_1, \beta_2, \beta_3$. So there exists $\tau > 0$ with $(\beta_1 - \tau, \beta_1) \cap Z = \phi$, $(\beta_2, \beta_2 + \tau) \cap Z = \phi$, and $(\beta_3, \beta_3 + \tau) \cap Z = \phi$. Since it does not belong to any of φ_α, χ_α and ψ_α for any $\alpha > \beta_1 - \tau$, $\alpha < \beta_2 + \tau$ and $\alpha < \beta_3 + \tau$, thus $\varphi(i) \leqslant \beta_1 - \tau$, $\chi(i) \geqslant \beta_2 + \tau$, and $\psi(i) \geqslant \beta_3 + \tau$. So, i does not belong to any of φ_α, χ_α and ψ_α. Therefore, the union is a neutrosophic left (resp. right) ideal of $\mathcal{E}$. The proof is now complete.

4. **Application**

In this section, we gave an application of neutrosophic gamma-near-rings in a voting, in which neutrosophic gamma-near-rings can be applied to account for imprecise preferences among voters. Hier, we assume that A, B, and C are three voters voting on a proposal. They can vote in favor or uncertainty or against the proposal. The result of voting is given in Table 2.

Table 2. Result of voting.

voters	fovar	uncertainty	against
A	0.7	0.4	0.6
B	0.5	0.7	0.8
C	0.6	0.7	0.9

Now, we create a mathematical simulation of the voting process. The set of voter $\{A, B, C\}$ represent by $Z_3 = \{0, 1, 2\}$ and we use neutrosophic gamma-near-ring with operations:

The first operation $*$ define as follows $a * b = \min(a, b) \forall a, b \in Z_a$ and represents a degree of support for the proposal. The second operation $\circledast$ define as follows $a \circledast b = \max(a, b) \forall a, b \in Z_2$ and represents a degree of disagreement with the proposal. let us calculate the voting outcome using neutrosophic gamma-near-ring with operations:

Table 3. The voting outcome.

Voting Outcome	favor	uncertainty	Against
support far the proposal	A	A	A
disagree for the proposal	B and C	B and C	B and C

Based on the neutrosophic gamma-near-ring operations, the proposal receives a lower degree of support and a higher degree of disagreement see Table 3. This can be interpreted as a ”not consensus” in favor of the proposal, considering the imprecision in voters’ preferences. This example illustrates how neutrosophic gamma-near-ring can be used to model and analyze voting games when voters have imprecise preferences. The operations help capture the degrees of support and disagreement, allowing for a more nuanced understanding of the voting outcome in situations where preferences are not clear-cut

5. **Conclusions**

In this article, the idea of neutrosophic ideals has been examined in the context of Γ-NR. Through a thorough analysis of these ideals, we have uncovered essential properties that govern their behavior. The characterizations of the neutrosophic ideals enabled a deeper understanding of their structural nature. In particular, we demonstrated the construction of a neutrosophic ideal using a combination of ideals.

The neutrosophic ideals in Γ-NR open new avenues for research and application in various mathematical and real-world scenarios. The insights gained from this study pave the way for further investigation of the intricate interplay between neutrosophic ideals and the underlying algebraic structures. In future work, we examine the applicability of the notion of neutrosophic ideal in many areas such as coding theory, graph theory, and quantum computing, and seek connections to other branches of mathematics. Furthermore, the results presented findings contribute to our understanding of neutrosophic ideals in Γ-near-rings, shedding light on their significance and potential implications. This study serves as a foundation for future studies in algebraic structures, including their applications.

Data Availability Statement

All data generated or analysed during this study are included in this article.

Declarations

Conflict of interest I am not in any conflict of interest.

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